

ANALYSIS OF TORQUE LOADED MEMBERS

ME 210: MECHANICS OF MATERIALS

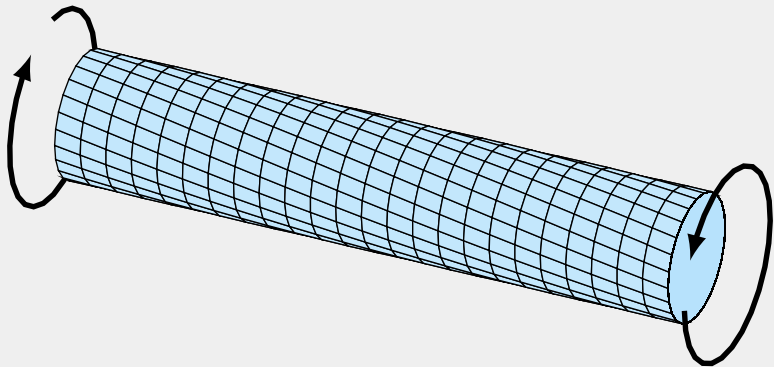
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WHAT IS TORSION?

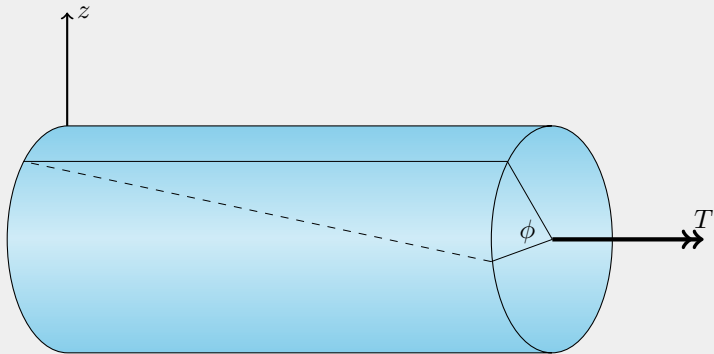
- Twisting of a straight bar loaded by torque (torsional moment)
- Twisting happens about its longitudinal axis

STATE OF STRESS IN TORSION



TORSIONAL DEFORMATION IN A CIRCULAR BAR

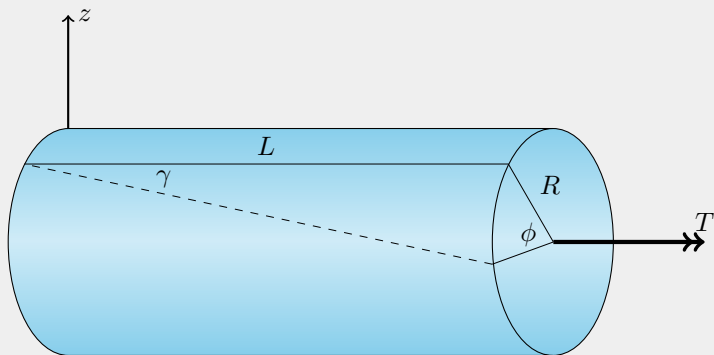
- When twisted, all cross-section remains circular and subjected to the same torque – *pure torsion*.



- ϕ is called the *angle of twist*, increasing linearly along the length of the bar

STATE OF STRAIN IN A TWISTED BAR

- Change in element angle is shear strain



$$\gamma = \frac{Rd\phi}{dx} = R\theta = \frac{R\phi}{L}$$

MAXIMUM SHEAR STRAIN IN A TWISTED BAR

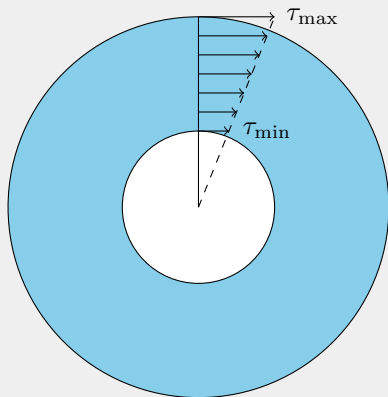
- Maximum shear strain happens at the outer surface

$$\gamma_{\max} = \frac{R\phi}{L}$$

- What about minimum shear strain?

$$\gamma_{\min} = \dots$$

STATE OF STRESS IN A CIRCULAR BAR UNDER TORSION



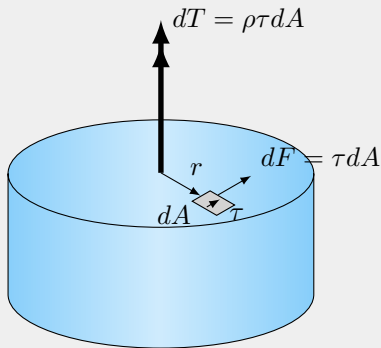
■ For linear elastic deformation

$$\tau = G\gamma$$

$$\tau_{\max} = G\gamma_{\max} = GR\frac{\phi}{L}$$

$$\tau = Gr\frac{\phi}{L} = \tau_{\max}\frac{r}{R}$$

TORSION FORMULA



$$dF = \tau dA$$

$$dT = r\tau dA = \frac{r^2 \tau_{\max}}{R} dA$$

$$\int_0^T dT = \frac{\tau_{\max}}{R} \int_A r^2 dA = \frac{\tau_{\max}}{R} J$$

Torsional Shear Stress Formula

$$\tau = \frac{Tr}{J}$$

POLAR MOMENT OF INERTIA: J

■ Solid cylindrical shaft

$$\begin{aligned} I_p &= \int_A r^2 dA \\ &= \int_0^{2\pi} \int_0^R r^2 r dr d\theta \\ &= \frac{\pi}{2} R^4 \end{aligned}$$

■ Hollow shaft: how do we do that?

EXAMPLE: MINIMUM SHAFT RADIUS



$$T_{\max} = 1735 \text{ N-m}$$

$$\tau_{\max} = 200 \text{ MPa}$$

$$r_{\min} = ?$$

SOLUTION: MINIMUM REQUIRED RADIUS

$$\tau = \frac{Tr}{J} = \frac{2T}{\pi r_{\min}^3}$$
$$r_{\min} = \left(\frac{2(1735)}{\pi(200 \times 10^6)} \right)^{1/3}$$
$$= 0.0177 = 1.77 \text{ cm}$$

Isn't that a bit small?

DEFORMATION UNDER TORSION: ANGLE OF TWIST ϕ

- Combine Hooke's law and Torsion formula

$$\tau_{\max} = GR\theta = \frac{TR}{J}$$

$$\theta = \frac{\phi}{L} = \frac{T}{GJ}$$

$$\phi = \frac{TL}{GJ} = \frac{T}{k_T}$$

- k_T is called the *torsional stiffness*

EXAMPLE: SHAFT DESIGN

A gasoline engine has a maximum torque output of 300 N-m. Your boss wants you to design a 2-m long shaft that is going to limit the angle of twist to $\phi \leq 0.1$ rad. The shaft should be made of medium carbon steel $G = 80$ GPa, $\tau_{\text{allow}} = 200$ MPa.

SOLUTION: SHAFT DESIGN

Two conditions: τ_{allow} and ϕ

$$\begin{aligned} R_{\tau} &= \left(\frac{2T}{\pi \tau_{\text{allow}}} \right)^{1/3} \\ &= \left(\frac{2(300)}{\pi(200 \times 10^6)} \right)^{1/3} \\ &= 9.85 \times 10^{-3} \text{ m} = 9.85 \text{ mm} \\ \phi &= \frac{TL}{GJ} = \frac{2TL}{G\pi R_{\phi}^4} \\ R_{\phi} &= \left(\frac{2(300)(2)}{(80 \times 10^9)\pi(0.1)} \right)^{1/4} \\ &= 0.0148 \text{ m} = 1.48 \text{ cm} \end{aligned}$$

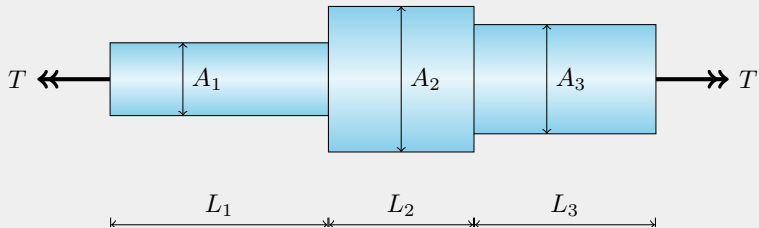
We will need to design based on the bigger requirement, 1.48 cm.

NONUNIFORM TORSION

- T , J , or G is not constant
 - ▶ Segments
 - ▶ Continuously varying

SEGMENTS OF CONSTANT VALUES

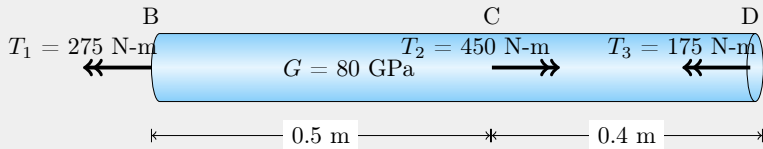
- Determine internal torques and corresponding deformations



$$\phi = \phi_1 + \phi_2 + \phi_3 + \dots$$

$$\phi = \sum_{i=1}^n \phi_i = \sum_{i=1}^n \frac{T_i L_i}{G_i J_i}$$

EXAMPLE: SHAFT WITH VARIOUS SEGMENTS



- Find τ_{\max} in each segments and ϕ_{BD} . Let $R = 1.5 \text{ cm}$.

SOLUTION

Use the method of section to determine torque within segment BC,

$$T_{BC} = T_1 = 275 \text{ Nm}$$

Torque within segment CD,

$$T_{CD} = T_3 = -175 \text{ Nm}$$

SOLUTION

The maximum shear stress in each segment is at the outer diameter.
We have

$$\tau_{\max} = \frac{Tr}{J} = \frac{2T}{\pi r^3}$$
$$(\tau_{\max})_{BC} = \frac{2(275 \text{ Nm})}{\pi(1.5 \times 10^{-2} \text{ m})^3} = 51.9 \text{ MPa}$$
$$(\tau_{\max})_{CD} = \frac{2(175 \text{ Nm})}{\pi(1.5 \times 10^{-2} \text{ m})^3} = 33 \text{ MPa}$$

SOLUTION

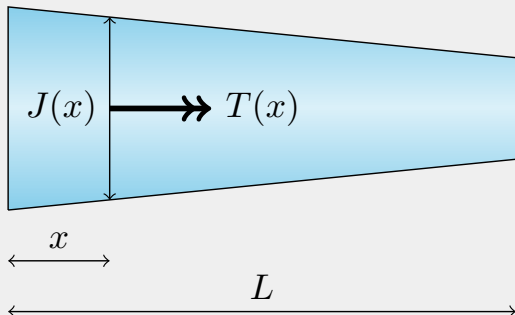
Angle of twist between B and D is the sum of the angles of twist in BC and CD.

$$\begin{aligned}\phi_{BD} &= \phi_{BC} + \phi_{CD} \\ J &= \frac{\pi r^4}{2} = \frac{\pi (1.5 \times 10^{-2} \text{ m})^4}{2} = 7.95 \times 10^{-8} \text{ m}^4 \\ \phi_{BC} &= \frac{T_{BC} L_1}{GJ} = \frac{(275 \text{ Nm})(0.5 \text{ m})}{(80 \text{ GPa})(7.95 \times 10^{-8} \text{ m}^4)} = 0.0216 \text{ rad} \\ \phi_{CD} &= \frac{T_{CD} L_2}{GJ} = \frac{(-175 \text{ Nm})(0.4 \text{ m})}{(80 \text{ GPa})(7.95 \times 10^{-8} \text{ m}^4)} = -0.0110 \text{ rad} \\ \phi_{BD} &= 0.0216 - 0.0110 = 0.0106 \text{ rad}\end{aligned}$$

Therefore, the bar twisted in the same direction as T_2 by 0.0106 rad.

CONTINUOUSLY VARYING TORQUE / SIZE / PROPERTIES

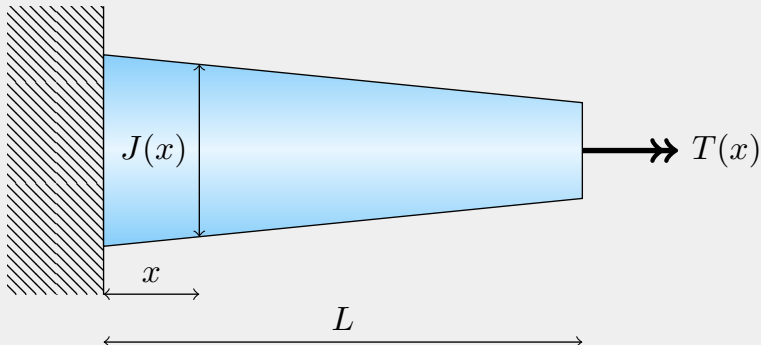
■ Back to integration



$$d\phi = \frac{T(x)dx}{G(x)J(x)}$$

$$\phi = \int_0^L \frac{T(x)dx}{GJ(x)}$$

EXAMPLE: CONTINUOUSLY VARYING SHAFT



- What is the total angle of twist ϕ ?

SOLUTION

$$\begin{aligned}\phi &= \int_0^L \frac{T dx}{GJ(x)} = \frac{T}{G} \int_0^L \frac{dx}{J(x)} \\&= \frac{T}{G} \int_0^L \frac{dx}{(\pi/2) \left(\frac{r_2 - r_1}{L} x + r_1 \right)^4} \\&= \frac{2TL}{3\pi G(r_2 - r_1)} \left(\frac{r_2 - r_1}{L} x + r_1 \right)^{-3} \bigg|_0^L \\&= \frac{2TL}{3\pi G(r_2 - r_1)} \left(-\frac{1}{r_2^3} + \frac{1}{r_1^3} \right)\end{aligned}$$

POWER TRANSMISSION THROUGH SHAFT

- The most important application of shaft is rotational power transmission

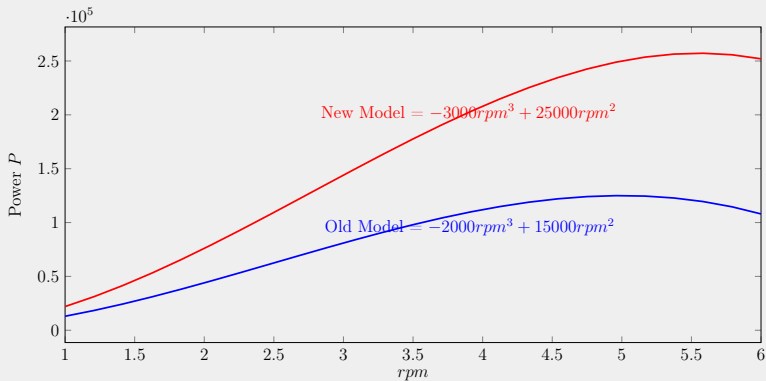
$$P = T\omega$$

- Engine power and speed are typically in *hp* and *rpm*

$$1 \text{ hp} = 746 \text{ W}$$

$$1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

EXAMPLE: SHAFT DESIGN FOR AN ENGINE



Is it safe to use the old shaft? $\tau_{allow} = 200 \text{ MPa}$.

SOLUTION

First order of business is determining maximum torque required.
From $P = T\omega$

$$T_{\text{old}} = -2000\left(\frac{2\pi}{60}\right)^3 \omega^2 + 15000\left(\frac{2\pi}{60}\right)^2 \omega$$

$$T_{\text{new}} = -3000\left(\frac{2\pi}{60}\right)^3 \omega^2 + 25000\left(\frac{2\pi}{60}\right)^2 \omega$$

SOLUTION FINDING MAX TORQUE IS EASY NOW

$$\begin{aligned}\frac{dT_{old}}{d\omega} &= 0 = -4000\left(\frac{2\pi}{60}\right)^3\omega + 15000\left(\frac{2\pi}{60}\right)^2 \\ \omega_{old}^* &= \frac{15 \times 60}{4 \times 2\pi} = 35.8 \text{ rad/s} = 342 \text{ rpm} \\ T_{old, \max} &= 2945 \text{ N-m}\end{aligned}$$

SOLUTION

Max torque for the new engine is

$$\begin{aligned}\frac{dT_{\text{new}}}{d\omega} = 0 &= -6000\left(\frac{2\pi}{60}\right)^3\omega + 25000\left(\frac{2\pi}{60}\right)^2 \\ \omega_{\text{new}}^* &= \frac{25 \times 60}{6 \times 2\pi} = 39.8 \text{ rad/s} = 380 \text{ rpm} \\ T_{\text{new, max}} &= 5454 \text{ N-m}\end{aligned}$$

Since the new model requires a larger torque, the old shaft will *NOT* work

SOLUTION

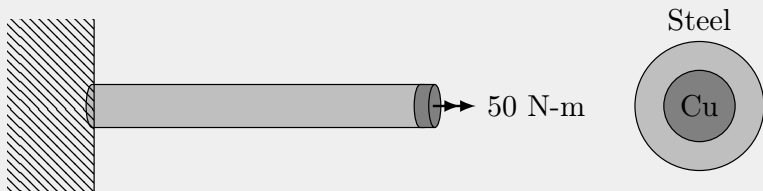
We can now design the new shaft using T_{new}

$$\tau_{\max} = \frac{TR}{J} = \frac{2T}{\pi R^3}$$
$$R = \left[\frac{2(5454)}{\pi(200 \times 10^6)} \right]^{1/3}$$
$$= 0.026 \text{ m} = 2.6 \text{ cm}$$

Statically Indeterminate Torque-Loaded Members

1. Equilibrium equation: torque
2. Compatibility: angle of twist ϕ constraints
3. Hooke's law: relate torque to angle of twist

EXAMPLE: COMPOUND SHAFT



$$G_{Cu} = 50 \text{ GPa} \quad G_{St} = 80 \text{ GPa}$$

1. What is the angle of twist at the end compared to the wall?
2. What is the equivalent shear modulus of the bar?

SOLUTION

This is a statically indeterminate problem, with the rigid disk acting as a constraint on the angle of twist of the steel and copper sections.

1: Equilibrium

$$T_{\text{cu}} + T_{\text{st}} = 50$$

SOLUTION

2: Compatibility – being constrained by the rigid disk, the two materials must rotate by the same amount ϕ

$$\phi_{\text{cu}} = \phi_{\text{st}} = \phi$$

3: Hooke's Law

$$\begin{aligned}\frac{T_{\text{cu}} L}{G_{\text{cu}} J_{\text{cu}}} &= \frac{T_{\text{st}} L}{G_{\text{st}} J_{\text{st}}} \\ T_{\text{cu}} &= \frac{50 T_{\text{st}} (\pi/2) (0.05^4)}{80 (\pi/2) (0.1^4 - 0.05^4)} \\ T_{\text{cu}} &= \frac{T_{\text{st}}}{24}\end{aligned}$$

SOLUTION

Plug back into equilibrium equation,

$$25T_{\text{cu}} = 50$$

$$T_{\text{cu}} = 2 \text{ N-m}$$

$$T_{\text{st}} = 48 \text{ N-m}$$

Once we obtained the torques, finding the angle of twist is easy:

$$\begin{aligned}\phi &= \frac{T_{\text{cu}} L}{G_{\text{cu}} J_{\text{cu}}} \\ &= \frac{2(1)}{50 \times 10^9 (\pi/2) 0.05^4} \\ &= 4.08 \times 10^{-6} \text{ rad}\end{aligned}$$

SOLUTION: EQUIVALENT SHEAR MODULUS

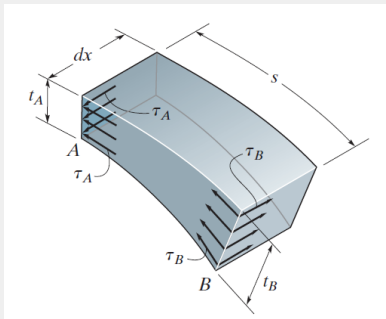
Similar to equivalent modulus for compound bar:

Given T , find modulus of a single material with same J and L that gives same ϕ

$$4.08 \times 10^{-6} = \frac{50(1)}{G_e (\pi/2) 0.1^4}$$

$$G_e = 78.1 \times 10^9 \text{ Pa} = 78.1 \text{ GPa}$$

TORSION IN THIN-WALLED TUBES



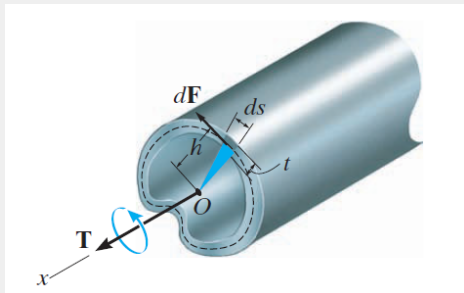
■ Equilibrium

$$\tau_A t_A dx = \tau_B t_B dx$$

$$\tau_A t_A = \tau_B t_B = q$$

- q is called *shear flow* and is constant over the cross section
- τ is maximum at the thinnest part

TORSION FORMULA FOR THIN-WALLED TUBES



$$dT = r dF = r q ds$$

$$T = q \oint r ds$$

$$T = 2A_m q$$

$$q = \tau t = \frac{T}{2A_m}$$

$$\tau = \frac{T}{2A_m t}$$

ANGLE OF TWIST IN THIN-WALLED TUBE

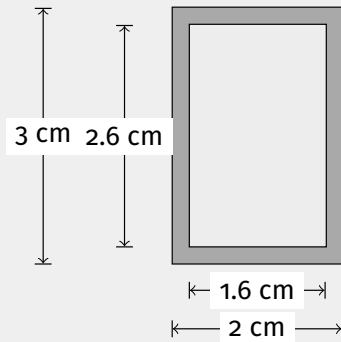
- Derived using energy method

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

- Intimidating, but actually quite simple
- For tubes with segments of constant thickness

$$\phi = \frac{TL}{4A_m^2 G} \left[\frac{s_1}{t_1} + \frac{s_2}{t_2} + \dots \right]$$

EXAMPLE: SHEAR STRESS AND ANGLE OF TWIST IN BOX STEEL



- If the box steel is 2 m long with 20 N-m torque applied, determine τ_{\max} and ϕ . Steel has $G = 80$ GPa

SOLUTION

Since the hollow section thickness is constant, τ is constant

$$\tau = \frac{T}{2A_m t} = \frac{20}{2(0.018 \times 0.028)(0.002)}$$
$$= 9.92 \text{ MPa}$$

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} = \frac{20(2)}{4(0.018 \times 0.028)^2 (80 \times 10^9)} \left[\frac{2(0.018 + 0.028)}{0.002} \right]$$
$$= 2.26 \times 10^{-2} \text{ rad}$$