

HW1 Definitions and Notation

CSE20F21

Due: Tuesday, October 5, 2021 at 11:00PM on Gradescope

In this assignment,

You will practice reading and applying definitions to get comfortable working with mathematical language. As a result, you can expect to spend more time reading the questions and looking up notation than doing calculations.

For all HW assignments:

Weekly homework may be done individually or in groups of up to 3 students. You may switch HW partners for different HW assignments. The lowest HW score will not be included in your overall HW average. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission.

All submitted homework for this class must be typed. Diagrams may be hand-drawn and scanned and included in the typed document. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions¹.

Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.
- You may not collaborate on homework with anyone other than your group members. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza (as private notes viewable only to the Instructors). You *cannot* use any online resources about the course content other than the class material from this

¹To use this template, copy the source file (extension `.tex`) to your working directory or upload to Overleaf.

quarter – this is primarily to ensure that we all use consistent notation and definitions we will use this quarter.

- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw1-definitions-and-notation”.

Resources: To review the topics you are working with for this assignment, see the class material from Week 0 and 1. We will post frequently asked questions and our answers to them in a pinned Piazza post.

Assigned questions

1. (*Graded for correctness*²) Each of the sets below is described using set builder notation or as a result of set operations applied to other known sets. Rewrite them using the roster method.

Remember our discussions of data-types: use clear notation that is consistent with our class notes and definitions to communicate the data-types of the elements in each set.

Sample response that can be used as reference for the detail expected in your answer:

The set $\{A\} \circ \{AU, AC, AG\}$ can be written using the roster method as

$$\{AAU, AAC, AAG\}$$

because set-wise concatenation gives a set whose elements are all possible results of concatenating an element of the left set with an element of the right set. Since the left set in this example only has one element (**A**), each of the elements of the set we described starts with **A**. There are three elements of this set, one for each of the distinct elements of the right set.

(a)

$$\{x \in S \mid rnalen(x) = 1\} \circ \{x \in S \mid rnalen(x) = 1\}$$

where S is the set of RNA strands and $rnalen$ is the recursively defined function that we discussed in class.

²This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

(b)

$$\{(r, g, b) \in C \mid r + g + b = 1\}$$

where $C = \{(r, g, b) \mid 0 \leq r \leq 255, 0 \leq g \leq 255, 0 \leq b \leq 255, r \in \mathbb{N}, g \in \mathbb{N}, b \in \mathbb{N}\}$ is the set that you worked with in Monday's review quiz.

(c)

$$\{a \in \mathbb{Z} \mid a \text{ div } 2 = a \text{ mod } 2\}$$

2. (*Graded for fair effort completeness*³)

- (a) In Wednesday's review quiz, you considered some attempted recursive definitions for the function with domain \mathbb{N} and with codomain \mathbb{Z} which gives 2^n for each n . Write out a correct recursive definition of this function.
- (b) How would your answer to part (a) change if we consider a new function with the same domain and rule but whose codomain is \mathbb{R} ?
- (c) How would your answer to part (a) change if we consider a new function with the same codomain and rule but whose domain is \mathbb{R} ?
- (d) Write a recursive definition of the function with domain \mathbb{Z}^+ , codomain \mathbb{Z}^+ and which gives $n!$ for each n . The $!$ symbol is the "factorial" symbol and means that we need to multiply n by each of the integers between it and 1 inclusive. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

3. (*Graded for correctness*) Recall the function d_0 which takes an ordered pair of ratings 3-tuples and returns a measure of the distance between them given by

$$d_0((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

Sample response that can be used as reference for the detail expected in your answers for this question:

To give an example of two 3-tuples that are d_0 distance 1 from each other, consider the 3-tuples $(1, 0, 0)$ and $(0, 0, 0)$. We calculate the function application:

$$d_0((1, 0, 0), (0, 0, 0)) = \sqrt{(1 - 0)^2 + (0 - 0)^2 + (0 - 0)^2} = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1,$$

which is the result required for this example.

³This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

- (a) Give an example of three 3-tuples

$$(x_{1,1}, x_{1,2}, x_{1,3})$$

$$(x_{2,1}, x_{2,2}, x_{2,3})$$

$$(x_{3,1}, x_{3,2}, x_{3,3})$$

that are all d_0 distance **greater than** 1 from each other. In other words, for each i and j between 1 and 3 (with $i \neq j$),

$$d_0((x_{i,1}, x_{i,2}, x_{i,3}), (x_{j,1}, x_{j,2}, x_{j,3})) > 1$$

Your answer should include **both** specific values for each example 3-tuple **and** a justification of your examples with (clear, correct, complete) calculations and/or references to definitions and connecting them with the desired conclusion.

To think about: how will you justify that the square root of a number is greater than 1? Are calculations from a calculator accurate enough to help us?

- (b) What is the range of values that results from applying the function d_0 to ordered pairs of 3-tuple ratings? That is, what are the smallest and largest possible results?

Your answer should include **both** specific values for the smallest and largest possible results **and** a justification of your answers with (clear, correct, complete) calculations and/or references to definitions and connecting them with the desired conclusion.

HW2 Numbers Due: Tuesday, October 12, 2021 at 11:00PM on Gradescope

In this assignment,

You will consider multiple number representations and how they connect to applications in computer science. You will also practice tracing and working with algorithms.

Instructions and academic integrity reminders for all homework assignments in CSE20 this quarter are on the class website and on the hw1-definitions-and-notations assignment.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw2-numbers”.

Resources: To review the topics you are working with for this assignment, see the class material from Friday of Week 1 and Monday and Wednesday of Week 2. We will post frequently asked questions and our answers to them in a pinned Piazza post.

Assigned questions

1. (*Graded for fair effort completeness*⁴) Pick a nonnegative integer between 100 and 1000 (inclusive) and express it using at least three different representations, at least two of which must be ones discussed in class. Include at least one trace of using procedure *baseb1* from page 16 of the Week 0 and 1 notes to calculate the base b_1 expansion of your number (for your choice of b_1) and include at least one trace of using procedure *baseb2* from page 16 of the Week 0 and 1 notes to calculate the base b_2 expansion of your number (for your choice of b_2).

Extra; not for credit: Consider choosing representations that might be useful in some way; what would make one representation more useful than another?

⁴This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

2. In this question, we will use recursive definitions to give a precise description of the set of strings that form octal (base 8) expansions and the values of those expansions. Let's start by defining the set of possible coefficients

$$C_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

- (a) (*Graded for fair effort completeness*) Fill in the recursive definition of the set of all strings that form octal expansions, S_8 :

Definition The set S_8 is defined (recursively) by:

Basis Step: If $x \in \underline{\hspace{4cm}}$, then $x \in S_8$
 Recursive Step: If $s \in S_8$ and $x \in C_8$, then $\underline{\hspace{4cm}} \in S_8$

- (b) (*Graded for correctness*⁵) Consider the function $v_8 : S_8 \rightarrow \mathbb{Z}^+$ defined recursively by

Basis Step: If $x \in C_8$, then $v_8(x) = x$.
 Recursive Step: If $s \in S_8$ and $x \in C_8$, then $v_8(sx) = 8v_8(s) + x$, where the input sx is the result of string concatenation and the output $8v_8(s)$ is the result of integer multiplication.

Calculate $v_8(104)$, including all steps in your calculation and justifications for them.

- (c) (*Graded for fair effort completeness*) It turns out⁶ that for any string u in S_8 , the value of the octal expansion $(u)_8$ equals $v_8(u)$. Using this fact, write an expression relating the value of $(u000)_8$ to the value of $(u)_8$ and justify it.
3. (*Graded for correctness*) Recall that, mathematically, a color can be represented as a 3-tuple (r, g, b) where r represents the red component, g the green component, b the blue component and where each of r, g, b must be from the collection $\{x \in \mathbb{N} \mid 0 \leq x \leq 255\}$.

As an **alternative** representation, in this assignment we'll use base 16 fixed-width expansions to represent colors as single numbers.

Definition: A **hex color** is a nonnegative integer less than or equal to 16777215. For n a hex color, we define its red, green, and blue components by first writing its base 16 fixed-width 6 expansion

$$n = (r_1 r_2 g_1 g_2 b_1 b_2)_{16,6}$$

and then defining $(r_1 r_2)_{16,2}$ is the red component, $(g_1 g_2)_{16,2}$ is the green component, and $(b_1 b_2)_{16,2}$ is the blue component.

⁵This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

⁶We'll be able to prove this in Week 7 or so, once we've talked about induction.

Sample response that can be used as reference for the detail expected in your answer:

In RGB codes⁷ white is represented as maximum red, maximum green, and maximum blue and so has $(FF)_{16,2}$ as each of these components. This means that the hex color for white is $(FFFFFF)_{16,6}$ which is the value

$$15 \cdot 16^5 + 15 \cdot 16^4 + 15 \cdot 16^3 + 15 \cdot 16^2 + 15 \cdot 16^1 + 15 \cdot 16^0 = 16^6 - 1 = 16777215$$

- (a) Write the hex color representing red (with no green or blue) in base 16 fixed-width 6 and also calculate its value (using usual mathematical conventions). Include your (clear, correct, complete) calculations.
- (b) Write the hex color representing green (with no red or blue) in base 16 fixed-width 6 and also calculate its value (using usual mathematical conventions). Include your (clear, correct, complete) calculations.
- (c) Write the hex color representing blue (with no red or green) in base 16 fixed-width 6 and also calculate its values (using usual mathematical conventions). Include your (clear, correct, complete) calculations.
- (d) The human eye can't distinguish between some hex colors because of physical limitations. Give an example of two hex colors c_1 and c_2 such that they look indistinguishable (the colors they represent are very very similar) but

$$c_1 - c_2 > 50000$$

Justify your choice with (clear, correct, complete) calculations and/or references to definitions, and connecting these calculations and/or definitions with the desired properties. Include squares with each of your two colors so that we can see how indistinguishable they are.

Pro tip: To show a color, you can use the following LaTeX source code:

```
\definecolor{UCSDaccent}{RGB}{0,198,215}
\textcolor{UCSDaccent}{\rule{1cm}{1cm}}
```

which produces



Notice that the code to define the color uses the decimal-like values for each of the red, green, and blue components. For the UCSD accent color we defined, the base 16 fixed-width 2 values are: red is $(00)_{16,2}$, green is $(C6)_{16,2}$, blue is $(D7)_{16,2}$.

Extra; not for credit: What does this mean about the choice of hex color for representing colors? What are advantages and disadvantages of this representation?

⁷You can use online tools to visualize the colors associated with different values for the red, green, and blue components, e.g. https://www.w3schools.com/colors/colors_rgb.asp.

4. (*Graded for correctness*) In class (Week 2 notes page 7), we discussed fixed-width addition. In this question we will look at fixed-width multiplication. The algorithm for fixed-width multiplication is to multiply using the usual long-multiplication algorithm (column-by-column and carry), and dropping all leftmost columns so the result is the same width as the input terms. For each of the examples below, consider whether this algorithm gives the correct value for the product of the two numbers, based on the way the bitstrings are interpreted.

Sample response that can be used as reference for the detail expected in your answer:

The fixed-width 5 multiplication of $[00101]_{2c,5}$ and $[00101]_{2c,5}$ does not give the correct value for the product, as we can see from the following calculation.

First, we calculate the values:

$$[00101]_{2c,5} = 0 \cdot (-2^4) + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 4 + 1 = 5$$

so the correct value for the product is $5 \cdot 5 = 25$, which cannot be represented in 2s complement width 5 (the largest positive number that can be represented in 2s complement width 5 is $[01111]_{2c,5} = 8 + 4 + 2 + 1 = 15$).

When we perform the fixed-width 5 multiplication algorithm:

$$\begin{array}{r} 00101 \\ \times 00101 \\ \hline 00101 \\ +00000 \\ +00101 \\ \hline 11001 \end{array}$$

we get $[11001]_{2c,5} = 1 \cdot (-2^4) + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = -16 + 8 + 1 = -7$, which is not the required value.

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- (a) Does the fixed-width 5 multiplication of $[11101]_{2c,5}$ and $[11011]_{2c,5}$ give the correct value for the product? Justify your answer with (clear, correct, complete) calculations and/or references to definitions, and connecting these calculations and/or definitions with your answer.
- (b) Does the fixed-width 5 multiplication of $[00100]_{2c,5}$ and $[11100]_{2c,5}$ give the correct value for the product? Justify your answer with (clear, correct, complete) calculations and/or references to definitions, and connecting these calculations and/or definitions with your answer.

HW3 Circuits and Logic Due: Tuesday, October 26, 2021 at 11:00PM on Gradescope

In this assignment,

You will consider how circuits and logic can be used to represent mathematical and technical claims. You will use propositional and predicate logic to evaluate these claims.

Instructions and academic integrity reminders for all homework assignments in CSE20 this quarter are on the class website and on the hw1-definitions-and-notations assignment.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw3-circuits-and-logic”.

Resources: To review the topics you are working with for this assignment, see the class material from Week 3 and Week 4. We will post frequently asked questions and our answers to them in a pinned Piazza post.

Assigned questions

1. Imagine a friend suggests the following argument to you: “The compound proposition

$$(x \rightarrow y) \rightarrow z$$

is logically equivalent to

$$x \rightarrow (y \rightarrow z)$$

because I can transform one to the other using the following sequence of logical equivalences like distributivity and associativity and using that $(p \rightarrow q) \equiv (\neg p \vee q)$:

$$(x \rightarrow y) \rightarrow z \equiv \neg(\neg x \vee y) \vee z \equiv \neg x \vee (\neg y \vee z) \equiv x \rightarrow (\neg y \vee z) \equiv x \rightarrow (y \rightarrow z)$$

so conditionals are associative just like conjunctions and disjunctions”.

- (a) (*Graded for correctness*⁸) Prove to your friend that they made a mistake by giving a truth assignment to the propositional variables where the two compound propositions $(x \rightarrow y) \rightarrow z$ and $x \rightarrow (y \rightarrow z)$ have different truth values. Justify your choice by evaluating these compound propositions using the definitions of the logical connectives and include enough intermediate steps so that a student in CSE 20 who may be struggling with the material can still follow along with your reasoning.
- (b) (*Graded for fair effort completeness*⁹) Help your friend find the problem in their argument by pointing out which step(s) were incorrect.

⁸This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

⁹This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

- (c) (*Graded for fair effort completeness*) Give **three** different compound propositions that are actually logically equivalent to (and not the same as)

$$x \rightarrow (y \rightarrow z)$$

Justify each one of these logical equivalences either by applying a sequence of logical equivalences or using a truth table. Notice that you can use other logical operators (e.g. $\neg, \vee, \wedge, \oplus, \rightarrow, \leftrightarrow$) when constructing your compound propositions.

Bonus; not for credit (do not hand in): How would you translate each of the equivalent compound propositions in English? Does doing so help illustrate why they are equivalent?

2. For each part of this question you will use the following input-output definition table with four inputs x_3, x_2, x_1, x_0

| x_3 | x_2 | x_1 | x_0 | out |
|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 |

- (a) (*Graded for fair effort completeness*) Construct a compound proposition that implements this input-output table and draw a combinatorial circuit corresponding to this compound proposition. We recommend the following steps:

- Draw symbols for the inputs on the left-hand-side and for the output on the right-hand side.
- Construct an expression for out using (some of) the inputs x_3, x_2, x_1, x_0 and the logic gates XOR, AND, OR, NOT. *Hint:* are normal forms helpful here? How do you choose which normal form to use?
- Draw and label the gates corresponding to the expression you construct, and connect appropriately with wires.

(b) (*Graded for correctness*) For each of the following predicates, consider whether this input-output table (and the logic circuit from part (a)) implements the rule for the predicate. If yes, explain why using the definitions of the operations involved and consider **all** possible inputs. If no, explain why not by providing specific example input values and using the input-output definition table to compute the logic circuit's output for this example input and then comparing with the value of the predicate in question to justify your example.

i. $P_1 : \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \rightarrow \{T, F\}$ given by

$$P_1((x_3, x_2, x_1, x_0)) = \begin{cases} T & \text{when } (x_3x_2x_1x_0)_{2,4} \leq 5 \\ F & \text{otherwise} \end{cases}$$

ii. $P_2 : \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \rightarrow \{T, F\}$ given by

$$P_2((x_3, x_2, x_1, x_0)) = \begin{cases} T & \text{when } (x_3x_2x_1x_0)_{2,4} \bmod 5 = 0 \\ F & \text{otherwise} \end{cases}$$

iii. $P_3 : \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \rightarrow \{T, F\}$ given by

$$P_3((x_3, x_2, x_1, x_0)) = \begin{cases} T & \text{when } (x_3x_2x_1x_0)_{2,4} = [x_3x_2x_1x_0]_{s,4} \\ F & \text{otherwise} \end{cases}$$

3. Recall the functions *mutation*, *insertion*, and *deletion* defined in class. We define the predicates:

Mut with domain $S \times S$ is defined by, for $s_1 \in S$ and $s_2 \in S$,

$$Mut((s_1, s_2)) = \exists k \in \mathbb{Z}^+ \exists b \in B(mutation((s_1, k, b)) = s_2)$$

Ins with domain $S \times S$ is defined by, for $s_1 \in S$ and $s_2 \in S$,

$$Ins((s_1, s_2)) = \exists k \in \mathbb{Z}^+ \exists b \in B(insertion((s_1, k, b)) = s_2)$$

Del with domain $\{s \in S \mid rnalen(s) > 1\} \times S$ is defined by, for $s_1 \in \{s \in S \mid rnalen(s) > 1\}$ and $s_2 \in S$,

$$Del((s_1, s_2)) = \exists k \in \mathbb{Z}^+ (deletion((s_1, k)) = s_2)$$

For each quantified statement below, **first** translate to an English sentence.

Then, negate the **whole** statement and rewrite this negated statement so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

The translations are graded for fair effort completeness.

The negations are graded for correctness. For negations: You do not need to justify your work for this part of the question. However, if you include correct intermediate steps, we might be able to award partial credit for an incorrect answer.

Sample response that can be used as reference for the detail expected in your answer: Consider the statement

$$\forall n \in \mathbb{Z}^+ \exists s \in S (L((s, n)) \wedge F_{\mathbf{A}}(s) \wedge BC((s, \mathbf{A}, n)))$$

Solution: English translation is

For each positive integer there is some RNA strand of that length that starts with \mathbf{A} and all of its bases are \mathbf{A} .

We obtain the negation using multiple applications of De Morgan's rule and logical equivalences.

$$\begin{aligned} & \neg \forall n \in \mathbb{Z}^+ \exists s \in S (L((s, n)) \wedge F_{\mathbf{A}}(s) \wedge BC((s, \mathbf{A}, n))) \\ & \equiv \exists n \in \mathbb{Z}^+ \neg \exists s \in S (L((s, n)) \wedge F_{\mathbf{A}}(s) \wedge BC((s, \mathbf{A}, n))) \\ & \equiv \exists n \in \mathbb{Z}^+ \forall s \in S \neg (L((s, n)) \wedge F_{\mathbf{A}}(s) \wedge BC((s, \mathbf{A}, n))) \\ & \equiv \exists n \in \mathbb{Z}^+ \forall s \in S (\neg L((s, n)) \vee \neg F_{\mathbf{A}}(s) \vee \neg BC((s, \mathbf{A}, n))) \end{aligned}$$

(a) First statement:

$$\forall s \in S (Mut((s, s)) \leftrightarrow Ins((s, s)))$$

(b) Second statement

$$\forall s_1 \in S \forall s_2 \in S \forall s_3 \in S ((Mut((s_1, s_2)) \wedge Mut((s_2, s_3))) \rightarrow Mut((s_1, s_3)))$$

(c) Third statement: we use S' to abbreviate $\{s \in S \mid rnalen(s) > 1\}$

$$\forall s_2 \in S' \exists s_1 \in S' (Del((s_1, s_2))) \wedge \neg \forall s_1 \in S' \exists s_2 \in S' (Del((s_1, s_2)))$$

Bonus; not for credit (do not hand in): For each statement above, is the statement or its negation true? How do you know?

HW4 Proofs and Sets Due: Tuesday, November 2, 2021 at 11:00PM on Gradescope

In this assignment,

You will analyze statements and determine if they are true or false using valid proof strategies. You will also determine if candidate arguments are valid.

Instructions and academic integrity reminders for all homework assignments in CSE20 this quarter are on the class website and on the hw1-definitions-and-notations assignment.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw4-proofs-and-sets”.

Resources: To review the topics you are working with for this assignment, see the class material from Week 5. We will post frequently asked questions and our answers to them in a pinned Piazza post.

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion:** To prove that $\forall x P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization:** To prove that $\forall x P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.
- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Strategies for conjunction:** To prove that $p \wedge q$ is true, have two subgoals: subgoal (1) prove p is true; and, subgoal (2) prove q is true. To prove that $p \wedge q$ is false, it's enough to prove that p is false. To prove that $p \wedge q$ is false, it's enough to prove that q is false.
- **Proof of Conditional by Direct Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume p is true and use that assumption to show q is true.
- **Proof of Conditional by Contrapositive Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume $\neg q$ is true and use that assumption to show $\neg p$ is true.
- **Proof of disjunction using equivalent conditional:** To prove that the disjunction $p \vee q$ is true, we can rewrite it equivalently as $\neg p \rightarrow q$ and then use direct proof or contrapositive proof.
- **Proof by Cases:** To prove q when we know $p_1 \vee p_2$, show that $p_1 \rightarrow q$ and $p_2 \rightarrow q$.

Assigned questions

1. Consider the predicate $Pr(x)$ over the set of integers, which evaluates to T exactly when x is prime. Consider the following statements.

| | |
|----------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| (i) $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z} (x \leq y \rightarrow Pr(y))$ | (v) $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z} (Pr(y) \rightarrow y \leq x)$ |
| (ii) $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z} (y \leq x \rightarrow Pr(y))$ | (vi) $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z} (Pr(y) \rightarrow x \leq y)$ |
| (iii) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} (x \leq y \rightarrow Pr(y))$ | (vii) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} (Pr(y) \rightarrow y \leq x)$ |
| (iv) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} (y \leq x \rightarrow Pr(y))$ | (viii) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} (Pr(y) \rightarrow x \leq y)$ |

- (a) (*Graded for correctness of choice and fair effort completeness in justification*¹⁰) Which of the statements (i) - (viii) is being **proved** by the following proof:

Choose $x = 1$, an integer, and we will work to show it is a **witness** for the existential claim. By universal generalization, **choose** e to be an **arbitrary** integer. Towards a **direct proof**, **assume** that $Pr(e)$ holds. We **WTS** that $1 \leq e$. By definition of the predicate Pr , since $Pr(e)$ is true, $e > 1$. By definition of \leq , this means that $1 \leq e$, as required and the claim has been proved. \square

Hint: it may be useful to identify the key words in the proof that indicate proof strategies.

- (b) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **disproved** by the following proof:

To disprove the statement, we will prove the universal statement that is logically equivalent to its negation. By universal generalization, **choose** e to be an **arbitrary** integer. We need to find a **witness** integer y such that $y \leq e$ and $\neg Pr(y)$. Notice that $e > 1 \vee e \leq 1$ is true, and we proceed in a **proof by cases**. **Case 1:** Assume $e > 1$ and **WTS** there is a witness integer y such that $y \leq e$ and $\neg Pr(y)$. Choose $y = 0$, an integer. Then, since by **case assumption** $1 < e$, we have $y = 0 \leq 1 \leq e$. Moreover, since $y = 0$, $y > 1$ is false and so (by the definition of Pr), the predicate Pr evaluated at y is false, as required to prove the **conjunction** $y \leq e$ and $\neg Pr(y)$. **Case 2:** Assume $e \leq 1$ and **WTS** there is a witness integer y such that $y \leq e$ and $\neg Pr(y)$. Choose $y = e - 1$, an integer (because subtracting 1 from the integer e still gives an integer). By definition of subtraction, $y = e - 1 \leq e$. Moreover, since by the **case assumption** $y = e - 1 \leq 1 - 1 = 0$, $y > 1$ is false. Thus, (by the definition of Pr), the predicate Pr evaluated at y is false. We have proved the **conjunction** $y \leq e$ and $\neg Pr(y)$ as required. Since each case is complete, the proof by cases is complete and the original statement has been disproved. \square

Hint: it may be useful to identify the key words in the proof that indicate proof strategies.

- (c) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z} \forall y \in \mathbb{Z} (x \neq y \rightarrow (Pr(x) \vee Pr(y)))$$

¹⁰Graded for correctness means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound. Graded for fair effort completeness means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

- (d) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and proof*) Translate the statement to English and then prove or disprove it

$$(\forall x \in \mathbb{Z} \text{ Pr}(x)) \oplus (\exists x \in \mathbb{Z} \text{ Pr}(x))$$

- (e) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z} \forall y \in \mathbb{Z} ((\text{Pr}(x) \wedge \text{Pr}(y)) \leftrightarrow \text{Pr}(x + y))$$

- (f) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z} (\text{Pr}(x) \rightarrow \exists y \in \mathbb{Z} (x < y \wedge \text{Pr}(y)))$$

2. Let $W = \mathcal{P}(\{1, 2, 3, 4, 5\})$.

Sample response that can be used as reference for the detail expected in your answers for this question:

To give a witness for the existential claim

$$\exists B \in W (B \in \{X \in W \mid 1 \in X\} \cap \{X \in W \mid 2 \in X\})$$

consider $B = \{1, 2\}$. To prove that this is a valid witness, we need to show that it is in the domain of quantification W and that it makes the predicate being quantified evaluate to true. By definition of set-builder notation and intersection, it's enough to prove that $\{1, 2\} \in W$ and that $1 \in \{1, 2\}$ and that $2 \in \{1, 2\}$.

- By definition of power set, elements of W are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of W .
- Also, by definition of the roster method, $1 \in \{1, 2\}$.
- Similarly, by definition of roster method, $2 \in \{1, 2\}$.

Thus $B = \{1, 2\}$ is an element of the domain which is in the intersection of the two sets mentioned in the predicate being quantified and is a witness to the existential claim. QED

- (a) (*Graded for correctness*) Give a witness to the existential claim

$$\exists X \in W (X \cup X = \emptyset)$$

Justify your example by explanations that include references to the relevant definitions.

- (b) (*Graded for correctness*) Give a counterexample to the universal claim

$$\forall X \in W \ (\{a \in X \mid a \text{ is even} \} \subsetneq X)$$

Justify your example by explanations that include references to the relevant definitions.

- (c) (*Graded for correctness*) Give a witness to the existential claim

$$\exists (X, Y) \in W \times W \ (X \cup Y = Y)$$

Justify your example by explanations that include references to the relevant definitions.

3. Recall our representation of movie preferences in a three-movie database using 1 in a component to indicate liking the movie represented by that component, -1 to indicate not liking the movie, and 0 to indicate neutral opinion or haven't seen the movie. We call Rt the set of all ratings 3-tuples. We defined the function $d_0 : Rt \times Rt \rightarrow \mathbb{R}$ which takes an ordered pair of ratings 3-tuples and returns a measure of the distance between them given by

$$d_0(((x_1, x_2, x_3), (y_1, y_2, y_3))) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

Another measure of the distance between a pair of ratings 3-tuples is given by the following function $d_1 : Rt \times Rt \rightarrow \mathbb{R}$ given by

$$d_1(((x_1, x_2, x_3), (y_1, y_2, y_3))) = \sum_{i=1}^3 |x_i - y_i|$$

- (a) For each of the statements below, first translate them symbolically (using quantifiers, logical operators, and arithmetic operations), then determine whether each is true or false by applying the proof strategies to prove each statement or its negation. (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*)
- i. For all ordered pairs of ratings 3-tuples, the value of the function d_0 is greater than the value of the function d_1 .
 - ii. The maximum value of the function d_1 is greater than the maximum value of the function d_0 .
- (b) (*Graded for correctness*) Write a statement about 3-tuples of movie ratings that uses the function d_1 and has at least one universal and one existential quantifier. Your response will be graded correct if all the syntax in your statement is correct.
- (c) (*Graded for fair effort completeness*) Translate the property you wrote symbolically in the last step to English. Indicate if it is true, false, or if you don't know (sometimes we can write interesting properties, and we're not sure if they are true or not!). Give informal justification for whether you think it is true/ false, or explain why the proof strategies we have so far do not appear to be sufficient to determine whether the statement holds.

HW5 Proofs and Induction Due: Tuesday, November 16, 2021 at 11:00PM on Gradescope

In this assignment,

You will work with recursively defined sets and functions and prove properties about them, practicing induction and other proof strategies.

Instructions and academic integrity reminders for all homework assignments in CSE20 this quarter are on the class website and on the hw1-definitions-and-notations assignment.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw5-proofs-and-induction”.

Resources: To review the topics you are working with for this assignment, see the class material from Weeks 5 through 7. We will post frequently asked questions and our answers to them in a pinned Piazza post.

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion:** To prove that $\forall x P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization:** To prove that $\forall x P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.
- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Strategies for conjunction:** To prove that $p \wedge q$ is true, have two subgoals: subgoal (1) prove p is true; and, subgoal (2) prove q is true. To prove that $p \wedge q$ is false, it's enough to prove that p is false. To prove that $p \wedge q$ is false, it's enough to prove that q is false.
- **Proof of Conditional by Direct Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume p is true and use that assumption to show q is true.
- **Proof of Conditional by Contrapositive Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume $\neg q$ is true and use that assumption to show $\neg p$ is true.
- **Proof of disjunction using equivalent conditional:** To prove that the disjunction $p \vee q$ is true, we can rewrite it equivalently as $\neg p \rightarrow q$ and then use direct proof or contrapositive proof.

- **Proof by Cases:** To prove q when we know $p_1 \vee p_2$, show that $p_1 \rightarrow q$ and $p_2 \rightarrow q$.
- **Proof by Structural Induction:** To prove that $\forall x \in X P(x)$ where X is a recursively defined set, prove two cases:
 - Basis Step: Show the statement holds for elements specified in the basis step of the definition.
 - Recursive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.
- **Proof by Mathematical Induction:** To prove a universal quantification over the set of all integers greater than or equal to some base integer b :
 - Basis Step: Show the statement holds for b .
 - Recursive Step: Consider an arbitrary integer n greater than or equal to b , assume (as the **induction hypothesis**) that the property holds for n , and use this and other facts to prove that the property holds for $n + 1$.
- **Proof by Strong Induction** To prove that a universal quantification over the set of all integers greater than or equal to some base integer b holds, pick a fixed nonnegative integer j and then:
 - Basis Step: Show the statement holds for $b, b + 1, \dots, b + j$.
 - Recursive Step: Consider an arbitrary integer n greater than or equal to $b + j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** $b, b + 1, \dots, n$, and use this and other facts to prove that the property holds for $n + 1$.
- **Proof by Contradiction**

To prove that a statement p is true, pick another statement r and once we show that $\neg p \rightarrow (r \wedge \neg r)$ then we can conclude that p is true.

Informally The statement we care about can't possibly be false, so it must be true.

Assigned questions

1. Recall the definitions from class about factoring and divisibility: when a and b are integers and a is nonzero, a **divides** b means there is an integer c such that $b = ac$. In this case, we say a is a **factor** of b , a is a **divisor** of b , b is a **multiple** of a , $a|b$. We define the function $PosFactors : \mathbb{Z}^+ \rightarrow \mathcal{P}(\mathbb{Z}^+)$ by

$$PosFactors(n) = \{x \in \mathbb{Z}^+ \mid x \text{ is a factor of } n\}$$

Sample calculation that can be used as reference for the detail expected in your answer when working with this function:

The function application $PosFactors(4)$ evaluates to

$$PosFactors(4) = \{1, 2, 4\}$$

because the only possible positive factors of 4 are 1, 2, 3, 4 (the positive integers less than or equal to 4) and when we divide we get:

$$\begin{array}{ll} 4 = 4 \cdot 1 + 0 & \text{so 4 is a factor of 4} \\ 4 = 3 \cdot 1 + 1 & \text{so 3 is not a factor of 4} \\ 4 = 2 \cdot 2 + 0 & \text{so 2 is a factor of 4} \\ 4 = 1 \cdot 4 + 0 & \text{so 1 is a factor of 4} \end{array}$$

-
- (a) (*Graded for correctness*¹¹) Give a witness that proves the statement

$$\exists x \in \mathbb{Z}^+ \forall y \in \mathbb{Z}^+ (x \in PosFactors(y))$$

Justify your choice of witness by explanations that include references to the relevant definitions.

- (b) (*Graded for correctness*) Give a counterexample that disproves the statement

$$\forall n \in \mathbb{Z}^+ (PosFactors(n) \subseteq PosFactors(n+1))$$

Justify your choice of counterexample by explanations that include references to the relevant definitions.

¹¹Graded for correctness means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

(c) (*Graded for fair effort completeness*) Consider the following attempted proof.

Attempted proof: For arbitrary integers a, b, c , assume towards a direct proof that $(a + b)|c$. We need to show that $a|c$ and $b|c$. Let n be the integer $c \text{ div } (a + b)$. Since $(a + b)|c$, by definition of divides, $n|c$ and n is an integer. Since $c = 1 \cdot n \cdot (a + b)$, $(n \cdot (a + b))|c$. Rewriting by distributing multiplication over addition, we have $na|c$ and $nb|c$. Since $a|na$ and $na|c$, we have $a|c$. Similarly, since $b|nb$ and $nb|c$, we have $b|c$. Thus, we have proved both conjuncts and the proof is complete.

Select the statement below that the attempted proof is trying to prove.

- (i) $\forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z} ((a|c \vee b|c) \rightarrow (a + b)|c)$
- (ii) $\forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z} ((a|b \wedge a|c) \rightarrow a|(b + c))$
- (iii) $\forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z} ((a + b)|c \rightarrow (a|c \wedge b|c))$

Identify the first major error in the attempted proof and explain why it is incorrect.

Next, disprove the statement the attempted proof was attempting to prove.

Extra practice; not for credit: prove or disprove the other two statements.

2. In this question, we'll consider the function which calculates the sum of the first n positive integers.

(a) (*Graded for fair effort completeness*¹²)

Give a recursive definition of this function, including domain, codomain and both the basis step and recursive step of the rule. That is, fill in the blanks

$$\text{sumOfFirst} : \underline{\text{domain}} \rightarrow \underline{\text{codomain}}$$

given by

Basis step : fill in basis step

Recursive step : fill in recursive step

Notation: Using summation, this function can be written $\text{sumOfFirst}(n) = \sum_{i=1}^n i$.

(b) (*Graded for fair effort completeness*) It turns out that the value of this function can also be calculated explicitly (without recursion)¹³. You will prove this by completing the proof of the identity

$$\forall n \in \mathbb{Z}^+ \left(\text{sumOfFirst}(n) = \frac{n(n+1)}{2} \right)$$

¹²Graded for fair effort completeness means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

¹³When the value of a function that is recursively defined can also be calculated without recursion, we call the formula that we can use to calculate the value without recursion the "closed form formula" for the function.

Fill in the missing parts of the proof of this statement:

Proof: We proceed by mathematical induction on the set of positive integers.

Basis Step: Choose $n = 1$ as the basis step. Using the Basis Step in the recursive definition of *sumOfFirst*, $\text{sumOfFirst}(1) = 1$. Plugging n into the RHS of the desired formula, $\frac{1(1+1)}{2} = \frac{2}{2} = 1$. Since LHS=RHS, the Basis step is complete.

Recursive Step: Consider an arbitrary $k \geq 1$. We assume (as the induction hypothesis) that fill in the blank here.

We want to show that $\text{sumOfFirst}(k + 1) = \frac{(k+1) \cdot ((k+1)+1)}{2}$.

Fill in the rest of the proof here.

- (c) (*Graded for fair effort completeness*) When calculating the runtime of an algorithm, nested for loops sometimes lead to program runtimes that involve the sum of the first n positive integers. To estimate the rate of growth of this runtime, it is useful to find an upper bound for this function in terms of a simpler function. Use the explicit formula from the earlier parts of this question and mathematical induction to prove

$$\forall n \in \mathbb{Z}^+ \quad (\text{sumOfFirst}(n) \leq n^2)$$

3. Recall the definition of linked lists that we discussed in class.

Define the function *count* which returns the number of occurrences of a datum in the list. Formally, $\text{count} : L \times \mathbb{N} \rightarrow \mathbb{N}$, where

Basis Step: If $m \in \mathbb{N}$, $\text{count}(\langle \rangle, m) = 0$

Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then

$$\text{count}(\langle (n, l), m \rangle) = \begin{cases} 1 + \text{count}(\langle l, m \rangle) & \text{if } n = m \\ \text{count}(\langle l, m \rangle) & \text{otherwise} \end{cases}$$

A mystery function is defined by $\text{mystery} : L \times \mathbb{N} \rightarrow L$, where

Basis Step: If $m \in \mathbb{N}$, $\text{mystery}(\langle \rangle, m) = \langle \rangle$

Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then

$$\text{mystery}(\langle (n, l), m \rangle) = \begin{cases} l & \text{if } n = m \\ \text{mystery}(\langle l, m \rangle) & \text{otherwise} \end{cases}$$

- (a) (*Graded for correctness*) Prove that

$$\forall m \in \mathbb{N} \exists l \in L \quad (\text{count}(\langle l, 20 \rangle) = m)$$

- (b) (*Graded for correctness*) Give an example input x to the function such that

$$\text{mystery}((x , 2)) = []$$

For full credit, include all intermediate steps of the function application that justifies your choice of x , with brief justifications for each.

- (c) (*Graded for correctness*) Evaluate the function application

$$\text{mystery}(((2, (0, (2, []))) , 2))$$

For full credit, include all intermediate steps of the function application, with brief justifications for each.

- (d) (*Graded for fair effort completeness for English statements and correctness in use of syntax for symbolic statements*) Describe the rule of the function *mystery* in English. Then, write a true statement that describes an invariant using both the functions *mystery* and *count*. Express this invariant both symbolically and in English.

HW6 Proofs, Numbers, and Cardinality Due: Tuesday, November 23, 2021 at 11:00PM on Gradescope

In this assignment,

You will practice determining and justifying whether statements are true in multiple contexts.

Instructions and academic integrity reminders for all homework assignments in CSE20 this quarter are on the class website and on the hw1-definitions-and-notations assignment.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw6-proofs-numbers-cardinality”.

Resources: To review the topics you are working with for this assignment, see the class material from Weeks 6 through 8. We will post frequently asked questions and our answers to them in a pinned Piazza post.

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion:** To prove that $\forall x P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization:** To prove that $\forall x P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.
- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Strategies for conjunction:** To prove that $p \wedge q$ is true, have two subgoals: subgoal (1) prove p is true; and, subgoal (2) prove q is true. To prove that $p \wedge q$ is false, it's enough to prove that p is false. To prove that $p \wedge q$ is false, it's enough to prove that q is false.
- **Proof of Conditional by Direct Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume p is true and use that assumption to show q is true.
- **Proof of Conditional by Contrapositive Proof:** To prove that the implication $p \rightarrow q$ is true, we can assume $\neg q$ is true and use that assumption to show $\neg p$ is true.
- **Proof of disjunction using equivalent conditional:** To prove that the disjunction $p \vee q$ is true, we can rewrite it equivalently as $\neg p \rightarrow q$ and then use direct proof or contrapositive proof.

- **Proof by Cases:** To prove q when we know $p_1 \vee p_2$, show that $p_1 \rightarrow q$ and $p_2 \rightarrow q$.
- **Proof by Structural Induction:** To prove that $\forall x \in X P(x)$ where X is a recursively defined set, prove two cases:
 - Basis Step: Show the statement holds for elements specified in the basis step of the definition.
 - Recursive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.
- **Proof by Mathematical Induction:** To prove a universal quantification over the set of all integers greater than or equal to some base integer b :
 - Basis Step: Show the statement holds for b .
 - Recursive Step: Consider an arbitrary integer n greater than or equal to b , assume (as the **induction hypothesis**) that the property holds for n , and use this and other facts to prove that the property holds for $n + 1$.
- **Proof by Strong Induction** To prove that a universal quantification over the set of all integers greater than or equal to some base integer b holds, pick a fixed nonnegative integer j and then:
 - Basis Step: Show the statement holds for $b, b + 1, \dots, b + j$.
 - Recursive Step: Consider an arbitrary integer n greater than or equal to $b + j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** $b, b + 1, \dots, n$, and use this and other facts to prove that the property holds for $n + 1$.
- **Proof by Contradiction**

To prove that a statement p is true, pick another statement r and once we show that $\neg p \rightarrow (r \wedge \neg r)$ then we can conclude that p is true.

Informally The statement we care about can't possibly be false, so it must be true.

Assigned questions

- Recall the definition of the set of linked lists from class, and some associated functions.

Basis Step: $[] \in L$

Recursive Step: If $l \in L$ and $n \in \mathbb{N}$, then $(n, l) \in L$

The length function $length : L \rightarrow \mathbb{N}$ is defined by

Basis Step: $length([]) = 0$

Recursive Step: If $l \in L$ and $n \in \mathbb{N}$, then $length((n, l)) = 1 + length(l)$

The function $prepend : L \times \mathbb{N} \rightarrow L$ is defined by

$$prepend((l, n)) = (n, l)$$

The function $append : L \times \mathbb{N} \rightarrow L$ is defined by

Basis Step: If $m \in \mathbb{N}$ then $append([], m) = (m, [])$

Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then $append((n, l), m) = (n, append(l, m))$

- (a) (*Graded for fair effort completeness*¹⁴) Fill in the blanks in the following proof of the statement

$$\forall l \in L \forall m \in \mathbb{N} (length(prepend((l, m))) = length(append((l, m))))$$

Proof: We proceed by structural induction on L .

Basis step: We need to show that

$$\forall m \in \mathbb{N} (length(prepend([], m)) = length(append([], m)))$$

Towards universal generalization, let m be an arbitrary natural number. Calculating:

$$LHS = \underline{\text{BLANK 1}}$$

$$RHS = \underline{\text{BLANK 2}}$$

Since $LHS = RHS$, the basis step is complete.

Recursive step: Consider an arbitrary: $l' \in L$, $n \in \mathbb{N}$, and we assume as the **induction hypothesis** that:

$$\underline{\text{BLANK 3}}$$

Our goal is to show that

$$\forall m \in \mathbb{N} (length(prepend((n, l'), m)) = length(append((n, l'), m)))$$

is also true. Let m be an arbitrary natural number. Calculating:

$$LHS = \underline{\text{BLANK 4}}$$

$$RHS = \underline{\text{BLANK 5}}$$

Since $LHS = RHS$, the recursive step is complete.

¹⁴Graded for fair effort completeness means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

- (b) (*Graded for correctness*¹⁵) Disprove the statement

$$\forall l \in L \forall m \in \mathbb{N} (\text{prepend}(l, m) = \text{append}(l, m))$$

- (c) (*Graded for correctness*) Determine whether the statement

$$\exists l \in L \exists m \in \mathbb{N} (\text{prepend}(l, m) = \text{append}(l, m))$$

is true or false, and justify your conclusion using valid proof strategies.

2. Recall the definition of the set of rational numbers,

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

We define the set of **irrational** numbers $\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$.

- (a) (*Translation graded for fair effort completeness; Witness graded for correctness*) Translate the statement to English and then give a witness that could be used to prove the statement

$$\exists x \in \mathbb{Q} \forall y \in \overline{\mathbb{Q}} (x \cdot y \in \mathbb{Q})$$

You do not need to justify your answer. However, if you include clear explanations, we may be able to give partial credit for an answer with some imprecision.

- (b) (*Translation graded for fair effort completeness; Witness graded for correctness*) Translate the statement to English and then give a counterexample that could be used to disprove the statement

$$\forall x \in \overline{\mathbb{Q}} (x > 0 \rightarrow x \geq 1)$$

You do not need to justify your answer. However, if you include clear explanations, we may be able to give partial credit for an answer with some imprecision.

- (c) (*Translation graded for fair effort completeness; Witness graded for correctness*) Translate the statement to English and then give a witness that could be used to prove the statement

$$\exists (x, y, z) \in \overline{\mathbb{Q}} \times \mathbb{Q} \times \mathbb{Q} (y \neq z \wedge x^y = z)$$

You do not need to justify your answer. However, if you include clear explanations, we may be able to give partial credit for an answer with some imprecision.

- (d) (*Graded for fair effort completeness*¹⁶) Fill in the blanks in the following argument.

Claimed statement: $\exists x \in \overline{\mathbb{Q}} \exists y \in \overline{\mathbb{Q}} \exists z \in \mathbb{Q} (x^y = z)$.

¹⁵Graded for correctness means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

¹⁶Fair effort completeness for this question means either attempting to correctly answer each part or to write a sentence or two on where you get stuck in your attempt to correctly answer the question.

- **Case 1:** We need to show that

Assume towards a direct proof that **(ii)**_____. We choose the witnesses $x = \sqrt{2}$, $y = \sqrt{2}$, $z = \sqrt{2}^{\sqrt{2}}$. By the theorem we proved in class, $\sqrt{2} \notin \mathbb{Q}$. Since $x = y = \sqrt{2}$, $x \in \overline{\mathbb{Q}}$ and $y \in \overline{\mathbb{Q}}$. By the assumption of this direct proof, $z \in \mathbb{Q}$. Thus, the witnesses we picked are in the required domains. Moreover, by definition, $z = x^y$, as required. Thus, the existential claim is proved and we have completed the direct proof required for this case.

By the assumption of this direct proof, $x \in \overline{\mathbb{Q}}$. As we mentioned above, $\sqrt{2} \notin \mathbb{Q}$ so $y \in \overline{\mathbb{Q}}$. Picking $p = 2, q = 1$, we observe that $z = \frac{2}{1}$ and since **(vi)**_____, $z \in \mathbb{Q}$. Thus, the three witnesses we picked are in the required domains. Calculating:

which proves that **(vii)**_____, and hence x, y, z are the required witnesses. Thus, the existential claim is proved and we have completed the direct proof required for this case.

3. Recall that A **hex color** is a nonnegative integer, n , that has a base 16 fixed-width 6 expansion

where $(r_1 r_2)_{16,2}$ is the red component, $(g_1 g_2)_{16,2}$ is the green component, and $(b_1 b_2)_{16,2}$ is the blue component. For notational convenience, we define the set $C = \{x \in \mathbb{N} \mid x < 16^6\}$. This is the set of possible hex colors because these are all numbers that have hexadecimal fixed-width 6 expansions.

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- (b) Consider the function $red : C \rightarrow C$ given by $red((r_1 r_2 g_1 g_2 b_1 b_2)_{16,6}) = (r_1 r_2 0000)_{16,6}$.
- (*Graded for correctness*) Determine whether red is one-to-one, and justify your conclusion using valid proof strategies.
 - (*Graded for correctness*) Determine whether red is onto, and justify your conclusion using valid proof strategies.
4. Consider the set of ratings in a 3-movie database $R = \{-1, 0, 1\} \times \{-1, 0, 1\} \times \{-1, 0, 1\}$ and the set of bases of RNA strands $B = \{\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{G}\}$.
- (*Graded for fair effort completeness*) Give a (well-defined) one-to-one function with domain R and codomain B or explain why there is no such function.
 - (*Graded for fair effort completeness*) Give a (well-defined) one-to-one function with domain B and codomain R or explain why there is no such function.
 - (*Graded for fair effort completeness*) Give a (well-defined) onto function with domain R and codomain B or explain why there is no such function.
 - (*Graded for fair effort completeness*) Give a (well-defined) onto function with domain B and codomain R or explain why there is no such function.

Sample calculation that can be used as reference for the detail expected in your answer when specifying functions and reasoning about their properties:

We give a (well-defined) function with domain R and codomain B that is neither one-to-one nor onto.

Define $g : R \rightarrow B$ by, for $(x_1, x_2, x_3) \in R$,

$$g((x_1, x_2, x_3)) = \begin{cases} \mathbf{A} & \text{if } x_1 = 1 \\ \mathbf{C} & \text{if } x_1 = 0 \\ \mathbf{G} & \text{if } x_1 = -1 \end{cases}$$

This function is well-defined because each ratings 3-tuples is mapped to a unique base. However, this function is not one-to-one, as we can see from the counterexample: $a = (1, 1, 1)$, $b = (1, 0, 0)$. These are ratings 3-tuples (in the domain) which are distinct (they disagree about the second and third movies) but

$$g(a) = g((1, 1, 1)) = \mathbf{A} = g((1, 0, 0)) = g(b)$$

because the two ratings agree on the first movie. Distinct domain elements getting mapped to the same codomain elements is a counterexample to injectivity.

The function g is also not onto, as we can see from the counterexample \mathbf{U} . This is an element of the codomain which is not $f(x)$ for any x in the domain, as we can see from the piecewise definition of g , where in no case do we have the output value \mathbf{U} .

Project Part 1 due 10/14/21; Part 2 due 11/4/21; Part 3 due 12/2/21

The project component of this class will be an opportunity for you to extend your work on assignments and explore applications of your choosing.

Why? To go deeper and explore the material from discrete math and how it relates to Computer Science.

How? During emergency remote instruction last academic year, we discovered that video assessment and some open-ended personalized projects help ensure fairness and can be less stressful for students than in-person midterm exams. Asynchronous project submission also gives flexibility and allows more physical distancing.

Your videos: We will delete all the videos we receive from you after assigning final grades for the course, and they will be stored in a university-controlled Google Drive directory only accessible to the course staff during the quarter. Please send an email to the instructor (minnes@eng.ucsd.edu) if you have concerns about the video / screencast components of this project or cannot complete projects in this style for some reason.

You may produce screencasts with any software you choose. One option is to record yourself with Zoom; a tutorial on how to use Zoom to record a screencast (courtesy of Prof. Joe Politz) is here:

https://drive.google.com/open?id=1KROMAQuTCk40zwrEFotlYSJJQdcG_GUU.

The video that was produced from that recording session in Zoom is here:

<https://drive.google.com/open?id=1MxJN6CQcXqIb0ekDYMxjh7mTt1TyRVM1>

What resources can you use?

This project must be completed individually, without any help from other people, including the course staff (other than logistics support if you get stuck with screencast).

You can use any of this quarter's CSE 20 offering (notes, readings, class videos, homework feedback). These resources should be more than enough. If you are struggling to get started and want to look elsewhere online, you must acknowledge this by listing and citing any resources you consult (even if you do not explicitly quote them). Link directly to them and include the name of the author / video creator and the reason you consulted this reference. The work you submit for the project needs to be your own. Again, you shouldn't need to look anywhere other than this quarter's material and doing so may result in definitions or notations that conflict with our norms in this class so think carefully before you go down this path.

The project has three parts.

- Part 1 of Project: due Thursday October 14
- Part 2 of Project: due Thursday November 4
- Part 3 of Project: due Thursday December 2

Part 1: due Thursday October 14

Written component

1. In Week 1, we discussed the mathematical definition of a function, namely that a **function** is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain.
 - (a) Write out the definition for an example function you make up. You can choose any example you like, so long as it is your own independent effort and it is not a function from a class example, homework, or Review Quiz in this class so far. Use the notation defined in class. Label and define the domain, codomain, and rule clearly.
 - (b) Calculate the result of applying your function to an element in its domain. Just like in homework, include (clear, correct, complete) calculations and/or references to definitions to present your function application.
2. In CSE 20 this quarter, we will be exploring the applications of discrete mathematics for core CS topics. The following videos introduce some of these topics and the work happening here at UCSD to explore them. Pick one of the following videos, watch it, and then write a few sentences answering the following:
 - (a) Which video did you watch? Why did you choose the video you watched?
 - (b) What followup question(s) would you like to ask the person in the video about their work? At least one of your followup questions should be about a technical aspect of the work that you would like to learn more about.

(Click on the video titles below for the links)

Bioinformatics and virology: *Niema Moshiri - Genome Sequence Alignment*

Human robotics interaction: *Angelique Taylor - Improving Human-Robot Interaction*

Computer vision: *Manmohan Chandraker: Giving Computers the Gift of Vision*

Data centers and energy efficiency: *Max Mellette: Improving Data Centers with Photonics*

Programming languages and data structures: *Nadia Polikarpova: Creating New Languages for Programming*

Machine learning (and surfing) for climate science: *Studying Climate Change Through Surfing with Smartfin - We Are CSE: Jasmine Simmons*

Video component

Presenting your reasoning and demonstrating it via screenshare are important skills that also show us a lot of your learning. Getting practice with this style of presentation is a good thing for you to learn in general and a rich way for us to assess your skills.

Prepare a 3-5 minute screencast video that starts with your face and your student ID for a few seconds at the beginning, and introduce yourself audibly while on screen. You don't have to be on camera for the rest of the video, though it's fine if you are. We are looking for a brief confirmation that it's you creating the video and doing the work submitted for the project.

Then, explain your work in question 1 of the written component. Discuss at least one potential mistake that someone solving a similar question should avoid (this could be a mistake you made while thinking about this problem or something you anticipate a classmate might struggle with); explain why the mistake is wrong and how to fix it.

Finally, explain any differences between your pre-survey description of "what a function is" and the mathematical definition of what a function is. You should have an email copy of your responses to the pre-survey, and you can refer to what you wrote in your explanation.

Use this Google form

<https://forms.gle/SLd8SrJdXR5HCLQr7> (click to follow link)

to directly upload a video file for this assignment. It should be a file that you can easily play on your system. One way you can determine if this is true is if you can store it on your Google Drive and play it from there, since that's how we will watch it.

Checklist (this is how we will grade Part 1 of the project)

- Question 1
 - The function definition is complete and uses correct notation, and is different from class, homework, and quiz examples.
 - The calculation of the function application is correct and is supported by clear, correct, complete justification.
- Question 2
 - (At least) one of the videos is mentioned and a reason for selecting it is included.
 - At least one technical question is described that is connected to the video selected.
- Video
 - Video loads correctly and is between 3 and 5 minutes. It includes your face and your student ID, and you introduce yourself audibly while on screen.
 - Video presents your solution for Question 1.
 - A potential mistake is presented and discussed.
 - The mathematical definition of function is compared to your response in the pre-survey.

Part 2: due Thursday November 4

Written component

1. In this part of the project, you will select one question from one of the review quizzes from 10/4/2021 (Monday of Week 1) to 10/29/2021 (Friday of Week 5) to revisit. Include the problem statement, why you picked this question (e.g. what is interesting about it, what is hard about it, or why you wanted to take a second look at it), and your solution.

- Question selection: you can pick any **one question** listed in the Review sections of the relevant notes documents, and you must address all of its parts.
- For each part of your chosen question: prepare a complete solution (you can use the homework solutions we post for guidance about the style). Your submission will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Your goal should be to convince the reader that your results and methods are sound. Imagine you are preparing these solutions for someone else taking CSE 20 who missed that week and is “catching up”.

2. In this part of the project, you’ll consider the importance of data types and precision in Computer Science. Read three articles

(1) This discussion of the causes of a wide-spread problem in published genomics papers

<https://www.nature.com/articles/d41586-021-02211-4> (Click to follow link)

from the journal Nature;

(2) this IEEE profile of Katherine Johnson,

<https://spectrum.ieee.org/the-institute/ieee-history/katherine-johnson-the-hidden-figures-mathematician-who-got-astronaut-john-glenn-into-space>

(Click to follow link) a NASA “computer” who calculated trajectories for early space exploration and who passed away in 2020; and

(3) this NASA report about the unsuccessful 1999 Mars Climate Orbiter mission

<https://solarsystem.nasa.gov/missions/mars-climate-orbiter/in-depth/>

In one or two sentences, summarize the main lesson you draw from each article.

Thinking back to your own experiences, give an example of when you used computers or Computer Science to help you **avoid** an error. Also, give an example when your use of computers or Computer Science **caused** an error.

What measures do you take to increase your confidence in the results of your own human and digital (i.e. machine) computation? Why do you think these are sufficient?

Video component

Presenting your reasoning and demonstrating it via screenshare are important skills that also show us a lot of your learning. Getting practice with this style of presentation is a good thing for you to learn in general and a rich way for us to assess your skills.

Prepare a 3-5 minute screencast video explaining your work in question 1 of the written component. During your solution presentation, point out at least one potential mistake that someone solving a similar question should avoid (this could be a mistake you made while thinking about this problem or something you anticipate a classmate might struggle with); explain why the mistake is wrong and how to fix it.

You do not need to include complete details of every part of your solution. It is up to you to choose what is most important so that you can stick to the timing guidelines and still have time to include discussing potential mistakes.

Include your face and your student ID (we'd like a photo ID that includes your name and picture if possible) for a few seconds at the beginning, and introduce yourself audibly while on screen. You don't have to be on camera the whole time, though it's fine if you are. We are looking for a brief confirmation that it's you creating the video/doing the work attached to the video.

Use this Google form

<https://forms.gle/SLd8SrJdXR5HCLQr7> (click to follow link)

to directly upload a video file for this assignment. It should be a file that you can easily play on your system. One way you can determine if this is true is if you can store it on your Google Drive and play it from there, since that's how we will watch it.

Checklist (this is how we will grade Part 2 of the project)

- Question 1
 - Selected review quiz question is labelled clearly, including the day it belongs to and the statement of the question.
 - Solution is complete: it addresses each part of the review quiz question selected.
 - Solution is correct: it clearly and correctly justifies the correct answer for each part of the question. You are welcome to check your answers with the Gradescope autograder (we will be reopening the review quizzes for this purpose). We will evaluate your submissions for the quality of your justification.
- Question 2
 - A key lesson from each of the three references is stated clearly and is relevant to the message of the articles. Supporting explanations are included.

- A specific example of an instance where using computers/ CS *caused* an error is described.
 - A specific example of an instance where using computers/ CS helped *avoid* an error is described.
 - Lesson(s) are drawn from the previous experiences.
 - Specific strategies for increasing confidence in computation are described and justified.
- Video
 - Video loads correctly and is between 3 and 5 minutes. It includes your face and your student ID, and you introduce yourself audibly while on screen.
 - Video presents your solution for Question 1.
 - A potential mistake is presented and discussed.

Part 3: due Thursday December 2

Written component

1. In this part of the project, you will analyze a quantified statement about RNA strands. The definitions for RNA strands are available in the class notes. Example quantified statements about RNA strands are in the homework. Complete the following:
 - (a) Write a quantified statement symbolically. Your quantified statement should satisfy **all** of the following requirements:
 - Have a nesting of quantifiers with at least one forall quantification and at least one existential quantification.
 - Have at least one negation and at least one binary logic operation (and, or, xor, if, iff).
 - Negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
 - Use **exactly one** of the predicates Mut, Ins, Del.
 - Not be a statement we have analyzed already in class materials.
 - (b) Translate your statement from part a. to English.
 - (c) Negate the whole statement from part a. and rewrite this negated statement so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
 - (d) Prove or disprove your statement from part a.
2. In this part of the project, you'll consider the impact Computer Science has on society. Read two articles:
 - (1) This policy piece about facial recognition software
<https://thehill.com/policy/technology/569543-federal-agencies-planning-to-expand-use-of-facial-recognition> (Click to follow link);
and
 - (2) this exploration of accessibility for visually impaired website users
<https://www.wsj.com/articles/colorblind-users-push-technology-designers-to-use-signals-beyond-color-11591351201> (Click to follow link).

Update (November 18): the Wall Street Journal about web design is now behind a pay wall. If you do not have access to it, please read the following two articles instead:
<https://webaim.org/articles/visual/colorblind> (Click to follow link) and <https://webaim.org/resources/designers/> (Click to follow link)

In one or two sentences, summarize the main lesson you draw from each article.

Give an example of an algorithm or computer system that you use or have been used by others to make some decision that affects you. The algorithms may be institutional or personal, formal or heuristic, and should output a specific result or decision. Explain your

answer, either by reference to your knowledge of the algorithm itself or by observations you make about outputs of the algorithm.

Give an example of how someone different from you might have a different experience with this algorithm. Support your example with lessons you learned in the readings, citing what you learned from the articles.

Video component

Presenting your reasoning and demonstrating it via screenshare are important skills that also show us a lot of your learning. Getting practice with this style of presentation is a good thing for you to learn in general and a rich way for us to assess your skills.

Prepare a 3-5 minute screencast video explaining your work in question 1 parts (c) and (d) of the written component (i.e. the negation and proof). During your solution presentation, point out at least one potential mistake that someone solving a similar question should avoid (this could be a mistake you made while thinking about this problem or something you anticipate a classmate might struggle with); explain why the mistake is wrong and how to fix it.

You do not need to include complete details of every part of your solution to these parts. It is up to you to choose what is most important so that you can stick to the timing guidelines and still have time to include discussing potential mistakes.

Include your face and your student ID (we'd like a photo ID that includes your name and picture if possible) for a few seconds at the beginning, and introduce yourself audibly while on screen. You don't have to be on camera the whole time, though it's fine if you are. We are looking for a brief confirmation that it's you creating the video/doing the work attached to the video.

Use this Google form

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to directly upload a video file for this assignment. It should be a file that you can easily play on your system. One way you can determine if this is true is if you can store it on your Google Drive and play it from there, since that's how we will watch it.

Checklist (this is how we will grade Part 3 of the project)

- Question 1
 - Quantified statement is clearly stated, is well-defined, syntactically correct, and meets all requirements.
 - Translation to English is clear, correct, and complete.

- The negation of the quantified statement is clearly stated, is well-defined, syntactically correct, and meets all requirements.
- The proof or disproof of the original statement is clear, correct, and complete.
- Question 2
 - A key lesson from each of the two references is stated clearly and is relevant to the message of the articles. Supporting explanations are included.
 - A specific example of an algorithm or computer systems that impacts you is described.
 - A description of how this algorithm or computer system might impact someone different from you differently is included, and is supported with references to one or both of the articles.
- Video
 - Video loads correctly and is between 3 and 5 minutes. It includes your face and your student ID, and you introduce yourself audibly while on screen.
 - Video presents your solution for Question 1 parts (c) and (d).
 - A potential mistake is presented and discussed.