CA4141

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Aula 14 (parte 1) - 12nov21

Série de Fourier de função par/ímpar

ATENÇÃO Assinar presença no App ou no Portal do aluno

Tabela Integrais trigonométricas e Série de Fourier

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx)dx, \; n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\sin nx)dx, \; n \geq 1$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (\; a_n \cos nx + b_n \sin nx \;)$$

A

$$\int_{-\pi}^{\pi} \cos(nx)dx = 0, \; n \geq 1$$

$$\int_{-\pi}^{\pi} \sin(nx)dx = 0, \; n \geq 1$$

B

$$\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0, \; n \geq 1$$

$$\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = (-1)^n \frac{4\pi}{n^2}, \; n \geq 1$$

C

$$\int_{-\pi}^{\pi} x\sin(nx)\,dx = (-1)^{n+1} \frac{2\pi}{n}, \; n \geq 1$$

$$\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0, \; n \geq 1$$

D

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx = 0, \; n, m \geq 1$$

E

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0, \; n \neq m, \; (n, m \geq 1)$$

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \pi, \; n \geq 1$$

F

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0, \; n \neq m, \; (n, m \geq 1)$$

$$\int_{-\pi}^{\pi} \sin^2(nx)dx = \pi, \; n \geq 1$$

Exercício Determinar a série de Fourier da função $f(x)$.

a) $f(x) = x, \quad -\pi \leq x \leq \pi$

Note que $f(x) = x$ é uma função **impar** logo a sua série de Fourier é uma série de **senos**. Assim, $a_n = 0, \ \forall n \geq 0.$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\sin nx)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \ \sin nx \, dx \quad \text{C}$$

$$= \frac{1}{\pi}(-1)^{n+1}\frac{2\pi}{n}$$

$$= (-1)^{n+1}\frac{2}{n}, \ n \geq 1$$

$$\boxed{b_n = (-1)^{n+1}\frac{2}{n}, \ n \geq 1}$$

$$\frac{\cancel{a_0}}{2} + \sum_{n=1}^{\infty}\left(\cancel{a_n} \cos nx + b_n \sin nx \right) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + b_5 \sin 5x + \cdots$$

$$2 \sin x + \frac{-2}{2}\sin 2x + \frac{2}{3}\sin 3x + \frac{-2}{4}\sin 4x + \frac{2}{5}\sin 5x - \cdots$$

Série de Fourier de $f(x) = x$

$$2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots + (-1)^{n-1}\frac{\sin(nx)}{n} + \cdots\right]$$

$$2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(nx)}{n}$$

b) $f(x) = x^2, \ -\pi \leq x \leq \pi$

Note que $f(x) = x^2$ é uma função **par** logo a sua série de Fourier é uma série de **cossenos**. Assim, $b_n = 0, \ \forall n \geq 1$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx \quad \textbf{B}$$

$$= \frac{1}{\pi}(-1)^n \frac{4\pi}{n^2}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = (-1)^n \frac{4}{n^2}, \ n \geq 1$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + \cancel{b_n} \sin nx \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + a_5 \cos 5x + \cdots$$

$$\frac{1}{2}\frac{2\pi^2}{3} + (-4)\cos x + \frac{4}{4}\cos 2x + \frac{-4}{9}\cos 3x + \frac{4}{16}\cos 4x + \frac{-4}{25}\cos 5x + \cdots$$

$$+ \frac{4(-1)^n}{n^2}\cos(nx) + \cdots$$

Série de Fourier de $f(x) = x^2$

$$\frac{\pi^2}{3} + 4\left[-\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \frac{\cos 4x}{16} - \frac{\cos 5x}{25} + \cdots + \cdots \right]$$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}$$

Exemplo Vamos **supor** que a série de Fourier da função $f(x) = x^2$ obtida no exercício anterior seja convergente para $f(x)$ para $-\pi \leq x \leq \pi$. Assim,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2}, \quad -\pi \leq x \leq \pi$$

$$x^2 = \frac{\pi^2}{3} + 4\left[\, -\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \frac{\cos 4x}{16} - \frac{\cos 5x}{25} + \cdots \,\right]$$

Calculando a igualdade acima para $x = \pi$:

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\, -\cos \pi + \frac{\cos 2\pi}{4} - \frac{\cos 3\pi}{9} + \frac{\cos 4\pi}{16} - \frac{\cos 5\pi}{25} + \cdots \,\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\, 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \,\right]$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Longrightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Conclusão $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

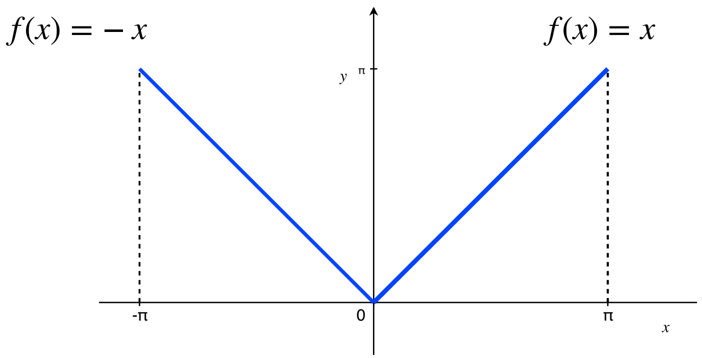
Exercício Vamos **supor** que a série de Fourier da função $f(x) = x^2$ obtida no exercício anterior seja convergente para $f(x)$ para $-\pi \leq x \leq \pi$. Assim,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}, \quad -\pi < x \leq \pi$$

Determinar a série de Fourier da função $\frac{x^3}{3} - \frac{\pi^2 x}{3}$.

Resposta $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4\sin nx}{n^3}$

Exercício Determinar os coeficientes de Fourier da função $f(x) = |x|$ com $-\pi \le x \le \pi$.



Note que $f(x) = |x|$ é uma função **par:**

$$f(-x) = |-x| = |x| = f(x)$$

logo a sua série de Fourier é uma série de **cossenos**.

$$\boxed{\therefore b_n = 0, \ \forall n \ge 1}$$

Para calcular os coeficientes a_n, vamos usar que

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_{0}^{\pi} xdx = \frac{1}{\pi}(x^2)\,|_0^{\pi} = \pi \qquad \boxed{\therefore a_0 = \pi}$$

Exercício (continuação) Determinar os coeficientes de Fourier da função $f(x) = |x|$ com $-\pi \leq x \leq \pi$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|(\cos nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x(\cos nx)dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right] - \left[0 + \frac{\cos(0)}{n^2}\right]\right\}$$

$$\int u\,dv = uv - \int v\,du \quad \text{(LIATE)}$$

$$\int \underbrace{x}_{u}\ \underbrace{\cos nx\,dx}_{dv} = \underbrace{x}_{u}\underbrace{\frac{\sin(nx)}{n}}_{v} - \int \underbrace{\frac{\sin(nx)}{n}}_{v}\underbrace{dx}_{du}$$

$$\int x\ \cos nx\,dx = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + c$$

$$a_n = \frac{2}{\pi}\ \left[\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\ \right]$$

$$\boxed{a_n = \frac{2}{\pi n^2}\ \left[\ \cos(n\pi) - 1\ \right]}$$

Exercício Determinar a série de Fourier da função $f(x) = |x|$ com $-\pi \leq x \leq \pi$.

$$b_n = 0,\ \forall n \geq 1$$

$$a_0 = \pi$$

Analisando o termo $a_n = \frac{2}{\pi n^2}\left[\cos(n\pi) - 1\right]$:

n par $\Longrightarrow \cos(n\pi) = 1$ e $a_n = 0$:

n ímpar $\Longrightarrow \cos(n\pi) = -1$ e $a_n = -\frac{4}{\pi n^2}$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + \cancel{b_n} \sin nx \right)$$

$$\frac{\pi}{2} + \sum_{n \text{ impar}}^{\infty} -\frac{4}{\pi n^2} \cos nx$$

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{1}{n^2} \cos nx$$

Série de Fourier de $f(x) = |x|$

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

$$\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + \cdots\right]$$