Introduction

The Butterworth filter is an IIR (Infinite Impulse Response) filter designed to achieve a maximally flat magnitude response in the passband. Unlike Chebyshev filters, it exhibits no ripples in either the passband or stopband. Although its roll-off is not the steepest, the smooth and monotonic response across both bands makes it one of the most widely used filters. The transition becomes progressively sharper with increasing filter order. Butterworth filters are typically chosen when a smooth frequency response is more important than an abrupt cut-off.

There are several types of Butterworth filters, which can be designed for different frequency responses, as shown below.

- Low Pass (LPF) which passes low frequencies while attenuates high frequencies.
- High Pass (HPF) which passes high frequencies while attenuates low frequencies.
- Band Pass (BPF) which passes frequencies included in a frequency band while attenuates all frequencies out the band.
- Band Stop (Notch) which rejects frequencies included in a frequency band while passes all frequencies out the band.

Magnitude - Squared Response

The squared magnitude response is given by the following equation:

$$|H(i\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$$

Where n is the order of the filter and ω_c the cut-off frequency.

At $\omega = \omega_c$: $|H(i\omega_c)|^2 = \frac{1}{2} \Longrightarrow |H(i\omega_c)| = \frac{1}{\sqrt{2}}$, which is universal for all Butterworth filters.

Frequency Response Characteristics

- Passband behavior

As $\omega \to 0$ (low-pass filter) $\Longrightarrow |H(j\omega)| \to 1$, which means that the filter has no ripples.

- Stopband behavior

As $\omega \rightarrow \infty$, $|H(j\omega)| \rightarrow 0$, Roll-off slope = −20n dB/decade−20ndB/decade.

- Transition

Gradual compared to Chebyshev filters with smoother impulse response (less ringing).

Filter Order and Selectivity

The order of the Butterworth filter (n), determines how sharp the transition is.

The general order formula is:

$$n \geq \frac{log_{10}(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1})}{2log_{10}(\frac{\omega_s}{\omega_p})}$$

where A_p is the maximum passband ripple (dB) which usually is around 1 dB, As is the minimum stopband attenuation, ω_p is the passband edge frequency, and lastly ω_s is the stopband edge frequency.

Each additional order increases slope by −20 dB/decade.

Poles of the Butterworth Filter

The filter's behavior comes from the location of poles in the complex s-plane. The poles lie on a circle of radius ω_c and they are equally spaced in angle, centered on the left half-plane.

For n-th order pole:

$$s_k = \omega_c e^{i(\frac{\pi}{2}+\frac{2k+1}{2n}\pi)}, \text{ for k = 0, 1, 2, 3, …, n – 1}$$

We only take left-half plane poles (stability).

Transfer Function Construction

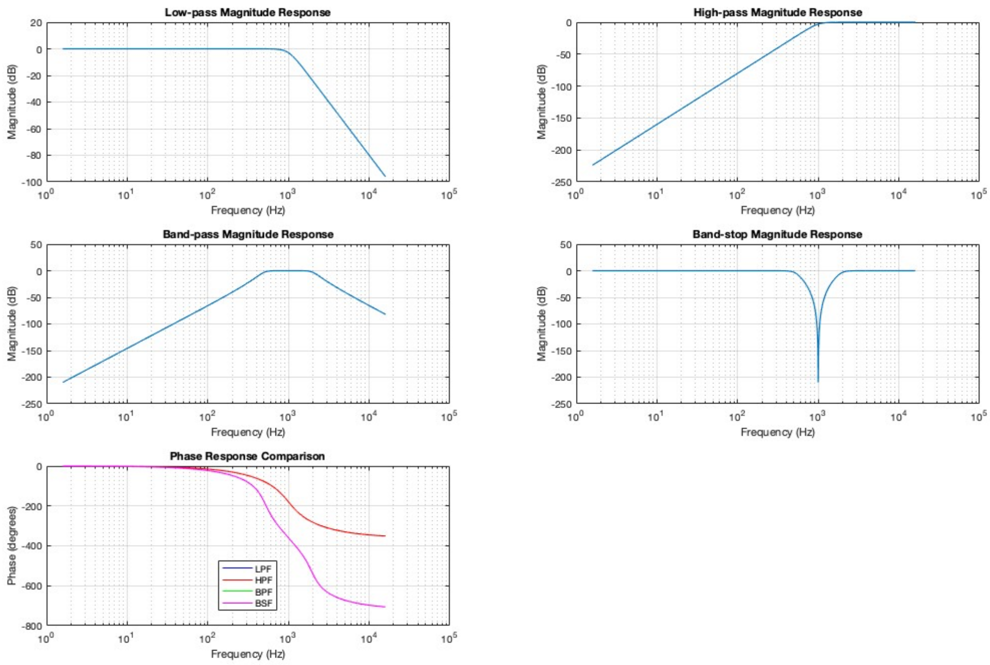
$$H(s) = \frac{K}{\prod_{k=1}^{n}(s-s_k)}$$

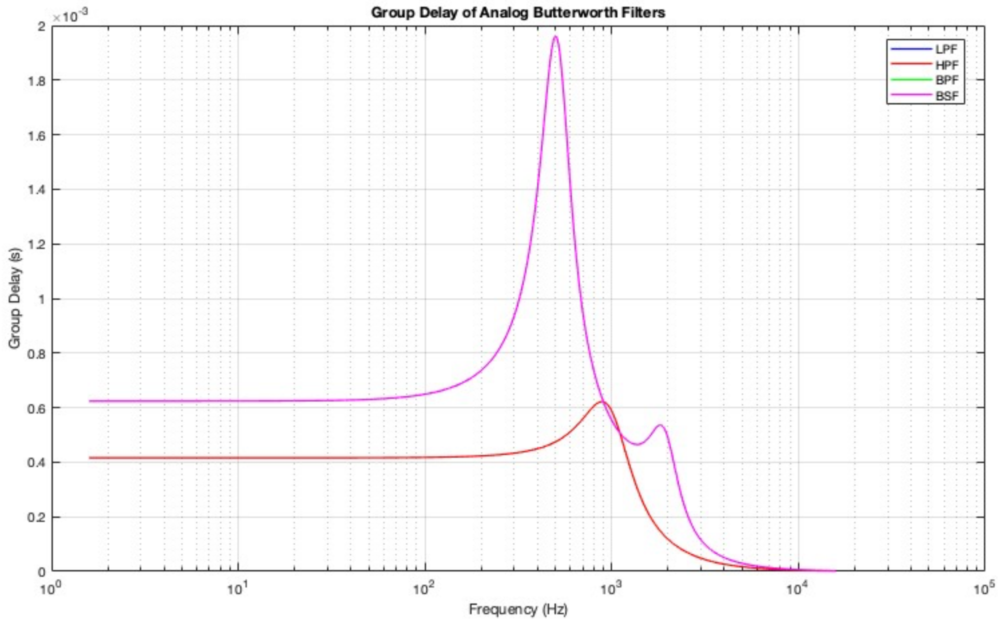
Where K is chosen so that $|H(0)| = 1$ for LPF.

Frequency Transformations

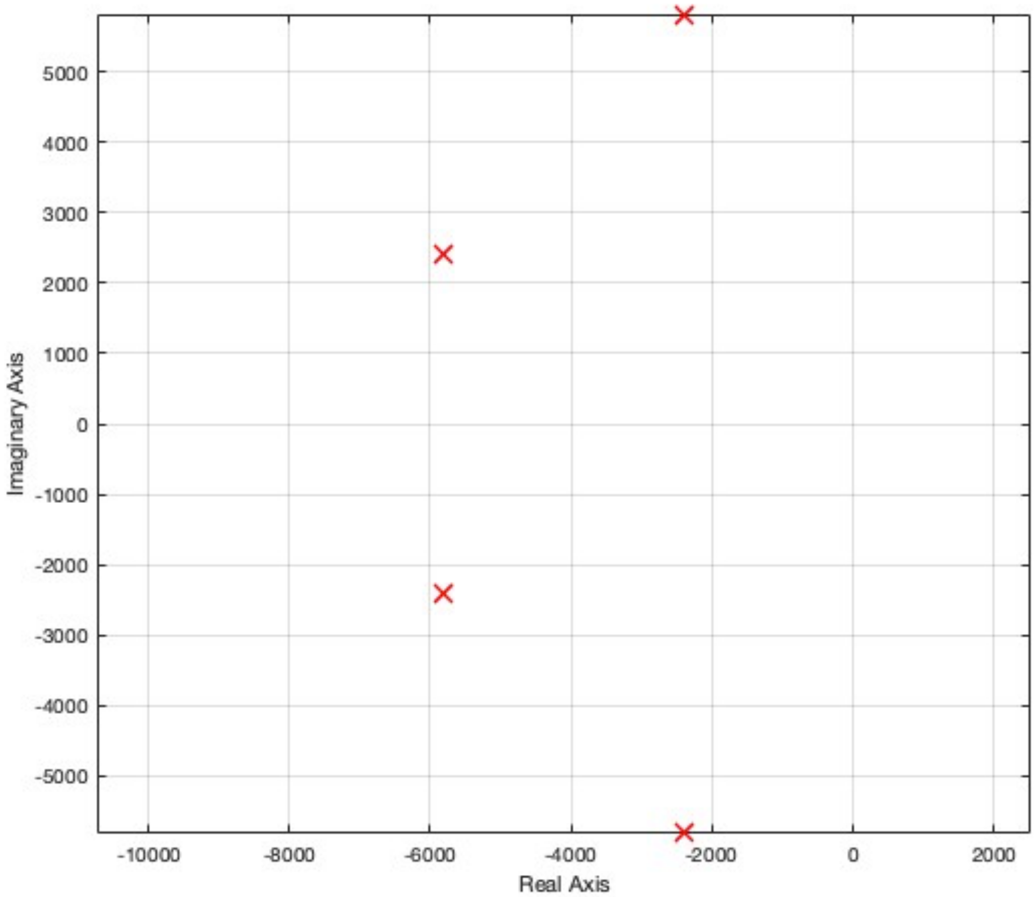
Low-pass: $s \to s/\omega_c$
High-pass: $s \to \omega_c/s$
Band-pass: $s \to s^2+\omega_0^2/B_s$
Band-stop: $s \to B_s/s^2+\omega_0^2$

MATLAB PLOTS: For Analog Butterworth Filters



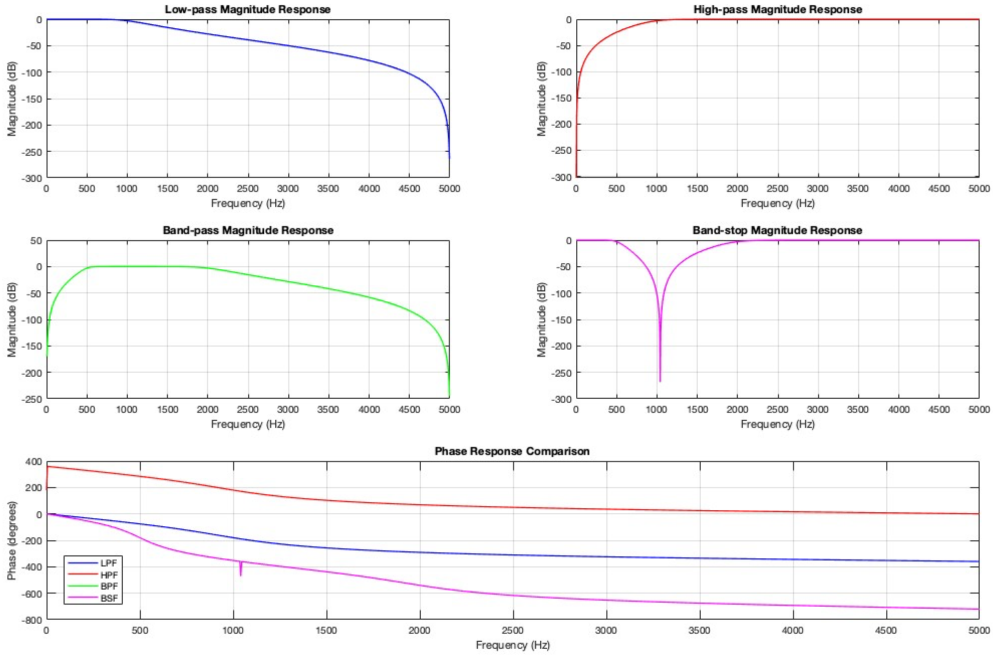


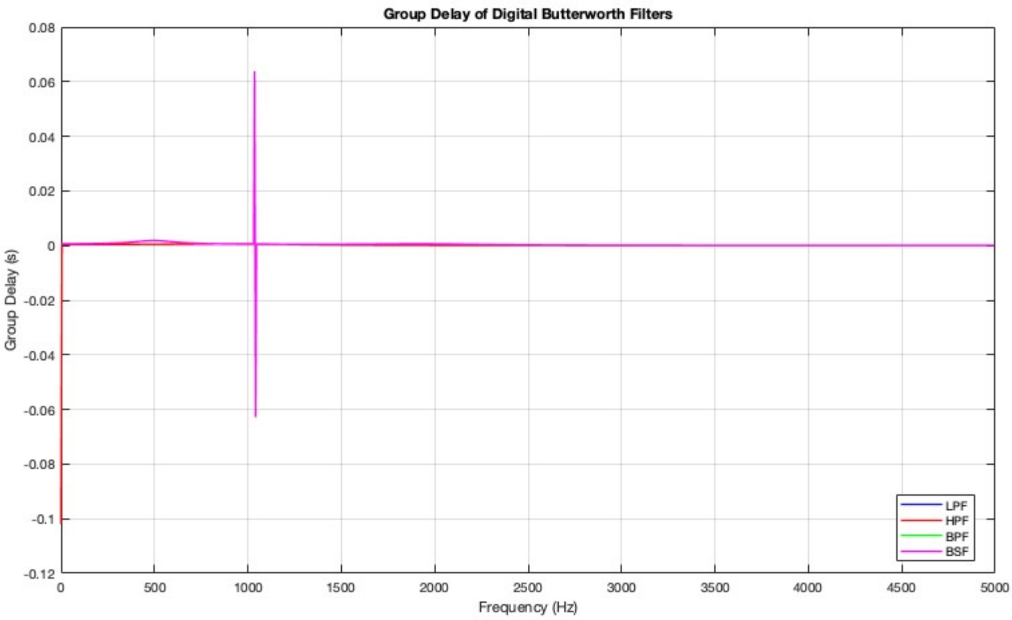
Group Delay of Analog Butterworth Filters
Group Delay (s)
Frequency (Hz)
LPF
HPF
BPF
BSF

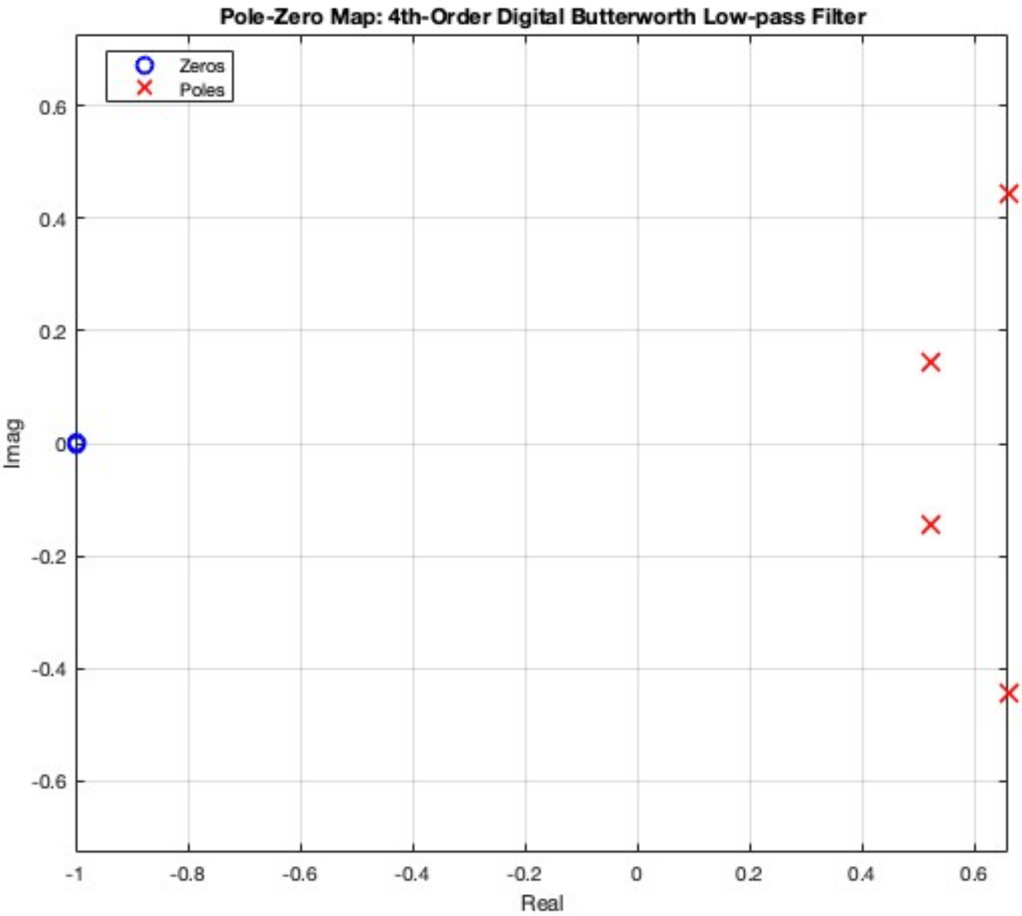


Imaginary Axis
Real Axis

MATLAB PLOTS: For Digital Butterworth Filters






Pole-Zero Map: 4th-Order Digital Butterworth Low-pass Filter
Zeros
Poles
Imag
Real
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6