CS 205 - Problem Set 4 - Induction

Sections 01-03 and 07-09

Due Date, Sunday November 13, 2022

Instructions

- Read Rosen 5.1-5.2 prior to completing this problem set
- This is an individual worksheet. NO group work is permitted.
- You are authorized to seek help from course staff ONLY
- This handout is also available in gradescope

your work may not be graded without you signing below
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Sign and PRINT your name : Akshaj Kammari
Your recitation section and/or day of the week : 7 (Tuesday)
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Problem 1

Prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n! = (n+1)! - 1$ for all positive integers $n \geq 1$ (not that $\cdot$ represents the multiplication - not decimal). Let $P(n) : 1.1! + 2.2! + 3.3! + ... + n.n! = (n+1)! - 1$.

1. Prove that P(1) is true

$P(n)$, $n=1$

$1 \cdot 1! = 1$ $\qquad$ $(1+1)! - 1 = 1$

$1 = 1$ $\qquad$ $\therefore P(1)$ is true

2. Show that $P(n) \rightarrow P(n+1)$.

Lets assume $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$

then $P(n+1) = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! + (n+1) \cdot (n+1)!$

$= (n+1)! - 1 + (n+1 \cdot (n+1)!)$

$= (n+1)! + (1 + n + 1) - 1$

$= (n+2)! - 1$

$= ((n+1)+1)! - 1$ ✓

Proof by Mathematical Induction

Problem 2

Suppose that a and b are real numbers with $0 < b < a$. prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$

Base Case: $n=1$

$a-b \leq a^0(a-b) \Rightarrow a-b \leq a-b$ ✓

Inductive Step: Assume $a^n - b^n \leq na^{n-1}(a-b)$ for $n+1$:

$$a^{n+1} - b^{n+1}$$
$$= a^{n+1} + a^n b - a^n b - b^{n+1}$$
$$= a^n(a-b) + b(a^n - b^n)$$

because $a > b$, then:

$$a^n(a-b) + b(a^n - b^n) \leq a^n(a-b) + a(a^n - b^n)$$
$$\leq a^n(a-b) + a \cdot na^{n-1}(a-b)$$
$$\leq a^n(a-b) + na^n(a-b)$$
$$\leq a^n + na^n(a-b)$$
$$\leq a^n(1+n)(a-b)$$
$$\leq (n+1)a^{(n+1)-1}(a-b)$$

$\therefore a^{n+1} - b^{n+1} \leq (n+1)a^n(a-b)$

$\Rightarrow a^n - b^n \leq na^{n-1}(a-b)$ for $\forall n \geq 1$

Proof by Induction

Problem 3

1. Find a formula for the sum

   $\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n(n+1)}$

   by examining the values of this expression for small values of n

$n=1: \frac{1}{1\cdot 2} = \frac{1}{2}$

$n=2: \frac{1}{2} + \frac{1}{2\cdot 3} = \frac{4}{6} = \frac{2}{3}$

$n=3: \frac{2}{3} + \frac{1}{3\cdot 4} = \frac{9}{12} = \frac{3}{4}$

$n=4: \frac{3}{4} + \frac{1}{4\cdot 5} = \frac{16}{20} = \frac{4}{5}$

$n=5: \frac{4}{5} + \frac{1}{5\cdot 6} = \frac{25}{30} = \frac{5}{6}$

$\boxed{\text{sum} = \frac{n}{n+1}}$

prove the formula you conjectured in first part by induction.

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$

Base Case: $n=1$

$$P(1) = \sum_{i=1}^{1} \frac{1}{1(2)} = \frac{1}{2} \quad \& \quad \frac{1}{1+1} = \frac{1}{2} \checkmark$$

Inductive Hypothesis: Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ for $\exists k \in n$

Inductive Step: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

$\therefore \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $\forall n \in \mathbb{Z} \geq 1$

Proof by Induction

Problem 4

1. Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps

3, 6, 9, 10, 12, 13, 15, 16, 18, 19, 20, 21, 22, 23, 24 ...

can form any amount after 18 cents

2. Prove your answer to first part by using strong induction.

Base Case: $P(18) = 6(3)$

Inductive Step: lets assume $m+8$ for $n \geq m \geq 0$
& all from 8 to $n+18$

$n=0: \quad 6(3) = 18$ ✓

$n=1: \quad 3(3) + 1(10) = 19$ ✓

$n \geq 2: \quad (n-2)+18$
$= n+16$

$\Rightarrow n+16 + 1(3)$
$= n+19$
$= (n+1)+18$

$\therefore$ $n+18$ cents can be formed for $\forall n \geq 0$
or $\forall n \geq 18$

Proof by Strong Induction

Problem 5

Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls. First state the problem as an induction argument and then construct a proof.

$P(n)$: n^{th} domino falls for $\forall n \geq 1$

Base Case: $n = 1, 2, \&\ 3$

Inductive Hypothesis: Assume $\forall n \leq k$, $P(n)$

- $P(k-2)$ true $\Rightarrow$ $P(n)$ true for $\forall n \leq k$
- $P(k+1)$ is 3 after $P(k-2)$
- $P(k+1)$ also true as 3 dominoes further also fall and $P(k-2)$ is true

$\therefore$ $P(n)$ is true for $\forall \geq 1$

Proof by Strong Induction