Analysis of Torque Loaded Members

ME 210: Mechanics of Materials

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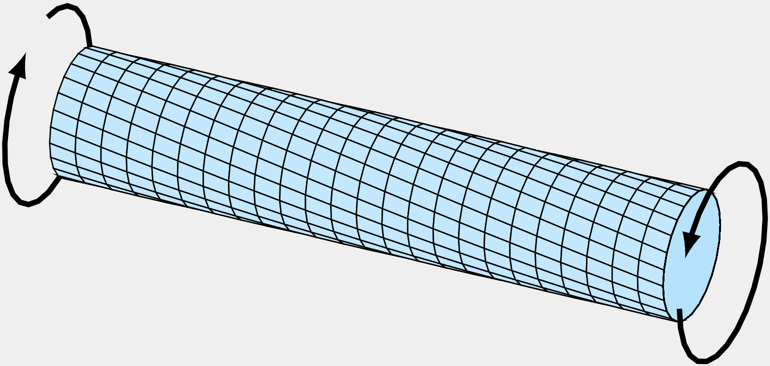
What is Torsion?

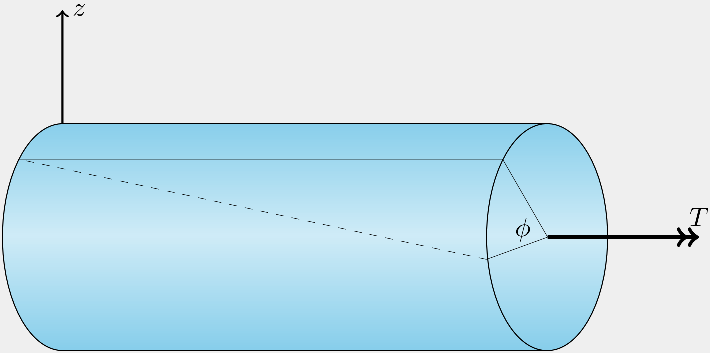
- Twisting of a straight bar loaded by torque (torsional moment)
- Twisting happens about its longitudinal axis

State of Stress in Torsion



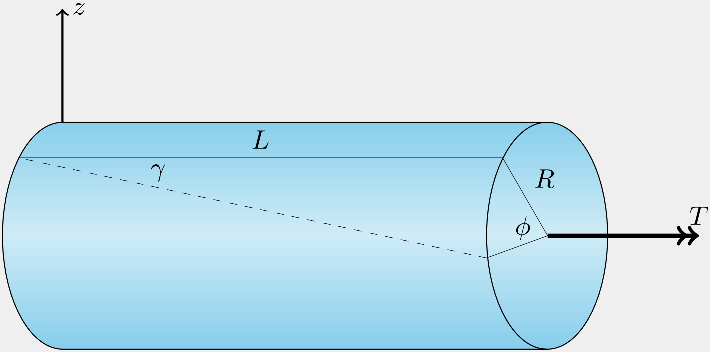
Torsional Deformation in a Circular Bar

- When twisted, all cross-section remains circular and subjected to the same torque – *pure torsion*.



- ϕ is called the *angle of twist*, increasing linearly along the length of the bar

- Change in element angle is shear strain



$$\gamma = \frac{R d\phi}{dx} = R\theta = \frac{R\phi}{L}$$

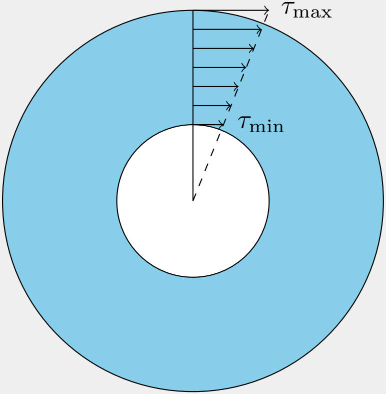
- Maximum shear strain happens at the outer surface

$$\gamma_{\max} = \frac{R\phi}{L}$$

- What about minimum shear strain?

$$\gamma_{\min} = \ldots$$

State of Stress in a Circular Bar under Torsion

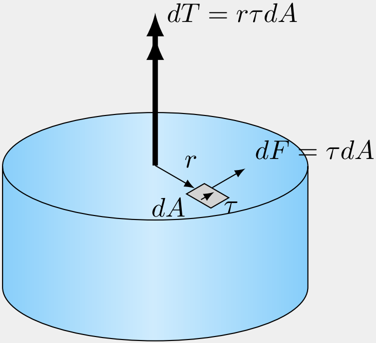


- For linear elastic deformation

$$\tau = G\gamma$$

$$\tau_{\max} = G\gamma_{\max} = GR\frac{\phi}{L}$$

$$\tau = Gr\frac{\phi}{L} = \tau_{\max}\frac{r}{R}$$



$$dF = \tau dA$$

$$dT = r\tau dA = \frac{r^2\tau_{max}}{R} dA$$

$$\int_0^T dT = \frac{\tau_{max}}{R}\int_A r^2 dA = \frac{\tau_{max}}{R} J$$

Torsional Shear Stress Formula

$$\tau = \frac{Tr}{J}$$

- Solid cylindrical shaft

$$
\begin{aligned}
I_p &= \int_A r^2 dA \\
&= \int_0^{2\pi} \int_0^R r^2 r dr d\theta \\
&= \frac{\pi}{2} R^4
\end{aligned}
$$

- Hollow shaft: how do we do that?

Example: Minimum Shaft Radius

T_{max} = 1735 N-m

τ_{max} = 200 MPa

r_{min} = ?

Solution: Minimum Required Radius

$$\tau = \frac{Tr}{J} = \frac{2T}{\pi r_{\min}^3}$$

$$r_{\min} = \left(\frac{2(1735)}{\pi(200 \times 10^6)}\right)^{1/3}$$

$$= 0.0177 = 1.77 \text{ cm}$$

Isn't that a bit small?

Deformation under Torsion: Angle of Twist ϕ

- Combine Hooke's law and Torsion formula

$$\tau_{\max} = GR\theta = \frac{TR}{J}$$

$$\theta = \frac{\phi}{L} = \frac{T}{GJ}$$
$$\phi = \frac{TL}{GJ} = \frac{T}{k_T}$$

- k_T is called the *torsional stiffness*

Example: Shaft Design

A gasoline engine has a maximum torque output of 300 N-m. Your boss wants you to design a 2-m long shaft that is going to limit the angle of twist to $\phi \leqslant 0.1$ rad. The shaft should be made of medium carbon steel $G = 80$ GPa, $\tau_{\text{allow}} = 200$ MPa.

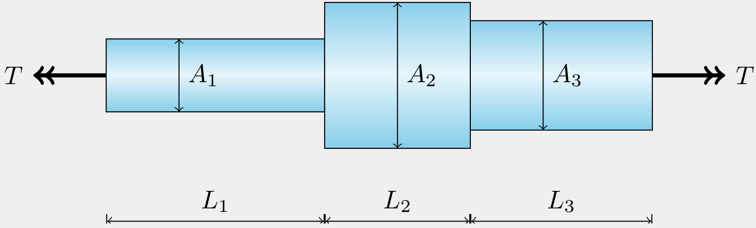
Two conditions: τ_{allow} and ϕ

$$
\begin{aligned}
R_\tau &= \left(\frac{2T}{\pi\tau_{\text{allow}}}\right)^{1/3} \\
&= \left(\frac{2(300)}{\pi(200\times10^6)}\right)^{1/3} \\
&= 9.85\times10^{-3}\text{ m} = 9.85\text{ mm} \\
\phi &= \frac{TL}{GJ} = \frac{2TL}{G\pi R_\phi^4} \\
R_\phi &= \left(\frac{2(300)(2)}{(80\times10^9)\pi(0.1)}\right)^{1/4} \\
&= 0.0148\text{ m} = 1.48\text{ cm}
\end{aligned}
$$

We will need to design based on the bigger requirement, 1.48 cm.

- T, J, or G is not constant
  - Segments
  - Continuously varying

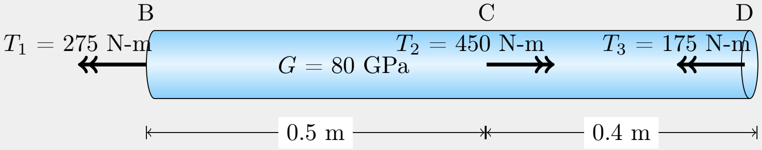
- Determine internal torques and corresponding deformations



$$\phi = \phi_1 + \phi_2 + \phi_3 + \ldots$$

$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

Example: Shaft with Various Segments



- Find τ_{max} in each segments and ϕ_{BD}. Let R = 1.5 cm.

Use the method of section to determine torque within segment BC,

$$T_{BC} = T_1 = 275 \text{ Nm}$$

Torque within segment CD,

$$T_{CD} = T_3 = -175 \text{ Nm}$$

The maximum shear stress in each segment is at the outer diameter. We have

$$\tau_{\max} = \frac{Tr}{J} = \frac{2T}{\pi r^3}$$

$$(\tau_{\max})_{BC} = \frac{2(275\ \mathrm{Nm})}{\pi(1.5 \times 10^{-2}\ \mathrm{m})^3} = 51.9\ \mathrm{MPa}$$

$$(\tau_{\max})_{CD} = \frac{2(175\ \mathrm{Nm})}{\pi(1.5 \times 10^{-2}\ \mathrm{m})^3} = 33\ \mathrm{MPa}$$

Angle of twist between B and D is the sum of the angles of twist in BC and CD.

$$\phi_{BD} = \phi_{BC} + \phi_{CD}$$

$$J = \frac{\pi r^4}{2} = \frac{\pi (1.5 \times 10^{-2}\ \text{m})^4}{2} = 7.95 \times 10^{-8}\ \text{m}^4$$

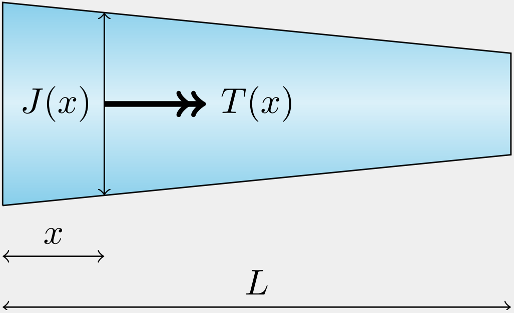
$$\phi_{BC} = \frac{T_{BC} L_1}{GJ} = \frac{(275\ \text{Nm})(0.5\ \text{m})}{(80\ \text{GPa})(7.95 \times 10^{-8}\ \text{m}^4)} = 0.0216\ \text{rad}$$

$$\phi_{CD} = \frac{T_{CD} L_2}{GJ} = \frac{(-175\ \text{Nm})(0.4\ \text{m})}{(80\ \text{GPa})(7.95 \times 10^{-8}\ \text{m}^4)} = -0.0110\ \text{rad}$$

$$\phi_{BD} = 0.0216 - 0.0110 = 0.0106\ \text{rad}$$

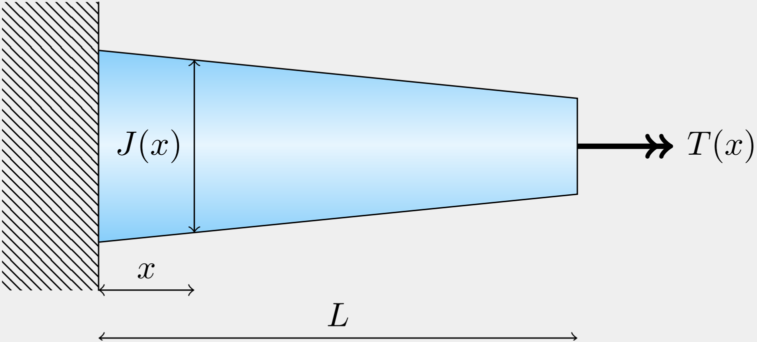
Therefore, the bar twisted in the same direction as T_2 by 0.0106 rad.

- Back to integration



$$d\phi = \frac{T(x)dx}{G(x)J(x)}$$
$$\phi = \int_0^L \frac{T(x)dx}{GJ(x)}$$

Example: Continuously Varying Shaft



- What is the total angle of twist ϕ?

Solution

$$\begin{aligned}
\phi &= \int_0^L \frac{T\,dx}{GJ(x)} = \frac{T}{G}\int_0^L \frac{dx}{J(x)} \\
&= \frac{T}{G}\int_0^L \frac{dx}{(\pi/2)\left(\frac{r_2 - r_1}{L}x + r_1\right)^4} \\
&= \frac{2TL}{3\pi G(r_2 - r_1)}\left(\frac{r_2 - r_1}{L}x + r_1\right)^{-3}\Bigg|_0^L \\
&= \frac{2TL}{3\pi G(r_2 - r_1)}\left(-\frac{1}{r_2^3} + \frac{1}{r_1^3}\right)
\end{aligned}$$

Power Transmission through Shaft

- The most important application of shaft is rotational power transmission

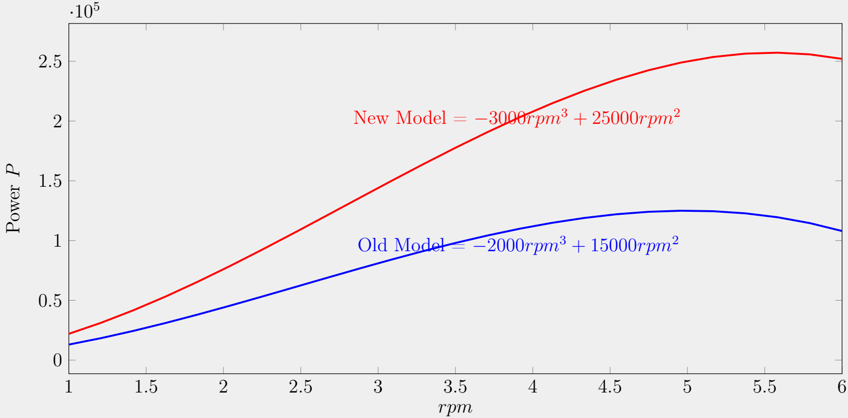
$$P = T\omega$$

- Engine power and speed are typically in *hp* and *rpm*

$$1 \text{ hp} = 746 \text{ W}$$

$$1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

Example: Shaft Design for an Engine



Is it safe to use the old shaft? τ_{allow} = 200 MPa.

Solution

First order of business is determining maximum torque required.
From $P = T\omega$

$$T_{\text{old}} = -2000(\frac{2\pi}{60})^3\omega^2 + 15000(\frac{2\pi}{60})^2\omega$$
$$T_{\text{new}} = -3000(\frac{2\pi}{60})^3\omega^2 + 25000(\frac{2\pi}{60})^2\omega$$

Solution Finding max torque is easy now

$$\frac{dT_{\text{old}}}{d\omega} = 0 = -4000\left(\frac{2\pi}{60}\right)^3\omega + 15000\left(\frac{2\pi}{60}\right)^2$$

$$\omega^*_{old} = \frac{15 \times 60}{4 \times 2\pi} = 35.8 \text{ rad/s} = 342 \text{ rpm}$$

$$T_{\text{old, max}} = 2945 \text{ N-m}$$

Solution

Max torque for the new engine is

$$
\begin{aligned}
\frac{dT_{\text{new}}}{d\omega} = 0 &= -6000\left(\frac{2\pi}{60}\right)^3\omega + 25000\left(\frac{2\pi}{60}\right)^2 \\
\omega^*_{new} &= \frac{25 \times 60}{6 \times 2\pi} = 39.8 \text{ rad/s} = 380 \text{ rpm} \\
T_{\text{new, max}} &= 5454 \text{ N-m}
\end{aligned}
$$

Since the new model requires a larger torque, the old shaft will *NOT* work

Solution

We can now design the new shaft using T_{new}

$$\tau_{max} = \frac{TR}{J} = \frac{2T}{\pi R^3}$$

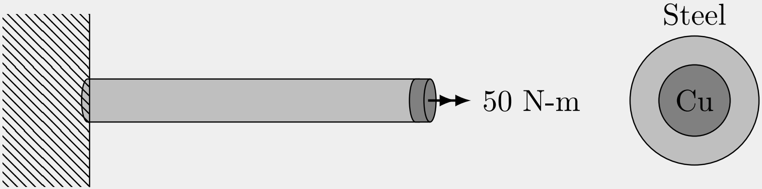
$$R = \left[\frac{2(5454)}{\pi(200 \times 10^6)}\right]^{1/3}$$

$$= 0.026 \text{ m} = 2.6 \text{ cm}$$

Statically Indeterminate Torque-Loaded Members

1. Equilibrium equation: torque
2. Compatibility: angle of twist ϕ constraints
3. Hooke's law: relate torque to angle of twist

Example: Compound Shaft



G_{Cu} = 50 GPa G_{St} = 80 GPa

1. What is the angle of twist at the end compared to the wall?
2. What is the equivalent shear modulus of the bar?

This is a statically indeterminate problem, with the rigid disk acting as a constraint on the angle of twist of the steel and copper sections.
1: Equilibrium

$$T_{cu} + T_{st} = 50$$

2: Compatibility – being constrained by the rigid disk, the two materials must rotate by the same amount ϕ

$$\phi_{cu} = \phi_{st} = \phi$$

3: Hooke's Law

$$\frac{T_{cu}L}{G_{cu}J_{cu}} = \frac{T_{st}L}{G_{st}J_{st}}$$

$$T_{cu} = \frac{50T_{st}(\pi/2)(0.05^4)}{80(\pi/2)(0.1^4 - 0.05^4)}$$

$$T_{cu} = \frac{T_{st}}{24}$$

Solution

Plug back into equilibrium equation,

$$\begin{aligned} 25T_{\text{cu}} &= 50 \\ T_{\text{cu}} &= 2 \text{ N-m} \\ T_{\text{st}} &= 48 \text{ N-m} \end{aligned}$$

Once we obtained the torques, finding the angle of twist is easy:

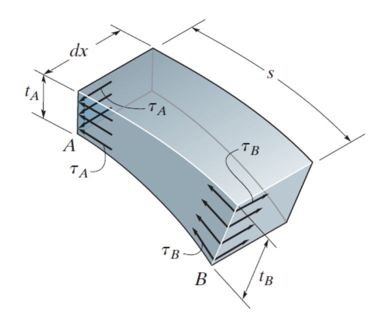
$$\begin{aligned} \phi &= \frac{T_{\text{cu}} L}{G_{\text{cu}} I_{\text{cu}}} \\ &= \frac{2(1)}{50 \times 10^9 (\pi/2) 0.05^4} \\ &= 4.08 \times 10^{-6} \text{ rad} \end{aligned}$$

Solution: Equivalent shear modulus

Similar to equivalent modulus for compound bar:
Given T, find modulus of a single material with same J and L that gives same ϕ

$$4.08 \times 10^{-6} = \frac{50(1)}{G_e(\pi/2)0.1^4}$$

$$G_e = 78.1 \times 10^9 \text{ Pa} = 78.1 \text{ GPa}$$

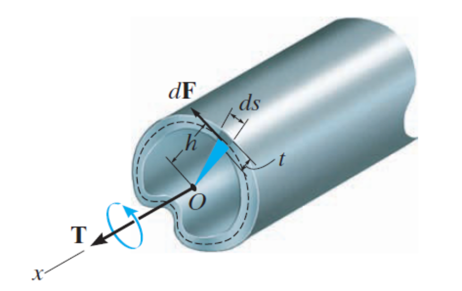


- Equilibrium

$$\tau_A t_A dx = \tau_B t_B dx$$
$$\tau_A t_A = \tau_B t_B = q$$

- q is called *shear flow* and is constant over the cross section
- τ is maximum at the thinnest part

Torsion Formula for Thin-walled Tubes



$$dT = r\,dF = rq\,ds$$

$$T = q \oint r\,ds$$

$$T = 2A_m q$$

$$q = \tau t = \frac{T}{2A_m}$$

$$\tau = \frac{T}{2A_m t}$$

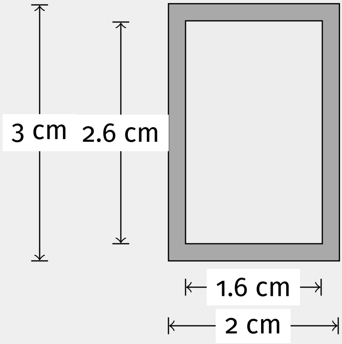
- Derived using energy method

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

- Intimidating, but actually quite simple
- For tubes with segments of constant thickness

$$\phi = \frac{TL}{4A_m^2 G} \left[\frac{s_1}{t_1} + \frac{s_2}{t_2} + \ldots \right]$$

Example: Shear Stress and Angle of Twist in Box Steel



- If the box steel is 2 m long with 20 N-m torque applied, determine τ_{max} and ϕ. Steel has G = 80 GPa

Since the hollow section thickness is constant, τ is constant

$$\begin{aligned}
\tau &= \frac{T}{2A_m t} = \frac{20}{2(0.018 \times 0.028)(0.002)} \\
&= 9.92 \text{ MPa} \\
\phi &= \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} = \frac{20(2)}{4(0.018 \times 0.028)^2(80 \times 10^9)} \left[\frac{2(0.018 + 0.028)}{0.002} \right] \\
&= 2.26 \times 10^{-2} \text{ rad}
\end{aligned}$$