Gear Design

ME 310: Mechanical Design

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Outline

Why Gears?

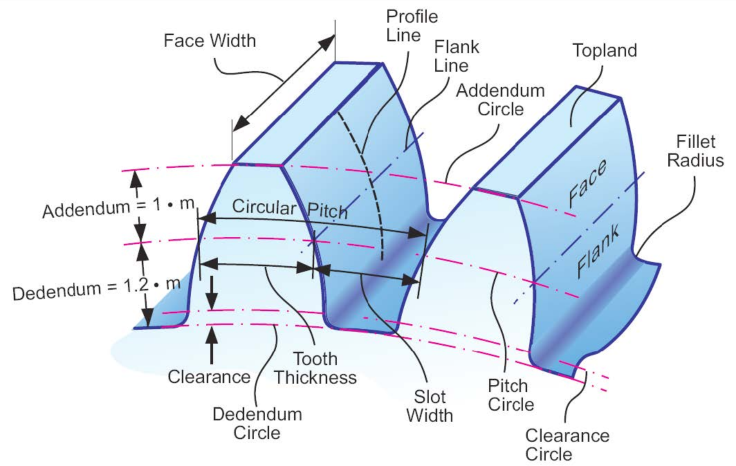
- High speed, low torque → low speed, high torque
- Speed: easy to get because voltage is easy
- Torque: hard to get because it requires large current

Principles of Gears

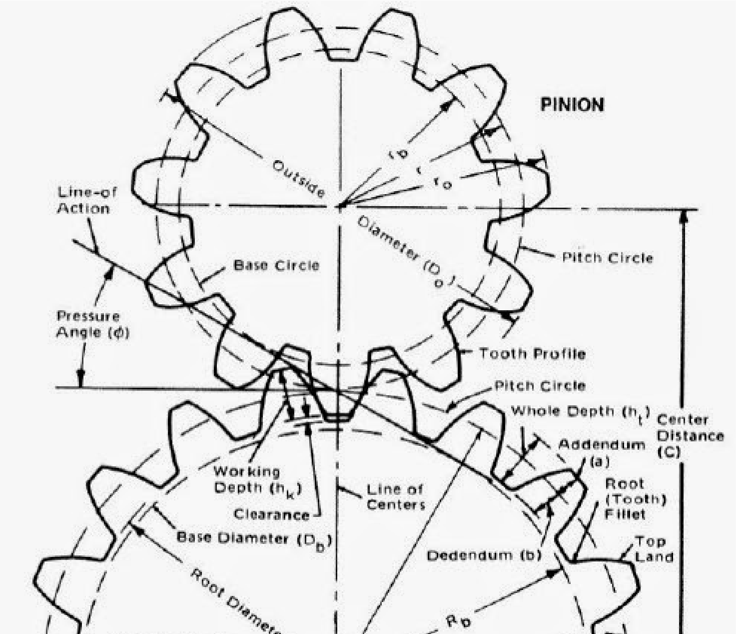
- Allow positive engagement between teeth
- High forces can be transmitted while in rolling contact
- Do not need friction to operate → high efficiency

Basic Law of Gearing

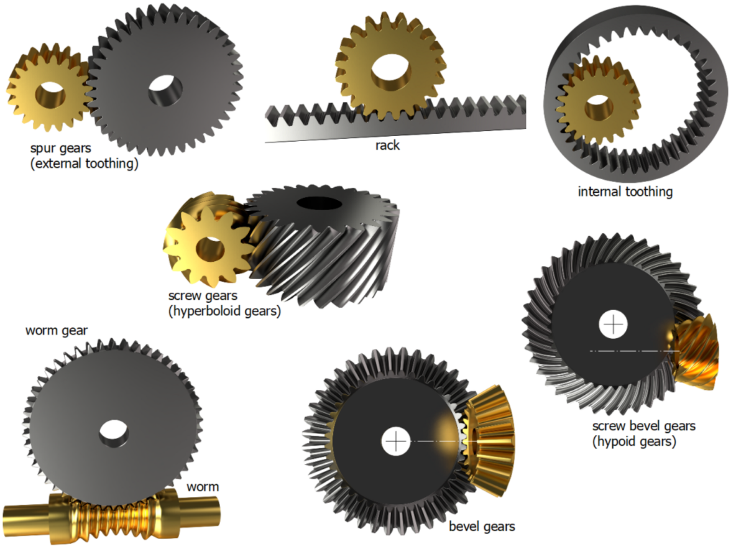
- Point of contact between two mating gears is always the same relative distances from the two centers.
- Any gear tooth profiles that follow the law of gearing will result in constant relative speed of rotation → involute profile



Gear Geometry



Gear Types


spur gears
(external toothing)
rack
internal toothing
screw gears
(hyperboloid gears)
worm gear
worm
bevel gears
screw bevel gears
(hypoid gears)

Gear Terminology

Pinion smaller of two gears, usually driving

Gear Larger of the two. Also called *wheel*. Usually driven.

Gear Materials

- Steel: medium-carbon steel + heat treatment + grinding
- Cast iron: surface fatigue > bending fatigue
- Nonferrous: bronzes → corrosion + wear resistant, low friction
- Nonmetallic: Nylon → low friction and weight + corrosion resistant, but low thermal conductivity

Gear Efficiency

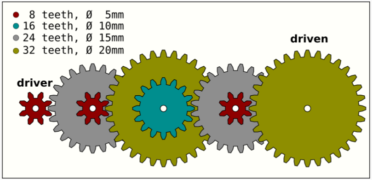
- With friction, gears are 90 - 95% efficient because of mostly rolling contact

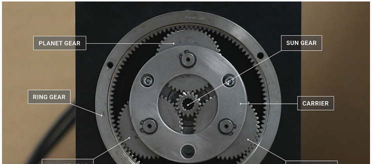
$$
\begin{aligned}
T_{\text{out}} &= \frac{\eta T_{\text{in}} d_{\text{out}}}{d_{\text{in}}} \\
\omega_{\text{out}} &= \frac{\omega_{\text{in}} d_{\text{in}}}{d_{\text{out}}} \\
P_{\text{out}} &= T_{\text{out}} \omega_{\text{out}} \\
RR &= \frac{\omega_{in}}{\omega_{out}} = \frac{z_{out}}{z_{in}} \approx \frac{T_{out}}{T_{in}}
\end{aligned}
$$

- RR = gear ratio

Outline

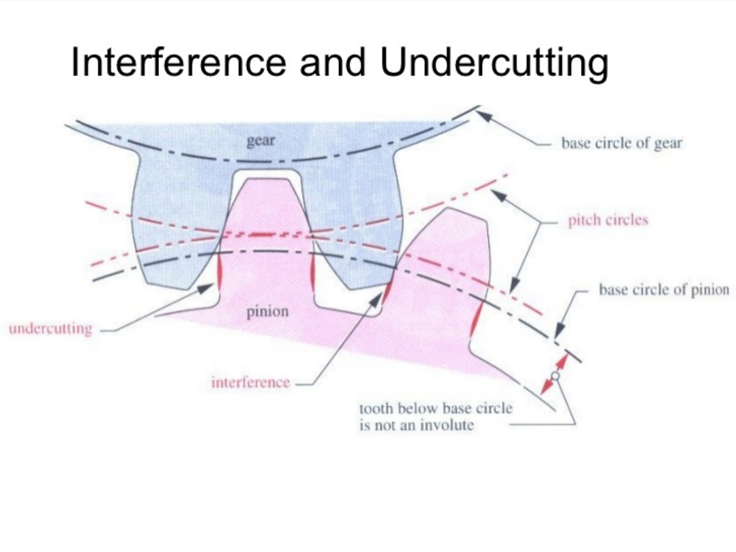
Gear Trains



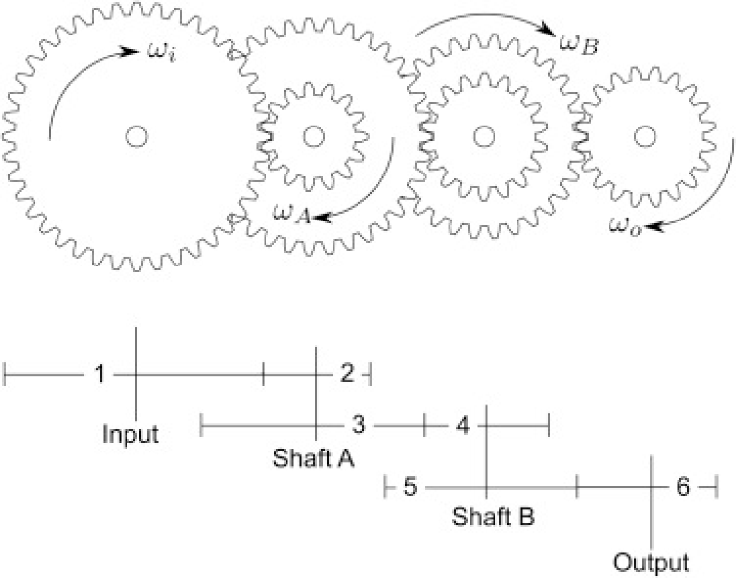


When large reduction is required

- Large gear + small pinion: simple, but large stress and interference
- Multiple pairs of gears and pinions: less simple, low stress, large space
- Planetary gears: complex, low stress, small space


Interference and Undercutting
gear
base circle of gear
pitch circles
base circle of pinion
pinion
undercutting
interference
tooth below base circle
is not an involute

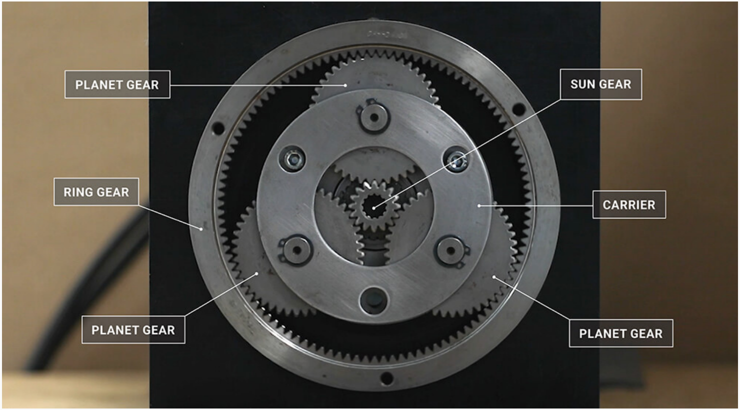
Normal Gear Trains



Planetary/Epicyclic Gear Train

- Planetary or epicyclic gears enable a high reduction ratio in small spaces

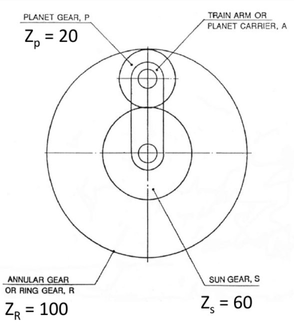
Planetary Gear Components



Planetary Gears: Torque, Forces, and Reduction Ratios

- Symmetry → no net force on shaft
- Multiple planet gears reduce individual torque/force
- Any combination of fixed, input, output gears
- 1 gear box -> multiple gear reduction ratios

Example

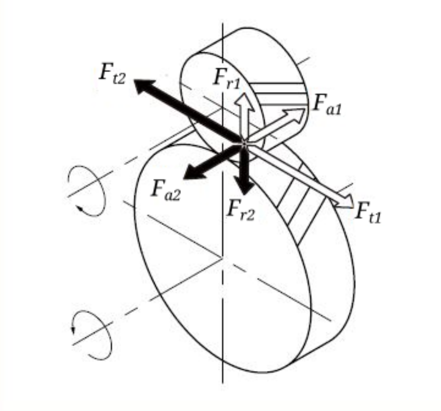


Fixed ring:

$$
\begin{aligned}
\omega_{\text{carrier}} &= 9 \\
\omega_{\text{planet}} &= (9)\,\frac{60/2 + 20/2}{20/2} = 36 \\
\omega_{\text{sun}} &= (36)\,\frac{20}{30} = 24 \\
RR &= 9/24 = 0.375
\end{aligned}
$$

Fixed	Input	Planet	Output	
Ring	Carrier 9	36	Sun 24	0.3
Sun	Carrier 9	36	Ring 14.4	0.6
Carrier	Sun 9	27	Ring 5.4	1.6

Gear Force Analysis: The Components

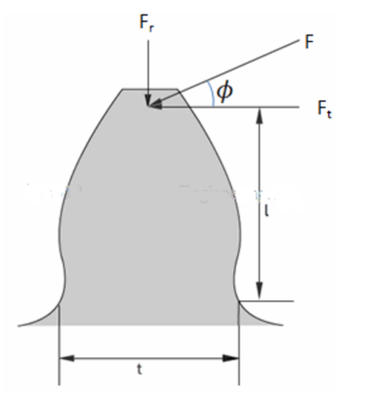


- F_t Tangential force (tangent to pitch circle)
- F_r Radial force (passing through gear center)
- F_a Axial force (parallel to axis of rotation)

$$\bar{F} = \bar{F}_t + \bar{F}_r + \bar{F}_a$$

Outline

Spur Gear Forces



ϕ pressure angle

v_t pitch line velocity

$$F_t = \frac{\text{Power}}{v_t}$$

$$F_r = F_t \tan\phi$$

$$F_a = 0$$

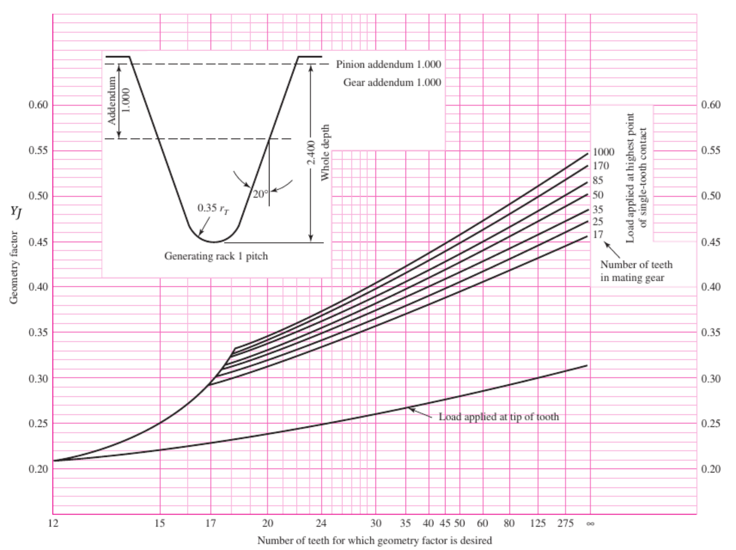
Spur Gear Stress

- Bending Stress → AGMA stress equation
- Consider tooth as a cantilever beam

$$\sigma = \frac{F_t}{b Y_J m} K_O K_m K_v$$

F_t	tangential force
b	face width
Y_J	geometry factor
m	module
K_O	overload factor
K_m	mounting factor
K_v	velocity factor

Geometry Factor: Y_J



Overload Factor: K_o

Power source	Driven Machine			
	Uniform	Light shock	Moderate shock	Heavy shock
Uniform	1.00	1.25	1.50	1.75
Light shock	1.20	1.40	1.75	2.25
Moderate shock	1.30	1.70	2.00	2.75

Power sources

Uniform Electric motor, constant-speed turbine

Light Water turbine, variable-speed drive

Moderate Multicylinder engine

Driven machine

Uniform Continuous generator

Light Fans, low-speed pumps, conveyors

Moderate high-speed pumps, compressors, heavy conveyers

Heavy rock crushers, punch press drivers

Mounting Factor: K_m

Characteristics of Support	Face Width (cm)			
	0 to 5	15	22.5	40
Accurate mountings, small bearing clearances, precision gears	1.3	1.4	1.5	1.8
Less rigid moutings, standard gears, full face contact	1.6	1.7	1.8	2.2
Less than full face contact	Over 2.2			

Velocity Factor: K_v

- Takes care of shock and impact loading

$$K_v = \left(\frac{A + \sqrt{200 v_t}}{A} \right)^B$$

$$A = 50 + 56(1 - B)$$

$$B = 0.25(12 - Q)^{2/3}$$

v_t pitch line velocity [m/s]

Q AGMA Quality Number

AGMA Recommended Quality Number: Q

v_t [m/s]	Q	Applications
0 - 4	6 - 8	Paper box making machine, cement, mill drives
4 - 10	8 - 10	Washing machine, printing press, computing mechanism
10 - 20	10 - 12	Automotive transmission, Antenna drive, propulsion drive
$\geqslant$ 20	12 - 14	Gyroscope

Gear Material Strength

$$S'_e = S_e C_L C_G C_S k_r k_t k_{ms}$$

S_e endurance limit

C_L load factor (= 1 for bending)

C_G gradient surface = 1

C_S surface factor (= 0.75 for machined surface)

k_r reliability factor

k_t temperature factor

k_{ms} median-stress factor (1 for two-way bending (followers), 1.4 for one-way bending (input or output))

Reliability Factor: k_r

Reliability (%)	k_r
50	1.000
90	0.897
99	0.814
99.9	0.753
99.99	0.702
99.999	0.659

Temperature Factor: k_t

$$k_t = \begin{cases} 1 & T \leqslant 160 \text{ F} \\ \dfrac{620}{460 + T} & T > 160 \text{ F} \end{cases}$$

General Guidelines

1. $RR \geqslant 1/6$
2. Use multi-stage gears for larger than $RR < 1/6$
3. $8m \leqslant b \leq 16m$
4. many small teeth $\gg$ few large teeth
5. few teeth $\rightarrow$ small gear, but be careful about interference
6. Avoid exact ratio $\rightarrow$ hunting tooth

Example

A pair of spur gears with face width $b = 3$ cm is used in a conveyor belt drive. The input motor has $\omega_{\max}$ of 200 rad/s. The pinion has 18 teeth. The conveyor has moderate shock and should be driven at 100 rad/s. The gears have pressure angles ϕ of 20°. Both pinion and gear has $m = 1$ cm. Determine the maximum power that the gears can transmit continuously with 1% chance of bending fatigue failure. Steel has $S_{ut} = 400$ MPa

Solution

Pitch line velicity v_t is

$$v_t = \omega R_{pitch} = \frac{\omega m z}{2}$$
$$= \frac{200(0.01)(18)}{2} = 18 \text{ m/s}$$

Determine number of teeth for mating gear (input pinion z_{in} = 18)

$$\frac{z_{out}}{z_{in}} = \frac{\omega_{in}}{\omega_{out}}$$
$$z_{out} = \frac{200(18)}{100} = 36 \text{ teeth}$$

Solution

Using $v_t = 18$ m/s, select $Q = 10$

$$B = 0.25(12 - 10)^{2/3} = 0.397$$

$$A = 50 + 56(1 - 0.397) = 83.77$$

$$K_v = \left(\frac{83.77 + \sqrt{200(18)}}{83.77}\right)^{0.397} = 1.24$$

Solution

For other factors

- b 0.03 (face width)
- Y_J 0.32 (18 pinion - 36 gear)
- K_O 1.5 (input - motor, output - moderate shock conveyor)
- K_m 1.6 (b = 3 cm, assuming standard gear + average mounting)
- K_v 1.24

Solution

The bending fatigue stress in the gear is

$$\begin{aligned}\sigma &= \frac{F_t}{bY_J m} K_O K_m K_v \\ &= \frac{F_t}{(0.03)(0.32)(0.01)}(1.5)(1.6)(1.24) \\ &= 31000 F_t\end{aligned}$$

Solution

Next, the material fatigue strength is

$$
\begin{aligned}
S_e' &= S_e C_L C_G C_S k_r k_t k_{ms} \\
&= (400 \times 10^6 (0.5))(1)(1)(0.75)(0.814)(1)(1.4) \\
&= 1.71 \times 10^8
\end{aligned}
$$

We can then find the maximum allowable tangential force

$$
\begin{aligned}
F_t &= \frac{1.71 \times 10^8}{31000} = 5516 \text{ N} \\
P &= T\omega = F_t v_t = 5516 \times 18 = 9.93 \times 10^4 \text{ W}
\end{aligned}
$$

Rack and Pinion

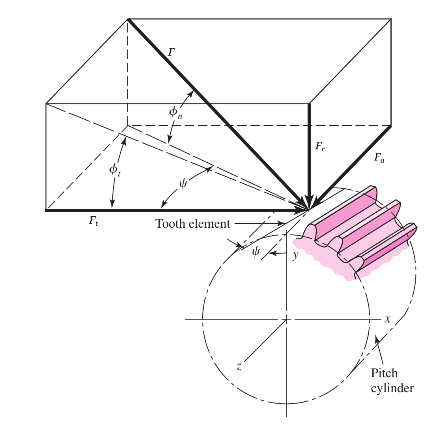
$$F_t = \frac{T}{R}$$
$$F_r = \frac{T \tan \phi}{R}$$

- Rack = linear gear
- Convert torque to force
- Cheaper but less accurate than power screw
- No mechanical advantages

Outline

Helical Gears

Helical Gear Force Analysis



ϕ_n normal pressure angle

ϕ_t tangential pressure angle

m_n normal module

m_t tangential module

ψ helix angle

$$F_t = \frac{\text{Power}}{v_t}$$

$$F_r = F_t \tan \phi_t$$

$$F_a = F_t \tan \psi$$

$$\tan \phi_n = \tan \phi_t \cos \psi$$

$$m_n = m_t \cos \psi$$

Why use Helical Gears?

- Smoother operations due to gradual teeth engagement
- Most common gears in automotive transmission

Design Equations

Same as spur gear equation with small modification

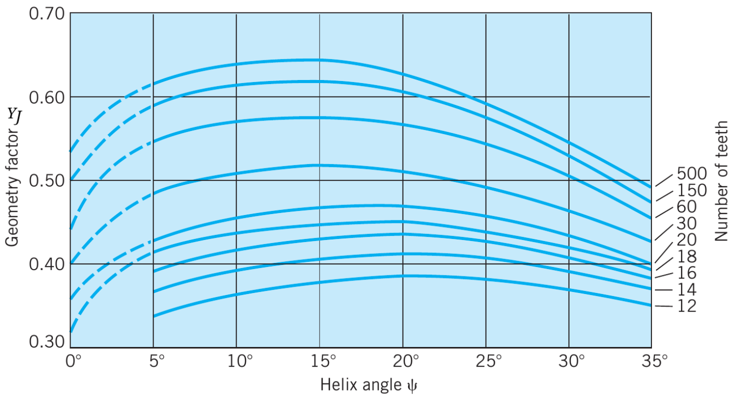
$$\sigma = \frac{F_t}{bY_J m_t} K_v K_o (0.93 K_m)$$

$$S_e' = S_e C_L C_G C_S k_r k_t k_{ms}$$

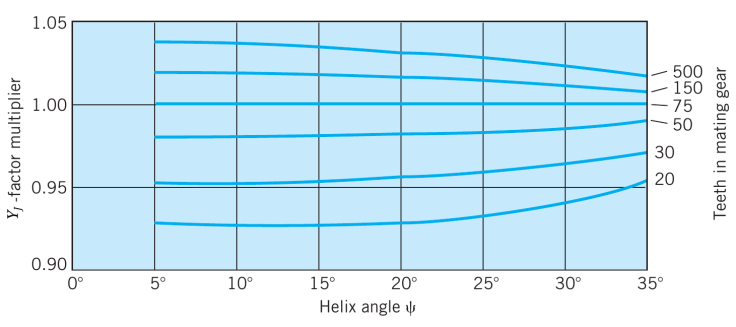
0.93 indicated helical gears less sensitivity to mounting factor

Y_J needs small modification for helical teeth

Geometry Factor: Y_J



Geometry Factor Multiplier: Y_J



Example: Meshing Helical Gears

A pair of meshing helical gears is connected at the input side to a 0.5-hp motor at 1800 rpm and to an output shaft at 600 rpm. The input gear has 18 teeth, $\phi_n = 20^\circ$, $m_n = 0.00173$, $\psi = 30^\circ$, $b = 2$ cm. From the given information, determine the pitch line velocity v_t, gear tooth forces $F_t, F_r,$ and F_a, and bending stress σ.

Solution: Meshing Helical Gears

Calculate tangential module from normal module, then pitch diameter and tangential velocity.

$$m_t = \frac{m_n}{\cos 30^\circ} = \frac{0.00173}{\cos 30^\circ} = 0.002$$

$$d = mz = 0.002(18) = 0.036 \text{ m}$$

$$v_t = \omega \frac{d}{2} = (1800)\frac{2\pi}{60}\frac{0.036}{2} = 3.4 \text{ m/s}$$

Solution: Meshing Helical Gears

Transmitted power only depends on tangential force, after which we can calculate axial and radial forces.

$$
\begin{aligned}
F_t &= \frac{\text{Power}}{v_t} = \frac{0.5(746)}{3.4} = 104\ \text{N} \\
\tan\phi_t &= \frac{\tan\phi_n}{\cos\psi} = \frac{\tan 20^\circ}{\cos 30^\circ} = 0.42 \\
\phi_t &= 22.8^\circ \\
F_r &= F_t \tan\phi_t = 104 \tan 22.8^\circ = 43.7\ \text{N} \\
F_a &= F_t \tan\psi = 104 \tan 30^\circ = 60\ \text{N}
\end{aligned}
$$

Solution: Meshing Helical Gears

$$\sigma = \frac{F_t}{bY_J m_t} K_v K_o (0.93K_m)$$

b = 0.02 m

For 18-teeth to 54-teeth mesh, Y_J = 0.99(0.42) = 0.416

Uniform-uniform input-output, K_o = 1

Solution: Meshing Helical Gears

For K_v, since $v_t = 3.57$ m/s, let $Q = 6$.

$$B = 0.25(12 - 6)^{2/3} = 0.825$$

$$A = 50 + 56(1 - 0.825) = 59.8$$

$$K_v = \left(\frac{59.8 + \sqrt{200 v_t}}{59.8}\right)^{0.825} = 1.36$$

For K_m, nothing specific about gears or mounting, let's go with the middle case for $b = 2$ cm. $K_m = 1.6$

Solution: Meshing Helical Gears

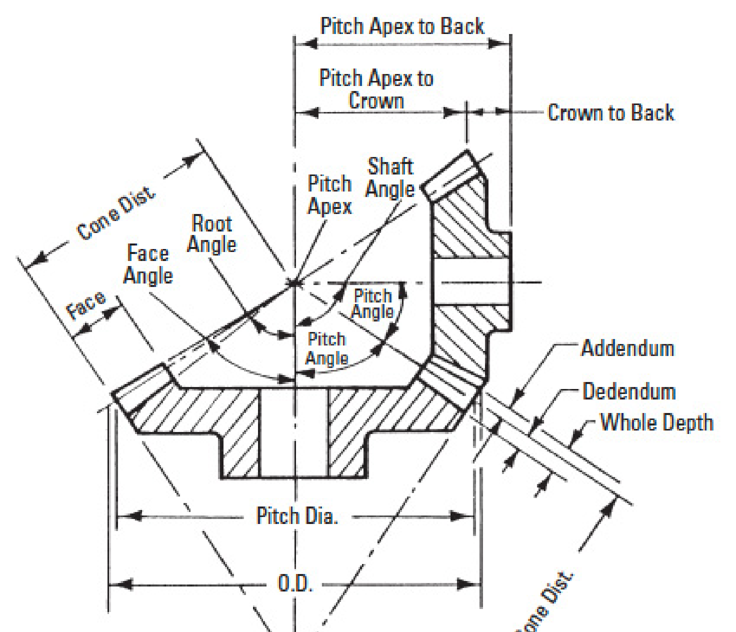
We can finally calculate σ

$$
\begin{aligned}
\sigma &= \frac{F_t}{bY_J m_t} K_v K_o (0.93 K_m) \\
&= \frac{104}{(0.02)(0.416)(0.002)} (1.36)(1)((0.93)1.6) \\
&= 1.26 \times 10^7 = 12.6 \text{ MPa}
\end{aligned}
$$

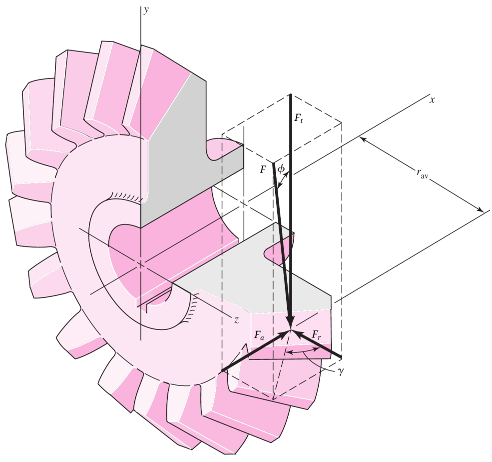
Outline

Bevel Gears

Bevel Gear Geometry



Bevel Gears



γ pitch angle

ϕ pressure angle

$$d_{av} = d - b \sin\gamma$$

$$v_{av} = \omega \frac{d_{av}}{2}$$

$$F_t = \frac{\text{Power}}{v_{av}}$$

$$F_a = F_t \tan\phi \sin\gamma$$

$$F_r = F_t \tan\phi \cos\gamma$$

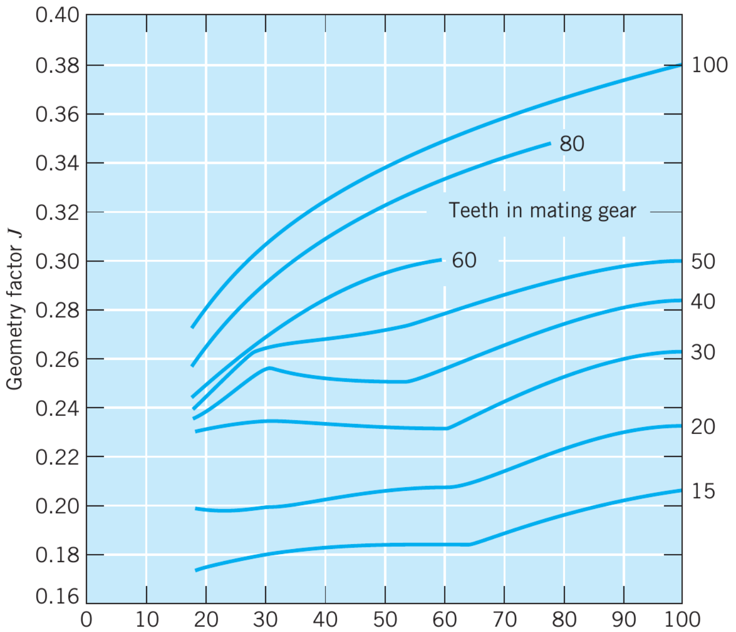
Design Equations

Same as spur gear equation with small modification

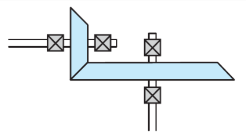
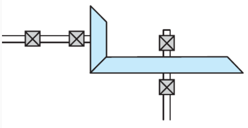
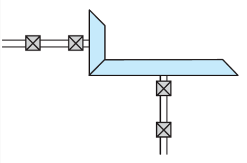
$$\sigma = \frac{F_t}{bY_J m} K_v K_o K_m$$

$$S_e' = S_e C_L C_G C_S k_r k_t k_{ms}$$

Geometry Factor: Y_J



Mounting Factor: K_m

Mounting type		Mounting Rigidity
Both straddle-mounted		1.0 to 1.25
straddle-overhung		1.1 to 1.4
Both overhung		1.25 to 1.5

Example: Bevel Gearset Design

Identical bevel gears has a module of 0.005 m/teeth, 25 teeth, 2-cm face width, and a 20° normal pressure angle. The gear quality is $Q = 7$. Both requires overhung mounting. The gears are made of ductile iron whose $S_e = 95$ MPa. Determine the power rating of the gearset at 600 rpm.

Solution: Bevel Gearset Design

For the stress side,

$$d_{av} = mz = 0.125 \text{ m}$$
$$v_t = \omega \frac{d_{av}}{2} = 600 \frac{2\pi}{60} \frac{0.125}{2} = 3.93 \text{ m/s}$$

For uniform-uniform loading, $K_o = 1$

$$B = 0.25(12 - 7)^{2/3} = 0.731$$
$$A = 50 + 56(1 - 0.731) = 65$$
$$K_v = \left(\frac{65 + \sqrt{200(3.93)}}{65} \right)^{0.731} = 1.3$$

For both-overhung mounting, $K_m = 1.5$

For 25-teeth pair, $Y_J = 0.22$

Solution: Bevel Gearset Design

Now, onto the strength side,

$$
\begin{aligned}
C_L &= 1 \text{ for bending} \\
C_s &= 0.75 \text{ for machined surface} \\
C_G &= 1
\end{aligned}
$$

No requirement on the reliablility. Let's be generous, give it 90% so that $k_r = 0.897$.

For normal operating temperature, $k_t = 1$.

For one-way bending, $k_{ms} = 1.4$.

Solution: Bevel Gearset Design

Set the two sides equal ($N_s = 1$), we have

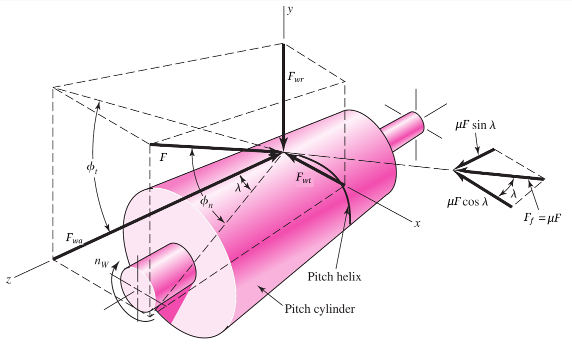
$$\frac{F_t}{(0.02)(0.22)(0.005)}(1.3)(1)(1.5) = 95 \times 10^6(1)(1)(0.75)(0.897)(1)(1.4)$$

$$F_t = 1009 \text{ N}$$

$$\text{Power} = F_t v_t = 1009(3.93) = 3965 \text{ W} = 5.31 \text{ hp}$$

Outline

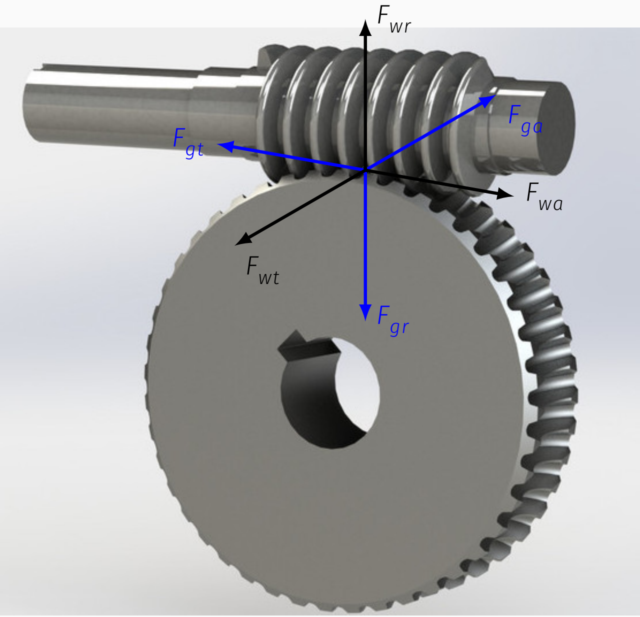
Worm Force Analysis



- With friction $F_f = \mu F$, λ = lead angle, ϕ_n = normal pressure angle

$$F_{wt} = F \cos \phi_n \sin \lambda + \mu F \cos \lambda = F_{ga}$$
$$F_{wr} = F \sin \phi_n = F_{gr}$$
$$F_{wa} = F \cos \phi_n \cos \lambda - \mu F \sin \lambda = F_{gt}$$

Worm Force Analysis II



Worm Efficiency

- Worm and worm gear velocities can be related by

$$\frac{v_g}{v_w} = \tan\lambda$$
$$v_s = \sqrt{v_w^2 + v_g^2} = v_w\sqrt{1+\tan^2\lambda}$$

- Efficiency η is

$$\begin{aligned}\eta &= \frac{F_{gt}v_g}{F_{wt}v_w} \\ &= \frac{\cos\phi_n\cos\lambda - \mu\sin\lambda}{\cos\phi_n\sin\lambda + \mu\cos\lambda}\tan\lambda \\ &= \frac{\cos\phi_n - \mu\tan\lambda}{\cos\phi_n + \mu\cot\lambda}\end{aligned}$$

Self-Locking

- Thread will lock itself (not backdrivable) when $F_{wt} \leqslant 0$

$$
\begin{aligned}
F_{wt} &= F\cos\phi_n \sin\lambda - \mu F \cos\lambda \leqslant 0 \\
\mu &\geqslant \cos\phi_n \tan\lambda
\end{aligned}
$$

- Note that when gear drives worm, friction direction is reversed (hence the negative sign)
- Desirable in cases where auto-braking is needed
- In systems with large inertia, sudden stop can break the worm tooth → alternative brake mechanism is needd

Worm Gear Design Equations

Worm gears have higher stresses than worm, so our main concern is designing the gear.

$$F_{gt,allow} = \frac{C_s d^{0.8} b C_m C_v}{75.948}$$

$F_{gt,allow}$	allowable gear force [N]
C_s	material factor
d	gear diameter [mm]
b	effective face width (actual width but less than $0.67d_w$) [mm]
C_m	ratio correction factor
C_v	velocity factor

C_s: Material factor

For center distance $C < 76.2$ mm

$$C_s = 720 + 0.000633C^3$$

For $C \geqslant 76.2$ mm

Sand-cast gears:

$$\begin{aligned} C_s &= 1000 && d \leqslant 63.5 \text{ mm} \\ C_s &= 1856.104 - 467.5454 \log d && d > 63.5 \text{ mm} \end{aligned}$$

Chilled-cast gears:

$$\begin{aligned} C_s &= 1000 && d \leqslant 203.2 \text{ mm} \\ C_s &= 2052.011 - 455.8259 \log d && d > 203.2 \text{ mm} \end{aligned}$$

Centrifugally-cast gears:

$$\begin{aligned} C_s &= 1000 && d \leqslant 635 \text{ mm} \\ C_s &= 1503.811 - 179.7503 \log d && d > 635 \text{ mm} \end{aligned}$$

Note: Use C and d in mm

C_m: Ratio correction factor

Depends on gear ratio,

$RR = \omega_i / \omega_o$

$$C_m = \begin{cases} 0.02\sqrt{-RR^2 + 40RR - 76} + 0.46 & 3 < RR \leqslant 20 \\ 0.0107\sqrt{-RR^2 + 56RR + 5145} & 20 < RR \leqslant 76 \\ 1.1483 - 0.00658RR & RR > 76 \end{cases}$$

C_v: Velocity factor

Depends on sliding velocity at mean worm diameter v_s:

$$C_v = \begin{cases} 0.659e^{-0.2165v_s} & 0 < v_s \leqslant 3.556 \text{ m/s} \\ 0.652v_s^{-0.571} & 3.556 < v_s \leqslant 15.24 \text{ m/s} \\ 1.098v_s^{-0.774} & v_s > 15.24 \text{ m/s} \end{cases}$$

Example: Worm gear speed reducer

A 2-hp, 1200-rpm motor drives a 60-rpm machanism by using a worm gear reducer. The gear is centrifugally-cast and has d = 20 cm. The worm has $\lambda = 12^\circ$, $\phi_n = 20^\circ$, and d_w = 5 cm. Assume $\mu = 0.1$, determine

1. all force components according to the rated power
2. power delivered to the driven mechanism
3. whether the drive is self-locking
4. safety factor of worm gear

Solution: Worm gear speed reducer

First, determine v_w to determine v_g

$$
\begin{aligned}
v_w &= \omega_w (d_w/2) = 3.14 \text{ m/s} \\
v_g &= v_w \tan\lambda = 3.14 \tan 12^\circ \\
&= 0.667 \text{ m/s}
\end{aligned}
$$

Power output at the worm gear is

$$
\begin{aligned}
\eta &= \frac{\cos\phi_n - \mu\tan\lambda}{\cos\phi_n + \mu\cot\lambda} = \frac{\cos 20^\circ - 0.1\tan 12^\circ}{\cos 20^\circ + 0.1\cot 12^\circ} = 0.65 \\
\text{Power}_g &= 0.65(2)(746) = 970 \text{ W} \\
F_{gt} &= \frac{\text{Power}_g}{v_g} = \frac{970}{0.667} \\
&= 1455 \text{ N}
\end{aligned}
$$

Solution: Worm gear speed reducer

The other forces can then be calculated.

$$F_{ga} = F_{wt} = \frac{\text{Power}_w}{v_w} = \frac{2(746)}{3.14} = 475 \text{ N}$$

To find $F_{wr} = F_{gr}$, we need first to find F, which we can solve from either F_{gt} or F_{ga}

$$\begin{aligned}
F_{ga} &= 475 = F\cos\phi_n \sin\lambda + \mu F\cos\lambda = F\left(\cos 20^\circ \sin 12^\circ + 0.1\cos 12^\circ\right) \\
475 &= 0.293F \\
F &= 1620 \text{ N} \\
F_{gr} &= F\sin\phi_n = 1620\sin 20^\circ = 554 \text{ N}
\end{aligned}$$

Solution: Worm gear speed reducer

Self locking

$$\begin{aligned} \mu &\geqslant \cos 20^\circ \tan 12^\circ \\ 0.1 &\geqslant 0.20 \end{aligned}$$

Nope! μ is too low to provide self-locking.

Solution: Worm gear speed reducer

Definition of safety factor

$$N_s = \frac{F_{gt,allow}}{F_{gt}}$$

Determine the allowable tangential force on worm gear and material factor

$$F_{gt,allow} = \frac{C_s d^{0.8} b C_m C_v}{75.948}$$

$$C = \frac{d_g}{2} + \frac{d_w}{2} = \frac{0.2 + 0.05}{2} = 0.125 \text{ m}$$

$$C_s = 1000$$

Solution: Worm gear speed reducer

Ratio correction factor

$$RR = 1200/60 = 20$$
$$C_m = 0.02\sqrt{-20^2 + 40(20) - 76} + 0.46 = 0.82$$

Velocity factor

$$v_s = v_g\sqrt{1 + \tan^2\lambda} = 3.14\sqrt{1 + \tan^2 12^\circ} = 3.21 \text{ m/s}$$
$$C_v = 0.659e^{-0.2165(3.21)} = 0.33$$

Solution: Worm gear speed reducer

Finally, the safety factor

$$F_{gt,allow} = \frac{1000(200)^{0.8}(0.67(50))(0.82)(0.33)}{75.948} = 8273 \text{ N}$$

$$N_s = \frac{8273}{1455} = 5.69$$