

## Netflix intro

What data should we encode about each Netflix account holder to help us make effective recommendations?

In machine learning, clustering can be used to group similar data for prediction and recommendation. For example, each Netflix user's viewing history can be represented as a  $n$ -tuple indicating their preferences about movies in the database, where  $n$  is the number of movies in the database. People with similar tastes in movies can then be clustered to provide recommendations of movies for one another. Mathematically, clustering is based on a notion of distance between pairs of  $n$ -tuples.

# Set operations

To define a set we can use the roster method, set builder notation, a recursive definition, and also we can apply a set operation to other sets.

## New! Cartesian product of sets and set-wise concatenation of sets of strings

**Definition:** Let  $X$  and  $Y$  be sets. The **Cartesian product** of  $X$  and  $Y$ , denoted  $X \times Y$ , is the set of all ordered pairs  $(x, y)$  where  $x \in X$  and  $y \in Y$

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

**Definition:** Let  $X$  and  $Y$  be sets of strings over the same alphabet. The **set-wise concatenation** of  $X$  and  $Y$ , denoted  $X \circ Y$ , is the set of all results of string concatenation  $xy$  where  $x \in X$  and  $y \in Y$

$$X \circ Y = \{xy \mid x \in X \text{ and } y \in Y\}$$

**Pro-tip:** the meaning of writing one element next to another like  $xy$  depends on the data-types of  $x$  and  $y$ . When  $x$  and  $y$  are strings, the convention is that  $xy$  is the result of string concatenation. When  $x$  and  $y$  are numbers, the convention is that  $xy$  is the result of multiplication. This is (one of the many reasons) why is it very important to declare the data-type of variables before we use them.

Fill in the missing entries in the table:

Set	Example elements in this set:			
$B$	A	C	G	U
	(A, C)	(U, U)		
$B \times \{-1, 0, 1\}$				
$\{-1, 0, 1\} \times B$				
			(0, 0, 0)	
$\{A, C, G, U\} \circ \{A, C, G, U\}$				
			GGGG	

# Defining functions

**New! Defining functions** A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain.

The domain and codomain are nonempty sets.

The rule can be depicted as a table, formula, or English description.

The notation is

“Let the function  $\text{FUNCTION-NAME: DOMAIN} \rightarrow \text{CODOMAIN}$  be given by  
 $\text{FUNCTION-NAME}(x) = \dots$  for every  $x \in \text{DOMAIN}$ ”.

or

“Consider the function  $\text{FUNCTION-NAME: DOMAIN} \rightarrow \text{CODOMAIN}$  given by  
 $\text{FUNCTION-NAME}(x) = \dots$  for every  $x \in \text{DOMAIN}$ ”.

Example: The absolute value function

**Domain**

**Codomain**

**Rule**

# Defining functions recursively

When the domain of a function is a *recursively defined set*, the rule assigning images to domain elements (outputs) can also be defined recursively.

Recall: The set of RNA strands  $S$  is defined (recursively) by:

$$\begin{array}{ll} \text{Basis Step:} & \mathbf{A} \in S, \mathbf{C} \in S, \mathbf{U} \in S, \mathbf{G} \in S \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then } sb \in S \end{array}$$

where  $sb$  is string concatenation.

**Definition** (Of a function, recursively) A function  $rnalen$  that computes the length of RNA strands in  $S$  is defined by:

$$\begin{array}{lll} & & rnalen : S \rightarrow \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rnalen(sb) = 1 + rnalen(s) \end{array}$$

The domain of  $rnalen$  is

The codomain of  $rnalen$  is

Example function application:

$$rnalen(\mathbf{ACU}) =$$

*Extra example:* A function  $basecount$  that computes the number of a given base  $b$  appearing in a RNA strand  $s$  is defined recursively:

$$\begin{array}{lll} & & basecount : S \times B \rightarrow \mathbb{N} \\ \text{Basis Step:} & \text{If } b_1 \in B, b_2 \in B & basecount((b_1, b_2)) = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases} \\ \text{Recursive Step:} & \text{If } s \in S, b_1 \in B, b_2 \in B & basecount((sb_1, b_2)) = \begin{cases} 1 + basecount((s, b_2)) & \text{when } b_1 = b_2 \\ basecount((s, b_2)) & \text{when } b_1 \neq b_2 \end{cases} \end{array}$$

$$basecount((\mathbf{ACU}, \mathbf{A})) = basecount((\mathbf{AC}, \mathbf{A})) = basecount((\mathbf{A}, \mathbf{A})) = 1$$

$$basecount((\mathbf{ACU}, \mathbf{G})) = basecount((\mathbf{AC}, \mathbf{G})) = basecount((\mathbf{A}, \mathbf{G})) = 0$$

*Extra example:* The function which outputs  $2^n$  when given a nonnegative integer  $n$  can be defined recursively, because its domain is the set of nonnegative integers.

# Why represent numbers

Modeling uses data-types that are encoded in a computer.

The details of the encoding impact the efficiency of algorithms we use to understand the systems we are modeling and the impacts of these algorithms on the people using the systems.

Case study: how to encode numbers?

## Base expansion definition

**Definition** For  $b$  an integer greater than 1 and  $n$  a positive integer, the **base  $b$  expansion of  $n$**  is

$$(a_{k-1} \cdots a_1 a_0)_b$$

where  $k$  is a positive integer,  $a_0, a_1, \dots, a_{k-1}$  are nonnegative integers less than  $b$ ,  $a_{k-1} \neq 0$ , and

$$n = \sum_{i=0}^{k-1} a_i b^i$$

Notice: *The base  $b$  expansion of a positive integer  $n$  is a string over the alphabet  $\{x \in \mathbb{N} \mid x < b\}$  whose leftmost character is nonzero.*

Base $b$	Collection of possible coefficients in base $b$ expansion of a positive integer
Binary ( $b = 2$ )	$\{0, 1\}$
Ternary ( $b = 3$ )	$\{0, 1, 2\}$
Octal ( $b = 8$ )	$\{0, 1, 2, 3, 4, 5, 6, 7\}$
Decimal ( $b = 10$ )	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Hexadecimal ( $b = 16$ )	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$ letter coefficient symbols represent numerical values $(A)_{16} = (10)_{10}$ $(B)_{16} = (11)_{10}$ $(C)_{16} = (12)_{10}$ $(D)_{16} = (13)_{10}$ $(E)_{16} = (14)_{10}$ $(F)_{16} = (15)_{10}$

# Base expansion examples

Common bases:	Binary $b = 2$	Octal $b = 8$	Decimal $b = 10$	Hexadecimal $b = 16$
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*Examples:*

$(1401)_2$

$(1401)_{10}$

$(1401)_{16}$

## Algorithm definition

<b>New!</b> An algorithm is a finite sequence of precise instructions for solving a problem.
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## Algorithm half

Algorithm for calculating integer part of half the input

```
1  procedure half( $n$ : a positive integer)
2     $r := 0$ 
3    while  $n > 1$ 
4       $r := r + 1$ 
5       $n := n - 2$ 
6    return  $r$  { $r$  holds the result of the operation}
```

$n$	$r$	$n > 1?$
6		

$n$	$r$	$n > 1?$
5		

## Algorithm log

Algorithm for calculating integer part of log

```
1  procedure log( $n$ : a positive integer)
2     $r := 0$ 
3    while  $n > 1$ 
4       $r := r + 1$ 
5       $n := \text{half}(n)$ 
6    return  $r$  { $r$  holds the result of the log operation}
```

$n$	$r$	$n > 1?$
8		

$n$	$r$	$n > 1?$
6		

$$2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8 \quad 2^4 = 16 \quad 2^5 = 32 \quad 2^6 = 64 \quad 2^7 = 128 \quad 2^8 = 256 \quad 2^9 = 512 \quad 2^{10} = 1024$$

## Division algorithm

**Integer division and remainders** (aka The Division Algorithm) Let  $n$  be an integer and  $d$  a positive integer. There are unique integers  $q$  and  $r$ , with  $0 \leq r < d$ , such that  $n = dq + r$ . In this case,  $d$  is called the divisor,  $n$  is called the dividend,  $q$  is called the quotient, and  $r$  is called the remainder. We write  $q = n \text{ div } d$  and  $r = n \text{ mod } d$ .

*Extra example:* How do **div** and **mod** compare to / and % in Java and python?

## Base expansion algorithms

## Two algorithms for constructing base $b$ expansion from decimal representation

**Most significant first:** Start with left-most coefficient of expansion

Calculating integer part of  $\log_b$

---

```

1 procedure logb(n, b: positive integers with  $b > 1$ )
2   r := 0
3   while  $n > 1$ 
4     r := r + 1
5     n := n div b
6   return r {r holds the result of the  $\log_b$  operation}
```

---

Calculating base  $b$  expansion, from left

```

1 procedure base1( $n, b$ : positive integers with  $b > 1$ )
2    $v := n$ 
3    $k := \log b(n, b) + 1$ 
4   for  $i := 1$  to  $k$ 
5      $a_{k-i} := 0$ 
6     while  $v \geq b^{k-i}$ 
7        $a_{k-i} := a_{k-i} + 1$ 
8        $v := v - b^{k-i}$ 
9   return  $(a_{k-1}, \dots, a_0)\{(a_{k-1} \dots a_0)_b \text{ is the base } b \text{ expansion of } n\}$ 

```

**Least significant first:** Start with right-most coefficient of expansion

Idea: (when  $k > 1$ )

$$\begin{aligned} n &= a_{k-1}b^{k-1} + \cdots + a_1b + a_0 \\ &= b(a_{k-1}b^{k-2} + \cdots + a_1) + a_0 \end{aligned}$$

so  $a_0 = n \bmod b$  and  $a_{k-1}b^{k-2} + \dots + a_1 = n \operatorname{div} b$ .

---

Calculating base  $b$  expansion, from right

```

1 procedure base2( $n, b$ : positive integers with  $b > 1$ )
2    $q := n$ 
3    $k := 0$ 
4   while  $q \neq 0$ 
5      $a_k := q \bmod b$ 
6      $q := q \div b$ 
7      $k := k + 1$ 
8   return  $(a_{k-1}, \dots, a_0) \{(a_{k-1} \dots a_0)_b \text{ is the base } b \text{ expansion of } n\}$ 

```

## Base expansion review

Find and fix any and all mistakes with the following:

(a)  $(1)_2 = (1)_8$

(b)  $(142)_{10} = (142)_{16}$

(c)  $(20)_{10} = (10100)_2$

(d)  $(35)_8 = (1D)_{16}$



# Base conversion algorithm

Recall the definition of base expansion we discussed:

**Definition** For  $b$  an integer greater than 1 and  $n$  a positive integer, the **base  $b$  expansion of  $n$**  is

$$(a_{k-1} \cdots a_1 a_0)_b$$

where  $k$  is a positive integer,  $a_0, a_1, \dots, a_{k-1}$  are nonnegative integers less than  $b$ ,  $a_{k-1} \neq 0$ , and

$$n = \sum_{i=0}^{k-1} a_i b^i$$

Notice: The base  $b$  expansion of a positive integer  $n$  is a string over the alphabet  $\{x \in \mathbb{N} \mid x < b\}$  whose leftmost character is nonzero.

Base $b$	Collection of possible coefficients in base $b$ expansion of a positive integer
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We write an algorithm for converting from base  $b_1$  expansion to base  $b_2$  expansion:

# Fixed width definition

**Definition** For  $b$  an integer greater than 1,  $w$  a positive integer, and  $n$  a nonnegative integer \_\_\_\_\_, the **base  $b$  fixed-width  $w$  expansion of  $n$**  is

$$(a_{w-1} \cdots a_1 a_0)_{b,w}$$

where  $a_0, a_1, \dots, a_{w-1}$  are nonnegative integers less than  $b$  and

$$n = \sum_{i=0}^{w-1} a_i b^i$$

# Fixed width example

Decimal $b = 10$	Binary $b = 2$	Binary fixed-width 10 $b = 2, w = 10$	Binary fixed-width 7 $b = 2, w = 7$	Binary fixed-width 4 $b = 2, w = 4$
$(20)_{10}$	(a)	(b)	(c)	(d)

# Fixed width fractional definition

**Definition** For  $b$  an integer greater than 1,  $w$  a positive integer,  $w'$  a positive integer, and  $x$  a real number the **base  $b$  fixed-width expansion of  $x$  with integer part width  $w$  and fractional part width  $w'$**  is  $(a_{w-1} \cdots a_1 a_0 . c_1 \cdots c_{w'})_{b,w,w'}$  where  $a_0, a_1, \dots, a_{w-1}, c_1, \dots, c_{w'}$  are nonnegative integers less than  $b$  and

$$x \geq \sum_{i=0}^{w-1} a_i b^i + \sum_{j=1}^{w'} c_j b^{-j} \qquad \text{and} \qquad x < \sum_{i=0}^{w-1} a_i b^i + \sum_{j=1}^{w'} c_j b^{-j} + b^{-w'}$$

3.75 in fixed-width binary, integer part width 2, fractional part width 8	
0.1 in fixed-width binary, integer part width 2, fractional part width 8	

```

[welcome $jshell
| Welcome to JShell -- Version 10.0.1
| For an introduction type: /help intro

[jshell> 0.1
$1 ==>

[jshell> 0.2
$2 ==>

[jshell> 0.1 + 0.2
$3 ==>

[jshell> Math.sqrt(2)
$4 ==>

[jshell> Math.sqrt(2)*Math.sqrt(2)
$5 ==>

[jshell> ]

```

Note: Java uses floating point, not fixed width representation, but similar rounding errors appear in both.

# Expansion summary

base $b$ expansion of $n$	base $b$ fixed-width $w$ expansion of $n$
For $b$ an integer greater than 1 and $n$ a positive integer, the <b>base <math>b</math> expansion of <math>n</math></b> is $(a_{k-1} \cdots a_1 a_0)_b$ where $k$ is a positive integer, $a_0, a_1, \dots, a_{k-1}$ are nonnegative integers less than $b$ , $a_{k-1} \neq 0$ , and $n = a_{k-1}b^{k-1} + \cdots + a_1b + a_0$	For $b$ an integer greater than 1, $w$ a positive integer, and $n$ a nonnegative integer with $n < b^w$ , the <b>base <math>b</math> fixed-width <math>w</math> expansion of <math>n</math></b> is $(a_{w-1} \cdots a_1 a_0)_{b,w}$ where $a_0, a_1, \dots, a_{w-1}$ are nonnegative integers less than $b$ and $n = a_{w-1}b^{w-1} + \cdots + a_1b + a_0$

# Negative int expansions

**Representing negative integers in binary:** Fix a positive integer width for the representation  $w$ ,  $w > 1$ .

	To represent a positive integer $n$	To represent a negative integer $-n$
Sign-magnitude	$[0a_{w-2} \cdots a_0]_{s,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $n = 17$ , $w = 7$ :	$[1a_{w-2} \cdots a_0]_{s,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $-n = -17$ , $w = 7$ :
2s complement	$[0a_{w-2} \cdots a_0]_{2c,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $n = 17$ , $w = 7$ :	$[1a_{w-2} \cdots a_0]_{2c,w}$ , where $2^{w-1} - n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $-n = -17$ , $w = 7$ :
Extra example: 1s complement	$[0a_{w-2} \cdots a_0]_{1c,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $n = 17$ , $w = 7$ :	$[1\bar{a}_{w-2} \cdots \bar{a}_0]_{1c,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ and we define $\bar{0} = 1$ and $\bar{1} = 0$ . Example $-n = -17$ , $w = 7$ :

# Calculating 2s complement

For positive integer  $n$ , to represent  $-n$  in 2s complement with width  $w$ ,

- Calculate  $2^{w-1} - n$ , convert result to binary fixed-width  $w - 1$ , pad with leading 1, or
- Express  $-n$  as a sum of powers of 2, where the leftmost  $2^{w-1}$  is negative weight, or
- Convert  $n$  to binary fixed-width, flip bits, add 1 (ignore overflow)

*Challenge: use definitions to explain why each of these approaches works.*

## Representing zero

**Representing 0:**

So far, the definitions of base expansions treat positive and negative integers. What about 0?

	To represent a <b>non-negative</b> integer $n$	To represent a <b>non-positive</b> integer $-n$
Sign-magnitude	$[0a_{w-2} \cdots a_0]_{s,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $n = 0$ , $w = 7$ :  (a)	$[1a_{w-2} \cdots a_0]_{s,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $-n = 0$ , $w = 7$ :  (b)
2s complement	$[0a_{w-2} \cdots a_0]_{2c,w}$ , where $n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $n = 0$ , $w = 7$ :  (c)	$[1a_{w-2} \cdots a_0]_{2c,w}$ , where $2^{w-1} - n = (a_{w-2} \cdots a_0)_{2,w-1}$ Example $-n = 0$ , $w = 7$ :  (d)

# Fixed width addition

**Fixed-width addition:** adding one bit at time, using the usual column-by-column and carry arithmetic, and dropping the carry from the leftmost column so the result is the same width as the summands. *Does this give the right value for the sum?*

$$\begin{array}{r} (1\ 1\ 0\ 1\ 0\ 0)_{2,6} \\ + (0\ 0\ 0\ 1\ 0\ 1)_{2,6} \\ \hline \end{array}$$

$$\begin{array}{r} [1\ 1\ 0\ 1\ 0\ 0]_{s,6} \\ + [0\ 0\ 0\ 1\ 0\ 1]_{s,6} \\ \hline \end{array}$$

$$\begin{array}{r} [1\ 1\ 0\ 1\ 0\ 0]_{2c,6} \\ + [0\ 0\ 0\ 1\ 0\ 1]_{2c,6} \\ \hline \end{array}$$

## Circuits basics

In a **combinatorial circuit** (also known as a **logic circuit**), we have **logic gates** connected by **wires**. The inputs to the circuits are the values set on the input wires: possible values are 0 (low) or 1 (high). The values flow along the wires from left to right. A wire may be split into two or more wires, indicated with a filled-in circle (representing solder). Values stay the same along a wire. When one or more wires flow into a gate, the output value of that gate is computed from the input values based on the gate's definition table. Outputs of gates may become inputs to other gates.

## Logic gates definitions

Inputs		Output
$x$	$y$	$x$ AND $y$
1	1	1
1	0	0
0	1	0
0	0	0



Inputs		Output
$x$	$y$	$x$ XOR $y$
1	1	0
1	0	1
0	1	1
0	0	0



Input	Output
$x$	NOT $x$
1	0
0	1



# Digital circuits basic examples

Example digital circuit:



Output when  $x = 1, y = 0, z = 0, w = 1$  is \_\_\_\_\_

Output when  $x = 1, y = 1, z = 1, w = 1$  is \_\_\_\_\_

Output when  $x = 0, y = 0, z = 0, w = 1$  is \_\_\_\_\_

Draw a logic circuit with inputs  $x$  and  $y$  whose output is always 0. *Can you use exactly 1 gate?*

## Half adder circuit

**Fixed-width addition:** adding one bit at time, using the usual column-by-column and carry arithmetic, and dropping the carry from the leftmost column so the result is the same width as the summands. In many cases, this gives representation of the correct value for the sum when we interpret the summands in fixed-width binary or in 2s complement.

For single column:

Input		Output	
$x_0$	$y_0$	$c_0$	$s_0$
1	1		
1	0		
0	1		
0	0		



## Two bit adder circuit

Draw a logic circuit that implements fixed-width 2 binary addition:

- Inputs  $x_0, y_0, x_1, y_1$  represent  $(x_1x_0)_{2,2}$  and  $(y_1y_0)_{2,2}$
- Outputs  $z_0, z_1, z_2$  represent  $(z_2z_1z_0)_{2,3} = (x_1x_0)_{2,2} + (y_1y_0)_{2,2}$  (may require up to width 3)

*First approach:* half-adder for each column, then combine carry from right column with sum of left column

Write expressions for the circuit output values in terms of input values:

$z_0 =$  \_\_\_\_\_

$z_1 =$  \_\_\_\_\_

$z_2 =$  \_\_\_\_\_



*Second approach:* for middle column, first add carry from right column to  $x_1$ , then add result to  $y_1$

Write expressions for the circuit output values in terms of input values:

$z_0 =$  \_\_\_\_\_

$z_1 =$  \_\_\_\_\_

$z_2 =$  \_\_\_\_\_

*Extra example* Describe how to generalize this addition circuit for larger width inputs.