

# **INTRODUCTION TO THEORIES OF FAILURE**

ME 210: MECHANICS OF MATERIALS

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# 1 Material Failure

## 2 Fracture & Yield

## 3 Fatigue

## 4 Buckling

- Conditions in which materials lose its load-carrying capacity
- 3 most prevalent failure modes
  - ▶ Fracture & Yield – excessive stress causing molecular bond breakdown
  - ▶ Fatigue – cumulative damage through repeated loadings
  - ▶ Buckling – geometric instability from compressive load

# OUTLINE

1 Material Failure

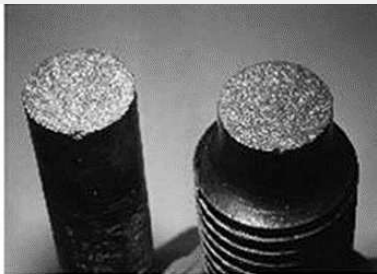
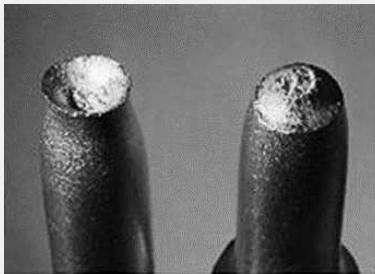
2 Fracture & Yield

3 Fatigue

4 Buckling

# FRACTURE & YIELD

- Sufficiently high stress to overcome intermolecular bonds
- Under uniaxial stress, a material fails when
- Brittle material - broken molecular bond leads to material separation – *Fracture*
- Ductile material - surface slip – *Yield*



# FRACTURE

- Material separation usually *perpendicular* to the direction of stress – predicted by normal stress
- Where or what is the maximum normal stress in a given material?  
–  $\sigma_1$  and  $\sigma_2$



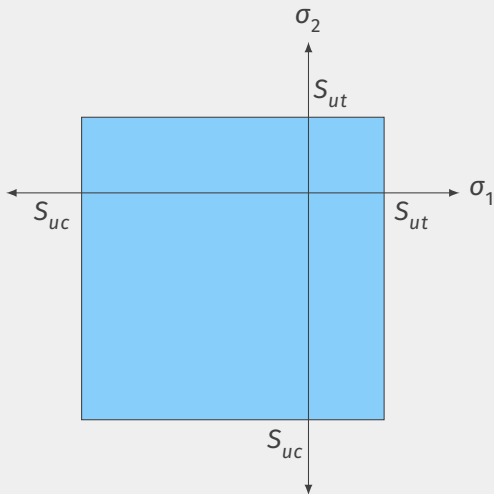
# MAXIMUM NORMAL STRESS THEORY

- In a nutshell:  
Determine if the maximum tensile and compressive stresses go beyond limit.
- What is the limit?

For  $\sigma_1 > \sigma_2$

$$\sigma_1 \leq S_{ut}$$

$$\sigma_2 \leq S_{uc}$$

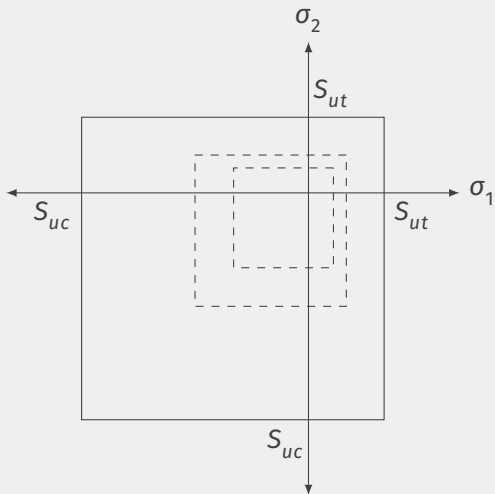


# DESIGN EQUATION FOR MNST

if  $\sigma_1 > \sigma_2$

$$\sigma_1 = \frac{S_{ut}}{N_s}$$

$$\sigma_2 = \frac{S_{uc}}{N_s}$$



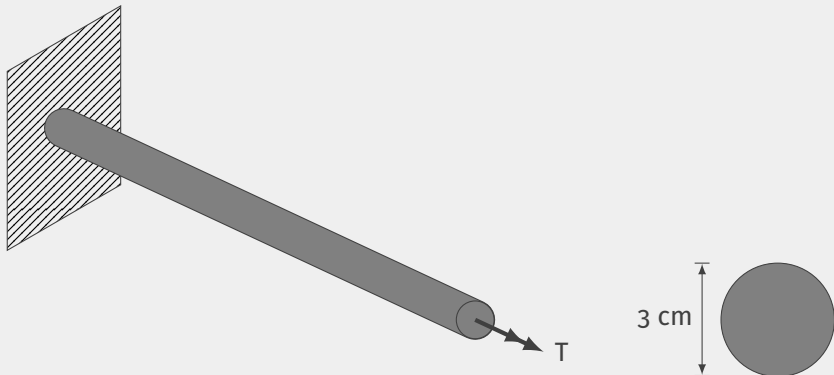


## EXAMPLE: THE MANY MODES OF CHALK FRACTURE

- How do you think a blackboard chalk would break under axial load? bending? torsion?



## EXAMPLE: BRITTLE SHAFT UNDER TORSION



- Determine maximum  $T$  when  $S_{ut} = 100$  MPa and  $S_{uc} = 200$  MPa

- Surface slip from material shearing – predicted by shear stress
- Two prevalent methods of determining yield
- Maximum Shear Stress Theory (MSST or Tresca criteria)
- Maximum Distortion Energy Theory (MDET or Von Mises criteria)

# MAXIMUM SHEAR STRESS THEORY: YIELD UNDER UNI-AXIAL STRESS

- The maximum shear stress in material cannot exceed max shear under uniaxial stress at yield – what?
- Under uniaxial stress, ductile material still fails due to shear stress, which can be derived as

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$\tau_{max} = \frac{S_y}{2}$$

# MAXIMUM SHEAR STRESS THEORY: SMALL SHEAR?

- What if material is in a state such that shear stress is small? Oh? Like how?
- When  $|\sigma_x - \sigma_y|$  and  $\tau_{xy}$  are small
- In that case, normal stresses cannot exceed  $S_y$

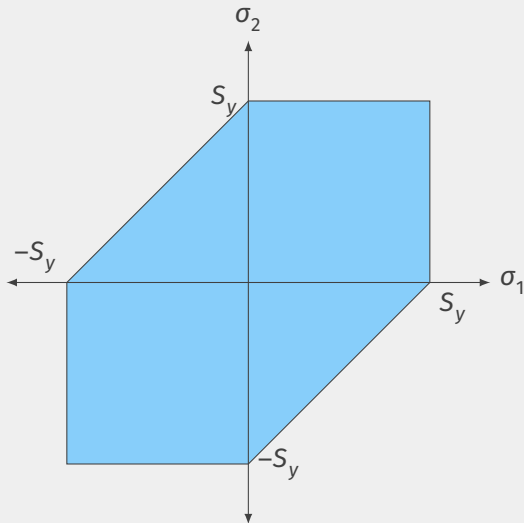
$$\sigma_{1,2} < S_y$$

# MAXIMUM SHEAR STRESS THEORY: IN SUMMARY

## ■ Combining the cases

$$\tau_{max} \leq \frac{S_y}{2}$$

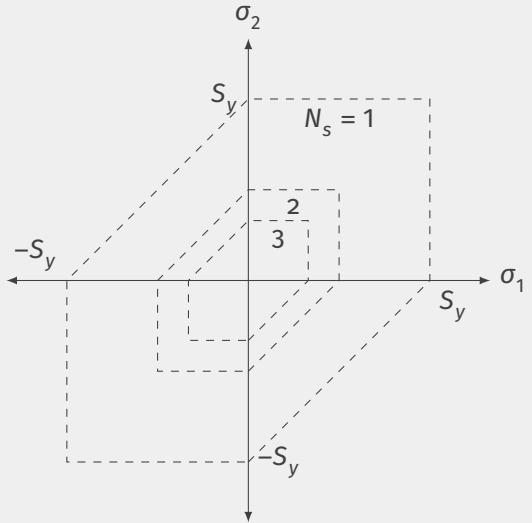
$$\sigma_{1,2} \leq S_y$$



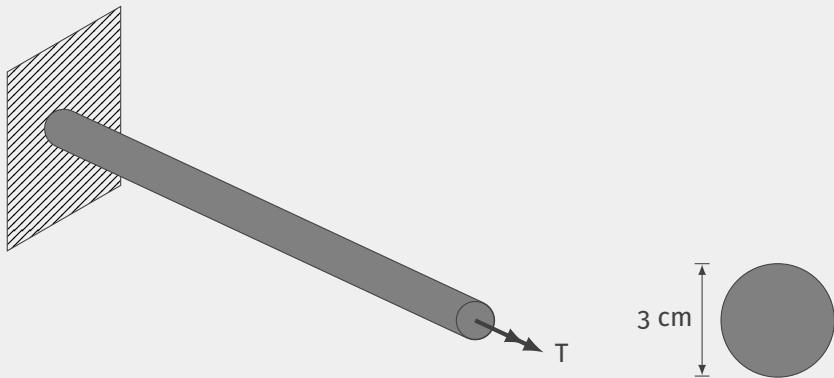
# DESIGN EQUATION FOR MSST

$$\tau_{max} = \frac{S_y}{2N_s}$$

$$\sigma_{1,2} = \frac{S_y}{N_s}$$



## EXAMPLE: DUCTILE SHAFT UNDER TORSION



- Determine maximum  $T$  when  $S_y = 100$  MPa
- Compare to brittle shaft problem



# DISTORTION ENERGY THEORY: CRITERIA

- A deformed material has two types of strain energy
- Dilatation strain energy → change in volume
- Distortion strain energy → change in shape → ductile failure
- Criteria for failure: distortion strain energy equal to that during uniaxial stress yield

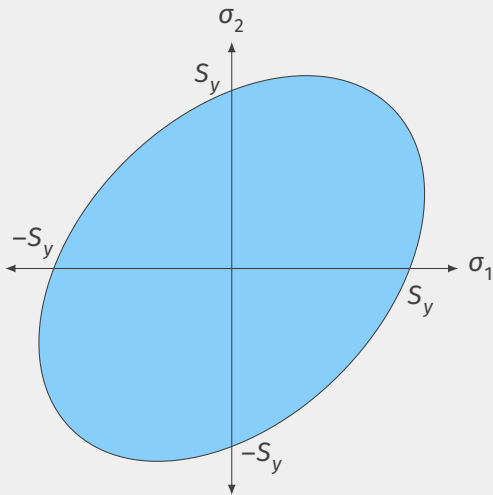
# MAXIMUM DISTORTION ENERGY THEORY: IN SUMMARY

- Material will fail when

$$\sigma_e = S_y$$

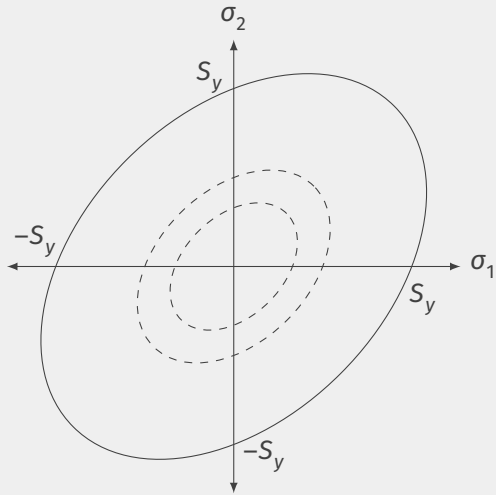
- where  $\sigma_e$  is the equivalent stress

$$\begin{aligned}\sigma_e &= \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2} \\ &= \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}\end{aligned}$$

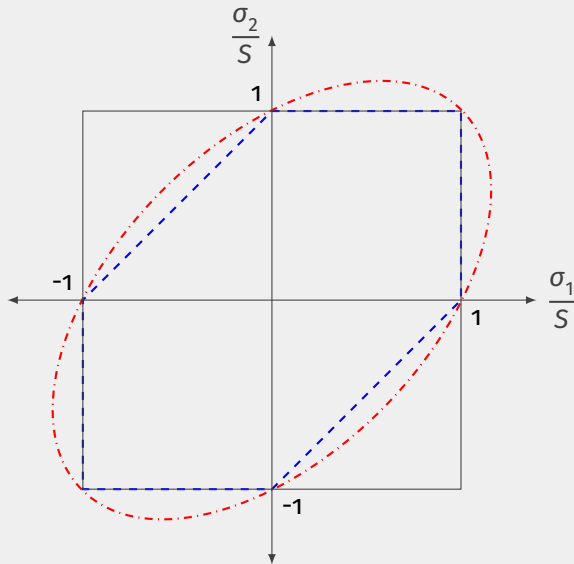


# DESIGN EQUATION FOR MDET

$$\sigma_e = \frac{S_y}{N_s}$$



# COMPARISON BETWEEN DIFFERENT CRITERIA



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**3 Fatigue**

4 Buckling

# FATIGUE



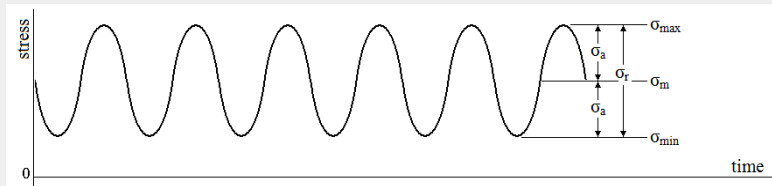
- Failure of material based on repeated or fluctuating loads
- Smaller than ultimate tensile stress
- Crack starts small and grows increasingly fast as load is repeated

# REPEATED LOADING

- Much like a periodic function, there are mainly two important features of a repeated loading concerning fatigue

- Stress amplitude  $\sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2}$

- Mean (average) stress  $\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2}$

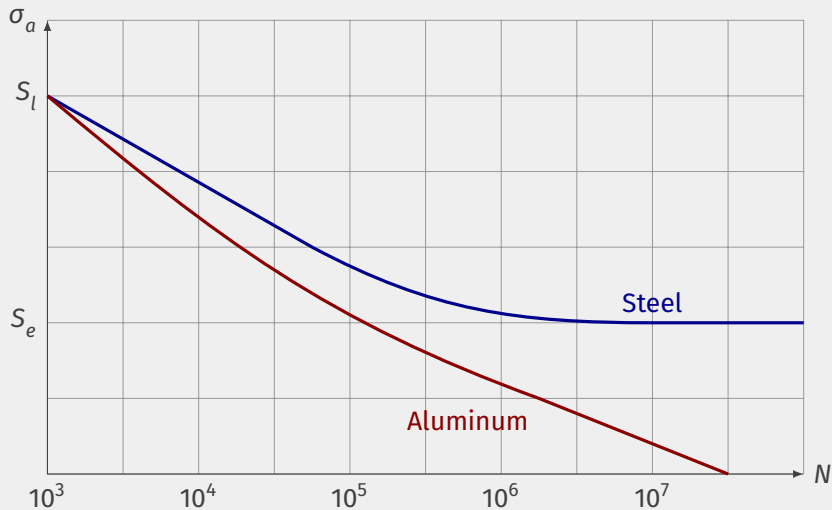


# TESTING FOR FATIGUE}





# S-N OR ENDURANCE DIAGRAM



# ENDURANCE LIMIT OR FATIGUE LIMIT

Endurance limit is the maximum stress amplitude for which the part will last  $10^7$  cycles

For steel

$$\text{Bending: } S_e \approx 0.5S_{ut}$$

$$\text{Axial: } S_e \approx 0.45S_{ut}$$

$$\text{Torsion: } S_e \approx 0.29S_{ut}$$

Aluminum does *not* have endurance limit (is that a good thing?)

# LOW CYCLE FATIGUE ( $N < 1000$ )

- Steel parts will last for  $\approx 1000$  cycles if the applied stress amplitude is

Bending:  $S_l \approx 0.9S_{ut}$

Axial:  $S_l \approx 0.75S_{ut}$

Torsion:  $S_l \approx 0.72S_{ut}$



## HIGH CYCLE FATIGUE ( $10^3 < N < 10^6$ )

- Steel parts will last for about  $N$  cycles if fatigue stress  $\sigma_a$

$$\sigma_a = 10^c N^b$$

$$b = -\frac{1}{3} \log \frac{S_l}{S_e}$$

$$c = \log \frac{S_l^2}{S_e}$$

## EXAMPLE: BAR UNDER CYCLIC AXIAL LOAD

A bar is made of steel with  $S_{ut} = 300$  MPa and is cyclically loaded axially between -200 and 200 MPa. Determine the useful life of this bar.

# MODIFIED ENDURANCE LIMIT

- Experiments find that endurance limits of materials vary due to their conditions

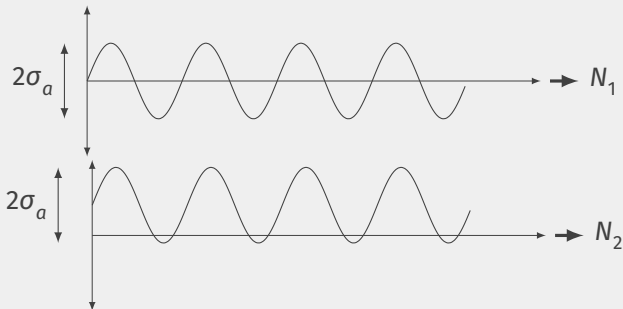
$$S'_e = K_f K_s K_t S_e$$

$K_f$  = finishing factor

$K_s$  = size factor

$K_t$  = thermal factor

# FATIGUE FAILURE WITH NONZERO AVERAGE STRESS



- $N_1 > N_2$ ?
- By how much?

# AVERAGE STRESS CORRECTION EQUATIONS

## ■ Soderberg relation

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{N_s}$$

## ■ Gerber relation

$$\frac{\sigma_a}{S_e} + \left( \frac{\sigma_m}{S_{ut}} \right)^2 = \frac{1}{N_s}$$

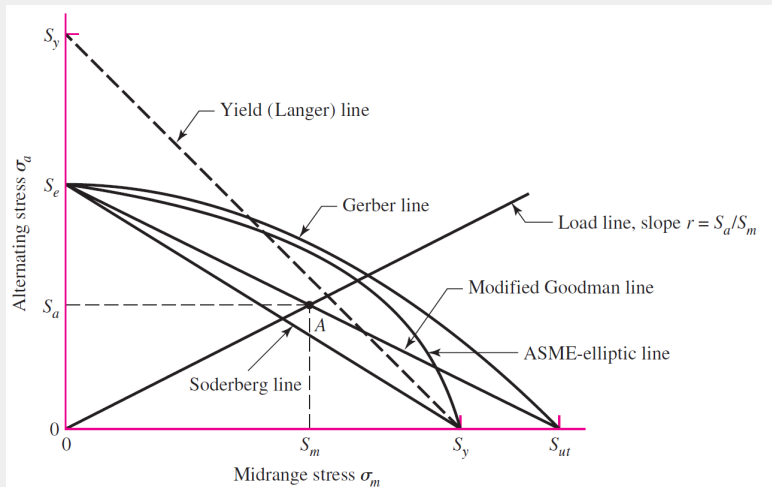
## ■ Goodman relation

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{N_s}$$

These are called *constant life lines*.

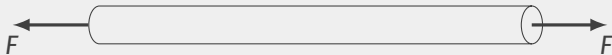


# AVERAGE STRESS CORRECTION COMPARISON



## EXAMPLE: BAR UNDER NONZERO AVERAGE STRESS

A cylindrical rod under a periodic tensile load between 0 and 8000 N. Material of the rod has  $S_y = 427$  MPa  $S_{ut} = 748$  MPa. Determine the suitable diameter of the rod. Use Soderberg relation.



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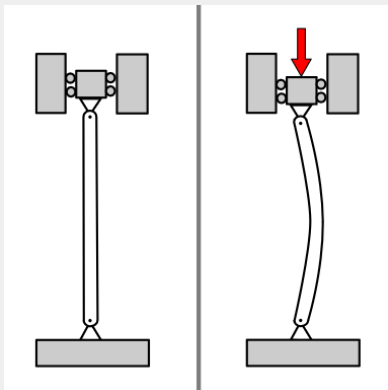
**4 Buckling**

# WHAT IS BUCKLING?

- Failure due to instability of column under compressive load
- Restoring moment < moment from compressive load



# GOVERNING EQUATION OF BUCKLING



$$M(x) = -Pv$$

$$EI \frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \left( \frac{P}{EI} \right) v = 0$$

# SOLVING FOR BUCKLING

The generalized solution takes the form

$$v(x) = A \cos \sqrt{\frac{P}{EI}}x + B \sin \sqrt{\frac{P}{EI}}x$$

Solving for  $A$  and  $B$ , we know that  $v(0) = 0$  and  $v(L) = 0$

$$v(0) = 0 = A \cos 0 + B \sin 0$$

$$A = 0$$

# BORING SOLUTION

$$v(L) = 0$$

$$B \sin \sqrt{\frac{P}{EI}} L = 0$$

$$B = 0$$

$$v(x) = 0 \rightarrow \text{trivial solution}$$

The solution is correct, but not helpful in determining the load  $P$   
So what's the interesting solution then?

# INTERESTING SOLUTION

Ok, so  $B$  can't be 0, which means

$$\sin \sqrt{\frac{P}{EI}} L = 0$$

$$\sqrt{\frac{P}{EI}} L = n\pi$$

$$P = \frac{n^2 \pi^2 EI}{L^2}, n = 1, 2, \dots$$

Now this is so much more helpful and interesting.



# CRITICAL LOAD AND CORRESPONDING MODE SHAPE

For  $n = 1$

$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

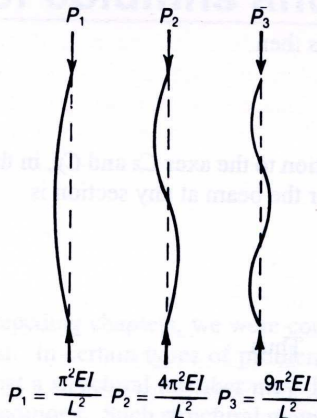
This is called *Euler load* or *Critical load*.

What does the corresponding buckled column look like?

$$v(x) = B \sin \sqrt{\frac{P}{EI}} x = B \sin \frac{n\pi x}{L}$$

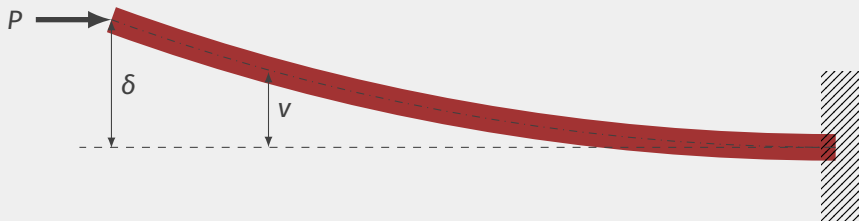
# MODE SHAPES FOR BUCKLED COLUMNS

## Buckling of columns and beams



**Figure 18.2** Modes of buckling of a pin-ended strut.

# BUCKLING IN FIXED - FREE SUPPORTS



$$M = P(\delta - v)$$

$$EI \frac{d^2v}{dx^2} = P(\delta - v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}\delta$$

# GOVERNING DIFFERENTIAL EQUATION SOLUTION

This equation is nonhomogeneous  $\rightarrow$  the solution consists of both a complementary and particular solutions.

$$v = C_1 \sin \left( \sqrt{\frac{P}{EI}} x \right) + C_2 \cos \left( \sqrt{\frac{P}{EI}} x \right) + \delta$$

Solving for constants of integration using  $v(x = 0) = 0$  and  $v(x = L) = \delta$ , we have that

$$\delta \cos \left( \sqrt{\frac{P}{EI}} L \right) = 0$$

The nontrivial solution indicates that

$$\cos\left(\sqrt{\frac{P}{EI}}L\right) = 0$$
$$\sqrt{\frac{P}{EI}}L = \frac{(2n+1)\pi}{2}$$

The smallest critical load ( $n = 0$ ) is

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

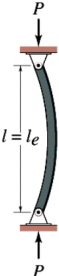

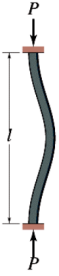
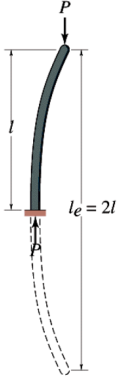
# GENERALIZED CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$
$$L_e = KL$$

$L_e$  = effective length

$K$  = constant depending on supports

# BUCKLING IN OTHER TYPES OF SUPPORTS

End condition description	Both ends pinned	One end pinned and one end fixed	Both ends fixed	One end fixed and one end free
Illustration of end condition				
Theoretical effective column length	$l_e = l$	$l_e = 0.7l$	$l_e = 0.5l$	$l_e = 2l$
AISC (1989)–recommended effective column length	$l_e = l$	$l_e = 0.8l$	$l_e = 0.65l$	$l_e = 2.1l$

# SLENDERNESS RATIO $\lambda$

- How *slender* is a column?

$$\lambda = \frac{KL}{r_g}$$

where  $r_g$  = radius of gyration =  $\sqrt{\frac{I}{A}}$

$$\begin{aligned}\sigma_{cr} &= \frac{P_{cr}}{A} \\ &= \frac{\pi^2 EI}{AL_e^2} \\ &= \frac{\pi^2 E}{\lambda^2}\end{aligned}$$



# BUCKLING DESIGN EQUATION

$$P_{allow} = \frac{P_{cr}}{N_s} = \frac{\pi^2 EI}{N_s L_e^2}$$
$$\sigma_{allow} = \frac{\sigma_{cr}}{N_s} = \frac{\pi^2 E}{N_s \lambda^2}$$

# EXAMPLE: BUCKLING OF A RAILROAD TRACK IN THE SUN



- What parameters do we need to determine required  $\Delta T$  to cause buckling?
- $E, A, I, L$ ?
- $\alpha$ ?