FORMULATION OF ENGAGEMENT AS LOCAL ANTICIPATION

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ABSTRACT

Quantifying player engagement ("fun") in game design remains a significant challenge. Traditional Flow theory emphasizes the balance between challenge and skill but provides qualitative rather than quantitative guidance, making systematic optimization difficult. Existing approaches rely on subjective evaluation methods that cannot predict engagement levels or establish mathematical optimality conditions. A rigorous quantitative metric is needed to enable systematic design analysis and optimization.

In this work, we introduce a *local anticipation* metric for a game state s, defined as

$$\mu(s) = \sum_{s'} P(s \to s')\, D(s'), \quad A(s) = \sqrt{\sum_{s'} P(s \to s')\big(D(s') - \mu(s)\big)^2}$$

where $P(s \to s')$ denotes the transition probability and $D(s')$ denotes the desirability of outcome s'. This metric captures *anticipation of meaningful future events* via the probability-weighted standard deviation of outcome values.

Key contributions:

1. **Derivation of Metric:** We derive the formulation from observed principles, yielding the Local Anticipation formulation.
2. **Fundamental Bounds:** We prove that in binary-payoff games, $A(s) \leq 0.5$, establishing a theoretical maximum for single-turn game.
3. **Optimally Engaging Single-Turn Games:** We demonstrate that coin toss achieves maximum anticipation (A=0.5) and prove that the coin toss is the optimal single-turn game design for maximizing anticipation.

Keywords Game Theory · Quantitative Game Design · Player Engagement · Decision Theory · Mathematical Optimization · Entertainment Computing

1 Introduction

Quantifying player engagement in game design remains a fundamental challenge. While established theories such as Flow theory [3] provide qualitative frameworks for understanding optimal experience, they lack the mathematical precision necessary for systematic design optimization and objective comparison of game mechanics.

This limitation constrains design innovation. Developers must rely on intuition-based approaches without rigorous methods to evaluate novel concepts. Current evaluation techniques require either subjective assessment (which cannot establish mathematical optimality) or full implementation for playtesting (making systematic design exploration prohibitively expensive). The absence of objective engagement metrics forces developers to favor established patterns over innovative mechanics.

We address this limitation by introducing a mathematical framework that quantifies engagement through *local anticipation*—the probability-weighted standard deviation of outcome desirabilities. This provides an objective metric computable directly from game state definitions.

We demonstrate the framework's analytical power through complete mathematical characterization of single-turn games, deriving fundamental engagement bounds and proving that coin toss represents the optimal single-turn game design for maximizing player anticipation.

2 Related Work

Flow theory, introduced by Csikszentmihalyi [3], identifies optimal experience as arising from the balance between challenge and skill levels. While Flow theory represents the most influential theoretical framework for understanding engagement in game design, it provides only qualitative guidance and cannot quantify the role of uncertainty or differentiate between various sources of engagement within optimal flow states. These limitations render Flow theory objectively insufficient for systematic game design optimization.

GameFlow [4] adapted flow theory into eight game-specific elements and demonstrated that structured criteria could distinguish between successful and failed games. Subsequent frameworks like PLEX [2] categorized 22 playful experiences, while the Player Experience Inventory [1] provided dual-level measurement validated across multiple studies. Despite theoretical sophistication, these approaches remain fundamentally qualitative assessment tools rather than mathematical optimization frameworks.

Existing game metrics typically measure player behavior (retention, progression) or subjective responses (surveys, reviews) rather than fundamental engagement properties. While valuable for post-launch optimization, these approaches cannot guide design decisions during development and lack the mathematical rigor necessary for systematic design exploration.

The absence of quantitative frameworks for engagement measurement has left game design without objective tools for evaluating novel mechanics or establishing mathematical optimality conditions, creating a fundamental gap between theoretical understanding and practical design needs.

3 Derivation

The Theory of Anticipation emerges from fundamental observations about the mathematical nature of engagement, establishing why engagement must be modeled through transition variance rather than static state properties.

3.1 Static States Generate No Engagement

Consider achieving a highly desirable game state—commanding lead, optimal positioning—then pausing the game indefinitely. Despite occupying the most advantageous position, this scenario generates zero engagement. This *paused game paradox* reveals that engagement cannot originate from state desirability alone.

Similarly, engagement vanishes when outcomes become certain: guaranteed victory or inevitable defeat both eliminate engagement regardless of how favorable the situation appears.

3.2 Engagement Requires Transition Uncertainty

Engagement emerges exclusively from uncertainty about meaningful state transitions. Consider a penalty-only scenario: choosing between losing 10 points (90% probability) versus losing 100 points (10% probability). Despite all outcomes being undesirable, the uncertainty about which penalty will occur creates measurable engagement.

This demonstrates that variance between potential outcomes drives engagement even when all outcomes represent losses.

3.3 Variance-Based Formulation

These observations establish that neither average desirability nor absolute outcome values determine engagement levels. Instead, the mathematical spread between possible outcomes creates engagement regardless of whether those outcomes represent gains or losses.

This motivates a variance-based approach: engagement should be measured through the probability-weighted dispersion of outcome desirabilities, necessitating the weighted standard deviation formulation that follows.

4 Local Anticipation

We model a game as a set of states S with transition probabilities $P(s \to s')$ and desirability function $D : S \to \mathbb{R}$. For any state $s \in S$, we define the expected desirability as:

$$\mu(s) = \sum_{s' \in S} P(s \to s') \cdot D(s') \tag{1}$$

The *local anticipation* at state s quantifies engagement through the probability-weighted standard deviation of outcome desirabilities:

$$A(s) = \sqrt{\sum_{s' \in S} P(s \to s') \cdot (D(s') - \mu(s))^2} \tag{2}$$

This formulation captures anticipation as the uncertainty about meaningful future outcomes. High anticipation occurs when transition probabilities are distributed across outcomes with significantly different desirabilities.

For single-turn games, we consider terminal states where $P(s \to s) = 1$ for all outcomes, reducing the analysis to the probability distribution over final desirabilities.

5 Enabling Objective Analysis

To eliminate any subjectivity, we apply this rule for desire values: winning states receive $D(\text{win}) = 1$ while all other states receive $D(\text{other}) = 0$. We term this the *canonical intrinsic desire*.

By using this canonical desire function, we remove any subjectivity. This enables comparison of various games under standardized conditions. While it is also possible to customize the desire function for experimental purposes, we've found that using the canonical intrinsic desire function was sufficient for most purposes we have encountered.

6 Theoretical Properties

We establish some properties of the local anticipation metric:

Theorem 1 (Anticipation Bound for Binary-Payoff Games). *For binary-payoff games, $A(s) \leq 0.5$ with equality achieved when $p = 0.5$.*

Proof. Consider a binary-payoff game with win probability p, where $D(\text{win}) = 1$ and $D(\text{lose}) = 0$.

The expected desirability is $\mu(s) = p$, so:

$$A(s) = \sqrt{p(1-p)^2 + (1-p)p^2} = \sqrt{p(1-p)}$$

To maximize $f(p) = p(1-p)$, we solve $f'(p) = 1 - 2p = 0$, giving $p = 1/2$.

Since $f''(p) = -2 < 0$, this is a maximum with $f(1/2) = 1/4$.

Therefore $A(s) \leq \sqrt{1/4} = 1/2$, with equality when $p = 1/2$.   □

Theorem 2 (Anticipation Bound for Canonical Single-Turn Games). *For any finite single-turn game under canonical desire assignment, $A(s) \leq 0.5$ with equality achieved when the probability of winning equals 0.5.*

Proof. Under canonical desire assignment, any multi-outcome game reduces to binary outcomes. Since all winning states have $D = 1$ and all non-winning states have $D = 0$, this creates win/not-win probabilities that reduce to Theorem 1, regardless of the number of outcomes.   □

7 Optimal Single-Turn Game Design

Given the theoretical maximum $A = 0.5$, we identify the coin toss as the optimal single-turn game for maximizing player anticipation in canonical settings.

Theorem 3 (Coin Toss as Optimal Single-Turn Game). *Among all canonical single-turn games, the fair coin toss achieves maximum local anticipation.*

Proof. From Theorem 2, the maximum anticipation for canonical games is achieved at $p = 0.5$ with $A = 0.5$. The fair coin toss has exactly these properties: $P(\text{heads}) = P(\text{tails}) = 0.5$, making it the maximizer of local anticipation among all canonical single-turn games. $\square$

The coin toss represents the solution to the optimization problem:

$$\max_{p \in [0,1]} \sqrt{p(1-p)}$$

8 Expansion of the Framework

8.1 Multi-Turn Games

Real games involve sequences of decisions extending beyond single transitions. It is intuitive to extend our framework to multi-turn games. For acyclic multi-turn games, it is trivial to compute expected desire values through dynamic programming, backpropagating from terminal conditions:

Definition 1 (Multi-Turn Desire Propagation). *Given terminal states $T \subseteq S$ with known desire values, we compute:*

$$D(s) = \sum_{s' \in S} P(s \to s') \cdot D(s') \tag{3}$$

for all non-terminal states in reverse topological order.

Then we can compute the anticipation for each state using the Local Anticipation formula. The algorithm takes $O(|S|)$ time and scales efficiently if state size is not too large.

8.2 Nested Decomposition of Anticipation

The most significant extension involves *nested anticipation analysis*, where anticipation values themselves become desire functions for higher-order calculations. This recursive structure produces Taylor-series-like anticipation components $A_1, A_2, A_3, \ldots$, each capturing different engagement depths.

Definition 2 (Hierarchical Anticipation). *Define the local anticipation function $\mathcal{A}(s; D)$ for state s with desire function D:*

$$\mathcal{A}(s; D) = \sqrt{\sum_{s'} P(s \to s') \cdot (D(s') - \mu_D(s))^2} \tag{4}$$

where $\mu_D(s) = \sum_{s'} P(s \to s') \cdot D(s')$.

The hierarchical components are then defined as:

$$A_1(s) = \mathcal{A}(s; D) \tag{5}$$
$$A_2(s) = \mathcal{A}(s; A_1) \tag{6}$$
$$A_n(s) = \mathcal{A}(s; A_{n-1}) \text{ for } n \geq 2 \tag{7}$$

This decomposition captures respective depths of engagement:

- A_1: Immediate excitement from direct outcomes
- A_2: Longer term anticipation from varied future possibilities
- A_{3+}: More longer term & deep anticipation from complex strategic interactions

8.3   The Game Ending Button Problem

Consider a thought experiment: provide players with a button that instantly ends the game with 50% probability of victory at any moment. This design achieves the theoretical maximum $A_1 = 0.5$ for all game states, as every state effectively becomes a coin-toss game.

However, such a game would be profoundly unengaging despite maximizing first-order anticipation. This paradox demonstrates why optimizing A_1 alone is insufficient for creating compelling experiences.

The game ending button game fails because it eliminates all higher anticipation components. More intuitively, it lacks strategic depth since only the game ending button matters, effectively making this game into consecutive single-turn coin-toss games. In short, this game lacks "depth."

The higher-order components effectively capture this notion of "depth." During our preliminary experimentation, the sum $(A_1 + A_2 + A_3 + \ldots)$ more precisely represented the quality of a game design than A_1 alone. This makes sense since human intuition naturally and automatically builds models and predicts outcomes to some degree, so it is natural to use combined sum of anticipation components to measure perceived anticipation rather than just A_1.

9   Future Work

9.1   More Standardization For Game Design Comparison

The Game Ending Button Problem demonstrates that multi-component analysis is essential for meaningful engagement measurement. We internally used sum of A_1 to A_5, however, there is yet no established methodology for optimal weighting of components or determining how many components are needed for standard comparison across games.

Do we need weightings on the components? How many components are reasonable for standard comparison?

Future research could focus on answering these questions.

9.2   Conjecture: Higher-Order Components May Capture Replayability

Consider a known phenomenon: a game appears very fun initially with no changes in anticipation A_1, however it gradually becomes boring over time despite unchanged mechanics. This phenomenon is often considered as evidence that "fun" is inherently subjective and therefore unquantifiable.

However, the hierarchical structure appears to address this elegantly. The decaying of engagement over time occurs because as the understanding of the game evolves, players become able to predict outcomes more accurately, reducing the perceived variance of significnat events within a game session.

A_2 is technically an anticipation of anticipation, indicating perceived variance after a player has mastered A_1 level patterns.

So we can now predict several properties according to this explanation. As players successfully model the game and master A_1 level hierarchy, A_2 now becomes the primary driver. When A_2 patterns become predictable, A_3 takes precedence, and so forth. Games with strong higher-order components will be able to maintain engagement longer through multiple mastery phases, explaining why strategically complex games like League of Legends demonstrate exceptional longevity compared to action focused games with shallow strategic depth.

While this interpretation aligns well with observations, more systematic analysis will be appreciated to establish the exact relationship between higher-order anticipation components and replayability. For example, is it logarithmic or linear, and the tendency of time needed for players to master each level of anticipation hierarchy.

9.3   Conjecture: Optimizing Higher-Order Components May Generate Narrative Structure

Optimizing higher-order anticipation creates natural trade-offs with immediate engagement. Maximizing A_2 requires variance in A_1 values. Since A_1 has an upper bound of 0.5, creating non-zero variance mathematically necessitates A_1 values lower than 0.5.

This constraint naturally suggests to design games with alternating high- and low-anticipation states, creating rhythmic engagement patterns. This mathematical requirement appears to indicate some known storytelling principles.

While not yet proven, our informal experiments suggest that optimization of multidimensional anticipation may converge on classical narrative structures such as Freytag's Pyramid or the Three-Act Structure. Further investigation on this conjecture could yield valuable insights.

9.4 Conjecture: Anticipation May Be Unbounded for Higher-Order Components

While $A_1 \leq 0.5$ for binary games, higher-order components may be unbounded. We propose this conjecture based on the observation that deeper engagement often requires temporal development. Creating profound emotional impact within short time frames appears fundamentally more difficult compared to given longer periods.

Following this pattern, while A_1 remains constrained to 0.5, A_2 may have higher bounds, A_3 may exceed A_2, and so forth. If this conjecture proves true, we can theoretically build games with arbitrarily high anticipation by making games longer and deeper, constrained only by session time and event density.

Validating this conjecture would have significant implications for game design industry. Proving this conjecture means, as opposed to current impression that game designs are saturated so room for innovation is almost exhausted, we may be able to create games with arbitrarily high anticipation with help of newer methodologies like computer-assisted design discovery.

9.5 Computer-Assisted Design Discovery

Components beyond A_2 exceed the domain where human intuition can effectively optimize. Optimizing A_2 necessarily requires sacrificing A_1 values, creating a complex combinatorial multi-dimensional optimization problem that surpasses human cognitive capacity for simultaneous consideration of multiple competing objectives.

This matches the observed phenomenon that designing games with strategic depth and long-term engagement is profoundly more difficult than creating immediately exciting and action-packed experiences.

Computer-assisted approaches offer systematic solutions to these limitations. By defining formal game models and employing genetic algorithms, gradient descent, and other optimization methods, we can discover novel game designs and optimal parameter configurations that navigate these complex trade-off spaces more effectively than human intuition alone.

We have shown the dynamic programming algorithms for computing nested components in $O(|S|)$ time for each component in above sections. Optimization of game design in this manner is readily feasible.

10 Limitations and Scope

This framework is best suited to games with well-defined objectives. It works well for competitive games with clear win conditions, especially competitive MOBAs, but performs poorly for games like Minecraft where exact win conditions don't exist.

Additionally, real games operate in real-time with prohibitively large state spaces. Practical application requires building idealized and simplified models for computational feasibility.

11 Conclusion

Based on observations that engagement emerges from uncertainty about meaningful outcomes, we derived the Local Anticipation formulation as the core of our framework. We proved that the optimal single-turn game under canonical settings is the fair coin toss, achieving maximum anticipation of 0.5.

We explored various extensions of the framework, including multi-turn games and nested anticipation components. The hierarchical decomposition of anticipation was especially notable, revealing interesting properties that match observed principles of game design and narrative structure.

Also, our experimentation shows promising results in using the framework as a design tool.

12   Applications

12.1   HpGame

HpGame models a simplified 1v1 fighting game where each player has health points and can attack with hit/miss probability. Players alternate turns until one reaches zero health.

Model Definition: Two players with HP $\in \{1, 2, 3, 4, 5\}$, with three equally likely transition outcomes per turn:

$$P(\text{Player 1 hits, Player 2 misses}) = \frac{1}{3} \tag{8}$$

$$P(\text{Both players hit}) = \frac{1}{3} \tag{9}$$

$$P(\text{Player 1 misses, Player 2 hits}) = \frac{1}{3} \tag{10}$$

Each successful hit deals 1 damage.

Results:

Player1 HP	Player2 HP	A_1	A_2	A_3	A_4	A_5	Sum
2	1	0.31	0.22	0.00	0.00	0.00	0.53
1	1	0.47	0.00	0.00	0.00	0.00	0.47
2	2	0.28	0.07	0.10	0.00	0.00	0.45

Table 1: Most engaging game states in baseline HpGame

Game Design Score: 0.176 (A_1 metric), 0.430 (sum A_1~A_5)

Analysis reveals the most engaging moments occur at close HP values with moderate win probabilities. The highest individual A_1 values occur at equal HP states (coin-toss game). The reason why A_1 is not 0.5 is because win probability is not exactly 0.5 due to the both attack (draw) scenario.

Also, it is remarkable that the sum of A_1~A_5 can exceed 0.5, as shown in the first row. This observation suggests potential for an unbounded conjecture regarding hierarchical anticipation components.

Note that Game Design Score(GDS) is averaged A_1-A_5 over all states, weighted by respective state's reachability from the initial state.

12.2   HpGame_Rage

We expanded HpGame with new "rage/crit" mechanics. This rage/crit system yielded the highest GDS in our trials.

Rage System:

- Rage accumulates when dealing or receiving damage

- Critical hits deal additional damage: damage $= 1 + (\text{rage} \times \text{rage\_multiplier})$ (default multiplier is 1)

- Rage persists and does not spend on critical hits, only increasing over time

- Players can land critical hits with fixed probability (10% in this experiment)

Model Definition:

The rage system expands the transition space to 8 possible outcomes per turn, where p_h = hit probability, p_c = critical hit probability, and p_m = miss probability. Note that we don't have both miss scenario to prevent cyclic states. This missing probability will be evenly distributed to the other outcomes.

$$P(\text{P1 hits, P2 misses}) = p_h \cdot p_m \tag{11}$$
$$P(\text{P1 crits, P2 misses}) = p_c \cdot p_m \tag{12}$$
$$P(\text{P1 misses, P2 hits}) = p_m \cdot p_h \tag{13}$$
$$P(\text{P1 misses, P2 crits}) = p_m \cdot p_c \tag{14}$$
$$P(\text{Both hit}) = p_h \cdot p_h \tag{15}$$
$$P(\text{P1 hits, P2 crits}) = p_h \cdot p_c \tag{16}$$
$$P(\text{P1 crits, P2 hits}) = p_c \cdot p_h \tag{17}$$
$$P(\text{Both crit}) = p_c \cdot p_c \tag{18}$$

Results:

HP_1	HP_2	Rage_1	Rage_2	A_1	A_2	A_3	A_4	A_5	Sum
4	3	2	3	0.26	0.31	0.22	0.11	0.04	0.94
3	4	3	3	0.27	0.30	0.21	0.11	0.03	0.92
3	3	4	4	0.27	0.29	0.17	0.06	0.02	0.81
3	3	3	4	0.27	0.29	0.17	0.06	0.02	0.81
3	3	3	2	0.27	0.29	0.17	0.06	0.02	0.81
3	3	3	3	0.27	0.29	0.17	0.06	0.02	0.81
3	3	2	3	0.27	0.29	0.17	0.06	0.02	0.81
3	3	4	3	0.27	0.29	0.17	0.06	0.02	0.81
4	3	2	2	0.23	0.26	0.19	0.10	0.04	0.82
4	3	3	2	0.23	0.26	0.19	0.10	0.04	0.82
5	3	2	2	0.18	0.24	0.21	0.13	0.06	0.82
4	2	2	3	0.28	0.25	0.16	0.06	0.02	0.77
4	2	4	4	0.28	0.25	0.16	0.06	0.02	0.77
4	2	3	4	0.28	0.25	0.16	0.06	0.02	0.77
4	2	4	3	0.28	0.25	0.16	0.06	0.02	0.77
4	2	3	3	0.28	0.25	0.16	0.06	0.02	0.77
3	5	2	2	0.14	0.23	0.19	0.12	0.06	0.74
3	4	2	2	0.20	0.23	0.18	0.10	0.03	0.74
3	4	2	3	0.20	0.23	0.18	0.10	0.03	0.74
2	4	3	3	0.29	0.22	0.15	0.06	0.02	0.74

Table 2: Top engaging states with rage mechanics

Game	A_1 Score	Sum(A_1~A_5)	Improvement
HpGame	0.176	0.430	–
HpGame_Rage	0.199	0.544	+26.5%

Table 3: Performance comparison: HpGame vs. HpGame_Rage

TBD the analysis

12.3 Parameter Optimization

We optimized the critical hit chance in HpGame_Rage to maximize engagement. Optimization was performed by simple brute-force search over 1% increments from 0% to 40% critical hit chance.

Key Findings:

- **Optimal critical chance: 13%** (score: 0.551)
- Clear inverted-U relationship, not monotonic
- Engagement degrades beyond 15% critical chance

This optimization demonstrates the framework's ability to discover non-obvious parameter values that maximize engagement, validating its utility for systematic game design.

Critical %	GDS	Critical %	GDS
0%	0.430	21%	0.529
1%	0.434	22%	0.524
2%	0.451	23%	0.519
3%	0.469	24%	0.514
4%	0.485	25%	0.509
5%	0.500	26%	0.503
6%	0.513	27%	0.498
7%	0.523	28%	0.493
8%	0.532	29%	0.487
9%	0.539	30%	0.482
10%	0.544	31%	0.476
11%	0.548	32%	0.471
12%	0.550	33%	0.466
13%	**0.551**	34%	0.461
14%	0.551	35%	0.456
15%	0.549	36%	0.451
16%	0.548	37%	0.446
17%	0.545	38%	0.441
18%	0.542	39%	0.436
19%	0.538	40%	0.432
20%	0.534		

Table 4: Critical hit probability optimization results

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