Least Eleatest proofs	Least	greatest	proofs
-----------------------	-------	----------	--------

Least greatest proofs
For a set of numbers X , how do you formalize "there is a greatest X " or "there is a least X "?
Prove or disprove: There is a least prime number.
Prove or disprove: There is a greatest integer.
Approach 1, De Morgan's and universal generalization:
Approach 2, proof by contradiction:
Extra examples: Prove or disprove that \mathbb{N} , \mathbb{Q} each have a least and a greatest element.

Gcd definition



Gcd examples

Why do we restrict to the situation where a and b are not both zero?

Calculate gcd((10, 15))

Calculate gcd((10,20))

Gcd basic claims

Claim : For any integers a, b (not both zero), $gcd((a, b)) \ge 1$.

Proof: Show that 1 is a common factor of any two integers, so since the gcd is the greatest common factor it is greater than or equal to any common factor.

Claim: For any positive integers a,b, gcd((a,b) $) \leq a$ and gcd((a,b) $) \leq b$.

Proof Using the definition of gcd and the fact that factors of a positive integer are less than or equal to that integer.

Claim: For any positive integers a, b, if a divides b then gcd((a, b)) = a.

Proof Using previous claim and definition of gcd.

Claim: For any positive integers a, b, c, if there is some integer q such that a = bq + c,

$$\gcd(\ (a,b)\)=\gcd(\ (b,c)\)$$

Proof Prove that an	$ay\ common\ divisor\ of\ a,$	b divides c and that any	$common\ divisor\ of\ b, c\ divides$	a.

Gcd lemma relatively prime

Lemma: For any integers p,q (not both zero), $\gcd\left(\left(\frac{p}{\gcd((p,q))},\frac{q}{\gcd((p,q))}\right)\right)=1$. In other words, can reduce to relatively prime integers by dividing by \gcd .

Proof:

Let x be arbitrary positive integer and assume that x is a factor of each of $\frac{p}{\gcd((p,q))}$ and $\frac{q}{\gcd((p,q))}$. This gives integers α , β such that

$$\alpha x = \frac{p}{\gcd((p,q))}$$
 $\beta x = \frac{q}{\gcd((p,q))}$

Multiplying both sides by the denominator in the RHS:

$$\alpha x \cdot gcd((p,q)) = p$$
 $\beta x \cdot gcd((p,q)) = q$

In other words, $x \cdot gcd(p,q)$ is a common divisor of p,q. By definition of gcd, this means

$$x \cdot gcd((p,q)) \le gcd((p,q))$$

and since gcd((p,q)) is positive, this means, $x \leq 1$.

Definitions

Term	Notation Example(s)	We say in English
sequence	x_1,\ldots,x_n	A sequence x_1 to x_n
summation	$\sum_{i=1}^{n} x_i \text{ or } \sum_{i=1}^{n} x_i$	The sum of the terms of the sequence x_1 to x_n
all reals	\mathbb{R}	The (set of all) real numbers (numbers on the number line)
all integers	\mathbb{Z}	The (set of all) integers (whole numbers including negatives, zero, and positives)
all positive integers	\mathbb{Z}^+	The (set of all) strictly positive integers
all natural numbers	N	The (set of all) natural numbers. Note : we use the convention that 0 is a natural number.
piecewise rule definition	$f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$	Define f of x to be x when x is nonnegative and to be $-x$ when x is negative
function application	f(7) $f(z)$ $f(g(z))$	f of 7 or f applied to 7 or the image of 7 under f f of z or f applied to z or the image of z under f f of g of z or f applied to the result of g applied to z
absolute value square root	$\begin{array}{c} -3 \\ \sqrt{9} \end{array}$	The absolute value of -3 The non-negative square root of 9

Defining sets

To define sets:

To define a set using **roster method**, explicitly list its elements. That is, start with { then list elements of the set separated by commas and close with }.

To define a set using **set builder definition**, either form "The set of all x from the universe U such that x is ..." by writing

$$\{x \in U \mid ...x...\}$$

or form "the collection of all outputs of some operation when the input ranges over the universe U" by writing

$$\{...x... \mid x \in U\}$$

We use the symbol \in as "is an element of" to indicate membership in a set.

Example sets: For each of the following, identify whether it's defined using the roster method or set builder notation and give an example element.