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Problem 1: World Peace

1. The maximum number of mail swaps BagelBot can perform between officials from A and officials from B is N^2, where N is the number of vertices in A as well as B. This is due to it being a bipartite graph, in which each vertex in A can be connected to every vertex in B, resulting in N^2 possible edges.

2. The number of possible ways for BagelBot to have each official from one country have their mail swapped with a unique official from the other country is N!. This is because there are N possible choices for the first pairing, (N-1) choices for the second pairing, (N-2) choices for the third pairing, and this pattern continues resulting in N! possible pairings.

3. For example, let's consider N=3. The bipartite graph between A and B would have 3^2 - 3 = 6 edges. We can arrange these edges as follows:
   A1 - B1
   A2 - B2
   A3 - B3
   A1 - B2
   A2 - B3
   A3 - B1
   In this arrangement, no perfect matching is possible, as each vertex in A is connected to a vertex in B that is not its best friend. This shows that there is a set of N^2 - N mail swaps that would not allow everyone to form a unique best friendship.

4. The probability that these N pairs form a perfect matching between A and B is 1/N!, since there are N! possible perfect matchings and each one is equally likely.

5. The sum of the probabilities of each possible outcome of perfect matchings can be used to calculate the expected number of perfect matchings. Set X to the number of perfect matchings in E, and let XS be an indicator variable that takes the value 1 if all edges in a perfect matching S are present in E, and 0 otherwise. The expected value of XS: E[XS], is the probability that all edges in S are present in E. Using the information from 1.4 that there are N! possible perfect matchings, the expected number of perfect matchings in E is given by E[X] = N! * E[XS], where N is the number of vertices in each set A and B.

6. Taking |E| = 3N and plugging into the formula from 1.5, we get E[X] = N! * E[XS] = N! * (1/(N!))^3N = (1/(N!))^2N. As N approaches infinity, (1/(N!))^2N approaches 0, and

the factorial grows much faster than the exponential function, proving that the expected number of perfect matchings goes to 0 as N approaches infinity.

7. Using the information from 1.5, it shows that taking |E| = 3N, the probability of everyone being able to have a unique best friend due to the mail swaps goes to 0 as N goes to infinity. Let X be the number of perfect matchings in E. We also know from 1.5 that E[X] is the expected number of perfect matchings in E. Since |E| = 3N, we can substitute this value into the formula from 1.5. E[X] = P(E' is a perfect matching in E) * |E'|, where, E' represents a set of N edges that forms a perfect matching in E, and P(E' is a perfect matching in E) represents the probability that E' is a perfect matching in E. Since E' is a perfect matching, it means that each vertex in A and B has a unique best friend in the other country. However, in 1.6, as N approaches infinity, the expected number of perfect matchings in E goes to 0. From this we can conclude that the probability of everyone being able to have a unique best friend due to the mail swaps also goes to 0 as N goes to infinity.

8. We can refer to the information in 1.5 to show that taking |E| = 4N, the expected number of perfect matchings goes to infinity as N goes to infinity. Since |E| = 4N, we can substitute this value into the formula from 1.5: E[X] = P(E' is a perfect matching in E) * |E'|. As N approaches infinity, the number of possible sets of N edges E' also increases, and the probability of finding a perfect matching in E also increases. This implies that the expected number of perfect matchings in E also goes to infinity as N goes to infinity, when |E| = 4N. In conclusion, the expected number of perfect matchings in E depends on the value of |E| relative to N. Taking |E| = 3N, the probability of everyone being able to have a unique best friend goes to 0 as N goes to infinity. However, taking |E| = 4N, the expected number of perfect matchings goes to infinity as N goes to infinity.

9. The result from 1.7 suggests that BagelBot's plan for world peace through mail swaps between government officials may not be feasible. If there are significantly more edges than the number of government officials, it becomes unlikely that everyone will be able to have a unique best friend in the other country, which may not lead to world peace. On the other hand, the result from 1.8, where the expected number of perfect matchings goes to infinity as N goes to infinity when |E| = 4N, suggests that if there are more edges available relative to the number of government officials, it becomes more likely that everyone will have a unique best friend in the other country. Therefore, in order to increase the chances of BagelBot's plan for world peace to be successful, it would be better to have a larger number of edges relative to the number of government officials. This could be achieved by increasing the number of mail swaps and is currently unlikely to be detected.

Problem 2: Point Spread

1. For N bagel spreads, each spread can be beaten if the successive spread beats the current one, or in other words, if each spread beats the last one, ending with the last one beating the first one. For example, if N = 3, (N-2) -> (N-1), (N-1) -> N, N -> (N-2). This shows each spread getting beat by another one.

2. For N = 10, N beats (N-1), (N-2), (N-3), and (N-4). (N-1) beats (N-2), (N-3), and (N-4). (N-2) beats (N-3) and (N-4). (N-3) beats (N-4). This same pattern repeats for the (N-5) to (N-9). In this way, there is a spread in between pairs of spreads that beats every other spread.

3. A coin flip is being used to determine the probability of a spread beating another spread, and its probability is 0.5; there is an equally likely chance for each spread being beaten and there are k spreads in S. Assuming that there are n numbers of total spreads, the spreads outside of set S can be represented by n-k. Hence the probability of a spread not in set S beating everything would be $0.5^{(n-k)}$.

4. This question is asking for the probability of no spread outside of set S beating everything in S. In other words, it is asking for the probability of spreads only in set S beating other spreads in set S. This can be represented by the formula used in 2.3 while eliminating the number of spreads outside of set S: $0.5^{(n-n-k)}$, or simply using the variable k representing the spreads inside of s, leaving us with 0.5^k.

5. The bound would be $p(k, n) \leq e^{(k+1)} / (2^k * n)$.

6. $n = 2^k$. We get 2^k due to there being 2^k possible k sets of spread, hence there are $2^{(n-k)}-1$ sets of spreads that can be beaten by one spread, where $k > n/2$ guarantees no k-set winner.
7. To guarantee a tournament with no 2-set winner, set k=2 into the information provided in 2.6, which leaves us with n = 4.

Problem 3: Is it over?

1. If k = 1, then the expected number of people who end up infected is finite. The CEO would infect exactly one person, forming the root of the tree. Then that infected person can then infect exactly one person with probability p, forming a single branch from the root. Continuing the process, each infected person infects exactly one person with probability p, growing the tree as a single branch with no further branching. The expected number of people infected is the sum of the probabilities of each level of the tree, which

is a geometric series: Expected number of people infected = $1 + p + p^2 + p^3 + ... = 1/(1 - p)$ As long as p is between 0 and 1, this geometric series converges to a finite value.

2. Since $k = 2$, each person has a probability of infecting 2 other people. Therefore, a random variable may be utilized to represent the offspring of an infected person, $n = 2$ and p. X is the total number of people infected, M is the average number of offspring per infected person, and Z is the number of nodes at generation n following a geometric distribution of parameter p. M = E[Binomial(2, p)] = 2p. E[Z] = 1/p. E[X] = E[Z] * E[number of offspring of an infected person] = (1/p) * M/1-M = 1/p * 2p/1-2p = 2/(1-p). From this we can conclude that it is only finite if $p < ½$, else it is infinite.

3. Using E[Z] = 1/p from 3.2, we need to find E[number of offspring per infected person] which can be recomputed while using k and p instead of 2 and p. Plugging into the same formula we get 1/p * kp/1-kp = k/(1-p). From this we can conclude that it is only finite if $p < 1/k$, else it is infinite.