

# **ANALYSIS OF MULTIAXIAL STRESS**

ME 210: MECHANICS OF MATERIALS

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# WHY SHOULD I CARE ABOUT THIS?

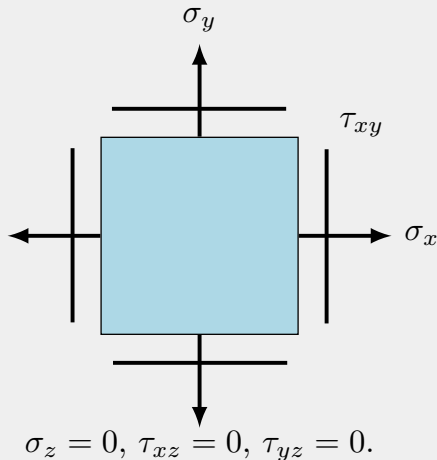
- Many components can be loaded by multiple forces/torques/moments
- How do we know if material will fail?
- We need to understand their states of stress first
  - ▶ Need to determine max normal and max shear stress

# DETERMINING MAX NORMAL AND SHEAR STRESSES

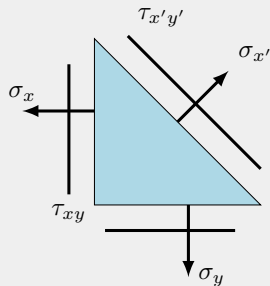
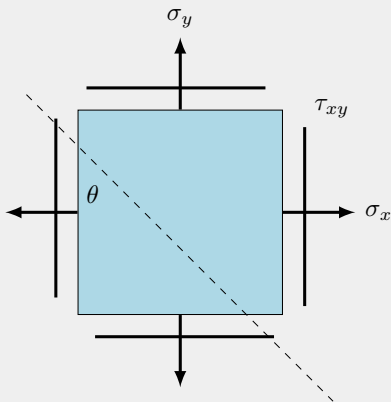
- Equivalent normal and shear stresses change with respect to orientation
- 2 types of simplification: *plane stress* and *plane strain*

# PLANE STRESS: WHAT'S THAT?

- Part of material with only *in-plane* normal and shear stresses
- No stress in *out-of-plane* direction



# STRESS TRANSFORMATION FOR PLANE STRESS



- Finding maximum normal and shear stresses and their direction
- Use equilibrium to solve

# EQUATIONS FOR STRESS TRANSFORMATION

$$\sum F_{x'} = 0;$$

$$\begin{aligned} \sigma_{x'} \Delta A - (\tau_{xy} \Delta A \sin \theta) \cos \theta - (\sigma_y \Delta A \sin \theta) \sin \theta \\ - (\tau_{xy} \Delta A \cos \theta) \sin \theta - (\sigma_x \Delta A \sin \theta) \cos \theta = 0 \end{aligned}$$

$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

$$\sum F_{y'} = 0;$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

■  $\sigma'_y = ?$

# WHERE ARE MY MAX STRESSES?

- Find the max

$$\frac{d\sigma'_x}{d\theta} = 0 = -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta$$

$$\frac{d\sigma'_y}{d\theta} = 0 = (\sigma_x - \sigma_y) \sin 2\theta - 2\tau_{xy} \cos 2\theta$$

- For both normal stress in x and y

$$2\theta_p = \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- $\theta_p$  is the *principal direction*

# GRAPHICAL REPRESENTATION OF STATE OF STRESS

- Rewrite stresses using double angle

$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

$$\vdots$$

$$\sigma_{x'} - \left( \frac{\sigma_x + \sigma_y}{2} \right) = \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

$$\vdots$$

$$\tau_{x'y'} = \left( \frac{\sigma_y - \sigma_x}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta$$



# GETTING THERE...

- Square both terms and add

$$\begin{aligned} \left[ \sigma_{x'} - \left( \frac{\sigma_x + \sigma_y}{2} \right) \right]^2 + \tau_{x'y'}^2 = \\ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 \cos^2 2\theta + 2 \left( \frac{\sigma_x - \sigma_y}{2} \right) \tau_{xy} \cos 2\theta \sin 2\theta + \tau_{xy}^2 \sin^2 2\theta \\ + \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 \sin^2 2\theta + 2 \left( \frac{\sigma_y - \sigma_x}{2} \right) \tau_{xy} \cos 2\theta \sin 2\theta + \tau_{xy}^2 \cos^2 2\theta \end{aligned}$$

# THE REPRESENTATION

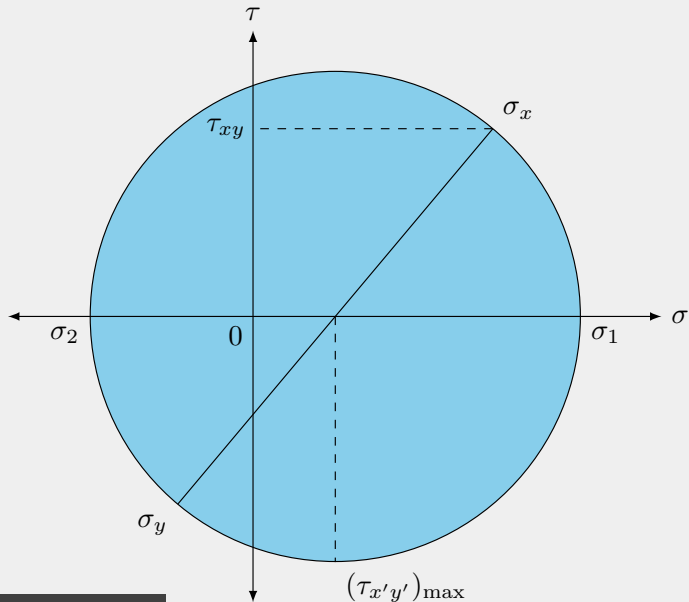
- Use trigonometry identities

$$\left[ \sigma_{x'} - \left( \frac{\sigma_x + \sigma_y}{2} \right) \right]^2 + \tau_{x'y'}^2 = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2$$

- What shape does that take?

$$\left( \sigma_{x'} - \sigma_{avg} \right)^2 + \tau_{x'y'}^2 = R^2$$

# MOHR'S CIRCLE



# PRINCIPAL STRESSES

- Maximum and minimum normal stresses = normal stresses at principal direction

$$\sigma_{x'}(\theta = \theta_p) = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{y'}(\theta = \theta_p) = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- $\sigma_1$  and  $\sigma_2$  are called *principal stresses*

# SHEAR STRESSES AT $\theta_p$

$$\tau_{x'y'}(\theta = \theta_p) = 0$$

- No shear stress in the principal direction, *ever!*
- Check on Mohr's circle
- So where's the maximum shear stress?

# MAXIMUM IN-PLANE SHEAR STRESS

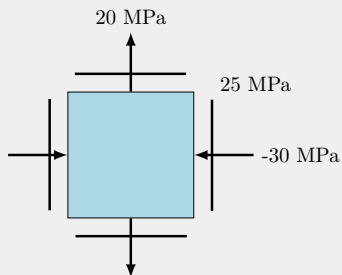
$$\frac{d\tau_{xy}}{d\theta} = 0 = 2 \left( \frac{\sigma_y - \sigma_x}{2} \right) \cos 2\theta - 2\tau_{xy} \sin 2\theta$$
$$\tan 2\theta_s = \frac{\sigma_y - \sigma_x}{2\tau_{xy}}$$

- $\theta_s$  is the maximum shear stress direction

$$\tau_{\max} = \tau_{xy}(\theta = \theta_s)$$

- $\tau_{\max}$  is the maximum in-plane shear stress

## EXAMPLE: STATE OF STRESS OF A SQUARE ELEMENT



■ First, find  $\tau_{\max}$

$$\begin{aligned}\tau_{\max} &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \sqrt{\left(\frac{-30 - 20}{2}\right)^2 + 25^2} \\ &= 35.4 \text{ MPa}\end{aligned}$$

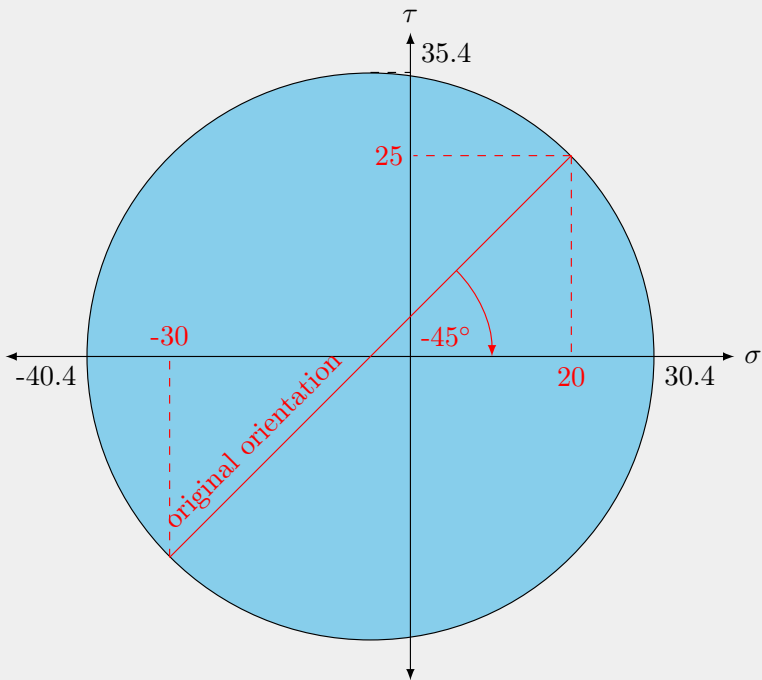
- Principal stresses are

$$\begin{aligned}\sigma_{1,2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{\sigma_x - \sigma_y}{2}^2 + \tau_{xy}^2} \\ &= \frac{-30 + 20}{2} \pm 35.4 \\ &= 30.4 \text{ MPa}, -40.4 \text{ MPa}\end{aligned}$$

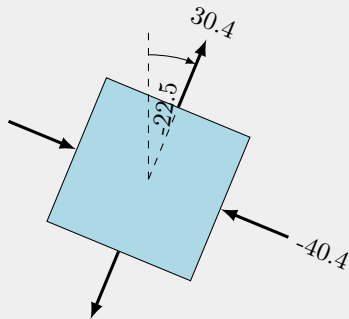
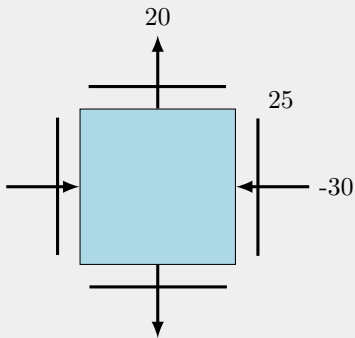
- Principal direction is

$$\begin{aligned}2\theta &= \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \\ &= -45^\circ \\ \theta &= -22.5^\circ\end{aligned}$$

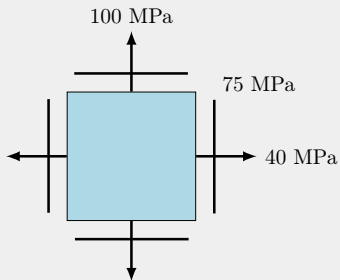




# ORIGINAL VS PRINCIPAL



## ANOTHER EXAMPLE:



■ First, find  $\tau_{\max}$

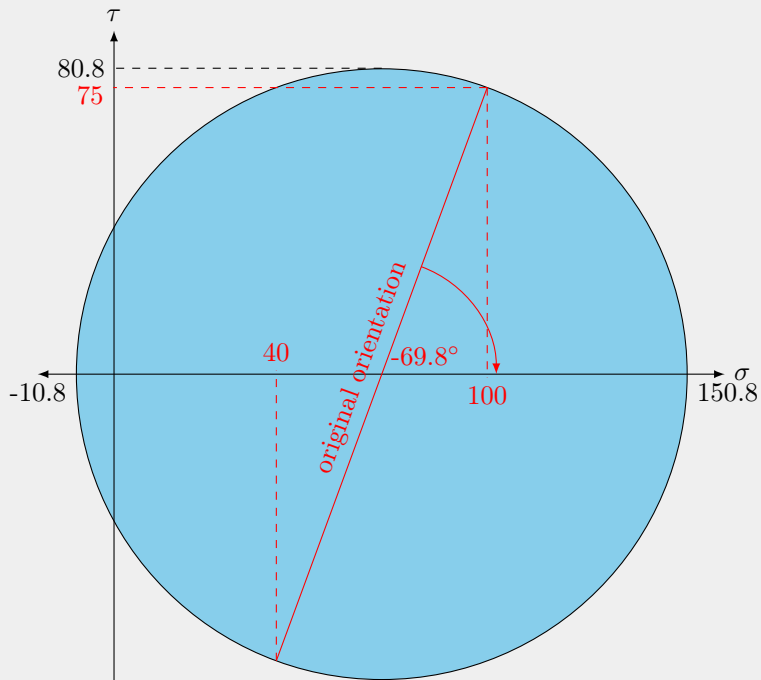
$$\begin{aligned}\tau_{\max} &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \sqrt{\left(\frac{40 - 100}{2}\right)^2 + 75^2} \\ &= 80.8 \text{ MPa}\end{aligned}$$

- Principal stresses are

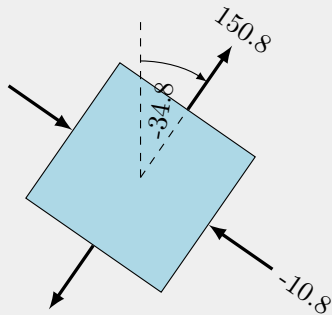
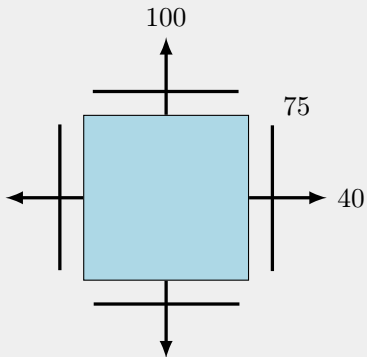
$$\begin{aligned}\sigma_{1,2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \frac{40 + 100}{2} \pm 80.8 \\ &= 150.8 \text{ MPa}, -10.8 \text{ MPa}\end{aligned}$$

- Principal direction is

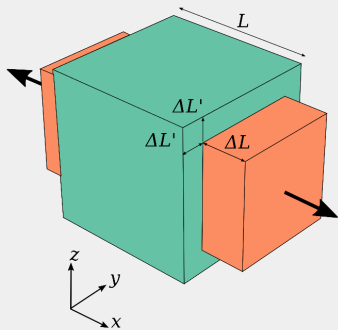
$$\begin{aligned}2\theta &= \tan^{-1} \frac{2(80.8)}{40 - 100} \\ &= -69.6^\circ \\ \theta &= -34.8^\circ\end{aligned}$$



# ORIGINAL VS PRINCIPAL



# HOOKE'S LAW FOR 3D STRESS



- Recall Poisson's ratio where

$$\nu = -\frac{\epsilon_{trans}}{\epsilon_{long}}$$

- In 3D materials, there are *two* transversal strains
  - For  $x$ , there are  $y$  and  $z$

# RETHINKING STRAINS IN 3D

	$\sigma_x$	$\sigma_y$	$\sigma_z$
$\epsilon_x$	$\frac{\sigma_x}{E}$	$-v \frac{\sigma_y}{E}$	$-v \frac{\sigma_z}{E}$
$\epsilon_y$	$-v \frac{\sigma_x}{E}$	$\frac{\sigma_y}{E}$	$-v \frac{\sigma_z}{E}$
$\epsilon_z$	$-v \frac{\sigma_x}{E}$	$-v \frac{\sigma_y}{E}$	$\frac{\sigma_z}{E}$



# VOLUME CHANGE

- Original volume of element  $V_0 = abc$
- Final volume is

$$\begin{aligned} V_{final} &= a (1 + \epsilon_x) b (1 + \epsilon_y) c (1 + \epsilon_z) \\ &\approx abc (1 + \epsilon_x + \epsilon_y + \epsilon_z) \end{aligned}$$

- Volume change is

$$V_{final} = abc (\epsilon_x + \epsilon_y + \epsilon_z)$$

# UNIT VOLUME CHANGE, $e$

- Also called *dilatation* or *volumetric strain*

$$e = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z$$

# SPHERICAL STRESS AND BULK MODULUS

- Spherical stress → same normal stresses in 3 axes

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \frac{1-2\nu}{E} \sigma_0$$

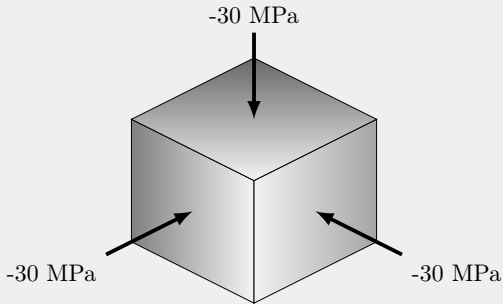
$$e = \frac{3(1-2\nu)}{E} \sigma_0$$

- Bulk modulus,  $K$ , is defined as

$$K = \frac{\sigma_0}{e} = \frac{E}{3(1-2\nu)}$$

## EXAMPLE: CUBE UNDER HYDROSTATIC PRESSURE

A  $0.25\text{-m}^3$  cube is being submerged under water where the water pressure is 30 MPa. If the material has  $E = 3 \text{ GPa}$  and Poisson's ratio of 0.3, find the final volume of the element.



## SOLUTION: CUBE UNDER HYDROSTATIC PRESSURE

Hydrostatic pressure situation represents the state where the pressure is equal in all direction. This means that we can assume that the element is under spherical stress. The unit volume change is

$$\begin{aligned} e &= \frac{3\sigma_o}{E} (1 - 2\nu) \\ &= \frac{3(-30 \times 10^6 \text{ Pa})(1 - 2(0.3))}{(3 \times 10^9 \text{ Pa})} \\ &= -0.012 \end{aligned}$$

## SOLUTION: CUBE UNDER HYDROSTATIC PRESSURE

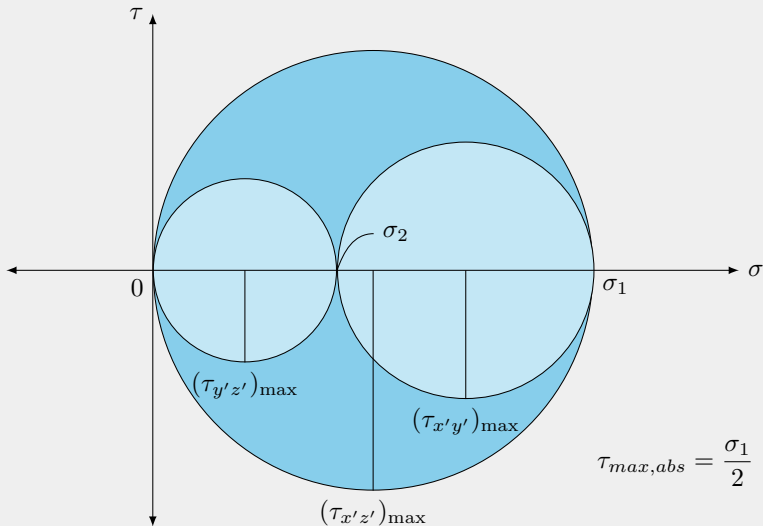
This means that the volume reduced by 1.2% (since hydrostatic pressure in this case is compressive), and thus the final volume is

$$\begin{aligned}V_f &= (1 + e)V \\&= 0.988(0.25 \text{ m}^3) \\&= 0.247 \text{ m}^3\end{aligned}$$

# ABSOLUTE MAXIMUM SHEAR STRESS

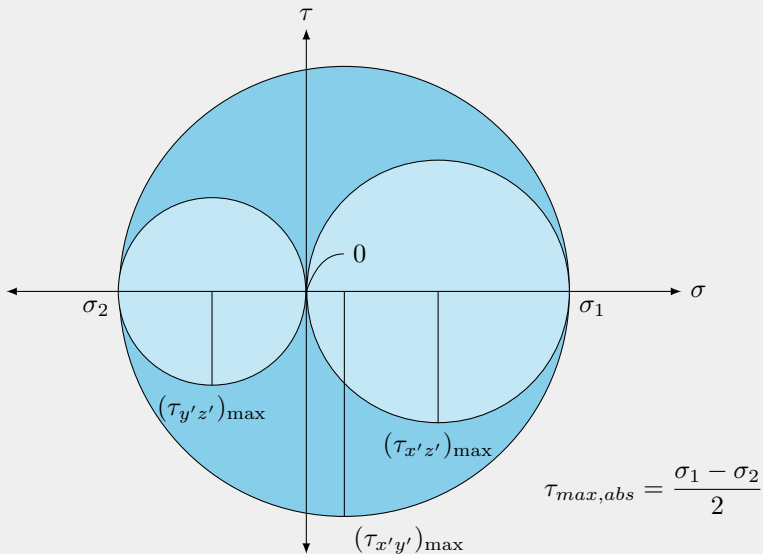
- In 3D plane stress problems, there are 3 principal stresses.
- The 3rd principal stress is 0.
- Use Mohr's circle to represent the relationship of all three.
- 2 cases:
  1.  $\sigma_1, \sigma_2 > 0$
  2.  $\sigma_1 > 0, \sigma_2 < 0$

## CASE I: STRESSES ARE OF THE SAME SIGN

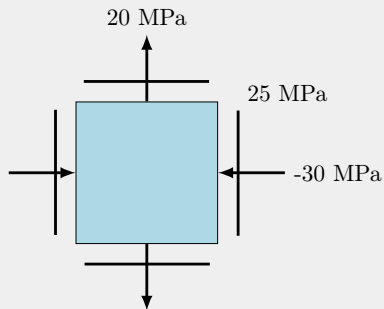




## CASE II: STRESSES ARE OF OPPOSITE SIGNS



## EXAMPLE



- From previous example, we have that  $\sigma_1 = 30.4$  MPa,  $\sigma_2 = -40.4$  MPa, and  $\tau_{\max} = 35.4$  MPa.
- Let us draw a Mohr's circle out of this.

