ĆW 1. ZAD DOM 1.

$$\begin{cases} T_{OT} = T_0 + A e^{-\beta t} \\ \frac{dT}{dt} = -\delta (T - T_{OT}) \end{cases}$$

$$\boxed{\frac{dT}{dt} = -\delta T + \delta T_0 + \delta A e^{-\beta t}}$$

RORJ: $\frac{dT}{dt} = -\delta T \rightarrow T_{RORJ} = C e^{-\delta t}$

RSRN: postulujemy $T_{RSRN} = D e^{-\beta t} + E$

$$\dot{T}_{RSRN} = -\beta D e^{-\beta t}$$

wtedy: $-\beta D e^{-\beta t} = -\delta D e^{-\beta t} - \delta E + \delta T_0 + \delta A e^{-\beta t}$

$$e^{-\beta t}(-\beta D + \delta D - \delta A) + \delta (E - T_0) = 0$$

$$\begin{cases} -\beta D + \delta D - \delta A = 0 \\ E - T_0 = 0 \end{cases} \rightarrow \begin{cases} D = \frac{\delta A}{\delta - \beta} \\ E = T_0 \end{cases}$$

$$T_{RSRN} = \frac{\delta A}{\delta - \beta} e^{-\beta t} + T_0$$

RORN = RORJ + RSRN $\rightarrow T(t) = C e^{-\delta t} + \frac{\delta A}{\delta - \beta} e^{-\beta t} + T_0$

warunek początkowy: $T(0) = T_0$

$$\cancel{T_0} = C + \frac{\delta A}{\delta - \beta} + \cancel{T_0}$$

$$C = -\frac{\delta A}{\delta - \beta}$$

ostatecznie: $\boxed{T(t) = T_0 + \frac{\delta A}{\delta - \beta}\left(e^{-\beta t} - e^{-\delta t}\right)}$

• $\alpha \gg \beta$: $\frac{\beta}{\alpha} \approx 0$ $\quad T = T_0 + Ae^{-\alpha t} \frac{1}{1-\frac{\beta}{\alpha}} \left[e^{\alpha t(1-\frac{\beta}{\alpha})} - 1 \right]$

$T \approx T_0 + Ae^{-\alpha t}\left[e^{\alpha t} - 1\right] = T_0 + A - Ae^{-\alpha t} = T_0 + A - Ae^{-\alpha t} + T_0 e^{-\alpha t} - T_0 e^{-\alpha t}$

$T \approx \left[T_0 - T_0 - A\right] e^{-\alpha T} + T_0 + A$ to jest rozwiązanie takie jak dla temperatury otoczenia równej $T_0 + A$ co jest sensowne, bo dla małych β, $T_{OT} = T_0 + Ae^{-\beta t} \approx T_0 + A$

• $\beta \gg \alpha$: $\frac{\beta}{\alpha} \gg 1$ $\quad T = T_0 + Ae^{-\alpha t} \frac{1}{1-\frac{\beta}{\alpha}} \left[e^{\alpha t(1-\frac{\beta}{\alpha})} - 1 \right] \approx$

$T \approx T_0 - \frac{A\alpha}{\beta} e^{-\alpha t}\left[e^{-\beta t} - 1\right] \approx T_0$ to jest sensowne, bo dla dużego β temperatura otoczenia szybko spada do T_0 więc temperatura ciała się nie zmienia

• $\beta \approx \alpha$

$T(t) = T_0 + \frac{\alpha A}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha t}\right) = T_0 + \frac{\alpha A}{\alpha-\beta} e^{-\alpha t}\left(e^{(\alpha-\beta)t} - 1\right) \approx T_0 + \frac{\alpha A}{\alpha - \beta} e^{-\alpha t}\left[1 + (\alpha-\beta)t - 1\right]$

$= T_0 + \frac{\alpha A}{\alpha-\beta} e^{-\alpha t}(\alpha-\beta)\cdot t = T_0 + \alpha \cdot A \cdot t\, e^{-\alpha t}$

Temp maksymalna dla $T' = 0$

$T' = \frac{\alpha A}{\alpha-\beta}\left[-\beta e^{-\beta t} + \alpha e^{-\alpha t}\right] = 0$

$\beta e^{-\beta t} = \alpha e^{-\alpha t}$

$e^{-\beta t} = \frac{\alpha}{\beta} e^{-\alpha t}$

$e^{-\beta t} = e^{-\alpha t} e^{\ln\frac{\alpha}{\beta}}$

$e^{-\beta t} = e^{\ln\frac{\alpha}{\beta} - \alpha t}$

$-\beta t = \ln\frac{\alpha}{\beta} - \alpha t$

$(\alpha - \beta)t = \ln\frac{\alpha}{\beta}$

$t = (\alpha-\beta)\ln\left(\frac{\alpha}{\beta}\right)$

ĆW 1. ZAD DOM 2

(1) temp. początkowa $= T_0$

temp. otoczenia $= T_{OT} = T_0 - \Delta T$

stosujemy rozwiązanie równania Newtona dla stałej temp. otoczenia

$$T(t) = (T_0 - T_{OT})e^{-\alpha t} + T_{OT} = (\cancel{T_0} - \cancel{T_0} + \Delta T)e^{-\alpha t} + T_0 - \Delta T =$$

$$= T_0 - \Delta T(1 - e^{-\alpha t})$$

$$T_1 = T(t_1) = T_0 - \Delta T(1 - e^{-\alpha t_1})$$

(2) temperatura początkowa $= T_1$

temperatura otoczenia $= T_{OT} = T_0 + \Delta T$

ponownie stosujemy rozwiązanie równania Newtona dla stałej temp. otoczenia

$$T(t) = (T_1 - T_{OT})e^{-\alpha t} + T_{OT} = [\cancel{T_0} - \Delta T(1 - e^{-\alpha t_1}) - \cancel{T_0} - \Delta T]e^{-\alpha t} + T_0 + \Delta T =$$

$$= \Delta T[e^{-\alpha t_1} - 2]e^{-\alpha t} + T_0 + \Delta T$$

oczekujemy $T(t_2) = T_0$

$$\Delta T[e^{-\alpha t_1} - 2]e^{-\alpha t_2} + \cancel{T_0} + \Delta T = \cancel{T_0} \quad // : \Delta T$$

$$[e^{-\alpha t_1} - 2]e^{-\alpha t_2} + 1 = 0 \quad / \cdot e^{\alpha t_2}$$

$$e^{\alpha t_2} = [2 - e^{-\alpha t_1}]$$

$$\alpha t_2 = \ln[2 - e^{-\alpha t_1}]$$

$$\boxed{t_2 = \frac{1}{\alpha}\ln[2 - e^{-\alpha t_1}]}$$

• $t_1 \to 0$

$$t_2 = \frac{1}{\alpha} \ln\left(2 - e^{-\alpha t_1}\right) \simeq \frac{1}{\alpha} \ln(2-1) = \frac{1}{\alpha} \ln(1) = 0$$

• $t_1 \to \infty$

$$t_2 = \frac{1}{\alpha} \ln\left(2 - e^{-\alpha t_1}\right) \simeq \frac{1}{\alpha} \ln(2)$$

Który czas krótszy?

$$\frac{t_2}{t_1} = \frac{1}{t_1} \cdot \frac{1}{\alpha} \ln\left[2 - e^{-\alpha t_1}\right] = \frac{\ln\left[2 - e^{-\alpha t_1}\right]}{\ln\left[e^{\alpha t_1}\right]}$$

$\ln(x)$ jest monotoniczna więc jeśli $t_2 > t_1$ to $\frac{2 - e^{-\alpha t_1}}{e^{\alpha t_1}} > 1$

wtedy $\frac{2 - e^{-\alpha t_1}}{e^{\alpha t_1}} > 1 \quad / \cdot e^{2\alpha t_1}$

$$2e^{\alpha t_1} - 1 > e^{2\alpha t_1} \quad \Big\} \; x = e^{2\alpha t_1}$$

$$2x - 1 > x^2$$

$0 > x^2 - 2x + 1$ czyli $(x-1)^2 < 0$

co jest oczywistą bzdurą, więc widzimy, że $t_2 > t_1$ jest nieprawdą. To oznacza, że

$$\boxed{\begin{array}{c} t_1 \geqslant t_2 \\ (\text{równość dla } t_1 = t_2 = 0) \end{array}}$$

ĆW 1. ZAD DOM 3

a) $T_0 = 273\,K$ $T_{OT} = 293\,K$ $T_1 = 287\,K$ $t_1 = 3600\,s$

Rozwiązanie równania Newtona dla stałej temp. otoczenia

$$T_1 = (T_0 - T_{OT})\, e^{-\alpha t} + T_{OT}$$

$$\downarrow$$

$$\frac{T_1 - T_{OT}}{T_0 - T_{OT}} = e^{-\alpha t} \quad / \ln()$$

$$\ln\left[\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right] = -\alpha t$$

$$\alpha = -\frac{1}{t} \ln\left[\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right]$$

$$\tau = \frac{1}{\alpha} = \frac{-t}{\ln\left[\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right]} = \frac{-3600\,s}{\ln\left[\frac{287-293}{273-293}\right]} = \frac{-3600\,s}{\ln\left[\frac{6}{20}\right]} \simeq 2990\,s \approx 50\,min$$

$$\boxed{\text{czas relaksacji } \tau \approx 50\,min}$$

b) $T_0 = 273\,K$ $T_{OT} = 293\,K$ $T_1 = 277\,K$ $\tau = 50\,min$

$$T_1 = (T_0 - T_{OT})\, e^{-\alpha t} + T_{OT}$$

$$\left(\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right) = e^{-\alpha t} \quad / \ln()$$

$$\ln\left(\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right) = -\frac{t}{\tau}$$

$$t = -\tau \ln\left(\frac{T_1 - T_{OT}}{T_0 - T_{OT}}\right) = \tau \ln\left(\frac{T_0 - T_{OT}}{T_1 - T_{OT}}\right) = 50\,min \cdot \ln\left(\frac{273-293}{277-293}\right) =$$

$$= 50\,min \ln\left(\frac{20}{16}\right) \approx 11\,min$$

$$\boxed{\text{optymalny czas } t \approx 11\,min}$$

ĆW 1. ZAD DOM 4

$$E_p(x) = D\left(e^{-\alpha(x-x_0)} - 1\right)^2$$

$$E_p(x) \approx E_p(x_0) + \frac{x-x_0}{1!}E_p'(x_0) + \frac{(x-x_0)^2}{2!}E_p''(x_0) + \frac{(x-x_0)^3}{3!}E_p'''(x_0)$$

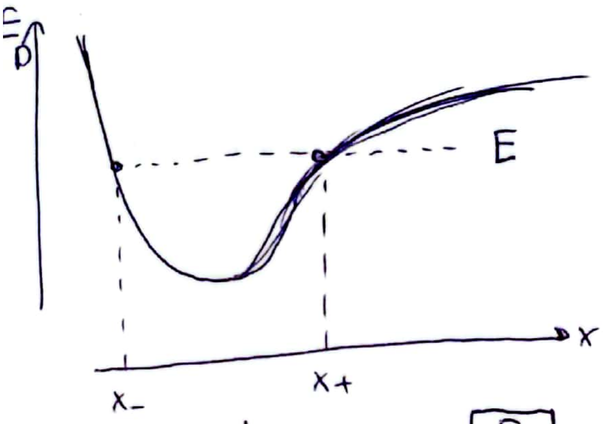
$$E_p'(x) = 2D\left(e^{-\alpha(x-x_0)} - 1\right)(-\alpha)e^{-\alpha(x-x_0)}$$
$$= -2\alpha D\, e^{-\alpha(x-x_0)}\left(e^{-\alpha(x-x_0)} - 1\right)$$

$$E_p''(x) = -2\alpha D\left[(-\alpha)e^{-\alpha(x-x_0)}\left(e^{-\alpha(x-x_0)} - 1\right) + e^{-\alpha(x-x_0)}(-\alpha)e^{-\alpha(x-x_0)}\right] =$$
$$= 2\alpha^2 D\left[e^{-\alpha(x-x_0)}\left(e^{-\alpha(x-x_0)} - 1\right) + e^{-\alpha(x-x_0)}e^{-\alpha(x-x_0)}\right] =$$
$$= 2\alpha^2 D\, e^{-\alpha(x-x_0)}\left[e^{-\alpha(x-x_0)} - 1 + e^{-\alpha(x-x_0)}\right] =$$
$$= 2\alpha^2 D e^{-\alpha(x-x_0)}\left[2e^{-\alpha(x-x_0)} - 1\right]$$

$$E_p'''(x) = 2\alpha^2 D\left\{(-\alpha)e^{-\alpha(x-x_0)}\cdot\left[2e^{-\alpha(x-x_0)} - 1\right] + e^{-\alpha(x-x_0)}\cdot 2(-\alpha)e^{-\alpha(x-x_0)}\right\}$$
$$= -2\alpha^3 D\left\{e^{-\alpha(x-x_0)}\left[2e^{-\alpha(x-x_0)} - 1\right] + 2e^{-\alpha(x-x_0)}e^{-\alpha(x-x_0)}\right\} =$$
$$= -2\alpha^3 D e^{-\alpha(x-x_0)}\left\{2e^{-\alpha(x-x_0)} - 1 + 2e^{-\alpha(x-x_0)}\right\} =$$
$$= -2\alpha^3 D e^{-\alpha(x-x_0)}\left[4e^{-\alpha(x-x_0)} - 1\right]$$

$$\left.\begin{aligned} E_p'(x_0) &= 0 \\ E_p''(x_0) &= 2\alpha^2 D \\ E_p'''(x_0) &= -6\alpha^3 D \end{aligned}\right\} \rightarrow E_p(x) \approx \alpha^2 D(x-x_0)^2 - \alpha^3 D(x-x_0)^3$$

$$E_p(x) \approx \alpha^2 D (x-x_0)^2 - \alpha^3 D (x-x_0)^3$$



(1) Na początku rozpatrzmy potencjał harmoniczny $E_p(x) \approx \alpha^2 D (x-x_0)^2$

$$E = \alpha^2 D (x-x_0)^2$$

$$\downarrow$$

$$\frac{E}{D\cdot\alpha^2} = (x-x_0)^2 \to |x-x_0| = \sqrt{\frac{E}{\alpha^2 D}}$$

czyli $x_\pm^h = x_0 \pm \sqrt{\frac{E}{\alpha^2 D}}$

(2) Teraz rozpatrujemy potencjał anharmoniczny z założeniem

$x_\pm = x_\pm^h + \varepsilon_\pm$ gdzie $\varepsilon_\pm$ to niewielkie poprawki

$$E = \alpha^2 D (x_\pm - x_0)^2 - \alpha^3 D (x_\pm - x_0)^3 =$$

$$= \alpha^2 D \left(\cancel{x_0} \pm \sqrt{\frac{E}{\alpha^2 D}} + \varepsilon_\pm - \cancel{x_0}\right)^2 - \alpha^3 D \left(\cancel{x_0} \pm \sqrt{\frac{E}{\alpha^2 D}} + \varepsilon_\pm - \cancel{x_0}\right)^3 =$$

$$= \alpha^2 D \left(\frac{E}{\alpha^2 D} \pm 2\varepsilon_\pm \sqrt{\frac{E}{\alpha^2 D}} + \cancel{\varepsilon_\pm^2}\right) - \alpha^3 D \left[\pm\left(\frac{E}{\alpha^2 D}\right)^{3/2} + 3\frac{E}{\alpha^2 D}\varepsilon_\pm \cancel{\pm \sqrt{\frac{E}{\alpha^2 D}}\varepsilon_\pm^2 \cdot 3 + \varepsilon_\pm^3}\right]$$

$$= \alpha^2 D \left(\frac{E}{\alpha^2 D} \pm 2\varepsilon_\pm \sqrt{\frac{E}{\alpha^2 D}}\right) - \alpha^3 D \left[\pm\left(\frac{E}{\alpha^2 D}\right)^{3/2} + 3\frac{E}{\alpha^2 D}\varepsilon_\pm\right] =$$

$$= E \pm 2\varepsilon_\pm \sqrt{\alpha^2 D E} \mp \alpha^3 D \left(\frac{E}{\alpha^2 D}\right)^{3/2} - 3\alpha E \varepsilon_\pm$$

$$\cancel{E} = \cancel{E} \pm 2\varepsilon_\pm \sqrt{\alpha^2 D E} \mp \frac{\alpha^3 D E^{3/2}}{\alpha^3 D^{3/2}} - 3\alpha E \varepsilon_\pm$$

$$\left(3\alpha E \mp 2\sqrt{\alpha^2 D E}\right)\varepsilon_\pm = \mp \left(\frac{E^3}{D}\right)^{1/2}$$

$$\varepsilon_\pm = \frac{\mp E^{3/2}/D^{1/2}}{3\alpha E \mp 2\sqrt{\alpha^2 D E}} = \frac{\mp E^{3/2}}{3\alpha E D^{1/2} \mp 2\alpha D E^{1/2}} = \frac{\mp E^{3/2}}{\alpha D E^{1/2}\left[3\left(\frac{E}{D}\right)^{1/2} \mp 2\right]}$$

$\left(\frac{E}{D}\right)^{1/2}$ jest małe $\to \varepsilon_\pm \frac{\mp E^{3/2}}{\mp 2\alpha D E^{1/2}} = \frac{E}{2\alpha D} \to \boxed{x_\pm = x_0 \pm \sqrt{\frac{E}{\alpha^2 D}} + \frac{E}{2\alpha D}}$

$\downarrow$

To bierze się z tego, że poprawka anharmoniczna jest mała tzn.

$$\alpha^2 D |x-x_0|^2 \gg \alpha^3 D |x-x_0|^3 \to \alpha|x-x_0| \ll 1 \to \alpha|x_\pm^h - x_0| \ll 1 \to \alpha\left|\sqrt{\frac{E}{\alpha^2 D}}\right| \ll 1$$

$$\left(\frac{E}{D}\right)^{1/2} \ll 1$$

ĆW 1. ZAD DOM 5

$$\begin{cases} L_s = L_{0s}(1+\alpha_s \Delta T) \\ L_m = L_{0m}(1+\alpha_m \Delta T) \\ L_s = L_m + d \quad \text{dla każdego } \Delta T \end{cases}$$

$$L_s - L_m - d = 0$$

$$L_{0s} + L_{0s}\alpha_s \Delta T - L_{0m} - L_{0m}\alpha_m \Delta T - d = 0$$

$$L_{0s} - L_{0m} - d = (-L_{0s}\alpha_s + L_{0m}\alpha_m)\Delta T \quad (*)$$

Ponieważ chcemy $L_s = L_m + d$ dla każdej ΔT to w szczególności $L_{0s} = L_{0m} + d$ czyli lewa strona równania $(*)$ wynosi 0. W takim razie prawa strona tego równania też wynosi 0 dla każdego ΔT

$$-L_{0s}\alpha_s + L_{0m}\alpha_m = 0$$

$$L_{0m} = \frac{\alpha_s}{\alpha_m} L_{0s}$$

$$\begin{cases} L_{0m} = \frac{\alpha_s}{\alpha_m} L_{0s} \\ L_{0s} - L_{0m} - d = 0 \end{cases} \rightarrow L_{0s} - \frac{\alpha_s}{\alpha_m} L_{0s} = d$$

$$L_{0s}\left(1 - \frac{\alpha_s}{\alpha_m}\right) = d$$

$$L_{0s} = \frac{d}{1 - \frac{\alpha_s}{\alpha_m}} = \frac{5\,cm}{1 - \frac{1{,}2}{1{,}6}} = \frac{5\,cm}{1/4} = 20\,cm$$

$$L_{0m} = \frac{\alpha_s}{\alpha_m} L_{0s} = \frac{3}{4} \cdot 20\,cm = 15\,cm$$

$$\boxed{\begin{aligned} L_{0s} &= 20\,cm \\ L_{0m} &= 15\,cm \end{aligned}}$$