

## Fundamental theorem proof

**Theorem:** Every positive integer *greater than 1* is a product of (one or more) primes.

**Before we prove, let's try some examples:**

$$20 =$$

$$100 =$$

$$5 =$$

**Proof by strong induction,** with  $b = 2$  and  $j = 0$ .

**Basis step:** WTS property is true about 2.

Since 2 is itself prime, it is already written as a product of (one) prime.

**Recursive step:** Consider an arbitrary integer  $n \geq 2$ . Assume (as the strong induction hypothesis, IH) that the property is true about each of  $2, \dots, n$ . WTS that the property is true about  $n + 1$ : We want to show that  $n + 1$  can be written as a product of primes. Notice that  $n + 1$  is itself prime or it is composite.

*Case 1:* assume  $n + 1$  is prime and then immediately it is written as a product of (one) prime so we are done.

*Case 2:* assume that  $n + 1$  is composite so there are integers  $x$  and  $y$  where  $n + 1 = xy$  and each of them is between 2 and  $n$  (inclusive). Therefore, the induction hypothesis applies to each of  $x$  and  $y$  so each of these factors of  $n + 1$  can be written as a product of primes. Multiplying these products together, we get a product of primes that gives  $n + 1$ , as required.

Since both cases give the necessary conclusion, the proof by cases for the recursive step is complete.

## Least greatest proofs

For a set of numbers  $X$ , how do you formalize “there is a greatest  $X$ ” or “there is a least  $X$ ”?

**Prove or disprove:** There is a least prime number.

**Prove or disprove:** There is a greatest integer.

*Approach 1, De Morgan’s and universal generalization:*

*Approach 2, proof by contradiction:*

*Extra examples:* Prove or disprove that  $\mathbb{N}$ ,  $\mathbb{Q}$  each have a least and a greatest element.

## Gcd definition

**Definition: Greatest common divisor** Let  $a$  and  $b$  be integers, not both zero. The largest integer  $d$  such that  $d$  is a factor of  $a$  and  $d$  is a factor of  $b$  is called the greatest common divisor of  $a$  and  $b$  and is denoted by  $\gcd(a, b)$ .

## Gcd examples

Why do we restrict to the situation where  $a$  and  $b$  are not both zero?

Calculate  $\gcd(10, 15)$

Calculate  $\gcd(10, 20)$

## Gcd basic claims

**Claim:** For any integers  $a, b$  (not both zero),  $\gcd(a, b) \geq 1$ .

**Proof:** *Show that 1 is a common factor of any two integers, so since the gcd is the greatest common factor it is greater than or equal to any common factor.*

**Claim:** For any positive integers  $a, b$ ,  $\gcd(a, b) \leq a$  and  $\gcd(a, b) \leq b$ .

**Proof** *Using the definition of gcd and the fact that factors of a positive integer are less than or equal to that integer.*

**Claim:** For any positive integers  $a, b$ , if  $a$  divides  $b$  then  $\gcd(a, b) = a$ .

**Proof** *Using previous claim and definition of gcd.*

**Claim:** For any positive integers  $a, b, c$ , if there is some integer  $q$  such that  $a = bq + c$ ,

$$\gcd(a, b) = \gcd(b, c)$$

**Proof** *Prove that any common divisor of  $a, b$  divides  $c$  and that any common divisor of  $b, c$  divides  $a$ .*

## Gcd lemma relatively prime

**Lemma:** For any integers  $p, q$  (not both zero),  $\gcd\left(\frac{p}{\gcd(p, q)}, \frac{q}{\gcd(p, q)}\right) = 1$ . In other words, can reduce to relatively prime integers by dividing by gcd.

**Proof:**

Let  $x$  be arbitrary positive integer and assume that  $x$  is a factor of each of  $\frac{p}{\gcd(p, q)}$  and  $\frac{q}{\gcd(p, q)}$ . This gives integers  $\alpha, \beta$  such that

$$\alpha x = \frac{p}{\gcd(p, q)} \qquad \beta x = \frac{q}{\gcd(p, q)}$$

Multiplying both sides by the denominator in the RHS:

$$\alpha x \cdot \gcd(p, q) = p \qquad \beta x \cdot \gcd(p, q) = q$$

In other words,  $x \cdot \gcd(p, q)$  is a common divisor of  $p, q$ . By definition of  $\gcd$ , this means

$$x \cdot \gcd(p, q) \leq \gcd(p, q)$$

and since  $\gcd(p, q)$  is positive, this means,  $x \leq 1$ .

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