Introduction to Theories of Failure

ME 210: Mechanics of Materials

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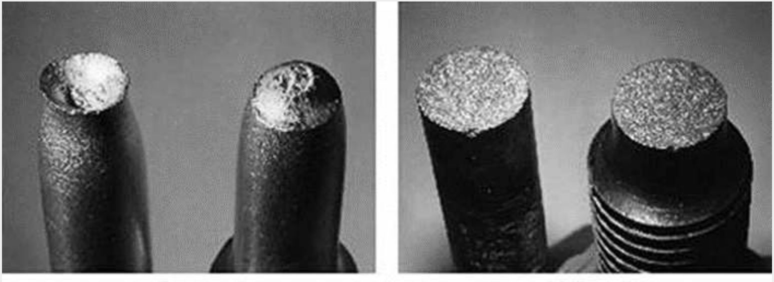
Material Failure

- Conditions in which materials lose its load-carrying capacity
- 3 most prevalent failure modes
  - Fracture & Yield – excessive stress causing molecular bond breakdown
  - Fatigue – cumulative damage through repeated loadings
  - Buckling – geometric instability from compressive load

Outline

Fracture & Yield

- Sufficiently high stress to overcome intermolecular bonds
- Under uniaxial stress, a material fails when
- Brittle material - broken molecular bond leads to material separation – *Fracture*
- Ductile material - surface slip – *Yield*



Fracture

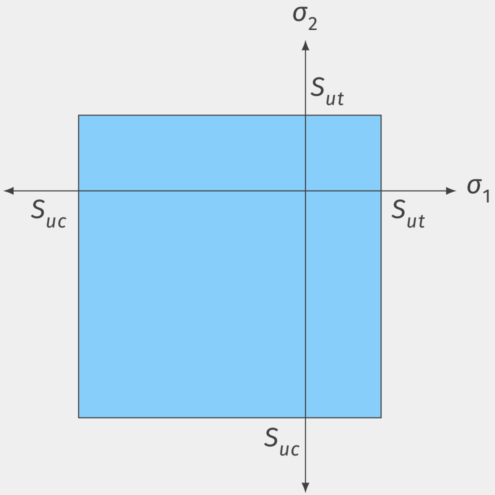
- Material separation usually *perpendicular* to the direction of stress – predicted by normal stress
- Where or what is the maximum normal stress in a given material? – σ_1 and σ_2

- In a nutshell: Determine if the maximum tensile and compressive stresses go beyond limit.
- What is the limit?

For $\sigma_1 > \sigma_2$

$$\sigma_1 \leqslant S_{ut}$$

$$\sigma_2 \leqslant S_{uc}$$

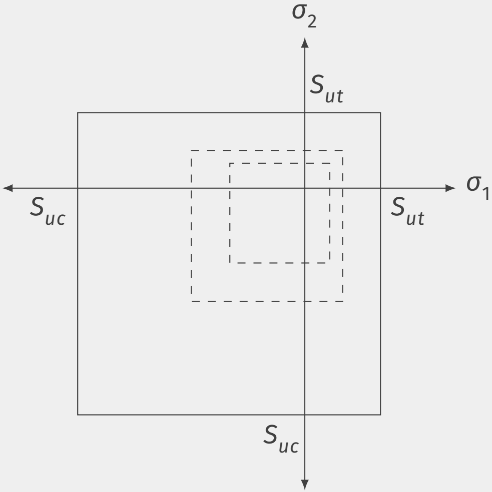


Design Equation for MNST

if $\sigma_1 > \sigma_2$

$$\sigma_1 = \frac{S_{ut}}{N_s}$$

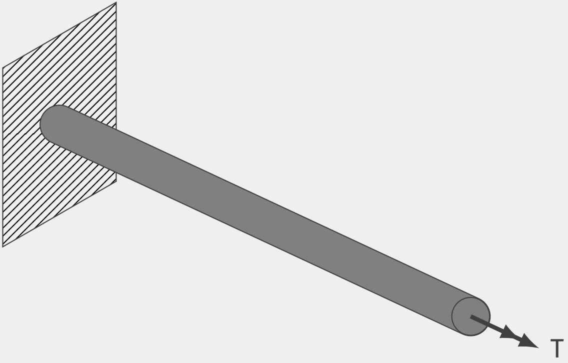
$$\sigma_2 = \frac{S_{uc}}{N_s}$$

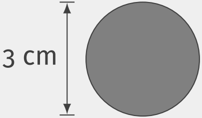


Example: The Many Modes of Chalk Fracture

- How do you think a blackboard chalk would break under axial load? bending? torsion?

Example: Brittle shaft under torsion





- Determine maximum T when S_{ut} = 100 MPa and S_{uc} = 200 MPa

Yield

- Surface slip from material shearing – predicted by shear stress
- Two prevalent methods of determining yield
- Maximum Shear Stress Theory (MSST or Tresca criteria)
- Maximum Distortion Energy Theory (MDET or Von Mises criteria)

Maximum Shear Stress Theory: Yield under Uniaxial Stress

- The maximum shear stress in material cannot exceed max shear under uniaxial stress at yield – what?
- Under uniaxial stress, ductile material still fails due to shear stress, which can be derived as

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \frac{S_y}{2}$$

Maximum Shear Stress Theory: Small Shear?

- What if material is in a state such that shear stress is small? Oh? Like how?
- When $|\sigma_x - \sigma_y|$ and τ_{xy} are small
- In that case, normal stresses cannot exceed S_y

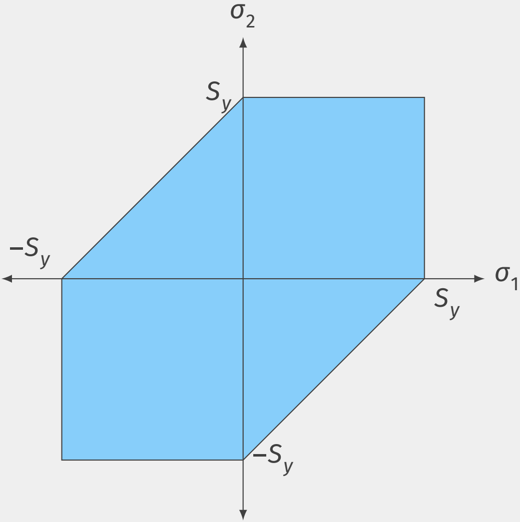
$$\sigma_{1,2} < S_y$$

Maximum Shear Stress Theory: In Summary

- Combining the cases

$$\tau_{max} \leqslant \frac{S_y}{2}$$

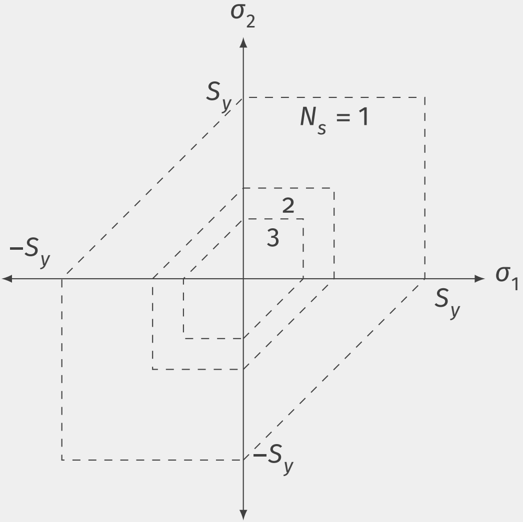
$$\sigma_{1,2} \leqslant S_y$$



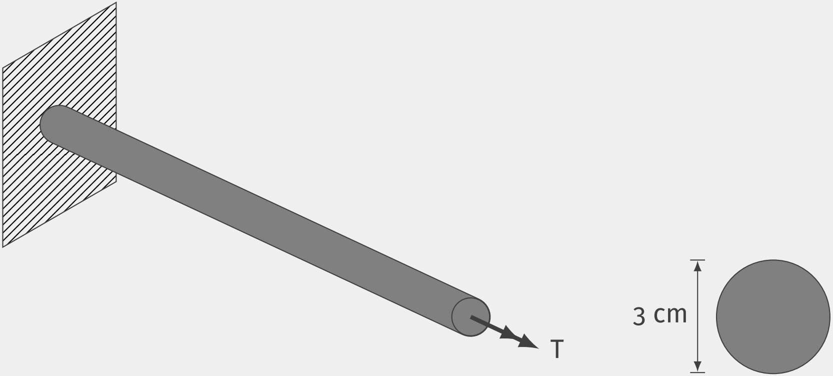
Design Equation for MSST

$$\tau_{max} = \frac{S_y}{2N_s}$$

$$\sigma_{1,2} = \frac{S_y}{N_s}$$



Example: Ductile shaft under torsion



- Determine maximum T when S_y = 100 MPa
- Compare to brittle shaft problem

Distortion Energy Theory: Criteria

- A deformed material has two types of strain energy
- Dilatation strain energy → change in volume
- Distortion srain energy → change in shape → ductile failure
- Criteria for failure: distortion strain energy equal to that during uniaxial stress yield

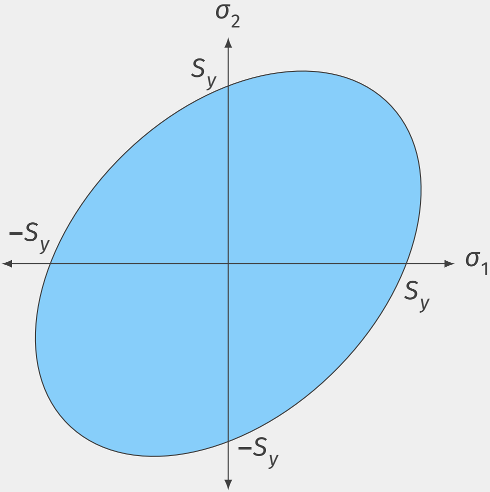
Maximum Distortion Energy Theory: In Summary

- Material will fail when

$$\sigma_e = S_y$$

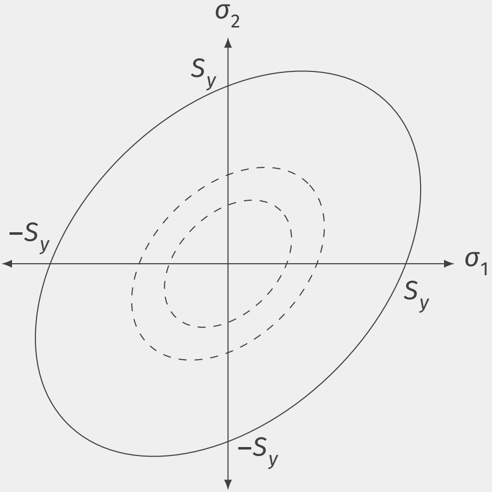
- where σ_e is the equivalent stress

$$\sigma_e = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$
$$= \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

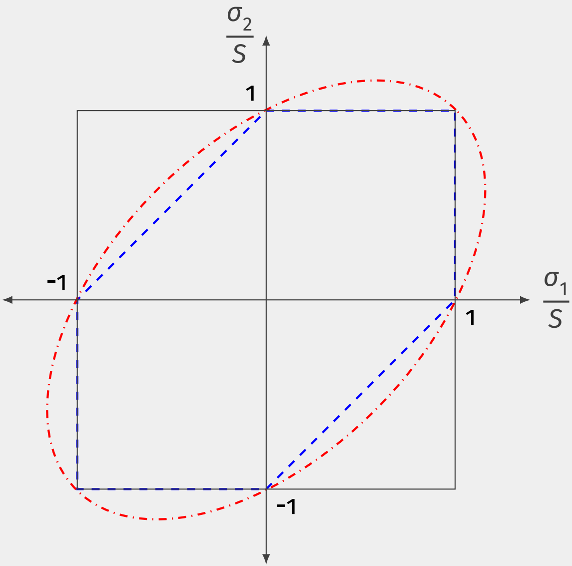


Design Equation for MDET

$$\sigma_e = \frac{S_y}{N_s}$$



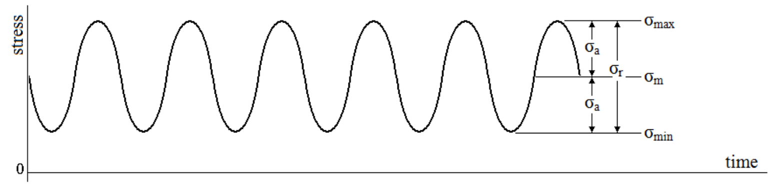
Comparison Between Different Criteria



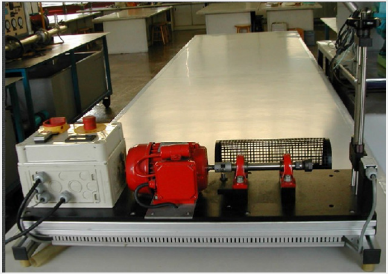
Outline

- Failure of material based on repeated or fluctuating loads
- Smaller than ultimate tensile stress
- Crack starts small and grows increasingly fast as load is repeated

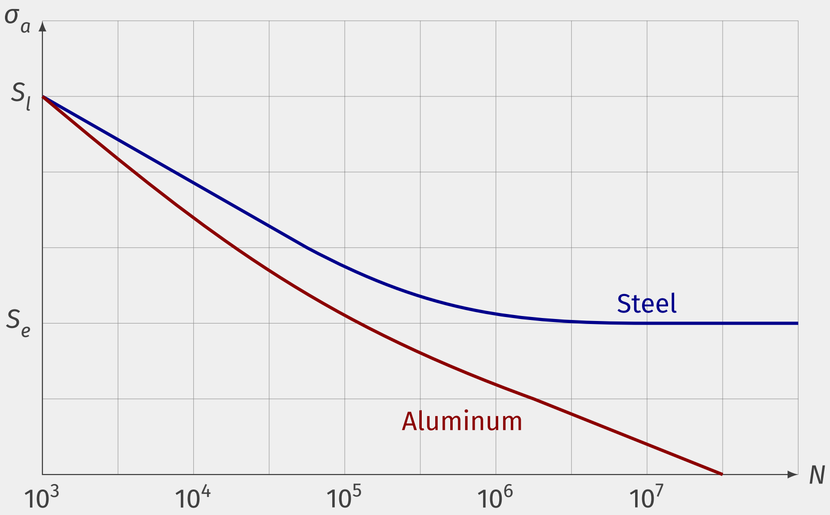
- Much like a periodic function, there are mainly two important features of a repeated loading concerning fatigue
- Stress amplitude $\sigma_a = \dfrac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2}$
- Mean (average) stress $\sigma_m = \dfrac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2}$



Testing for Fatigue}



S-N or Endurance Diagram



Endurance Limit or Fatigue Limit

Endurance limit is the maximum stress amplitude for which the part will last 10^7 cycles
For steel

$$\begin{aligned} \text{Bending: } & S_e \approx 0.5 S_{ut} \\ \text{Axial: } & S_e \approx 0.45 S_{ut} \\ \text{Torsion: } & S_e \approx 0.29 S_{ut} \end{aligned}$$

Aluminum does *not* have endurance limit (is that a good thing?)

Low cycle fatigue ($N < 1000$)}

- Steel parts will last for ≈ 1000 cycles if the applied stress amplitude is

$$
\begin{aligned}
\text{Bending: } & S_l \approx 0.9 S_{ut} \\
\text{Axial: } & S_l \approx 0.75 S_{ut} \\
\text{Torsion: } & S_l \approx 0.72 S_{ut}
\end{aligned}
$$

High cycle fatigue ($10^3 < N < 10^6$)

- Steel parts will last for about N cycles if fatigue stress σ_a

$$\sigma_a = 10^c N^b$$
$$b = -\frac{1}{3}\log\frac{S_l}{S_e}$$
$$c = \log\frac{S_l^2}{S_e}$$

A bar is made of steel with S_{ut} = 300 MPa and is cyclically loaded axially between -200 and 200 MPa. Determine the useful life of this bar.

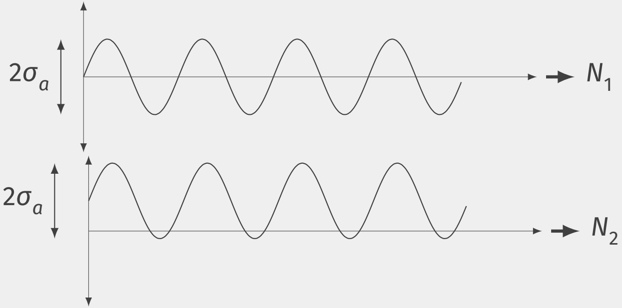
Modified Endurance Limit

- Experiments find that endurance limits of materials vary due to their conditions

$$S_e' = K_f K_s K_t S_e$$

$$\begin{aligned} K_f &= \text{finishing factor} \\ K_s &= \text{size factor} \\ K_t &= \text{thermal factor} \end{aligned}$$

Fatigue Failure with Nonzero Average Stress



- $N_1 > N_2$?
- By how much?

- Soderberg relation

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{N_s}$$

- Gerber relation

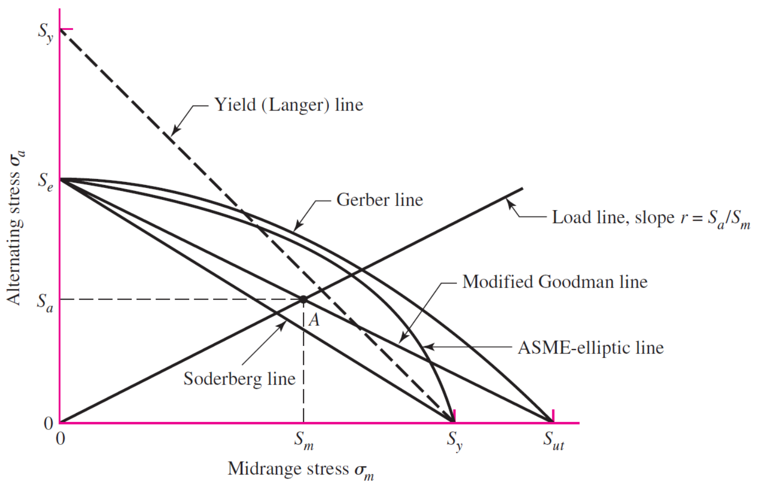
$$\frac{\sigma_a}{S_e} + \left(\frac{\sigma_m}{S_{ut}}\right)^2 = \frac{1}{N_s}$$

- Goodman relation

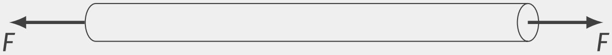
$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{N_s}$$

These are called *constant life lines*.

Average Stress Correction Comparison



A cylindrical rod under a periodic tensile load between 0 and 8000 N. Material of the rod has S_y = 427 MPa S_{ut} = 748 MPa. Determine the suitable diameter of the rod. Use Soderberg relation.

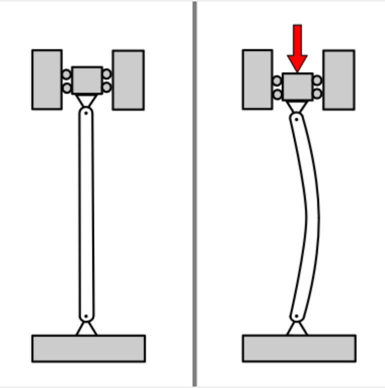


Outline

What is Buckling?

- Failure due to instability of column under compressive load
- Restoring moment < moment from compressive load

Governing Equation of Buckling



$$M(x) = -Pv$$

$$EI\frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0$$

Solving for Buckling

The generalized solution takes the form

$$v(x) = A\cos\sqrt{\frac{P}{EI}}x + B\sin\sqrt{\frac{P}{EI}}x$$

Solving for A and B, we know that $v(0) = 0$ and $v(L) = 0$

$$v(0) = 0 = A\cos 0 + B\sin 0$$

$$A = 0$$

Boring Solution

$$v(L) = 0$$

$$B \sin\sqrt{\frac{P}{EI}}L = 0$$

$$B = 0$$

$$v(x) = 0 \rightarrow trivialsolution$$

The solution is correct, but not helpful in determining the load P
So what’s the interesting solution then?

Ok, so B can't be 0, which means

$$\sin\sqrt{\frac{P}{EI}}L = 0$$

$$\sqrt{\frac{P}{EI}}L = n\pi$$

$$P = \frac{n^2\pi^2 EI}{L^2}, n = 1, 2, \ldots$$

Now this is so much more helpful and interesting.

Critical Load and Corresponding Mode Shape

For $n = 1$

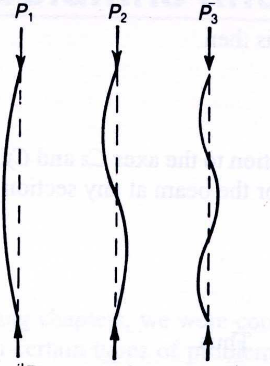
$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

This is called *Euler load* or *Critical load*.
What does the corresponding buckled column look like?

$$v(x) = B \sin \sqrt{\frac{P}{EI}} x = B \sin \frac{n\pi x}{L}$$

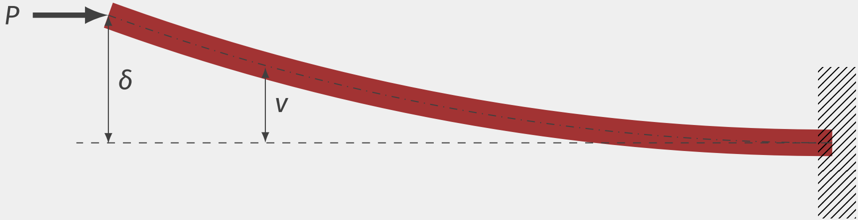
Mode Shapes for Buckled Columns

Buckling of columns and beams



$$P_1 = \frac{\pi^2 EI}{L^2} \quad P_2 = \frac{4\pi^2 EI}{L^2} \quad P_3 = \frac{9\pi^2 EI}{L^2}$$

Figure 18.2 Modes of buckling of a pin-ended strut.



$$M = P(\delta - v)$$

$$EI\frac{d^2v}{dx^2} = P(\delta - v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}\delta$$

Governing Differential Equation Solution

This equation is nonhomogeneous → the solution consists of both a complementary and particular solutions.

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) + \delta$$

Solving for constants of integration using $v(x = 0) = 0$ and $v(x = L) = \delta$, we have that

$$\delta \cos\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

The nontrivial solution indicates that

$$\cos\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

$$\sqrt{\frac{P}{EI}}L = \frac{(2n+1)\pi}{2}$$

The smallest critical load ($n = 0$) is

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

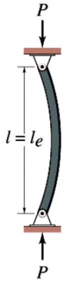
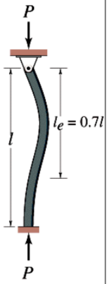
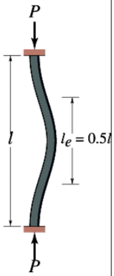
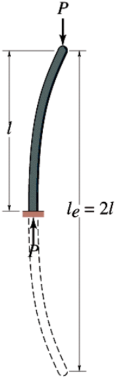
Generalized Critical Load

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

$$L_e = KL$$

L_e = effective length
K = constant depending on supports

Buckling in Other Types of Supports

End condition description	Both ends pinned	One end pinned and one end fixed	Both ends fixed	One end fixed and one end free
Illustration of end condition				
Theoretical effective column length	$l_e = l$	$l_e = 0.7l$	$l_e = 0.5l$	$l_e = 2l$
AISC (1989)– recommended effective column length	$l_e = l$	$l_e = 0.8l$	$l_e = 0.65l$	$l_e = 2.1l$

Slenderness Ratio λ

- How *slender* is a column?

$$\lambda = \frac{KL}{r_g}$$

where r_g = radius of gyration = $\sqrt{\frac{I}{A}}$

$$\begin{aligned}\sigma_{cr} &= \frac{P_{cr}}{A} \\ &= \frac{\pi^2 EI}{AL_e^2} \\ &= \frac{\pi^2 E}{\lambda^2}\end{aligned}$$

Buckling Design Equation

$$P_{allow} = \frac{P_{cr}}{N_s} = \frac{\pi^2 EI}{N_s L_e^2}$$

$$\sigma_{allow} = \frac{\sigma_{cr}}{N_s} = \frac{\pi^2 E}{N_s \lambda^2}$$

Example: Buckling of a Railroad Track in the Sun

- What parameters do we need to determine required ΔT to cause buckling?
- E, A, I, L?
- α?