The function of 2-Dimensional Representations in the Construction of 3-Dimensional Solids

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Abstract

This project uses design-based research methods to look at the challenges middle school age students face when they are asked to create three-dimensional (3d) models from their corresponding orthographic projections. This research looked to understand what knowledge is explicitly gained in orthographic projections and what is hidden in the semiotic structure of this two-dimensional (2d) representation. Using a guided interview, two sixth grade girls were tasked to find meaning in orthographic projection drawings - this project hopes to discover where in this mental translation do students discover meaning or reach cognitive impasses. The findings imply that unit measurements and continuous surfaces are easy to recognize from simple observation, but through features and offset planes need prior knowledge in order to grasp the full meaning behind a set of orthographic projections.

1. INTRODUCTION (Design Rationale)
1.1 Design problem (background and objectives)

On the popular instagram account @thingsihavedrawn, Tom Curtis makes Photoshopped renditions of the animals and creatures his kids doodle. The account's humor comes from the proportional and corporeal inaccuracies of the children's drawings and their hilarious, almost disturbing, consequences in their father's photo-realistic interpretations. But what these images do highlight is the profound ability that most children possess – the ability to recognize details on objects in the real, 3d world and create 2d representations of them (N. Sinclair et al., 2016).

Figure 1. Tim Curtis's Photoshop interpretation of a drawing of a monkey done by one of his children.

Most children refine their visualization capabilities over time, but it's a skill that most acquire informally. It isn't emphasized in most school curriculums, but it's a skill that necessitates a significant amount of formal training (Ben-Chaim, Lappan, & Houang, 1989).

The Common Core State Standards Initiative, the English language arts and mathematics curriculum adopted by a majority of the United States, began to tackle this issue. It asks students to be able to visualize and manipulate 2d drawings and their corresponding 3d objects when they have reached high school age. Nevertheless, there's still more to that must be done to adequately integrate spatial visualization into primary and secondary education.

With the move to computer-aided design (CAD) software like AutoCAD, Rhino, and Fusion 360, for drafting, the importance of traditional spatial visualization skills are falling out of practice (Lieu & Sorby, 2017). Although this ability is commonly tied to engineering and design, fields like radiology, geology and medicine call for individuals who are proficient in recognizing 2d representations of 3d objects (Gagnier & Shipley, 2016).

To complete this project, I propose to conduct design-based research on how these students interpret orthographic projections of an object. Using MathLink cubes as both a 2d and 3d meditative tool, I will conduct a guided interview, to see how students reason with the images and create the 3d objects they represent. As an interviewer, I aim to take on the role of "reflective practitioner" that observes and notes on the mental processes that take place during these kinds of tasks (Abrahamson, 2012). I also hope to play the role of teacher that unfolds the semiotic potential of these orthographic drawings (Mariotti, 2009). In doing so, I hope to deconstruct how the sign of orthographic image informs, mediates, and facilitates the construction of the objects.

After the interviews are completed, I will then analyse a series of moments from these interviews to gain a deeper insight into the underlying cognitive and perceptual processes the children undergo in completing this task.

1.2 Previous solutions and their problems

There is a split in how this subject is tackled. In one end of the arena, there are lessons that are limited to a screen or a piece of paper. Traditional orthographic projection exercises (figure 2) will ask students to use the orthographic projections to create isometric, or sometimes perspective, drawings. The issue with this approach is that students are limited to the 2d plane. There's no point in this exercise in which students are able to interface with a 3d object.

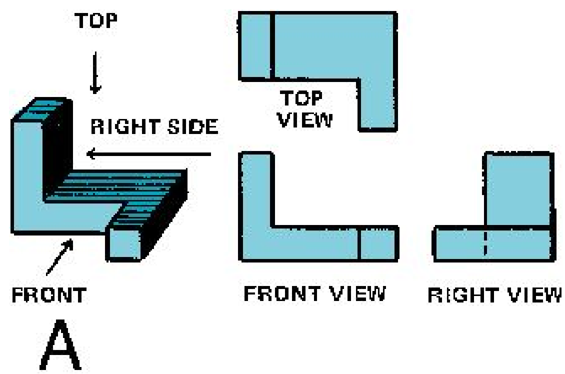


Figure 2. A diagram of an orthographic projection and how it derives its meaning.

There is an exercise that asks students to visualize a container being filled with a liquid (Mathematics Assessment Project, www.map.mathshell.org). The student is then asked to imagine what the surface of this liquid looks like. While this method does employ the imagination of a tangible situation in which to learn, the exercise is still limited to mental manipulations, worksheets, and drawn representations - there is no venture into the third dimension. Despite the fact that they utilize transparent, isometric drawings, manipulated in real time, interactive instructional videos, like the ones Khan Academy produces, also possess this 2d limitation. Online games move towards more interactive mediums, but they are still limited to the two dimensionality of screens; they lack any kind of tactilite interaction on the part of the user.

There are 3d tools and toys that aim to teach students about these concepts, but they are limiting in certain regards. For example, there is an exercise that involves placing a clear plastic box on top of a geometric object and tracing what the user sees onto each of the boxes faces. While this is an approach that does section off an object into discrete views, there is a limitation on the size of object you can put in the box. Additionally, teachers have utilized building cubes like MathLink blocks to allow students to build out orthographic projections. While this is a really scalable exercise, it doesn't allow for the creation of curved or organic shapes. Each of these solutions aim to give meaning to 2d representations of 3d objects, but they lack certain steps that allows students to apply their skills to other objects.

1.3 Cognitive domain analysis

In order to unpack the difficulties of building 3d models from their 2d representations, spatial visualization must first be defined. Nikolina Kovacevic's paper, S*patial Reasoning in Mathematics*, references Clements and Battista's 1992 definition of the term as a "set of cognitive processes by which mental representation for spatial objects, relationships and transformations are constructed and manipulated" (2019). Although this skill is particularly relevant in geometric tasks, *Designing Spatial Visual Tasks for Research* references Koedinger's 1992 works that speak to how "visualization can be an effective and efficient means of condensing information and increasing its accessibility (Sinclair, Mamolo, & Whiteley, 2011). Not only can this skill prove beneficial in synthesizing information, individuals who have mastered this skill show a higher proficiency in other mathematics related tasks (Ben-Chaim, Lappan, & Houang, 1989).

Difficulties that students face while performing spatial visualization skills have roots in cultural, sociological, and cognitive areas. Difficulties have arisen from the amount of visual information students are able to process to the visual cultures students have grown up in (Sinclair, Mamolo, & Whiteley, 2011). For the scope of this project, we will only be focusing on the cognitive difficulties students face when dealing with 2d to 3d translations.

Visual perception alone has been noted as a deterrent to those who are beginning to learn spatial visualization skills. Sinclair's review on the teaching of geometry brings up Gal and Linchevski's 2010 application of Duval's notion of figural apprehension. As they studied children's difficulties with geometry, they argued that "such difficulties … emerge because students may not "go further from the first glance of a geometrical figure which is the result of visual perception [which] may even hinder the use and development of geometric concepts and properties" (2016). Widder, Berman and Koichu note a similar finding from Kali and Orion's 1996 works that, "found that many students rely solely on external visual information, and fail to "penetrate" the 2-D sketches and construct desirable 3-D mental representations" (2014). This research shows that untrained observation leaves a student wandering because they aren't yet privy to, or trained to notice, the important information present in 2d representations.

But for most, spatial visualization is an informally acquired skill, not emphasized very much beyond geometry curriculums. Nevertheless, this skill requires a lot of formal training to master its meaning (Ben-Chaim, Lappan, & Houang, 1989). Widder, Berman and Koichu's paper on spatial geometry and dynamic geometry software references Parzysz's 1988 observation that, when creating 3d models, from 2d representations, students find difficulty in imagining an item in space and are, often times, unaware of the information that they are losing when they make these translations (2018). This phenomenon occurs when the students aren't made aware of what information is presented to them in the 2d representations to begin with. Without proper training, they can be blind to the rich meanings that these 2d representations possess.

To make use of these 2d representations, there is a "collateral knowledge" which must be brought to it. Collateral knowledge is a corpus of background knowledge that "is necessary both to interpret signs and to use signs to generate knowledge, that is to distinguish objects, to structure our experiences, to organize interaction, and so on" (Hoffman & Roth, 2007). Students must be taught this collateral knowledge to be able to unwrap the meaning behind the semiotic structures that's put into the symbols of the 2d shapes that represent their 3d counterparts.

Orthographic projections are a kind of semiotic structure, or semiotic mediation, that is leveraged to communicate information about a 3d object. Vygotsky's 1978 definition of "semiotic mediation" is explained in *Artifacts and signs after a Vygotskian perspective: the role of the teacher* as "which sees knowledge-construction as a consequence of instrumented activity where signs emerge and evolve within social interaction," (Mariotti, 2009). The use of these semiotic structures is important in that they are standardized across disciplines and have the capacity to carry a lot of meaning in a relatively small amount of symbols. Samet Okumus and Karen Hollebrands's 2019 paper synthesis Duvals' work on semiotics in mathematics as characters that take the form of "figural units not the points or the straight lines" (Okumus & Hollebrand, 2019). Each discipline has its own set of semiotic conventions, but being able to move across them is what equips students with a conceptual understanding of the discipline (Okumus & Hollebrands, 2019).

Effective learning of spatial visualization skills is seen when students are challenged with modes of learning that extend beyond paper and pencil (Sinclair, Mamolo, & Whiteley, 2011). in *Recent research on geometry education: an ICME-13 survey team report*, Sinclair's team finds that "there is reason to conjecture that developing dynamic and haptic forms of visuospatial reasoning would have a positive impact on children's learning." In incorporating a tactile medium into exercises, researchers more easily able to teach students spatial visualization skills (N. Sinclair et al., 2016). Okumus and Hollebrands's use of gesture to make 3d objects takes an embodied action route to teaching spatial visualization skills (2019). Additionally, the filling of clear objects with rice, water and other substances alongside a series of guided questions has also been an effective route to teaching spatial visualization skills (M. Sinclair et al., 2011).

1.4 Conjecture (Implementing theory to practice)

If students were given the opportunity to utilize building materials when learning to interpret orthographic drawings, their ability to complete the task would be facilitated by the incorporation of an intermediary tool that can function both in the 2nd and 3rd dimension.

Additionally, facilitating the understanding of the orthographic projections will create a kind of "collateral knowledge" in which to discuss the construction of the object (Hoffmann & Roth, 2007).

1.5 Design solution (the creative innovation)

Chase and Abrahamson, in The Giant Steps for Algebra project, leverage a technique called "reverse scaffolding" to encourage students to discover meaning on their own (Abrahamson, 2014). This method affords the learner the opportunity to build meaning as they learn instead of having to derive meaning from the tools given to them. In this project, the reverse scaffolding technique is used to understand how the participants derive meaning from orthographic projections when they are presented in isolation.

In order to develop an interview guide, a series of three situated intermediary learning objectives (SILOs), illustrated in figure 3, were developed in order to breakdown the kinds of elements learners would need to be privy to to gain a full understanding of the mental translation of 2d, orthographic projections to 3d objects(Abrahamson, Chase, Kumar, & Jain, 2014).

Each SILO builds off the previous one, so the following SILOs are presented in the order in which the students are inferred to learn them. a.) **Representing physical units in images:** understand that the drawing represents a 3x3 grid and that a square indicates a block unit. b.)

Continuity of faces: Shapes in the drawing represent continuous surfaces. Even though the face consists of discrete blocks, represented by the dotted red lines, the continuity of shape means they all make up the same surface. c.) **Relative position of surfaces:** if a surface is not continuous, it means that the surface outside of the boundary lies on a different plane. For example, the surface indicated by the pink box lies farther than the surface that's not.

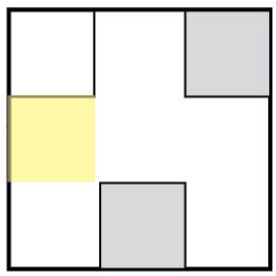
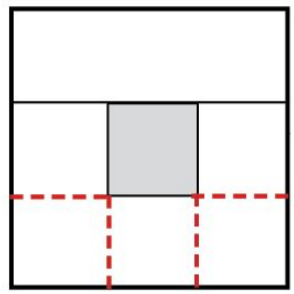
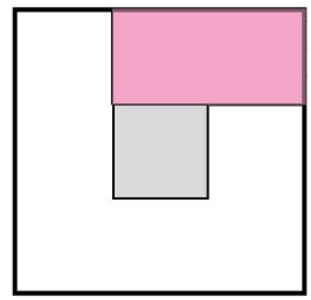
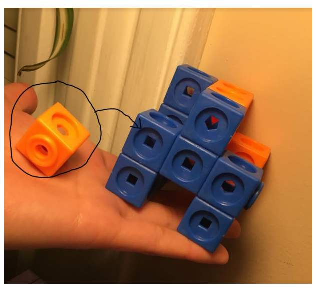
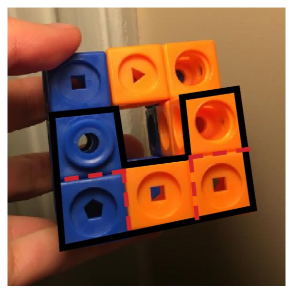
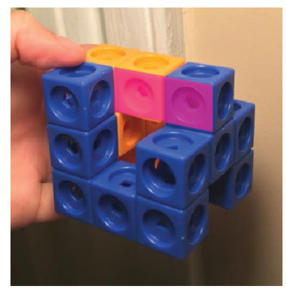
	SILO a	SILO b	SILO c
Orthographic projection card			
Corresponding MathLink cube block			

Figure 3. Illustrations and corresponding models of the SILOs described.

From these SILOs, an interview guide was developed (see appendix item 2). It uses MathLink cubes as an intermediary medium to facilitate the understanding of a series of orthographic projections. As the participants are guided through the activity by the interviewer, they are brought to each SILO with focused, yet open-ended questions that encourage the interviewee to discover the SILO on their own.

1.6 Research Question

To guide my research, I have developed the following research question to understand what is and is not implicit in orthographic projection drawings. How does the symbolic nature of 2d, orthographic projections drawings affect the subsequent construction, by new learners, of the 3d solids they represent?

2. Methods

2.1 Participants

The two participants I will be examining in this paper are Arya and Sasha. They are two 12 years old girls who live in Berkeley, California. They attend the sixth grade together and they both come from college educated parents. The two girls were interviewed together for approximately an hour and 12 minutes.

2.2 Materials

The following is a list of materials I utilized during the interview process:

- 3x3" 3d printed, plastic cube with unique geometric features cut out of it (figure 4).
- 5 cards (figure 5) representing 5 orthographic projections of the cube in figure four .
- MathLink Cubes in a variety of colors for building out the orthographic projections
- Tape, glue stick, pipe cleaners, sticky foam, tape, and star stickers
    a. There was no particular purpose for these materials except to facilitate creation if the participants felt inspired
- 1 3x3" cardboard cube

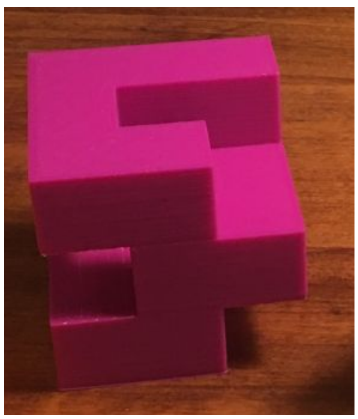


Figure 4: 2 sides of the cube used to make the orthographic views.

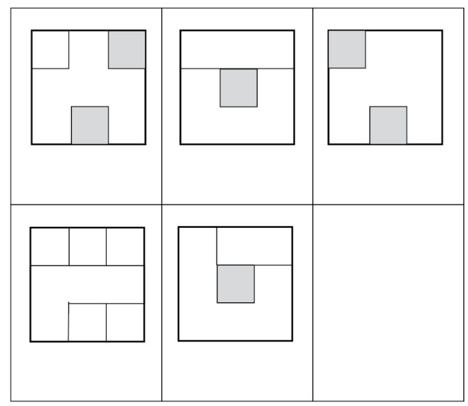


Figure 5: 2-dimensional drawings the 3D object pictured in figure 4. The cards represent the object's left, right, front, back, and top geometries.

2.3 Procedure

The interview begins with introductions. The participants are given a brief summary of the academic background and motivations of the interviewer. Then, a general description of the materials present are given - see materials list above for a comprehensive list. The participants are then told that they are being filmed. They are directed to where the camera is located and told that they can end the interview or the filming at any time.

The next set of questions look to understand how the main building tools are already understood by the participants. They are shown the orthographic cards and the MathLink cubes and asked if they have seen or interacted with these things before.

The interviewer then asks a series of questions that look to unpack the meaning of the cards with the participants. These questions point to some of the symbols used in the cards to communicate surface features. The participants are then asked to build out what they see on a card of their choosing with the MathLink cubes. The participants are also asked to think about how the 5 cards relate to each other.

Then the interviewer moves the participants towards building a 3D model from the cards. The participants are encouraged to see the cards as parts of a web that could be folded into a box.

After they make this realization, they are asked to place the cards on the 3x3" cardboard box in an order that they see fit.

Next, they are tasked to build a MathLink block model in accordance with the order that they placed the cards on the cardboard box. Lastly, the participants are given the model that the cards represent and asked to match the cards to the faces of the model.

2.4 Data Gathered

The following data was collected in an interview with Sasha and Arya who are described in the participants section of this paper. This section will not cover the entirety of the interview. Instead, the entire interview can be accessed through the link in appendix item 1.

In the beginning of the interview, the orthographic projection cards are laid out in front of the two girls. They were then asked what they looked like. Arya took the cards and began to arrange them in a particular order, explaining that "if you put these all together, they probably look like a maze or something." Sasha then takes one of the cards in her hand and points to different features on it. She says that it looks like "the top view, of like, a bedroom." In these preliminary moments, both of the girls saw these cards as a type of scheme or plan - some kind of a schematic that represented something in the real, 3d world.

The girls were then asked to choose one of the cards to build out using the MathLink cubes. In doing this activity, both girls acknowledged the features on the card in their MathLink build. In Sasha's model (figure 6) , although she only choose two colors, she intentionally chose to represent all of the square shaped partitions as one color. Arya had three partitions on her card versus Sasha's six, so she was able to designate a unique color to each of her sections.

Figure 6. Sasha's black and white MathLink model of the card (bottom left of image) she chose

The girls both used 9 cubes, arranged in a 3x3 square to build their respective card. When asked how they came to this decision, Sasha responded by pointing to a square partition on the card, calling it one of the "blocked off ones". She goes onto describe them as "single cubes." She correlates the single square unit to one cube shaped MathLink block and decides to base her units off of that association, showing her ability to see how a 2d square could represent a 3d cube.

Then, the five cards were presented to the girls. They were asked to arrange them how they saw fit. The girls took a few minutes to arrange the cards, making comments back and forth about what seemed to make sense to them. They didn't explain *how* certain arrangements made sense to them, they just vocalized the sentiment of "it does/doesn't make sense."

After they agreed on a certain arrangement, Arya began to map out a path along the cards with her finger. They were looking at what features could constitute an "entrance" or a complete "path." Earlier in the interview they had mentioned that they were taking a game design course. For several minutes, the girls brainstormed what kinds of rules and features they would incorporate into this game. While this was a creative interpretation of the cards, it wasn't pertinent to the goals of the interview. The girls were steered away from this maze model with the next question.

The girls were then asked to make any kind of 3d shape from the cards. They were encouraged to use the building materials around them to add dimensionality to the paper cards. Arya stacked the paper cards on top of themselves and said "you could just stack them together and that would still be 3d, right?" This moment illustrates an initial hesitance to move out of the 2d nature of the paper cards. By creating a stack of them, she technically meets the requirements of the task without challenging herself to think of the cards as something more than a paper card.

The girls experiment with propping the cards up like a tent and figuring out a form with popsicle sticks. Sasha then handles the cards for a second, arranging two cards to be parallel to each other. She mutters, "we can make four walls," before exclaiming, "we can make a cube!" Sasha grabs the cardboard box behind her as she makes this realization. Much like how she matched a single square unit on the cards to the block, she was able to match one square card to the side of the cardboard cube.

They were asked to place the cards on the cardboard box. Arya initially began sticking the cards onto random sides of the box. The interviewer then stepped in to ask them to arrange the cards in an order that they thought would make sense. At this point, the girls returned to the maze game that they had been thinking about in the earlier part of the interview.

The girls were then instructed to create every card out of MathLink cubes. They performed this task with ease, much like they did when asked to do this in the beginning of the interview. They used the same color-coding pattern, but it didn't remain consistent across all 5 cards. They treated each card with its own color coding system, distinguishing different partitions with different colors.

The interviewer then asks the girls to connect each individual face they created to make a cube. As they're putting the faces together, the interviewer asks, "do you think there's a way to arrange the cards [so] that when you connect them together, the patterns stay the same?" Sasha holds two faces together, lets out a sigh and says, "oh gosh." She then says that she likes them (the cards) "being random." At this moment, Sasha admits to the lack of scheme behind their arrangement. Although they were following the rules of their maze game, those rules were being made up along a more narrative track without any systematic rules to go along with it.

The last task the girls are presented with is one in which they are given the model that the cards represent and they are asked to tape the cards onto the sides they represent. Sasha picked up the model and looked at the side that wasn't represented by one of the five cards. As she tried to match a card with the feature that she described as an "equal side," she was met with no match. She ended up just finding a card with the closest match and moving on. The next side she was able to identify was because she matched the dark squares with through features of the model. In a three minute period, the girls were able to correctly match all the cards to their corresponding faces.

2.5 Data Analysis

A point in this interview that was difficult to reconcile was the point of the maze. Is a maze 2d or 3d? In most cases it's 2d. For a majority of the interview, the girls' apparent understanding of the cards were predominantly informed by this maze game that they were formulating in their heads. Therefore, their 3d understanding of the cards was often times deterred by the maze game that they had thought of in the preliminary stages of the interview. They guided their decisions on the configurations of the cards on the basis of matching features locally and following a narrative

If the girls were interviewed individually, this may not have come up so often. They shared the common experience of this game design class so they were able to bounce ideas off of each other when something came to mind.

It was routinely noticed that it was easier to impose the cards onto pre-existing features than it was to construct the feature itself. The girls took 3 minutes to match the cards to the faces of the model that it corresponded with. When they were asked to build out a single face of the card, it took them a bit longer. This could be due to the actual construction aspect, but the girls were also seen making design decisions along the way.

It was also noted that features of the card have distinct partitioned features. Throughout the build, these partitions were always interpreted as surface features. Their actual meaning - offset planes on the model - wasn't realized until the model itself was presented to the girls. Nevertheless, once asked to combine the 6 individual sides together, the girls realized how all the cards were related to each other. It was easy for them, especially Sasha, to recognize how offset planes appeared in the orthographic projections.

3. RESULTS

Through the course of this interview, it became apparent that the cards exist as a sign: a sign in which all its meaning isn't immediately evident by simply looking at it. It uses symbols and figure to represent features like offset planes and through holes. The meanings of these symbols and figure are conventions that need to be taught in order to communicate construction instructions using this medium.

Additionally, it was also observed that children seem to have a working knowledge of what schematics look like. Sasha noted that one of the orthographic projections looked like a birds eye view of a bedroom. Arya noted that the cards could also be interpreted as a top-down view of a maze.

What came up over and over again was evidence that reaffirmed that children possess the ability to match 2D features to their corresponding 3D features. It was clear when Sasha and Arya built 3x3 boxes to match their chosen card. This was also supported by the ease in which the girls were able to place the orthographic projections on their corresponding face of the model.

It was also found that, when given a 2D image, there isn't a motive to move outside of that level of dimensionality. Given the 5 cards, the girls didn't think to form a 3D solid until prompted to do so by questions from the interviewer. Even then, their thinking was difficult to push past the paradigm of this maze game that they had derived.

A profound moment in this interview was realizing that building tools only have the dimensionality that the task prescribes to them. The MathLink cubes were used almost entirely for their surface features - color and square faces. When asked to build a 3d cube, the girls had a hard time reconciling the surface features they had built with the continuity of the overall shape.

4. CONCLUSIONS

The scope of this paper was limited by sample size. My participants were both 6th grade females. In the future, I would like to test this on 6th, 7th, and 8th grade students to get the full scope of students who are both learning and have just learned this content.

I'd also like to see how different socioeconomic groups perform on this task since it does play such a significant role in determining American's quality of education. Even though Common Core is intended to be a nation-wide curriculum, its implementation varies greatly from district to district. Another social factor I'd like to examine is the role of gender in the performance of this task. In the past, some evidence has shown that men are more predisposed to spatial reasoning skills. Although this has been disproven in more recent publications, I think it would be due diligence to conduct these interviews with a wide variety of gender identities.

In the future, I'd like to restructure my interview guide in order to allow the interviewer to play more of a guiding role throughout the interview. In the interview with Sasha and Anya, I allowed them to a lot of room for explanation which I believe led to more distraction than discovery.

In order to play a more guiding role, I'd like to add more instructional, guiding content to the interview materials. This would include adding labels to the cards indicating which side of the object they represent. In the future, I'd like to skip over the construction of a possible web and move faster towards the girls placing the cards on the 3x3" cardboard box. I'd like to keep in mind, when given more information in the onset, how do students do in completing this task?

As more of an intellectual exploration, I'd like to use a few minutes at the end of future interviews to look at the construction of amorphous, organic shapes. How would students fair if given playdough or modeling clay instead of discrete blocks? Would this make for a more holistic learning experience or leave more room for confusion.

5. REFLECTION

This semester, I was impressed by the progress I was able to make. Going from ideation to user testing was an incredible journey. One of the most influential papers I read in this class were the ones pertaining to the Giant Steps for Algebra project. Although I still want to incorporate the method of reverse scaffolding in my interview protocol, I think I need to refine it more for my audience.

Next semester, I'd like to do more interviews with an interview that's been modified with the points laid out in the conclusions portion of this paper. I'd also like to begin developing an educational, possibly narrative-based, game around these findings.

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Appendix

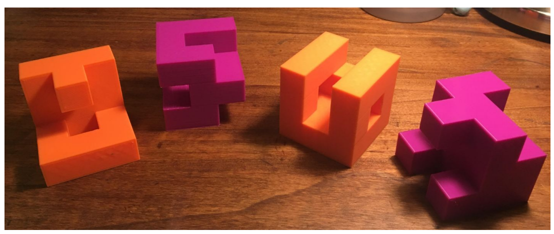
1. Link to entire interview with Sasha and Anya:
https://drive.google.com/file/d/1CANiYh_oUx3pLxAgGsrnxr5y7Thjwh2w/view?usp=sharing

2. Interview Guide:

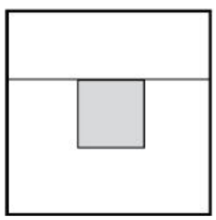
Interview Guide: Epistemic Grounds

Materials presented:

- Cube object with unique, cut out features
    - Ex: One of these four boxes



- 6 cards representing the 6 sides of the above cube
    - Ex:



- MathLink Cubes
- Basic prototyping materials
    - Tape, glue stick, pipe cleaners, sticky foam
- 1 cardboard cube the same size as above cube

What we say/do	Why we say/ do it	Possible Responses	How to respond to those responses
Hi, I'm from UC Berkeley. Thanks for helping me out with this project. You're going to get some diagrams and some blocks. I'm going to ask you a few questions about the activity. Along the way, please ask any questions you think of.	Introduction of myself and the project	Is UC Berkeley hard?	Sometimes, but it's also very fun and rewarding.
I've set up a camera over there [*point to the camera/ make clear where it is*]. It's recording this interview so we don't miss out on anything important you might say. Let me know if you want me to stop	Make students comfortable; Let them understand their rights as volunteer	I don't want my interview recorded.	That's totally fine. Is it okay if I only audio record it? If not, can my friend here take notes?

recording during the activity. After we're done, if you think of something you want taken out, just send me an email and I'll take it out of the footage.	participants.		
Please feel free to use any of the items on the table at any time to help you to explain or to think about things.	Let students know it's fine to touch and use materials: paper, crayons, pennies, dice, etc.	Why do you have sticky foam and pom poms?	I don't know what you might need. They are here to help you explain any ideas you might have that can't be said in words.
Understand how the main building tools are already understood			
[show the cards, put them in an arbitrary pattern] What do these cards look like? Have you seen them before? What do you think they mean? What do you think we're going to do with them?	Get a feel for what the participant knows about the cards	These cards look like a bunch of squares. I think they might be a picture of something. Can I move them around?	Hmm, can you say more about that? What about these shapes? What do you think that something might be? Of course! Please do.
[point to blocks on table] Have you ever played with these blocks before? If so, when? What kind of activities have you done with them? What do they remind you of? What do you think we can do with these	Get a feel for what the participant knows about the blocks	I played with these blocks in the third grade. They remind me of legos.	That's great! You're going to be an expert builder. Interesting, how are they different from Legos?
Delve into the meaning in the cards			
How do you think these cards relate to the blocks in front of you? Why did you say that?	Hint at a connection between the blocks and the cards. Understand the connection the participant made.	I think they are instructions on how to build something. Maybe the boxes were traced to draw these pictures?	Interesting, what makes you think that? That's quite possible. Could you show me how you'd trace the blocks?
[point to a single, arbitrary card] How would you make this card out of these blocks? Could you make this card for me?	Ask participant to engage with both the image and the MathLink cubes. Translate from image to object going from 2d to	How many blocks do I need?	It's up to you.

	2.5d		
What do the gray boxes mean?	Distinguish between gray and white boxes.	I think white might mean x color block and gray might mean y color block.	Interesting observation. Let's keep that in mind and keep going.
How did you decide to separate this surface/unit from this surface/unity?	Understand how the decided to partition the sections indicated in the card.	Because they are different shapes.	How so? What makes you think that?
How did you decide how many blocks to make this card out of?	Understand how they define a "unit".	I used all the blocks in front of me so there wouldn't be any left.	Do you think you could make the same shapes, but smaller? bigger?
Begin to build a 3D model from the cards. Deconstruct the processes behind it.			
How would you arrange these 6 cards? Why?	See if they make the connection between the 6 cards being 6 distinct faces of 1 block.	*[arranges cards in a line]*	Is there an order that you placed the cards in?
What kinds of 3D shapes could these 6 cards make? Feel free to use the supplies on the table if want to.	Lead the student into thinking about these shapes as a web. Remind them that there are construction resources. Encourage 3D thinking with tape & other objects.	[*bends the cards to make a cylinder]*	Why did you decide to make a curved surface?
Could you arrange the cards on this box *[present the cardboard box]*?	Make a jump from the 2-D web to a cube	I don't know how I'd do that.	There's no right answer, just give it your best guess.
[*after the arrangement is done]* Can you talk me through your process?	Get a feel for the participants processes.	I don't really know. I just taped these squares to a box.	Well, what was the first card that you attached?
How do the faces relate to each other? How did you decide to arrange the edges? What do the gray squares mean to you?	Understand the particulars of connecting the schematics.	I put these cards here because they look similar.	What about the cards are the same to you?
[questions dependent on the individual features of the participant's model]	Getting detail specific. Here's an	Well I chose this side to be the top	What about the card made it "easy".

Talk to me about this corner. Can you tell me more about the orientation of your box? Where's the top, bottom, etc.?	opportunity to bring attention to some of the flaws in the build and some of the successes.	because it was the easiest to think about.	
What was the easiest part to do? Why? What was the hardest part? Why?	What do they have an intuition for? What is left to be learned?	The easiest part was glueing the cards on The hardest part was deciding where this card would go..	Amazing, well I'm glad you had fun. Was there a part about arranging the cards that was easy for you? Why is that cards difficult to place?
Building a MathLink block model from the cards on the cardboard box.			
Now using these blocks, can you build out the plan you made on the cardboard box?	Bringing the plans from semi-3D to fully 3D?	Sure. Do I need to follow the cardboard box exactly?	What about the box do you not want to make?
The following questions are repeated from above because they touch on similar features of the activity. [*after the arrangement is done]* Can you talk me through your process?	Get a feel for the participants processes.	I don't know I just did it.	Well, can you talk about the first couple of blocks you put together. How did you decide on that feature?
How do the faces relate to each other? How did you decide to arrange the edges? What do the gray squares mean to you?	Understand the particulars of connecting the schematics.	Well all these faces have x number of lines, so I put them next to each other.	Why are the number of lines on a card important?
How did you decide to make this face? Why is this face in front of/behind this one?	Draw attention to the arrangement of surfaces.	This one is in front because it is bigger.	Why is it that bigger things need to be in the front?
[questions dependent on the individual features of the participant's model] Talk to me about this corner. Can you tell me more about the orientation of your box? Where's the top, bottom, etc.?	Getting detail specific. Here's an opportunity to bring attention to some of the flaws in the build and some of the successes.	This side is the bottom because it's easier for the box to stand up this way.	So if that's the bottom, what's the left side? How did you connect those two sides?
What was the easiest part to do? Why? What was the hardest part? Why?	What do they have an intuition for? What is left to be learned?	The hardest part is understanding where surfaces are. The easiest part	How did you get around that problem? Good to know! Are

		knowing where the holes are because of the gray.	there other features that made building this block easier?