HW4 Proofs and Sets

CSE20F21

Sample Solutions

Assigned questions

1. Consider the predicate $Pr(x)$ over the set of integers, which evaluates to T exactly when x is prime. Consider the following statements.

   (i) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ x \leq y \to Pr(y)\ )$

   (ii) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ y \leq x \to Pr(y)\ )$

   (iii) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ (\ x \leq y \to Pr(y)\ )$

   (iv) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ (\ y \leq x \to Pr(y)\ )$

   (v) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ Pr(y) \to y \leq x\ )$

   (vi) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ Pr(y) \to x \leq y\ )$

   (vii) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ (\ Pr(y) \to y \leq x\ )$

   (viii) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ (\ Pr(y) \to x \leq y\ )$

   (a) (*Graded for correctness of choice and fair effort completeness in justification* [1]) Which of the statements (i) - (viii) is being **proved** by the following proof:

   > Choose $x = 1$, an integer, and we will work to show it is a **witness** for the existential claim. By universal generalization, **choose** e to be an **arbitrary** integer. Towards a **direct proof**, **assume** that $Pr(e)$ holds. We **WTS** that $1 \leq e$. By definition of the predicate Pr, since $Pr(e)$ is true, $e > 1$. By definition of $\leq$, this means that $1 \leq e$, as required and the claim has been proved. $\square$

   Hint: it may be useful to identify the key words in the proof that indicate proof strategies.

   Solution: The key word "witness" indicates proving an existential claim, so the options so far are (i) , (ii) , (v), (vi). The next sentence proceeds by universal generalization, so the next logical structure is that of universal quantification, which is

[1] Graded for correctness means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound. Graded for fair effort completeness means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers.

still consistent with each of the four options so far. The next proof strategy used is that of direct proof, where the assumption is that $Pr(e)$. This means we are proving a conditional statement whose hypothesis is $Pr(...)$, the assertion that a number is prime (where the number is represented by the variable being universally quantified, because e was chosen as the arbitrary element in the universal generalization strategy). The only statements that match this are (v) and (vi), and we notice that e is replacing y in the predicate. Next, the proof works to show $1 \leq e$, where 1 was our candidate witness choice to replace x and e is the arbitrary element replacing y. This means that the goal in the direct proof is $x \leq y$, so the conditional statement originally looked like $Pr(y) \to x \leq y$, namely option (vi).

(b) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **disproved** by the following proof:

> To disprove the statement, we will prove the universal statement that is logically equivalent to its negation. By universal generalization, **choose** e to be an **arbitrary** integer. We need to find a **witness** integer y such that $y \leq e$ and $\neg Pr(y)$. Notice that $e > 1 \vee e \leq 1$ is true, and we proceed in a **proof by cases**. **Case 1**: Assume $e > 1$ and **WTS** there is a witness integer y such that $y \leq e$ and $\neg Pr(y)$. Choose $y = 0$, an integer. Then, since by **case assumption** $1 < e$, we have $y = 0 \leq 1 \leq e$. Moreover, since $y = 0$, $y > 1$ is false and so (by the definition of Pr), the predicate Pr evaluated at y is false, as required to prove the **conjunction** $y \leq e$ and $\neg Pr(y)$. **Case 2**: Assume $e \leq 1$ and **WTS** there is a witness integer y such that $y \leq e$ and $\neg Pr(y)$. Choose $y = e - 1$, an integer (because subtracting 1 from the integer e still gives an integer). By definition of subtraction, $y = e - 1 \leq e$. Moreover, since by the **case assumption** $y = e - 1 \leq 1 - 1 = 0$, $y > 1$ is false. Thus, (by the definition of Pr), the predicate Pr evaluated at y is false. We have proved the **conjunction** $y \leq e$ and $\neg Pr(y)$ as required. Since each case is complete, the proof by cases is complete and the original statement has been disproved. □

Hint: it may be useful to identify the key words in the proof that indicate proof strategies.

Solution: The original strategy this argument starts with matches our strategy for disproving existential claims. We will trace the argument to discover which universal claim is being proved, then negate it and use De Morgan to recover the original existential claim being disproved. Proceeding by universal generalization and then finding a witness means the logical structure of the statement being claimed is $\forall ... \exists y$. The goal for the rest of the proof is stated to be "$y \leq e$ and $\neg Pr(y)$" which would be the negation of the conditional statement $y \leq e \to Pr(y)$. This matches the predicate in the statement (ii).

(c) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ x \neq y \to (Pr(x) \vee Pr(y))\ )$$

Solution: This statement translates to "for all integers x, y, when $x \neq y$ then at least one of x, y is prime". This statement is **false**, as we can see from the counterexample $x = 1$, $y = 4$: these are both integers so they're candidate counterexamples, $1 \neq 4$ so the hypothesis of the conditional is true, and $Pr(1)$, $Pr(4)$ are both false (as we mentioned in class), so the conclusion of the conditional is false. Namely, the predicate being universally claimed is false at this element of the domain so we have found a counterexample.

(d) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and proof*) Translate the statement to English and then prove or disprove it

$$(\ \forall x \in \mathbb{Z}\ Pr(x)\ ) \oplus (\ \exists x \in \mathbb{Z}\ Pr(x)\ )$$

Solution: This statement translates to "either all integers are prime or at least one integer is prime, and not both". This statement is **true**, as we can see by proving that exactly one of the universal statements is true. First, notice that $\forall x \in \mathbb{Z}\ Pr(x)$ is false, using the counterexample $x = 1$ (an integer that is not prime). Second, notice that $\exists x \in \mathbb{Z}\ Pr(x)$ is true, using the witness $x = 2$ (an integer that is prime). Thus, the exclusive or evaluates to true.

(e) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ (\ (\ Pr(x) \wedge Pr(y)\ ) \leftrightarrow Pr(x+y)\ )$$

Solution: This statement translates to "the sum of two integers is prime if and only if both of them are prime". This statement is **false**, as we can see from the counterexample $x = 2, y = 2$. These are integer values and evaluating the predicate gives that $Pr(2) \wedge Pr(2)$ is true (since 2 is prime so each conjunct is true) while $Pr(2 + 2)$ is false (since 2 is a positive factor of 4 that is neither 1 nor 4) so the biconditional statement is false.

(f) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*) Translate the statement to English and then prove or disprove it

$$\forall x \in \mathbb{Z}\ (\ Pr(x) \to \exists y \in \mathbb{Z}\ (\ x < y \wedge Pr(y)\ )$$

Solution: This statement translates to "Every prime is less than another prime". This statement is **true**. The full proof requires showing that there are infinitely many primes, which we will do in class in a couple of weeks. The idea is that we can always find a number that doesn't have any of the positive integers less than it (other than 1) as a factor.

2. Let $W = \mathcal{P}(\{1,2,3,4,5\})$.

Sample response that can be used as reference for the detail expected in your answers for this question:

To give a witness for the existential claim

$$\exists B \in W \ (B \in \ \{X \in W \mid 1 \in X\} \cap \{X \in W \mid 2 \in X\} \ )$$

consider $B = \{1,2\}$. To prove that this is a valid witness, we need to show that it is in the domain of quantification W and that it makes the predicate being quantified evaluate to true. By definition of set-builder notation and intersection, it's enough to prove that $\{1,2\} \in W$ and that $1 \in \{1,2\}$ and that $2 \in \{1,2\}$.

- By definition of power set, elements of W are subsets of $\{1,2,3,4,5\}$. Since each element in $\{1,2\}$ is an element of $\{1,2,3,4,5\}$, $\{1,2\}$ is a subset of $\{1,2,3,4,5\}$ and hence is an element of W.
- Also, by definition of the roster method, $1 \in \{1,2\}$.
- Similarly, by definition of roster method, $2 \in \{1,2\}$.

Thus $B = \{1,2\}$ is an element of the domain which is in the intersection of the two sets mentioned in the predicate being quantified and is a witness to the existential claim. QED

(a) (*Graded for correctness*) Give a witness to the existential claim

$$\exists X \in W \ (\ X \cup X = \emptyset \ )$$

Justify your example by explanations that include references to the relevant definitions.

Solution: To give a witness for this existential claim consider $X = \emptyset$. To prove that this is a valid witness, we need to show that it is in the domain of quantification W and that it makes the predicate being quantified evaluate to true. Since W is a power set of a set and we saw in class that the empty set is an element of every power set (because it is a subset of every set), $\emptyset \in W$. Now, we calculate $\emptyset \cup \emptyset$. By definition of union, $\emptyset \cup \emptyset = \{x \mid x \in \emptyset \lor x \in \emptyset\} = \{x \mid F \lor F\} = \{x|F\} = \emptyset$, as required.

(b) (*Graded for correctness*) Give a counterexample to the universal claim

$$\forall X \in W \ (\{a \in X \mid a \text{ is even}\} \subsetneq X \ )$$

Justify your example by explanations that include references to the relevant definitions.

Solution: To give a counterexample for this universal claim consider $X = \{2,4\}$. To prove that this is a valid counterexample, we need to show that it is in the domain of quantification W and that it makes the predicate being quantified evaluate to false. Since $2 \in \{1,2,3,4,5\}$ and $4 \in \{1,2,3,4,5\}$, by definition of subsets,

$X \subseteq \{1,2,3,4,5\}$. Thus, by definition of the power set, $X \in \mathcal{P}(\{1,2,3,4,5\})$ so by definition of W, $X \in W$. By definition of set builder notation, $\{a \in X \mid a \text{ is even}\}$ is the subset of X that includes only the elements of X that are even. Since 2 and 4 are both even, by definition of X, $\{a \in X \mid a \text{ is even}\} = X$. By definition of proper subsets, $\{a \in X \mid a \text{ is even}\}$ is not a proper subset of X, or in other words, $\{a \in X \mid a \text{ is even}\} \subsetneq X$ is false, as required.

(c) (*Graded for correctness*) Give a witness to the existential claim

$$\exists (X,Y) \in W \times W \ (\ X \cup Y = Y\ )$$

Justify your example by explanations that include references to the relevant definitions.

Solution: A witness will be an ordered pair each of whose components is a subset of $\{1,2,3,4,5\}$ and which make the property true. Consider $(X,Y) = (\{1,2\},\{1,2,3\})$. This is an ordered pair and since $1 \in \{1,2,3,4,5\}$, $2 \in \{1,2,3,4,5\}$, and $3 \in \{1,2,3,4,5\}$, $X \subseteq \{1,2,3,4,5\}$ and $Y \subseteq \{1,2,3,4,5\}$ so $(X,Y) \in W \times W$. By definition of union,

$$\begin{aligned} X \cup Y &= \{a \mid a \in \{1,2\} \lor a \in \{1,2,3\}\} \\ &= \{a \mid a = 1 \lor a = 2 \lor a = 1 \lor a = 2 \lor a = 3\} \quad \text{by definition of roster method} \\ &= \{a \mid a = 1 \lor a = 2 \lor a = 3\} \quad \text{since } p \lor p \equiv p \\ &= \{1,2,3\} = Y \qquad \text{by definition of roster method} \end{aligned}$$

3. Recall our representation of movie preferences in a three-movie database using 1 in a component to indicate liking the movie represented by that component, -1 to indicate not liking the movie, and 0 to indicate neutral opinion or haven't seen the movie. We call Rt the set of all ratings 3-tuples. We defined the function $d_0 : Rt \times Rt \to \mathbb{R}$ which takes an ordered pair of ratings 3-tuples and returns a measure of the distance between them given by

$$d_0(\ (\ (x_1,x_2,x_3),(y_1,y_2,y_3)\ )\ ) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$

Another measure of the distance between a pair of ratings 3-tuples is given by the following function $d_1 : Rt \times Rt \to \mathbb{R}$ given by

$$d_1(\ (\ (x_1,x_2,x_3),(y_1,y_2,y_3)\ )\ ) = \sum_{i=1}^{3} |x_i - y_i|$$

(a) For each of the statements below, first translate them symbolically (using quantifiers, logical operators, and arithmetic operations), then determine whether each is true or false by applying the proof strategies to prove each statement or its negation. (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and of the proof*)

i. For all ordered pairs of ratings 3-tuples, the value of the function d_0 is greater than the value of the function d_1.

Solution: We translate the statement to

$$\forall x \in Rt \times Rt \; (\; d_0(x) > d_1(x) \;)$$

This statement is **false**, as we can see from the counterexample $x = (\,(1,0,-1),(1,0,-1)\,) \in Rt \times Rt$. Using the definitions of the functions

$$d_0(x) = \sqrt{(1-1)^2 + (0-0)^2 + (-1--1)^2} = \sqrt{0+0+0} = 0$$

and

$$d_1(x) = |1-1| + |0-0| + |-1--1| = 0+0+0 = 0$$

Since $d_0(x) = d_1(x)$, it is not the case that $d_0(x) > d_1(x)$.

ii. The maximum value of the function d_1 is greater than the maximum value of the function d_0.

Solution: We translate the statement to

$$\exists x_0 \; \exists x_1 \; (\; \forall y(\; d_0(x_0) \geq d_0(y)\;) \wedge \forall y(\; d_1(x_1) \geq d_1(y)\;) \wedge d_1(x_1) > d_0(x_0) \;)$$

(where all the quantifiers are over the domain $R_t \times R_t$). Notice that the logical structure of this statement is "there are two values where the first is the input to d_0 that gives the maximum value of d_0, the second is the input to d_1 that gives the maximum value of d_1, and this maximum value of d_1 is greater than or equal to this maximum value of d_0". In particuar, we used quantifiers to define how to calculate maximum values.

This statement is true because the maximum values for each of d_1 and d_0 are achieved when the inputs to the function have opposite ratings in each component and we can calculate the maximum for d_1 in this case will be 6 whereas for d_0 it will be $\sqrt{12} = 2\sqrt{3} < 6$ (because $\sqrt{3} < 3$).

With more details: we will show that $x_0 = x_1 = (\;(1,1,1),(-1,-1,-1)\;)$ will witness the existential claim. To do so, we have three goals:

- Goal 1: we need to show that $\forall y(\; d_0(x_0) \geq d_0(y))$
- Goal 2: we need to show that $\forall y(\; d_1(x_1) \geq d_1(y))$
- Goal 3: we need to show that $d_1(x_1) > d_0(x_0)$

To work towards these goals, it will be useful to calculate

$$d_1(x_1) = |1--1| + |1--1| + |1--1| = 2+2+2 = 6 = 2 \cdot 3$$

$$d_0(x_0) = \sqrt{(1--1)^2 + (1--1)^2 + (1--1)^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Since $\sqrt{3} < 3$, these calculations prove that $d_1(x_1) > d_0(x_0)$ and thus Goal 3 has been shown. To prove Goals (1) and (2), we notice that, since each component of a ratings 3-tuple must be -1, 0, or 1, the maximum absolute difference between component values is 2. Moreover, since these difference are added up (in d_1) or squared and then added up (in d_0), the function value for any ordered pair of ratings 3-tuples will be less than or equal to the values we calculated for x_0 and x_1.

(b) (*Graded for correctness*) Write a statement about 3-tuples of movie ratings that uses the function d_1 and has at least one universal and one existential quantifier. Your response will be graded correct if all the syntax in your statement is correct.

Solution: Consider the statement $\forall x \in Rt\ \exists y \in Rt\ (\ d_1(\ (x, y)\ ) = 0\ )$.

(c) (*Graded for fair effort completeness*) Translate the property you wrote symbolically in the last step to English. Indicate if it is true, false, or if you don't know (sometimes we can write interesting properties, and we're not sure if they are true or not!). Give informal justification for whether you think it is true/ false, or explain why the proof strategies we have so far do not appear to be sufficient to determine whether the statement holds.

Solution: The statement we wrote translates to "for each ratings 3-tuple there is some ratings 3-tuple that is $d1$ distance zero from it." This statement is true because, for arbitrary ratings 3-tuple x we can choose the witness ratings 3-tuple $y = x$ and then use the definition of d_1 to calculate that the output of the function will be zero since $y = x$.