Analysis of Axially Loaded Members

ME 210: Mechanics of Materials

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Outline

Axially Loaded Member

- Back to the basic: 1D stress-strain relationship
- Load is applied along the axis
- Load passes through the centroid
- For now, ignore lateral deformation

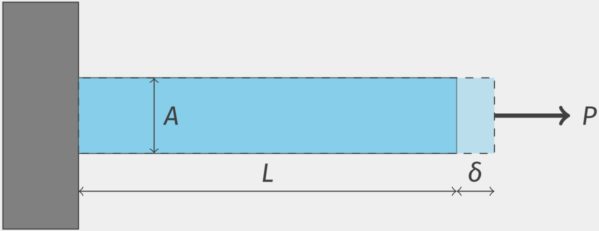
2 Types of Problems in Statics

- Statically determinate: equilibrium equation is all you need
- Statically Indeterminate Problems
  - equilibrium isn't enough

Outline

Statically Determinate Axially Loaded Members

- Simplest of them all
- Well, simplest doesn't really mean simple
- All you need is equilibrium equation... and a smart way to use it



- For any member with constant cross section

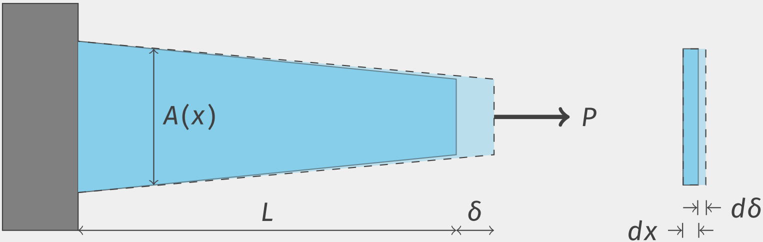
$$\sigma(x) = \frac{P}{A} \varepsilon(x) = \frac{\delta}{L}$$

Hooke's Law

$$\sigma(x) = E\varepsilon$$

$$\frac{P}{A} = E\frac{\delta}{L}$$

$$\delta = \frac{PL}{EA}$$



$$\sigma(x) = E(x)\varepsilon(x)$$

$$\frac{P(x)}{A(x)} = E(x)\frac{d\delta}{dx}$$

$$\delta = \int_0^L d\delta = \int_0^L \frac{P(x)}{A(x)E(x)}dx$$

Specific Conditions

- Multiple loads over multiple cross sections

$$\delta = \sum \frac{P_i L_i}{E_i A_i}$$

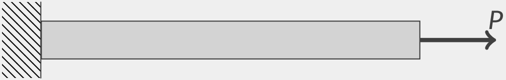
Deformation Analysis Steps

1. Determine force in each section of member
2. Determine properties of member
3. Use appropriate formula to solve

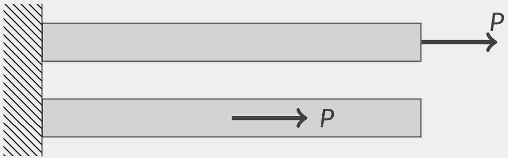
Finding Internal Forces

1. Convert ALL supports into support forces and moments
2. Draw boundary between free end and cross section of interest
3. Use equilibrium equation to find forces/loads at cross section

- P = 500 N, A = 2.5 cm^2, L = 2 m, δ = ?

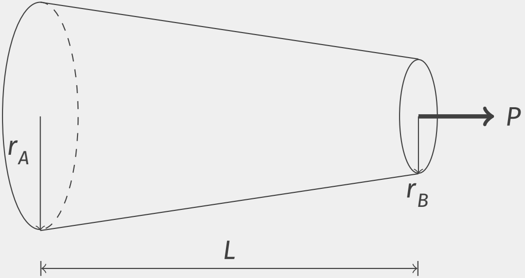


Example: Same bars, Same forces, Different locations



Example: Deformation of a continuously variable bar

- What is the elongation of the bar?



- load is constant throughout the length of the cylinder
- area is not constant
- we must use

$$\delta = \int_0^L d\delta = \int_0^L \frac{Pdx}{EA(x)}$$

- need to write area as a function of length $A = A(x)$

- Since the cylinder is actually a cone, radius r of area A is a linear function of x

$$r = mx + c$$

  where m is the slope and c is the y-intercept
- we know that at $x = 0, r = r_A$ and at $x = L, r = r_B$

$$\begin{aligned} m &= \frac{dr}{dx} = \frac{\Delta r}{\Delta x} = \frac{r_B - r_A}{L} \\ c &= r(x = 0) = r_A \\ r &= \frac{r_B - r_A}{L} x + r_A \end{aligned}$$

- We can now evaluate the integral

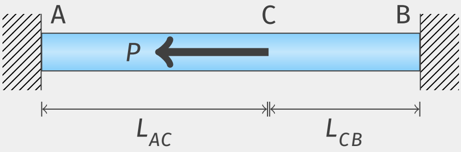
$$
\begin{aligned}
\delta &= \int_0^L \frac{P dx}{E\pi\left(\frac{r_B - r_A}{L}x + r_A\right)^2} \\
&= -\frac{PL}{\pi E\left(r_B - r_A\right)}\left[\frac{1}{\frac{r_B - r_A}{L}x + r_A}\right]_0^L \\
&= -\frac{PL}{\pi E\left(r_B - r_A\right)}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \\
&= \frac{PL}{\pi E r_A r_B}
\end{aligned}
$$

Outline

- For statically determinate problems, equilibrium is sufficient
- When members are constrained, equilibrium is not enough

- Fortunately, constraints typically provide geometric conditions → compatibility equations
- Constraints → restriction of deformation

Example: Basic Bar between Two Walls



- Equilibrium equation

$$F_B + F_A - P = 0$$

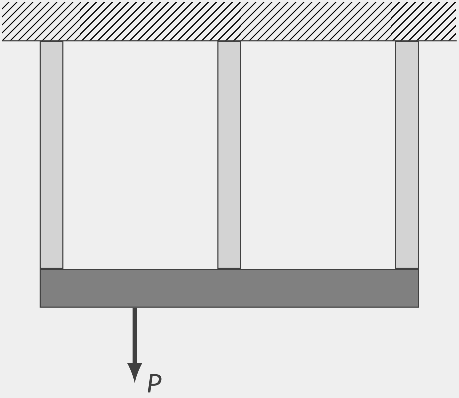
- Compatibility equation: fixed between two walls = no deformation

$$\delta_{AB} = 0$$
$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$
$$F_A = P\frac{L_{CB}}{L}; F_B = P\frac{L_{AC}}{L}$$

Solving SI Problems

1. Draw FBD of member(s)
2. Write equilibrium equation(s)
3. Consider geometry restrictions or constrains
4. Express them in compatibility equations
5. Apply Hooke's law to compatibility equations and solve

Example: Three Bars attached to Rigid Beam



- What are the forces in the bars if they have the same E, A?

Assuming all internal forces are tensile, they all point upward on the rigid beam.
Equilibrium:

$$\sum F_y = 0$$
$$F_1 + F_2 + F_3 = P$$

$$\sum M = 0$$
$$F_2 L + F_3(2L) = P\frac{L}{2}$$
$$F_2 + 2F_3 = \frac{P}{2}$$

Rigid beam can tilt, but not bend. Use similar triangles, compatibility equation is:

$$\frac{\delta_1 - \delta_3}{2L} = \frac{\delta_2 - \delta_3}{L}$$
$$\delta_1 - \delta_3 = 2\delta_2 - 2\delta_3$$
$$\delta_1 = 2\delta_2 - \delta_3$$

Convert compatibility to force equation using Hooke's Law

$$\frac{F_1 L}{AE} = 2\frac{F_2 L}{AE} - \frac{F_3 L}{AE}$$
$$F_1 = 2F_2 - F_3 F_2 = \frac{P}{2} - 2F_3$$
$$F_1 = 2F_2 - F_3 = P - 5F_3$$

Now, substitute this into the first equilibrium equation.

$$F_1 + F_2 + F_3 = P - 5F_3 + \frac{P}{2} - 2F_3 + F_3 = P$$

$$6F_3 = \frac{P}{2}$$

$$F_3 = \frac{P}{12}$$

$$F_1 = P - 5\frac{P}{12} = \frac{7P}{12}$$

$$F_2 = \frac{P}{2} - \frac{2P}{12} = \frac{P}{3}$$

Thermal Strains: Oh It's Baaaccccckkkk!!!!

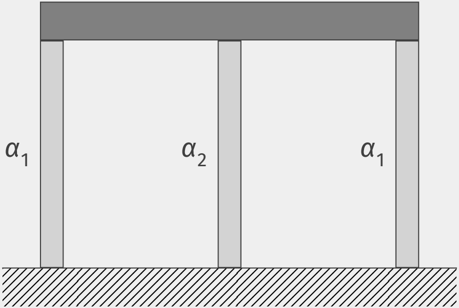
- Remember this?

$$
\begin{aligned}
\frac{1}{L}\frac{dL}{dT} &= \alpha \\
\varepsilon_T &= \alpha\Delta T = \alpha\left(T_2 - T_1\right)
\end{aligned}
$$

Temperature + Force = Pain

- Thermal stress → temperature change while constrained
- Combination of mechanical and thermal loads
- How do we deal with all dis shite?
- Superposition! → a fancy way of saying just add them up

Example: Thermal Stress in Fastened Bars



- What are the forces in each beam when the temperature is changed by ΔT?

Equilibrium equations

$$\sum F_y = 0$$
$$F_1 + F_2 + F_3 = 0$$

Using symmetry,

$$F_1 = F_3$$
$$F_2 = 2F_3 = 2F_1$$

Or using moment equilibrium

$$\sum M = 0$$
$$F_2 L + F_3(2L) = 0$$
$$F_2 = -2F_3$$

Moment equilibrium about the right side of the rigid beam gives the same equation.

$$F_2 = -2F_1 = -2F_3$$

With symmetry, all bars undergo identical deformation. Compatibility:

$$\delta_1 = \delta_2 = \delta_3$$

Hooke's Law:

$$\frac{F_1 L}{AE} + \alpha_1 \Delta T L = \frac{F_2 L}{AE} + \alpha_2 \Delta T L$$

Substituting $F_1 = F_2/2$, we have

$$F_2 = \frac{2}{3}\left(\alpha_1 - \alpha_2\right)\Delta TAE$$
$$F_1 = F_3 = \frac{1}{3}\left(\alpha_2 - \alpha_1\right)\Delta TAE$$

We have previously assumed all internal forces tensile, so if the sign comes out positive, the force *is* tensile. It is compressive otherwise. For $\alpha_1 > \alpha_2$ and $\Delta T > 0$

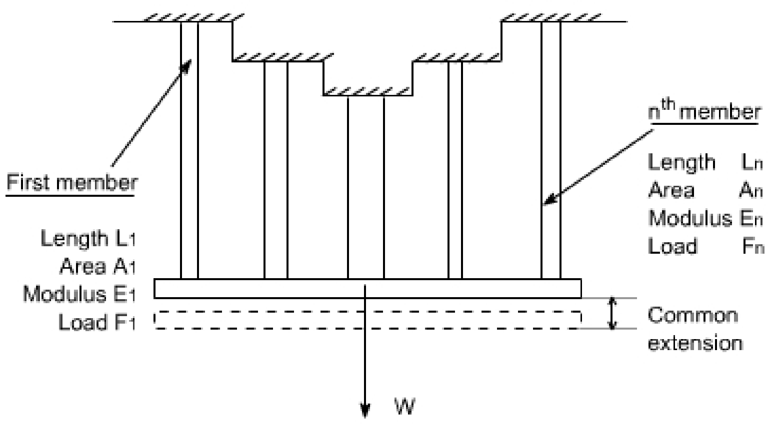
$$F_2 > 0 \text{ and } F_1, F_3 < 0$$

When the members are heated the left and right members will try to expand more than the middle one due to their higher coefficients of thermal expansion α_1. However, because of the rigid beam restriction, the left and right members are squeezed down, while the middle part is pulled. Other situations can be analyzed with similar logic.

Compatibility Equation

- One member: wall or support limits deformation
- More members: walls or attachment to rigid parts dictates deformation
- Check symmetry

Outline



- Multiple members that share the same deformations
- A special case of statically indeterminate problem

Force in Member of Compound Bar

- For any member

$$\delta = \delta_i = \frac{F_i L_i}{E_i A_i}$$

$$F_i = \frac{\delta E_i A_i}{L_i}$$

- Equilibrium equation

$$W = \sum F_i = \sum \frac{\delta E_i A_i}{L_i}$$

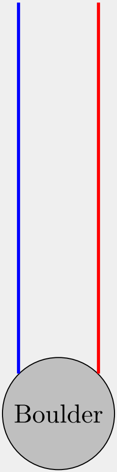
- Fraction of force in member i

$$\frac{F_i}{W} = \frac{\frac{E_i A_i}{L_i}}{\sum \frac{E_i A_i}{L_i}}$$

- Modulus of the compound bar, if members have the same length

$$\begin{aligned}
W &= F_1 + F_2 + \ldots \\
\sigma\left(A_1 + A_2 + \ldots\right) &= \sigma_1 A_1 + \sigma_2 A_2 + \ldots \\
\frac{\sigma}{\varepsilon}\left(A_1 + A_2 + \ldots\right) &= \frac{\sigma_1}{\varepsilon} A_1 + \frac{\sigma_2}{\varepsilon} A_2 + \ldots \\
E_c\left(A_1 + A_2 + \ldots\right) &= E_1 A_1 + E_2 A_2 + \ldots \\
E_c &= \frac{\sum EA}{\sum A}
\end{aligned}$$

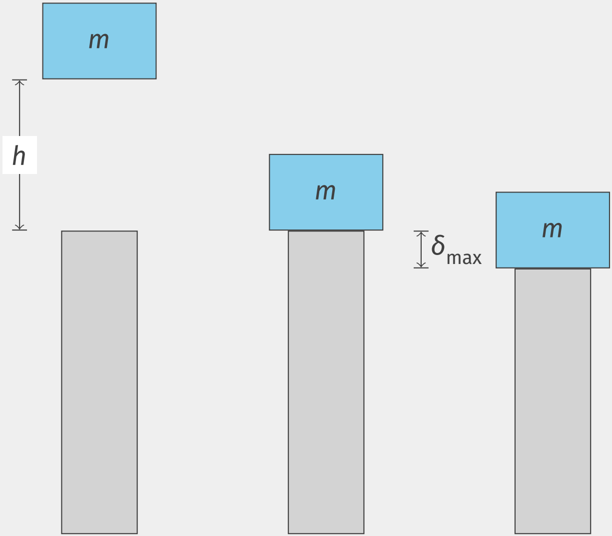
Example: Helping out a Friend



- Two equal length and cross section cables: steel (E = 210 GPa) and copper (E = 80 GPa)
- Boulder weighs 200 kg
- Who's carrying heavier load?

Outline

Object under Impact Loading



Maximum Deformation under Impact Loading

- Weight dropped from h

$$W\left(h+\delta_{\max}\right)=\frac{EA\delta_{\max}^{2}}{2L}$$

$$\delta_{\max}=\frac{WL}{EA}+\left[\left(\frac{WL}{EA}\right)^{2}+2h\left(\frac{WL}{EA}\right)\right]^{1/2}$$

- But what is $\frac{WL}{EA}$?

Max Deformation Compared to Static Load Deformation

- Max deformation in terms of static deformation

$$\delta_{\max} = \delta_{st} + \left[\delta_{st}^2 + 2h\delta_{st}\right]^{1/2}$$

- When $h \gg \delta_{st}$

$$\delta_{\max} = \sqrt{2h\delta_{st}} = \sqrt{\frac{mv^2L}{EA}}$$

Maximum Stress from Impact Loading

- Since $\delta = \dfrac{\sigma L}{E}$

$$\sigma_{\text{max}} = \frac{E\delta_{\text{max}}}{L}$$

$$\sigma_{\text{max}} = \frac{W}{A} + \left[\left(\frac{W}{A}\right)^2 + \frac{2hE}{L}\frac{W}{A}\right]^{1/2}$$

$$\sigma_{\text{max}} = \sigma_{st} + \left[\left(\sigma_{st}\right)^2 + \frac{2hE}{L}\sigma_{st}\right]^{1/2}$$

- If h is large

$$\sigma_{\text{max}} = \sqrt{\frac{2hE}{L}\sigma_{st}} = \sqrt{\frac{mv^2E}{AL}}$$

Determine the maximum allowable mass m that you can drop from the height h of 3 m a concrete block with A = 1 cm^2, E = 80 GPa, and L = 0.5 m so that the the resultant stress is below σ_{allow} = 10 MPa and δ_{max} < 3 mm.

Solution

2 conditions: σ_{allow} vs δ_{max}

σ_{allow}:

$$\sigma_{allow} = \sigma_{max} = \frac{W}{A} + \left[\left(\frac{W}{A}\right)^2 + \frac{2hE}{L}\frac{W}{A}\right]^{1/2}$$

$$10 \times 10^6 = \frac{W}{1 \times 10^{-4}} + \left[\left(\frac{W}{1 \times 10^{-4}}\right)^2 + \frac{2(3)(80 \times 10^9)}{0.5}\frac{W}{1 \times 10^{-4}}\right]^{1/2}$$

$$W = 0.01\ \text{N}$$

$$m = 0.01/10 = 0.001\ \text{kg}$$

We can also use the approximation (as a 3-m drop is pretty high, probably much higher than δ_{st}). Instead,

$$\begin{aligned}
\sigma_{\max} &= \sqrt{\frac{2hE}{L}\sigma_{st}} \\
10 \times 10^6 &= \sqrt{\frac{2(3)(80 \times 10^9)}{0.5}\sigma_{st}} \\
\sigma_{st} &= 104 \\
\frac{mg}{A} &= \frac{m(10)}{1 \times 10^{-4}} = 104 \\
m &= 0.00104 \text{ kg}
\end{aligned}$$

So we have essentially obtained the same answer.

Solution δ_{max}:

$$\delta_{max} = 0.003 = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2h\left(\frac{WL}{EA}\right)\right]^{1/2}$$

$$0.003 = \frac{W(0.5)}{80 \times 10^9 (10^{-4})} + \left[\left(\frac{W(0.5)}{80 \times 10^9 (10^{-4})}\right)^2 + 2h\left(\frac{W(0.5)}{80 \times 10^9 (10^{-4})}\right)\right]^{1/2}$$

$$W = 24 \text{ N}$$

$$m = 24/10 = 2.4 \text{ kg}$$

Solution Let's again try the short method and compare.

$$
\begin{aligned}
\delta_{\max} &= \sqrt{2h\delta_{st}} \\
0.003 &= \sqrt{2(3)\delta_{st}} \\
\delta_{st} &= 1.5 \times 10^{-6} = \frac{WL}{EA} \\
W = mg &= \frac{1.5 \times 10^{-6}(80 \times 10^{9})(1 \times 10^{4})}{0.5} = 24 \\
m &= 24/10 = 2.4 \text{ kg}
\end{aligned}
$$

The smaller of the two loads, 0.001 kg is our final answer.