1. if $f(n) = O(g(n))$ and $g(n) = O(h(n))$

f grows no faster than g g grows no faster than h

a) $f(n) + g(n) = \Theta(g(n))$

f + g grow at the same rate as g

To prove $f(n) + g(n) = \Theta(g(n))$, we can approach this by proving $f(n) + g(n) = \Omega(g(n))$ to be true at the same time as $f(n) + g(n) = O(g(n))$.

In other words, if $f+g \geq g$ and $f+g \leq g$, then $f+g = g$.

$\underline{f(n) + g(n) = \Omega(g(n))}$

$0 \leq f(n) \leq g(n)$ for all $n \geq n_0$ (given)

assuming for a constant c, $c \cdot g(n) \geq 0$

it is possible that $\frac{c \cdot (f(n) + g(n))}{c} \geq \frac{c \cdot g(n)}{c}$

$1 \cdot (f(n) + g(n)) \geq 1 \cdot g(n) \rightarrow f(n) + g(n) = \Omega g(n)$

$\underline{f(n) + g(n) = O(g(n))}$

for constants c_1 and c_2, there may be $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$

assuming $f(n) \geq 0$ as well as $c_1 \cdot g(n) = \Omega(g(n))$ it is possible that $\frac{c_2 \cdot (f(n) + g(n))}{c_2} \leq \frac{c_2 \cdot g(n)}{c_2}$

$1 \cdot (f(n) + g(n)) \leq 1 \cdot g(n)$

$\rightarrow f(n) + g(n) = O\, g(n)$

therefore, by the definition of Θ

$f(n) + g(n) = \Theta(g(n))$ is satisfied

b) $f(n) + g(n) = O(h(n))$

f + g grow no faster than h

Using the definition of O, as well as the information given that $f(n) = O\, g(n)$ and $g(n) = O(h(n))$:

$f(n) = O(g(n))$ if there exists a constant c & $n_0 > 0$ where $n \geq n_0$, $0 \leq f(n) \leq c \cdot g(n)$

$g(n) = O(h(n))$ if there exists a constant c & $n_0 > 0$ where $n \geq n_0$ $0 \leq g(n) \leq c \cdot h(n)$

Therefore, $0 \leq \frac{f(n)}{c} + \frac{g(n)}{c} \leq g(n)$

$0 \leq \frac{f(n)}{c} + \frac{g(n)}{c} \leq c_2 \cdot h(n)$

$0 \leq f(n) + g(n) \leq c \cdot c_2 \cdot h(n)$

This can be simplified into $f(n) + g(n) = O(h(n))$, satisfying the proof.

2. consider the functions $n!$, 2^n, n^d, $\log n$, $\log^* n$

(for some positive constant d)

a) $f(n) = \theta(f(n-1))$

$n!$, 2^n, n^d, $\log n$, $\log^* n$

b) $f(n) = \theta(f(\frac{n}{2}))$

2^n, n^d, $\log n$, $\log^* n$

c) $f(n) = \theta(f(\sqrt{n}))$

2^n, n^d, $\log n$, $\log^* n$

d) $f(n) = \theta(f(\log n))$

$\log n$, $\log^* n$

3. prove $\log n! = \Theta(n \log n)$ using upper and lower bounds of $n!$ from the quiz.

upper bound: n^n lower bounds: $(\frac{n}{2})^{\frac{n}{2}}$, $2^{\frac{n}{2}}$

Step	Reason
$n! \leq n^n$	given upper bound
$\log(n!) \leq \log(n^n)$	log of both sides
$\log(n!) \leq n \log(n)$	property of logs
$(\frac{n}{2})^{(\frac{n}{2})} \leq n!$ $2^{(\frac{n}{2})} \leq n!$	given lower bounds
$\log((\frac{n}{2})^{(\frac{n}{2})}) \leq \log(n!)$ $\log(2^{(\frac{n}{2})}) \leq \log(n!)$	log of both sides
$(\frac{n}{2})\log(\frac{n}{2}) \leq \log(n!)$ $(\frac{n}{2})\log(2) \leq \log(n!)$ $(\frac{1}{2}) n \log n \leq \log(n!)$	property of logs

$\left(\frac{1}{2}\right) n \log n \leq \log(n!) \leq n \log(n)$ | simplify

Since both sides of the inequality have the same growth, linear in n, it is safe to assume that log(n!) grows at the same rate as nlog(n).

4. Let $f(n) = \sum_{i=1}^{n} b^i$ for some constant $b > 0$

a) $f(n) = \theta(1)$ if $b < 1$

the sum of this function can be written as $b^n - \frac{1}{b-1}$. taking $\lim_{n \to \infty} b^n - \frac{1}{b-1}$. since $b < 1$, the b^n converges to 0, and the $\frac{-1}{b-1}$ part simplifies down to some constant, $= \theta(1)$.

b) $f(n) = \theta(n)$ if $b = 1$

the pattern of the summation of this would be $1^1 + 1^2 + 1^3 + \ldots$ which would be the same as n, hence $= \theta(n)$

c) $f(n) = \theta(b^n)$ if $b > 1$

the same expression from part a can be used: $b^n - \frac{1}{b-1}$. when we do $\lim_{n \to \infty} b^n - \frac{1}{b-1}$, however, since $b > 1$, the $b^n \to \infty$, deeming the $-\frac{1}{b-1}$ irrelevant, hence $= \theta(b^n)$.

5. Indicate whether the following functions are true or false. Justify.

a) $2^n = \Omega(4^{\sqrt{n}})$

Lets assume $n \geq n_0$. simplifying $4^{\sqrt{n}} = 4^{\frac{n}{2}} = 2^{2 \cdot \frac{n}{2}} = 2^n$

$2^n = \Omega(2^n) \Rightarrow 2^n \geq 2^n$. this statement is true.

b) $n^{\log n} = O(2^n)$

both sides of the equation can be represented in terms of e.

$n^{\log n} = e^{(\log n)^2}$ and $2^n = e^{n \log 2} \Rightarrow n^{\log n} \leq 2^n \Rightarrow n^{\log n} = O(2^n)$

this statement is true.

c) $\log(\log(n!)) = \theta(\log((\log n)!))'$

$\log(\log n!)$ grows slower compared to $n!$, therefore it must grow slower than $\log((\log n)!)$. since it is slower

$\log(\log(n!)) \neq \theta(\log((\log n)!))$. this statement is false.

d) $n^{\log(\log n)} = \theta((\log n)^{\log n})$

$\log(\log n) \leq \log n$, so $n^{\log(\log n)} < (\log n)^{\log n}$. since $n^{\log(\log n)}$ is slower, $n^{\log(\log n)} \neq \theta((\log n)^{\log n})$. this statement is false

e) $4^{\log n} = \Omega(2^{\sqrt{n}})$

$4^{\log n} = 2^{2 \log n} = 2^{\log n^2} = n^2 \Rightarrow n^2 = \Omega(2^{\sqrt{n}}) = n^2 \geq 2^{\sqrt{n}}$,

which is correct. this statement is true.

f) $n2^n = O(3^n)$

$n2^n$ grows exponentially, but 3^n grows at an exponentially faster rate than $n2^n$, leaving $n2^n$ to be bounded by 3^n. this statement is true.

g) $n^{0.1} = \Theta((\log n)^{10})$

$n^{0.1}$ grows very slow and cannot be compared to an exponential growth such as $(\log n)^{10}$, hence $n^{0.1} \neq \Theta((\log n)^{10})$.
this statement is false.

h) $n! = O(2^n)$

$n!$ grows at an exponential rate just as 2^n
this statement is true.

i) $n \log \log n = \Omega(n^{0.9} + n(\log n)^2)$

$n \log \log n$ grows much slower than $n^{0.9} + n(\log n)^2$;
$n \log \log n \neq \Omega(n^{0.9} + n(\log n)^2)$.
this statement is false.

6. Extra Credit

a) $\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \leq \frac{1}{1} + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right),$$ hence $\frac{1}{i} \leq \log n$

b) $\sum_{i=1}^{n} \frac{1}{i} = \Omega(\log n)$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \geq 0 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right),$ hence $\frac{1}{2} \geq \log n$