Analysis of Multiaxial Stress

ME 210: Mechanics of Materials

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Why Should I Care about This?

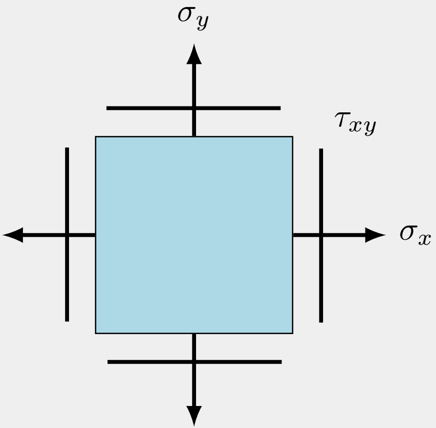
- Many components can be loaded by multiple forces/torques/moments
- How do we know if material will fail?
- We need to understand their states of stress first
  - Need to determine max normal and max shear stress

Determining Max Normal and Shear Stresses

- Equivalent normal and shear stresses change with respect to orientation
- 2 types of simplification: *plane stress* and *plane strain*

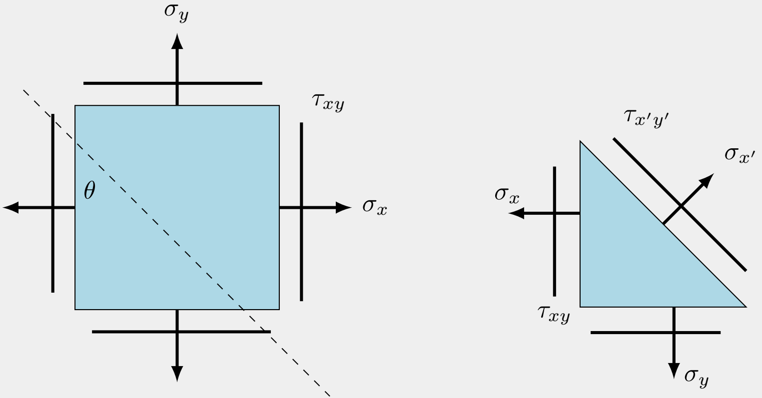
Plane Stress: What's That?

- Part of material with only *in-plane* normal and shear stresses
- No stress in *out-of-plane* direction



$$\sigma_z = 0,\ \tau_{xz} = 0,\ \tau_{yz} = 0.$$

Stress Transformation for Plane Stress



- Finding maximum normal and shear stresses and their direction
- Use equilibrium to solve

$$\sum F_{x'} = 0;$$

$$\sigma_{x'}\Delta A - (\tau_{xy}\Delta A \sin\theta)\cos\theta - (\sigma_y \Delta A \sin\theta)\sin\theta$$
$$- (\tau_{xy}\Delta A \cos\theta)\sin\theta - (\sigma_x \Delta A \sin\theta)\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sum F_{y'} = 0;$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

- $\sigma_y' = ?$

Where are My Max Stresses?

- Find the max

$$\frac{d\sigma_x'}{d\theta} = 0 = -\left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

$$\frac{d\sigma_y'}{d\theta} = 0 = \left(\sigma_x - \sigma_y\right)\sin 2\theta - 2\tau_{xy}\cos 2\theta$$

- For both normal stress in x and y

$$2\theta_p = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- θ_p is the *principal direction*

- Rewrite stresses using double angle

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$
$$\vdots$$
$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x + \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$\vdots$$
$$\tau_{x'y'} = \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

- Square both terms and add

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 =$$

$$\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 \cos^2 2\theta + 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\tau_{xy}\cos 2\theta \sin 2\theta + \tau_{xy}^2 \sin$$

$$+\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 \sin^2 2\theta + 2\left(\frac{\sigma_y - \sigma_x}{2}\right)\tau_{xy}\cos 2\theta \sin 2\theta + \tau_{xy}^2 \cos$$

The Representation

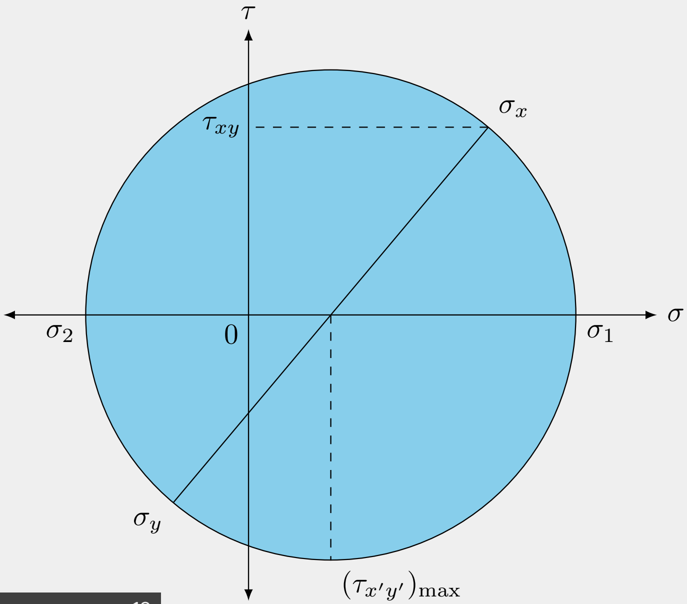
- Use trigonometry identities

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

- What shape does that take?

$$\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

Mohr's Circle



- Maximum and minimum normal stresses = normal stresses at principal direction

$$\sigma_{x'}(\theta = \theta_p) = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{y'}(\theta = \theta_p) = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- σ_1 and σ_2 are called *principal stresses*

Shear Stresses at θ_p

$$\tau_{x'y'}(\theta = \theta_p) = 0$$

- No shear stress in the principal direction, *ever!*
- Check on Mohr's circle
-
- So where's the maximum shear stress?

Maximum In-plane Shear Stress

$$\frac{d\tau_{xy}}{d\theta} = 0 = 2\left(\frac{\sigma_y - \sigma_x}{2}\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

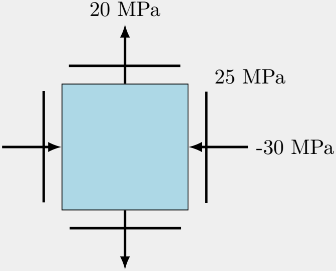
$$\tan 2\theta_s = \frac{\sigma_y - \sigma_x}{2\tau_{xy}}$$

- θ_s is the maximum shear stress direction

$$\tau_{\max} = \tau_{xy}(\theta = \theta_s)$$

- $\tau_{\max}$ is the maximum in-plane shear stress

Example: State of Stress of a Square Element



- First, find τ_{max}

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{-30 - 20}{2}\right)^2 + 25^2}$$

$$= 35.4 \text{ MPa}$$

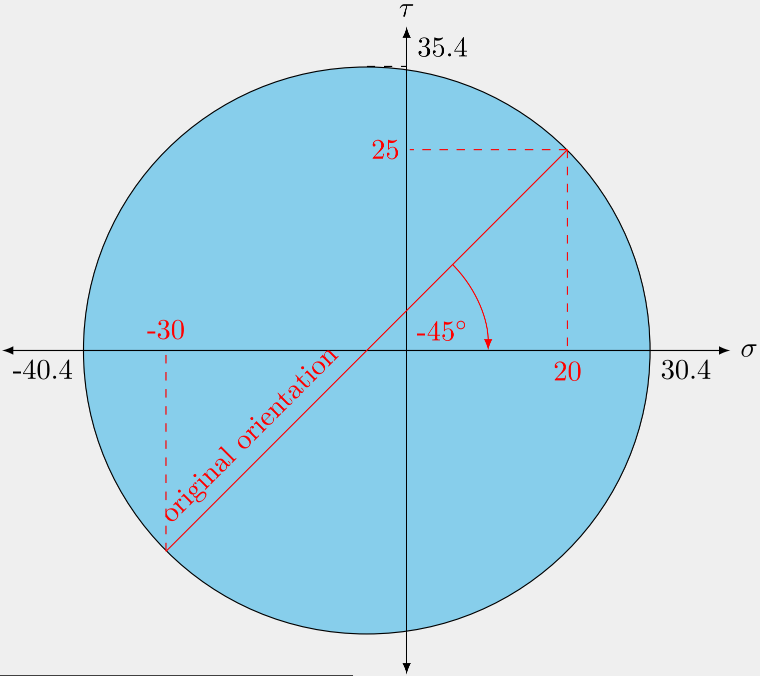
Continue

- Principal stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{\sigma_x - \sigma_y}{2}^2 + \tau_{xy}^2}$$
$$= \frac{-30 + 20}{2} \pm 35.4$$
$$= 30.4 \text{ MPa}, -40.4 \text{ MPa}$$

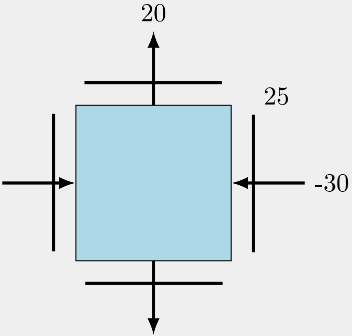
- Principal direction is

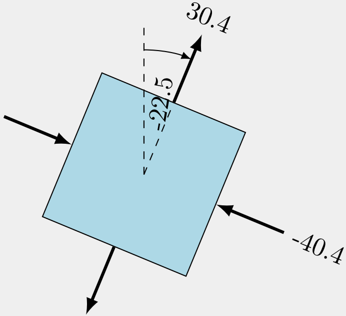
$$2\theta = \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$
$$= -45^\circ$$
$$\theta = -22.5^\circ$$



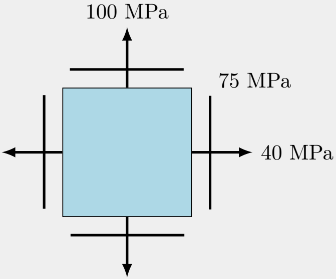
τ
35.4
25
-30
-45°
σ
-40.4
20
30.4
original orientation

Original vs Principal





Another Example:



- First, find τ_{max}

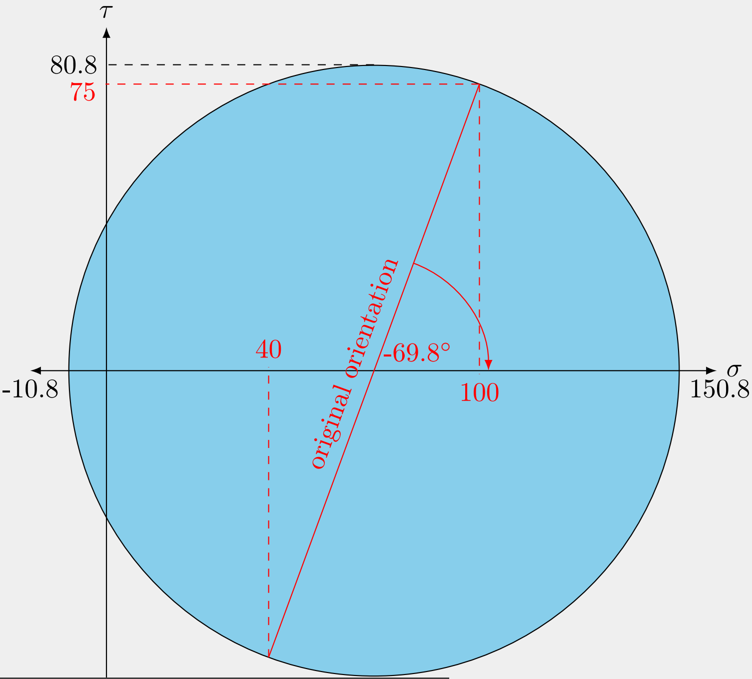
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= \sqrt{\left(\frac{40 - 100}{2}\right)^2 + 75^2}$$
$$= 80.8 \text{ MPa}$$

- Principal stresses are

$$\begin{aligned}\sigma_{1,2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{\sigma_x - \sigma_y}{2}^2 + \tau_{xy}^2} \\ &= \frac{40 + 100}{2} \pm 80.8 \\ &= 150.8 \text{ MPa}, -10.8 \text{ MPa}\end{aligned}$$

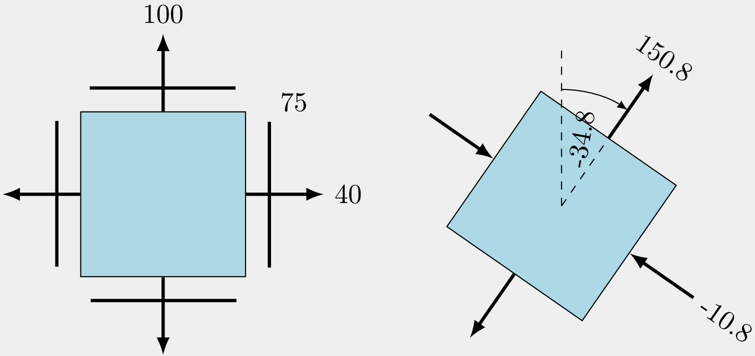
- Principal direction is

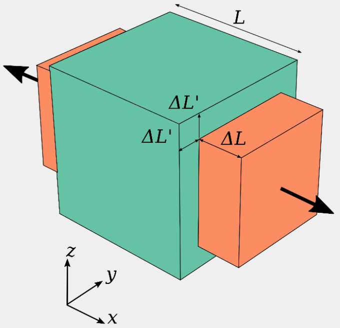
$$\begin{aligned}2\theta &= \tan^{-1} \frac{2(80.8)}{40 - 100} \\ &= -69.6^{\circ} \\ \theta &= -34.8^{\circ}\end{aligned}$$



τ
80.8
75
40
-69.8°
original orientation
σ
-10.8
100
150.8

Original vs Principal





- Recall Poisson's ratio where

$$\nu = -\frac{\varepsilon_{trans}}{\varepsilon_{long}}$$

- In 3D materials, there are *two* transversal strains
  - For x, there are y and z

Rethinking Strains in 3D

	σ_x	σ_y	σ_z
ε_x	$\frac{\sigma_x}{E}$	$-\nu\frac{\sigma_y}{E}$	$-\nu\frac{\sigma_z}{E}$
ε_y	$-\nu\frac{\sigma_x}{E}$	$\frac{\sigma_y}{E}$	$-\nu\frac{\sigma_z}{E}$
ε_z	$-\nu\frac{\sigma_x}{E}$	$-\nu\frac{\sigma_y}{E}$	$\frac{\sigma_z}{E}$

- Original volume of element $V_0 = abc$
- Final volume is

$$\begin{aligned} V_{final} &= a\left(1+\varepsilon_x\right) + b\left(1+\varepsilon_y\right) + c\left(1+\varepsilon_z\right) \\ &\approx abc\left(1+\varepsilon_x+\varepsilon_y+\varepsilon_z\right) \end{aligned}$$

- Volume change is

$$V_{final} = abc\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)$$

Unit Volume Change, e

- Also called *dilatation* or *volumetric strain*

$$e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Spherical Stress and Bulk Modulus

- Spherical stress → same normal stresses in 3 axes

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \frac{1 - 2v}{E}\sigma_0$$

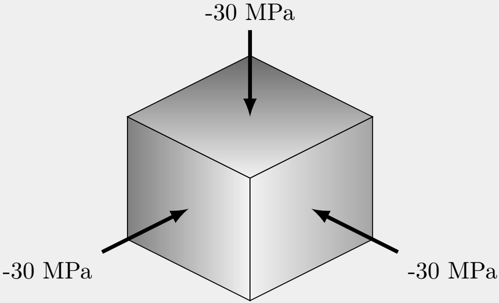
$$e = \frac{3(1 - 2v)}{E}\sigma_0$$

- Bulk modulus, K, is defined as

$$K = \frac{\sigma_0}{e} = \frac{E}{3(1 - 2v)}$$

Example: Cube under hydrostatic pressure

A 0.25-m^3 cube is being submerged under water where the water pressure is 30 MPa. If the material has E = 3 GPa and Poisson's ratio of 0.3, find the final volume of the element.



Solution: Cube under hydrostatic pressure

Hydrostatic pressure situation represents the state where the pressure is equal in all direction. This means that we can assume that the element is under spherical stress. The unit volume change is

$$\begin{aligned} e &= \frac{3\sigma_o}{E}(1-2v) \\ &= \frac{3(-30\times10^6\ \text{Pa})(1-2(0.3))}{(3\times10^9\ \text{Pa})} \\ &= -0.012 \end{aligned}$$

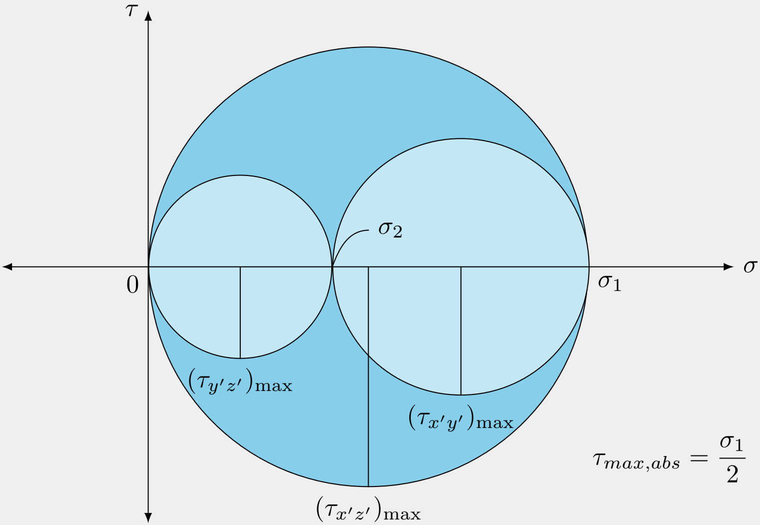
This means that the volume reduced by 1.2% (since hydrostatic pressure in this case is compressive), and thus the final volume is

$$\begin{aligned} V_f &= (1+e)V \\ &= 0.988(0.25\ \text{m}^3) \\ &= 0.247\ \text{m}^3 \end{aligned}$$

Absolute Maximum Shear Stress

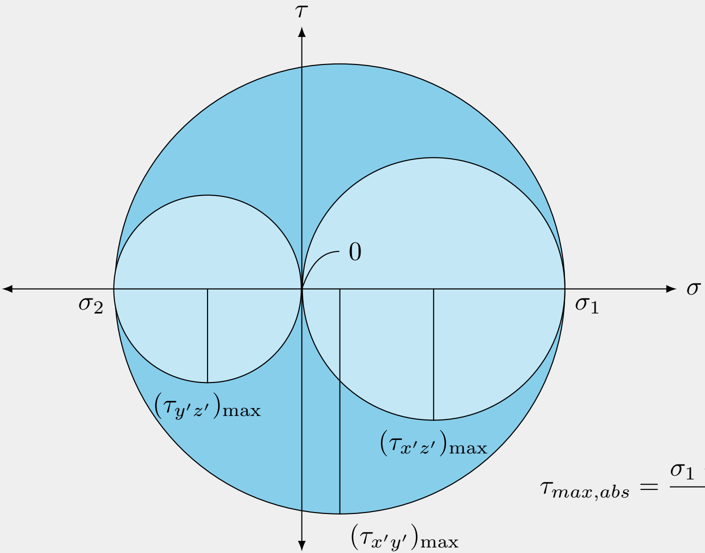
- In 3D plane stress problems, there are 3 principal stresses.
- The 3rd principal stress is 0.
- Use Mohr's circle to represent the relationship of all three.
- 2 cases:
    1. $\sigma_1, \sigma_2 > 0$
    2. $\sigma_1 > 0, \sigma_2 < 0$

Case I: Stresses are of the Same Sign



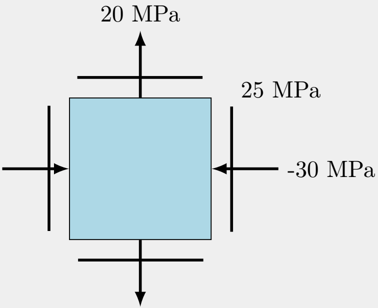
$$\tau_{max,abs} = \frac{\sigma_1}{2}$$

Case II: Stresses are of Opposite Signs

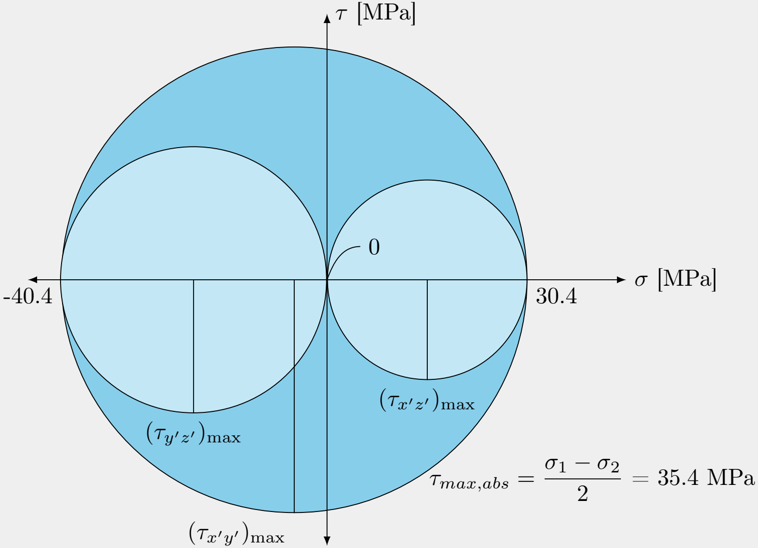


$$\tau_{max,abs} = \frac{\sigma_1 - \sigma_2}{2}$$

Example



- From previous example, we have that σ_1 = 30.4 MPa, σ_2 = -40.4 MPa, and τ_{max} = 35.4 MPa.
- Let us draw a Mohr's circle out of this.


τ [MPa]
0
σ [MPa]
-40.4
30.4
$(\tau_{x'z'})_{\max}$
$(\tau_{y'z'})_{\max}$
$\tau_{max,abs} = \dfrac{\sigma_1 - \sigma_2}{2} = 35.4$ MPa
$(\tau_{x'y'})_{\max}$