# Analysis of Members under Bending

ME 210: Mechanics of Materials

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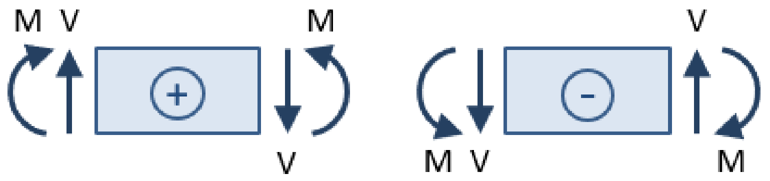
# Outline

# What is Bending?

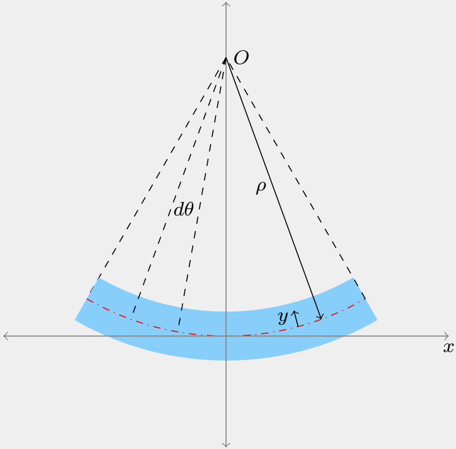
- Curving of a long, slender member under moment or force
- Caused by bending moment: moment whose direction is perpendicular to member axis

- Shear force and bending moment follow these sign conventions



# Beam Curvature

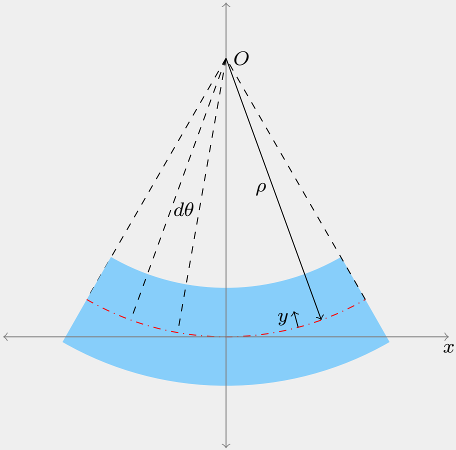
- In bending, beam curves in response to bending moment




$$\kappa = \frac{1}{\rho}$$

- Assume beam is thin compared to its curvature ($h \ll \rho$)
- This way, curvature of beam is constant throughout thickness

# Longitudinal Strain in Beam




- Longitudinal strain vary across thickness

$$\varepsilon_x = -\frac{y}{R} = -\kappa y$$

- No strain where $y = 0 \rightarrow$ *Neutral Axis*
- So where is this $R$?

# Stress in Beam Bending

- Again, assuming linear elastic deformation

$$\sigma_x = E\varepsilon_x = -\frac{Ey}{R}$$

- With this, we can now find $R$

# In Search of $R$

- Any arbitrary cross section must be at rest

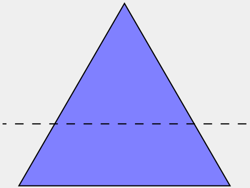
$$\int_A \sigma_x dA = -\int_A E\kappa y dA = 0$$

- For uniform material and *thin* beam ($\kappa$ is constant)

$$\int_A y dA = 0$$

- What does this equation remind you of?

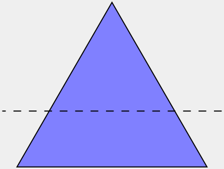
# Example: Neutral Axis of a Triangular Cross Section



$$\int_A y dA = 0$$

cut triangle into thin horizontal strips.
strip width $w$ is a function of $y$, so

$$w = b\left(1 - \frac{y}{h}\right)$$



$$
\begin{aligned}
0 &= \int_0^h w(y - y_{NA})dy \\
&= \int_0^h b\left(1 - \frac{y}{h}\right)(y - y_{NA})dy \\
&= \frac{b}{h}\int_0^h (hy - hy_{NA} - y^2 + yy_{NA})dy \\
&= \frac{b}{h}\left(h\frac{y^2}{2} - hy_{NA}y - \frac{y^3}{3} + y_{NA}\frac{y^2}{2}\right)_0^h
\end{aligned}
$$

divide through with $y$ and substitute with $h$ and 0

$$
\begin{aligned}
0 &= \frac{h^2}{2} - hy_{NA} - \frac{h^2}{3} + \frac{y_{NA}h}{2} \\
y_{NA} &= \frac{h}{3}
\end{aligned}
$$

# Outline

$$dM = -\sigma_x y dA$$

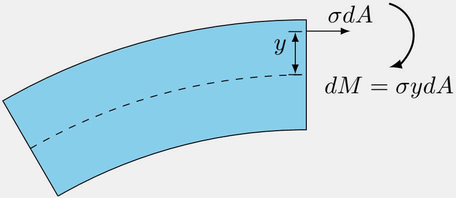
$$M = -\int_A \sigma_A y dA = \int_A \kappa E y^2 dA$$

$$M = \kappa E I$$

$$I = \int_A y^2 dA$$

$$\sigma_x = -E\kappa y$$

$$\kappa = -\frac{\sigma_x}{Ey}$$




$$\kappa = \frac{M}{EI}$$

$$\kappa = -\frac{\sigma_x}{Ey}$$

$$\sigma_x = -\frac{My}{I}$$

# Moment of Inertia, $I$

$$I = \int_A y^2 dA$$

- Represent resistance of cross section to bending
- Useful formulae

$$I_{\text{rect}} = \frac{bh^3}{12}$$
$$I_{\text{circ}} = \frac{\pi R^4}{4}$$

# Section Modulus $S$

- Max stress occurs furthest away from $NA$

$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1}$$
$$\sigma_2 = -\frac{Mc_2}{I} = -\frac{M}{S_2}$$

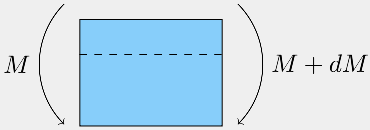
- Useful in choosing beam sections from catalog

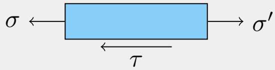
$$S_1 = \frac{I}{c_1}$$
$$S_2 = \frac{I}{c_2}$$

# Nonuniform Bending

- In *uniform* or *pure* bending, $M$ is constant along beam length
  - From applying bending moment directly
- Most beams are loaded by lateral forces
  - Gives rise to internal shear forces and stresses

$$V = \frac{dM}{dx}$$

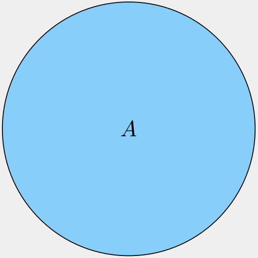


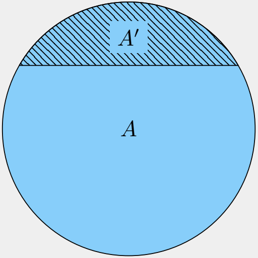

$$
\begin{aligned}
\tau t dx &= \int_{A'} \sigma' dA' - \int_{A'} \sigma dA' \\
&= \int_{A'} \frac{(M + dM)y}{I} dA' - \int_{A'} \frac{My}{I} dA' \\
&= \int_{A'} \frac{dMy}{I} dA'
\end{aligned}
$$

$$
\begin{aligned}
\tau &= \frac{1}{It} \left( \frac{dM}{dx} \right) \int_{A'} y dA' \\
&= \frac{VQ}{It}
\end{aligned}
$$

# What is Q?




$$
\begin{aligned}
Q &= \text{ first moment of area} \\
&= \int_{A'} y dA' \neq \int_{A} y dA \\
&= A' \bar{y}'
\end{aligned}
$$




$$
\begin{aligned}
Q &= \text{ first moment of area} \\
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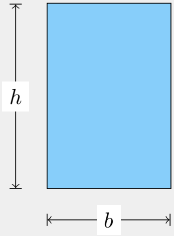
&= A' \bar{y}'
\end{aligned}
$$

# Example: Shear Stress Distribution in Rectangular Beam




$$
\begin{aligned}
Q(y) &= A'\bar{y}' \\
&= \left[\left(\frac{h}{2} - y\right) b\right] \left[y + \frac{\frac{h}{2} - y}{2}\right] \\
&= \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)
\end{aligned}
$$

# Solution

$$
\begin{aligned}
Q_{\text{max}} &= Q(y=0) \\
&= \frac{bh^2}{8} \\
\tau_{\text{max}} &= \frac{VQ}{It} \\
&= \frac{V\frac{bh^2}{8}}{\frac{bh^3}{12}b} \\
&= \frac{3V}{2bh} = \frac{3V}{2A}
\end{aligned}
$$

# Example: Maximum shear stress in a circular cross-sectioned beam

A cantilever beam with length $L$ and radius $r$ has a force $P$ applied at the middle. Determine the location and magnitude of maximum shear stress.

- Maximum shear force in the beam is anywhere from the fixed end to the middle
- Maximum shear stress occurs at NA
- Anywhere along NA from the fixed end to the middle has max shear stress

- Now, for the magnitude

$$\tau = \frac{VQ}{Ib} = \frac{P((\pi/2)r^2(4r/3\pi))}{(\pi/4)r^4(2r)}$$
$$= \frac{4P}{3\pi r^2} = \frac{4P}{3A}$$

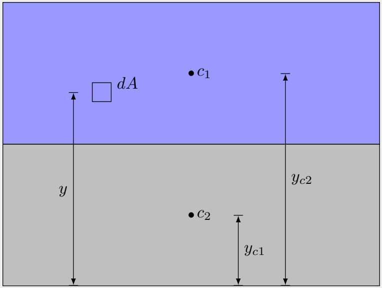
# Outline

- Two or more materials ($E_1 \neq E_2 \neq \ldots$)
- Deformation (and strain) remain unchanged

$$\varepsilon_x = -\frac{y}{R} = -\kappa y$$

- But where is the new neutral axis?
- Since sectional property is no longer uniform → neutral axis will move

# Neutral Axis of a Composite Beam




$$\int_{A1} \sigma_{x1} dA + \int_{A2} \sigma_{x2} dA = 0$$

$$-\int_{A1} E_1 \kappa y dA - \int_{A2} E_2 \kappa y dA = 0$$

$$-E_1 \int_{A1} y dA - E_2 \int_{A2} y dA = 0$$

$$E_1 y_{c1} A_1 + E_2 y_{c2} A_2 = 0$$

This works for more than 2 materials as well!

- Same equation applies, only different results

$$
\begin{aligned}
M &= -\int_A \sigma y dA \\
&= -\int_{A1} \sigma_{x1} y dA - \int_{A2} \sigma_{x2} y dA \\
&= E_1 \int_{A1} \kappa y^2 dA + E_2 \int_{A2} \kappa y^2 dA \\
&= \kappa \left(E_1 I_1 + E_2 I_2\right)
\end{aligned}
$$

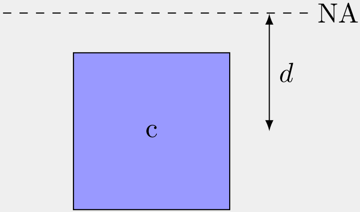
# What about $I$?

- Well, usually neutral axis goes through centroid

$$I = I_c = \int_A y^2 dA$$

- In composite beams, this is no longer true → *parallel axis theorem*

$$I = I_c + Ad^2$$



- From moment-curvature relationship

$$\kappa = \frac{M}{E_1 I_1 + E_2 I_2}$$

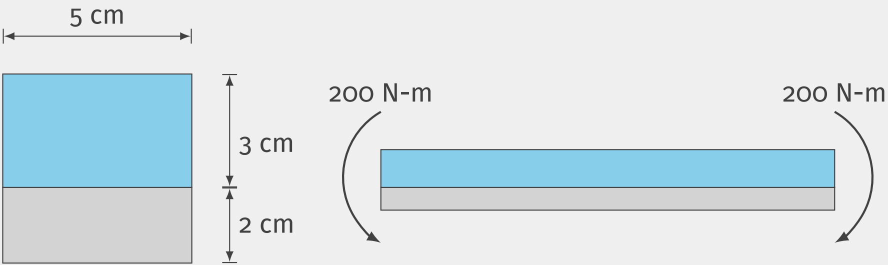
- From stress-curvature relationship

$$\sigma_x = -E\kappa y$$

$$\sigma_{x1} = -\frac{M y E_1}{E_1 I_1 + E_2 I_2}$$

$$\sigma_{x2} = -\frac{M y E_2}{E_1 I_1 + E_2 I_2}$$

- Top layer $E$ = 200 MPa. Bottom layer $E$ = 400 MPa. Determine $\sigma_{max}$ in tension and compression.



To determine stress, first we need to find neutral axis

$$
\begin{aligned}
0 &= 200 \times 10^6 (3 \times 5)(3.5 - y_{NA}) + 400 \times 10^6 (2 \times 5)(1 - y_{NA}) \\
0 &= 21 - 6y_{NA} + 8 - 8y_{NA} \\
y_{NA} &= 2.07 \text{ cm}
\end{aligned}
$$

Calculating the area moment of inertia of each cross section, we have

$$
\begin{aligned}
I_1 &= \frac{1}{12}(0.05)(0.03)^3 + (0.05)(0.03)(0.035 - 0.0207)^2 \\
&= 4.19 \times 10^{-7}\ \mathrm{m}^4 \\
I_2 &= \frac{1}{12}(0.05)(0.02)^3 + (0.05)(0.02)(0.0207 - 0.01)^2 \\
&= 1.48 \times 10^{-7}\ \mathrm{m}^4
\end{aligned}
$$

The maximum tensile stress, occurring on the top surface of the beam, is

$$
\begin{aligned}
\sigma_{\text{max},tensile} &= \frac{MyE_1}{E_1 I_1 + E_2 I_2} \\
&= \frac{200(0.05 - 0.0207)(200 \times 10^6)}{200 \times 10^6 (4.19 \times 10^{-7}) + 400 \times 10^6 (1.48 \times 10^{-7})} \\
&= 8.19 \text{ MPa}
\end{aligned}
$$

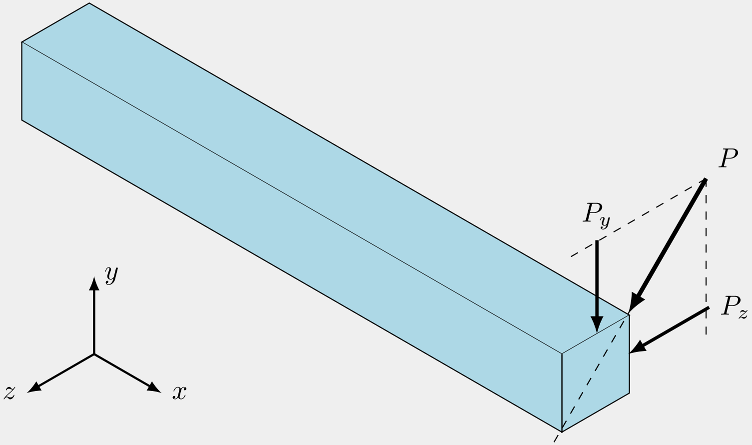
# Solution

Similarly for the maximum compressive stress at the bottom surface is

$$
\begin{aligned}
\sigma_{\text{max compressive}} &= \frac{M y E_2}{E_1 I_1 + E_2 I_2} \\
&= \frac{200(2.07 \times 10^{-2})(400 \times 10^{6})}{200 \times 10^{6}(4.19 \times 10^{-7}) + 400 \times 10^{6}(1.48 \times 10^{-7})} \\
&= 11.6 \text{ MPa}
\end{aligned}
$$

# Outline

- Normally load is applied along a symmetrical axis ($y$ or $z$)
- What if it isn't?



# Superposition to the Rescue

- When faced with a tough problem, break it down into smaller, easier problems

- Using superposition

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

- Neutral axis is

$$\tan\beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y}$$

# Outline

# Why Do We Care about Deflection?

- Stress is not always the limiting factor
- Especially important in load-bearing structure and flexure design

# Methods of Evaluation

- Direct Integration: beam curvature
- Energy Method: strain energy

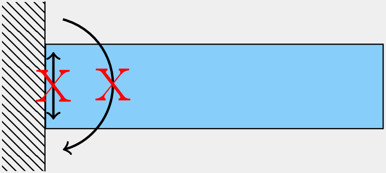
$$\frac{M}{EI} = \kappa = \frac{d^2 v}{dx^2}$$

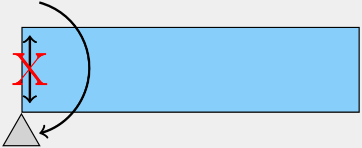
where $v$ is the deflection function at point $x$ along the beam

# Boundary Conditions

- Indefinite integrals give constants of integration
  - second order equations → 2 constants
- Need to apply knowledge about end conditions

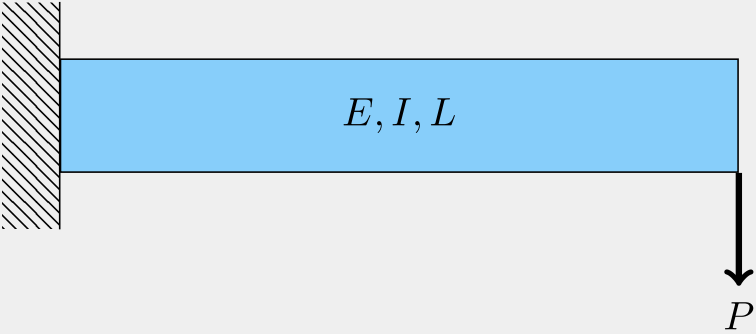
# Typical End Conditions




- Fixed end
  - No deflection: $v = 0$
  - No rotation: $\frac{dv}{dx} = 0$
- Simple support
  - No deflection: $v = 0$
  - Free rotation:

# Example: Deflection Curve of a Cantilever Beam




$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$M(x) = -P(L - x)$$

Determine the deflection curve by integrating twice.

$$EI\frac{d^2v}{dx^2} = M(x) = -P(L - x)$$
$$EI\frac{dv}{dx} = \frac{P}{2}(L - x)^2 + C_1$$

The fixed end does not allow rotation, and so $\frac{dv}{dx} = 0$ at $x = 0$.

$$C_1 = -\frac{PL^2}{2}$$

$$EI\frac{dv}{dx} = \frac{P}{2}(L-x)^2 - \frac{PL^2}{2}$$

$$EIv = -\frac{P}{6}(L-x)^3 - \frac{PL^2x}{2} + C_2$$

At the fixed end, there is no vertical deflection, i.e. $v = 0$ at $x = 0$

$$C_2 = \frac{PL^3}{6}$$

$$v = -\frac{P}{6EI}(L-x)^3 - \frac{PL^2x}{2EI} + \frac{PL^3}{6EI}$$

# Solution

The maximum deflection is at the free end. Its value is

$$v(L) = -\frac{PL^3}{2EI} + \frac{PL^3}{6EI} = -\frac{PL^3}{3EI}$$

Note that the negative sign means the deflection is downward.

- Elastic deformation: way to store energy → strain energy
- Deflection can be derived from stored energy
- How do we evaluate strain energy of a bent beam?

# Strain Energy in Beam

- For a small part of bent beam with length $dx$ and bending angle $d\theta$

$$d\theta = \kappa dx$$

- We also know from moment-curvature relationship that

$$\kappa = \frac{M}{EI}$$

- Assume 100% external work to internal energy transfer

$$dW = dU = \frac{M d\theta}{2}$$

$$U = \int_0^L \frac{M^2 dx}{2EI}$$

- Deflection $v_i$ where load $P_i$ is applied is equal to

$$\begin{aligned} v_i &= \frac{\partial U}{\partial P_i} \\ &= \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right) dx \end{aligned}$$

# Example: Deflection of a Cantilever Beam

$$M(x) = -P(L-x)$$

$$\delta_i = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right) dx$$

$$
\begin{aligned}
\delta_i &= \frac{1}{EI}\int_0^L -P(L-x)[-(L-x)]dx \\
&= \frac{P}{EI}\left[L^2x - x^2L + \frac{x^3}{3}\right]_0^L \\
&= \frac{PL^3}{3EI}
\end{aligned}
$$

- same answer as that of the direct integration methods
- opposite sign though???
- using energy method, positive means deflection is in the same direction as the force → down!