Optimization for Machine Learning

Lecture 7: Distributed Optimization II

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CISPA – `https://cms.cispa.saarland/optml24/`

May 28, 2024

Group Project

- Work on a research question (related to the course) in a small team!
- Present your result with a **poster** and a short (3 page) **report**.
- A list of project ideas is available here.
  - pick one project (or propose your own)

Hints:

- state the research question/hypothesis clearly!
- only include claims that are supported by evidence you provide!
- the contact person can help you! (but **not** last-minute!)

Group Project Timeline

- Group registration between May 28 – June 4 (register on CMS)
  - groups of 2–3
- before June 18: get in touch with the contact person and schedule a meeting!
  - read the related literature
  - prepare a list of **research goals and tasks**
- before June 25: meet with your contact person
  - zoom meeting, 30-60min, can also be in-person
  - **discuss your research plan**
  - **ask questions** about things you do not understand
- July 16: Poster presentation (& suggested report submission)
  - (note that the **poster printing deadline** is a bit earlier, TBA!)
- July 26: last possible date to submit the report

There will be no exercise sheets in the weeks of June 25/July 2 — you can also discuss the project in the exercise session.

Group Project Grading

- (50%) methodology and execution
  - are claims clearly stated and verified by evidence?
  - is the research question related to the course?
- (50%) presentation of the results
  - poster & poster presentation
  - final report
- bonus points for highly creative questions, interesting results, outstanding presentations, etc.

If the project is not passed, there will be an option to hand in a revised version in August.

Asynchronous SGD

(wrapping up)

Hogwild!

Input: $\mathbf{x}_0 \in \mathbb{R}^d$, stepsize γ, accessible memory location to store $\mathbf{x} \in \mathbb{R}^d$

At iteration t (in parallel):

$$
\begin{aligned}
&\mathbf{x}_t \leftarrow \mathbf{x} && \text{(inconsistent read of the memory } \mathbf{x}\text{)} \\
&\mathbf{g}_t = \mathbf{g}(\mathbf{x}_t) && \text{(stochastic gradient)} \\
&\textbf{for } i \in [d] && \text{(atomic coordinate write)} \\
&\quad [\mathbf{x}]_i := [\mathbf{x}]_i - \gamma[\mathbf{g}_t]_i &&
\end{aligned}
$$

Theorem

Theorem (Lecture-7).1 ([SK20, SMJ21])

Let $f : \mathbb{R}^d \to R$ be L-smooth with $F_0 = f(\mathbf{x}_0) - f^\star$. Then there exists a stepsize $\gamma \leq \gamma_{\text{crit}} := \frac{1}{10L(M+\tau)}$ such that after T steps of delayed SGD (with atomic vector operations):

$$\min_{t \leq T} \mathbb{E} \|\nabla f(\mathbf{x}_t)\|^2 = \mathcal{O}\left(\frac{F_0 L(M+\tau)}{T} + \frac{\sqrt{LF_0\sigma^2}}{\sqrt{T}}\right).$$

- Mini-Batch SGD can be seen as a variant of delayed SGD with $\tau = b$.
- We recover the mini-batch SGD result when considering the same number of gradient computations, i.e. $T \to Tb$ and replacing $\tau \to b$).

Proof I

The main ingredient for the proof is to define a virtual sequence $\tilde{\mathbf{x}}_t$ of iterates, $\tilde{\mathbf{x}}_0 = \mathbf{x}_0$, defined as [MPP+17, SK20]

$$\tilde{\mathbf{x}}_{t+1} = \tilde{\mathbf{x}}_t - \gamma \mathbf{g}_t .$$

Lemma (Lecture-7).2 (Decrease)

For $\gamma \leq \gamma_{\mathrm{crit}}$ *it holds*

$$\mathbb{E} f(\tilde{\mathbf{x}}_{t+1}) \leq \mathbb{E} f(\tilde{\mathbf{x}}_t) - \frac{\gamma}{4} \|\nabla f(\mathbf{x}_t)\|^2 + \frac{\gamma^2 L \sigma^2}{2} + \frac{\gamma L^2}{2} \mathbb{E} \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|^2$$

Lemma (Lecture-7).3 (Difference)

For $\gamma \leq \gamma_{\mathrm{crit}}$ *it holds* *here* $(t-\tau)_+ = \max\{0, t-\tau\}$

$$\mathbb{E} \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|^2 \leq \frac{1}{50 L^2 \tau} \sum_{k=(t-\tau)_+}^{t-1} \mathbb{E} \|\nabla f(\mathbf{x}_k)\|^2 + \frac{\gamma}{5L} \sigma^2 .$$

Proof II

Plug (Difference) into (Decrease), re-arrange, and divide by γ:

$$\frac{1}{4}\mathbb{E}\left\|\nabla f(\mathbf{x}_t)\right\|^2 \leq \frac{1}{\gamma}\left(\mathbb{E}f(\tilde{\mathbf{x}}_t) - \mathbb{E}f(\tilde{\mathbf{x}}_{t+1})\right) + \frac{\gamma L\sigma^2}{2} + \frac{1}{100\tau}\sum_{k=(t-\tau)_+}^{t-1}\mathbb{E}\left\|\nabla f(\mathbf{x}_k)\right\|^2 + \frac{\gamma L\sigma^2}{10}$$

Now we average over T. Note that the highlighted $\left\|\nabla f(\mathbf{x}_k)\right\|^2$ terms appear at most τ times.

$$\frac{1}{4T}\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\mathbf{x}_t)\right\|^2 \leq \frac{\Delta}{\gamma T} + \gamma L\sigma^2 + \frac{1}{100T}\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\mathbf{x}_t)\right\|^2$$

Note that $\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\mathbf{x}_t)\right\|^2$ appears on both sides, with $\frac{1}{4T} - \frac{1}{100T} \geq \frac{1}{5T}$.

$$\frac{1}{5T}\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\mathbf{x}_t)\right\|^2 \leq \frac{F_0}{\gamma T} + \gamma L\sigma^2\,.$$

Now the result follows by tuning γ (in the same way as before).

Proof of Lemma (Decrease)

This follows our standard path, with one small trick. By L-smoothness (at $\tilde{\mathbf{x}}_t$):

$$\begin{aligned}\mathbb{E}[f(\tilde{\mathbf{x}}_{t+1}) &\leq \mathbb{E}[f(\tilde{\mathbf{x}}_t)] - \gamma \nabla f(\tilde{\mathbf{x}}_t)^\top \mathbf{g}_t + \tfrac{\gamma^2 L}{2} \mathbb{E}\left\|\mathbf{g}_t\right\|^2 \\ &\leq \mathbb{E}[f(\tilde{\mathbf{x}}_t)] - \gamma \nabla f(\tilde{\mathbf{x}}_t)^\top \nabla f(\mathbf{x}_t) + \frac{\gamma^2 L}{2}\left((M+1)\left\|\nabla f(\mathbf{x}_t)\right\|^2 + \sigma^2\right)\end{aligned}$$

Now:

$$\begin{aligned}-\nabla f(\tilde{\mathbf{x}}_t)^\top \nabla f(\mathbf{x}_t) &= -\left(\nabla f(\tilde{\mathbf{x}}_t) - \nabla f(\mathbf{x}_t) + \nabla f(\mathbf{x}_t)\right)^\top \nabla f(\mathbf{x}_t) \\ &= -\left\|\nabla f(\mathbf{x}_t)\right\|^2 - \left(\nabla f(\tilde{\mathbf{x}}_t) - \nabla f(\mathbf{x}_t)\right)^\top \nabla f(\mathbf{x}_t) \\ &\leq -\left\|\nabla f(\mathbf{x}_t)\right\|^2 + \tfrac{1}{2}\left\|\nabla f(\mathbf{x}_t)\right\|^2 + \tfrac{1}{2}\left\|\nabla f(\tilde{\mathbf{x}}_t) - \nabla f(\mathbf{x}_t)\right\|^2 \\ &\leq -\tfrac{1}{2}\left\|\nabla f(\mathbf{x}_t)\right\|^2 + \tfrac{L^2}{2}\left\|\tilde{\mathbf{x}}_t - \mathbf{x}_t\right\|^2\end{aligned}$$

where we used $(-\mathbf{a}^\top \mathbf{b}) \leq \frac{1}{2}\left\|\mathbf{a}\right\|^2 + \frac{1}{2}\left\|\mathbf{b}\right\|^2$.

Now with $\gamma \leq \frac{1}{2L(M+\tau)} \leq \frac{1}{2L(M+1)}$:

$$\mathbb{E}[f(\tilde{\mathbf{x}}_{t+1}) \leq \mathbb{E}[f(\tilde{\mathbf{x}}_t)] - \frac{\gamma}{4}\left\|\nabla f(\mathbf{x}_t)\right\|^2 + \frac{\gamma^2 L \sigma^2}{2} + \frac{\gamma L^2}{2}\left\|\tilde{\mathbf{x}}_t - \mathbf{x}_t\right\|^2$$

Proof of Lemma (Difference)

Note that $\mathbf{x}_t = \mathbf{x}_0 - \gamma \sum_{k \in \mathcal{I}_t} \mathbf{g}_k$ and $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \gamma \sum_{k=0}^{t-1} \mathbf{g}_k$ and define $\xi_k = \mathbf{g}_k - \nabla f(\mathbf{x}_k)$, with $\mathbb{E}[\xi_k] = 0$.

Then

$$\mathbb{E}\left\|\tilde{\mathbf{x}}_t - \mathbf{x}_t\right\|^2 = \gamma^2 \mathbb{E}\left\|\sum_{k \in \mathcal{J}_t} \mathbf{g}_k\right\|^2 \leq 2\gamma^2 \mathbb{E}\left\|\sum_{k \in \mathcal{J}_t} \nabla f(\mathbf{x}_k)\right\|^2 + 2\gamma^2 \mathbb{E}\left\|\sum_{k \in \mathcal{J}_t} \xi_k\right\|^2$$

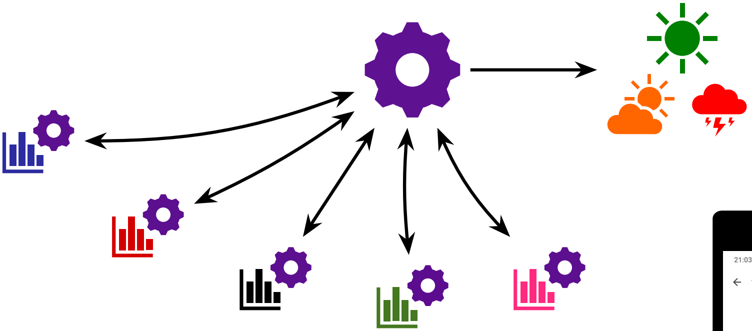
$$\leq 2\gamma^2 \tau \sum_{k=(t-\tau)_+}^{t-1} \left\|\nabla f(\mathbf{x}_k)\right\|^2 + 2\gamma^2 M \sum_{k=(t-\tau)_+}^{t-1} \left\|\nabla f(\mathbf{x}_k)\right\|^2 + 2\gamma^2 \tau \sigma^2$$

Where we used

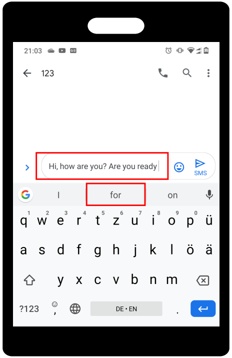
- $\left\|\nabla f(\mathbf{x}_k) + \xi_k\right\|^2 \leq 2\left\|\nabla f(\mathbf{x}_k)\right\|^2 + 2\left\|\xi_k\right\|^2$
- $\left\|\sum_{k=1}^{\tau} \mathbf{a}_k\right\|^2 \leq \tau \sum_{k=1}^{\tau} \left\|\mathbf{a}_k\right\|^2$
- $\mathbb{E}\left\|\sum_{k=1}^{\tau} \xi_k\right\|^2 = \sum_{k=1}^{\tau} \mathbb{E}\left\|\xi_k\right\|^2$ (independent noise)

Federated Learning

Example: Federated Learning [MMR$^+$17, KMea21]



- private data stays on device
- server coordinates training and aggregates focused updates



Training Objective

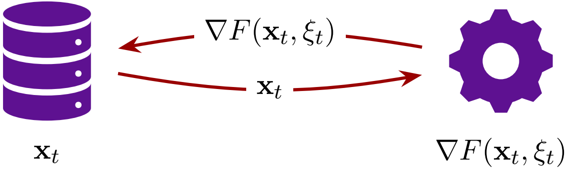
$$\min_{\mathbf{x}\in\mathbb{R}^d} \left[f(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} \underbrace{f_i(\mathbf{x})}_{\text{data } \mathcal{D}_i \text{ on client } i} \right] \qquad f_i(\mathbf{x}) = \begin{cases} \mathbb{E}_{\xi\sim\mathcal{D}_i} F(\mathbf{x},\xi) \\ \frac{1}{m}\sum_{j=1}^{m} f_{ij}(\mathbf{x}) \end{cases}$$

- Collaboratively solve **a (joint)** machine learning problem
- **efficiently**, in terms of:
  - computation (stochastic gradients, mini-batches),
  - communication (server $\leftrightarrow$ client).

Other very relevant scenarios:

• personalization • heterogenity • privacy • robustness

Communication Bottleneck



algorithm	rounds	gradients (total)
mini-batch SGD batch size $b = 1$	$\mathcal{O}\left(\frac{\sigma^2}{n\mu\epsilon} + \frac{L}{\mu}\log\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\frac{\sigma^2}{\mu\epsilon} + \frac{nL}{\mu}\log\frac{1}{\epsilon}\right)$
mini-batch SGD batch size b	$\mathcal{O}\left(\underbrace{\frac{\sigma^2}{nb\mu\epsilon}}_{\ll} + \underbrace{\frac{L}{\mu}\log\frac{1}{\epsilon}}_{=}\right)$	$\mathcal{O}\left(\frac{\sigma^2}{\mu\epsilon} + \frac{nbL}{\mu}\log\frac{1}{\epsilon}\right)$

Local SGD

To parallelize or not to parallelize?

Which algorithm is "better"?

- mini-batch SGD with batch size b and T iterations,
- SGD with bT iterations?

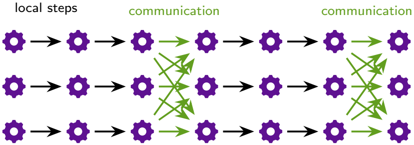
(Both can access the same #oracle calls, $C = bT$). Answer: it depends (on ϵ).

Thought experiment: assume $b \to \infty$ or $T \to \infty$, while keeping C constant.

- Mini-batch SGD ($b \to \infty$) will perform $T = \frac{C}{b} \leq 1$ iteration (and stay at $\mathbf{x}_0$)
- SGD with $b = 1$ will perform $T \to \infty$ steps (potentially converging to $\mathbf{x}^\star$).

Is there a way to "interpolate" between the two extremes?

Local SGD



Notation/Setting

- n machines
- $f_i(\mathbf{x})$ denote the function (data) available locally at node i
- local gradient oracle $\mathbb{E}[\mathbf{g}^{(i)}(\mathbf{x})] = \nabla f_i(\mathbf{x})$, $\forall i \in [n]$, with bounded variance:

$$\mathbb{E}\left\|\mathbf{g}^{(i)}(\mathbf{x}) - \nabla f_i(\mathbf{x})\right\|^2 \leq \sigma^2$$

- let $\mathbf{x}_t^{(i)} \in \mathbb{R}^d$ denote the local iterate at node i, $\forall i \in [n]$

Local SGD

Input: $\mathbf{x}_0 \in \mathbb{R}^d$, $\mathbf{x}_0^{(i)} = \mathbf{x}_0$, $\forall i \in [n]$, stepsize γ, $\tau \geq 1$ (number of local steps)

At iteration t (in parallel on all nodes $i \in [n]$):

$\mathbf{g}_t^i = \mathbf{g}^{(i)}(\mathbf{x}_t^{(i)})$ (stochastic gradient locally on each node)

if $t+1$ is a multiple of τ :

$$\mathbf{x}_{t+1}^{(i)} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_t^{(i)} - \gamma \mathbf{g}_t^i\right) \qquad \text{(global averaging)}$$

otherwise:

$$\mathbf{x}_{t+1}^{(i)} = \mathbf{x}_t^{(i)} - \gamma \mathbf{g}_t^i \qquad \text{(local step)}$$

Homegeneous Functions

For simplicity, assume

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_n(\mathbf{x})$$

(note that in general $\mathbf{g}_t^i \neq \mathbf{g}_t^j$ for $i \neq j$).

- This means stochastic gradients are uniformly sampled from the whole dataset (similar as for mini-batch SGD).
- For $\tau = 1$ Local SGD is identical to mini-batch SGD with batch size $b = n$.

Theorem (Lecture-7).4 (Homegeneous Case, [Sti19, KLB⁺20])

Let $f_i \colon \mathbb{R}^d \to \mathbb{R}$ be L-smooth, $\forall i \in [n]$ and $f_i = f_j$, $\forall i, j \in [n]$, with $\Delta = f(\mathbf{x}_0) - f^\star$. Then there exists a stepsize $\gamma \leq \gamma_{\text{crit}} := \frac{1}{20L\tau}$ such that after T steps (that is, T/τ communication rounds) of Local SGD it holds

$$\min_{t \leq T} \mathbb{E} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 = \mathcal{O}\left(\frac{\Delta L \tau}{T} + \frac{(\Delta L \sigma)^{2/3} \tau^{1/3}}{T^{2/3}} + \frac{\sqrt{L \Delta \sigma^2}}{\sqrt{Tn}} \right),$$

with $\bar{\mathbf{x}}_t := \frac{1}{n} \sum_{i=1}^n \mathbf{x}_t^{(i)}$.

Discussion

- Linear speedup if $\sigma^2 > 0$: the variance decreases linearly in the number of oracle calls (Tn). This is optimal.
- The deterministic optimization term (the term not depending on σ^2) is impacted by τ (similarly as with mini-batch SGD).
  - ideally, we would have hoped to see there $\mathcal{O}\left(\frac{\Delta L}{T}\right)$ (= progress in every iteration) vs. $\mathcal{O}\left(\frac{\Delta L \tau}{T}\right)$ (= progress every communication round)
- The theorem shows almost the same convergence as for mini-batch SGD, up to the higher order $\mathcal{O}(T^{-2/3})$ term.
  - There is no clear winner: see also [WPS+20].

Performance in Practice [LSPJ20]

ResNet-20 on CIFAR-10 (IID data)

	Top-1 acc.	local gradients	communication
Mini-batch SGD ($n = 16$, $\tau = 128$)	92.5%	2048	-
Mini-batch SGD ($n = 16$, $\tau = 1024$)	76.3%	16384	$\div$ 8
Local-SGD ($n = 16$, $\tau = 8 \times 128$)	92.0%	16384	$\div$ 8

Proof I

We will again use the virtual sequence technique. As virtual sequence we consider $\bar{\mathbf{x}}_t$ (note that the average is not computed in every iteration).

Lemma (Lecture-7).5 (Decrease)

For $\gamma \leq \frac{1}{4L}$ it holds

$$\mathbb{E}f(\bar{\mathbf{x}}_{t+1}) \leq Ef(\bar{\mathbf{x}}_t) - \frac{\gamma}{4}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \gamma^2 L\frac{\sigma^2}{n} + \frac{\gamma L^2}{n}\sum_{i=1}^n \mathbb{E}\left\|\mathbf{x}_t^{(i)} - \bar{\mathbf{x}}_t\right\|^2$$

Lemma (Lecture-7).6 (Difference)

For $\gamma \leq \gamma_{\text{crit}} = \frac{1}{20L\tau}$, with the notation for $R_t = \frac{1}{n}\sum_{i=1}^n \left\|\bar{\mathbf{x}}_t - \mathbf{x}_t^{(i)}\right\|^2$, it holds

$$\mathbb{E}R_t \leq \frac{1}{20L^2\tau}\sum_{j=(t-1)-k}^{t-1} \mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_j)\right\|^2 + 5\gamma^2\tau\sigma^2$$

where $(t-1)-k$ denotes the index of the last communication round ($k \leq \tau - 1$).

Proof II

Plug (Difference) into (Decrease), re-arrange and divide by γ:

$$\frac{1}{4}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 \leq \frac{1}{\gamma}\left(\mathbb{E}f(\bar{\mathbf{x}}_t) - \mathbb{E}f(\bar{\mathbf{x}}_{t+1})\right) + \gamma L\frac{\sigma^2}{n} + \frac{1}{20\tau}\sum_{j=(t-1)-k}^{t-1}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_j)\right\|^2 + 5\gamma^2 L^2 \tau\sigma^2$$

Now we divide by T and sum over $t = 0, \dots, T-1$:

$$\frac{1}{4T}\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 \leq \frac{1}{\gamma T}\sum_{t=0}^{T-1}\left[\left(\mathbb{E}f(\bar{\mathbf{x}}_t) - \mathbb{E}f(\bar{\mathbf{x}}_{t+1})\right) + \frac{1}{20T}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2\right] + \gamma^2 L\frac{\sigma^2}{n} + 5\gamma L^2\tau\sigma^2$$

Note that $\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2$ appears on both sides, with $\frac{1}{4T} - \frac{1}{20T} = \frac{1}{5T}$.

$$\frac{1}{5T}\sum_{t=0}^{T-1}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 \leq \frac{\Delta}{\gamma T} + \gamma L\frac{\sigma^2}{n} + 5\gamma^2 L^2\tau\sigma^2 \,.$$

Now the result follows by tuning γ (see Exercise Sheet 7).

Proof of Lemma (Decrease)

By L-smoothness:

$$\mathbb{E} f(\bar{\mathbf{x}}_{t+1}) \leq \mathbb{E} f(\bar{\mathbf{x}}_t) - \frac{\gamma}{n} \sum_{i=1}^n \nabla f(\bar{\mathbf{x}}_t)^\top \mathbf{g}_t^i + \frac{\gamma^2 L}{2} \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{g}_t^i \right\|^2$$

$$\leq \mathbb{E} f(\bar{\mathbf{x}}_t) - \frac{\gamma}{n} \sum_{i=1}^n \nabla f(\bar{\mathbf{x}}_t)^\top \nabla f_i(\mathbf{x}_t^{(i)}) + \frac{\gamma^2 L \sigma^2}{2n} + \frac{\gamma^2 L}{2} \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_i(\mathbf{x}_t^{(i)}) \right\|^2$$

Similarly as we have seen before, by adding and subtracting $\nabla f(\bar{\mathbf{x}}_t)$:

$$-\frac{1}{n} \sum_{i=1}^n \nabla f(\bar{\mathbf{x}}_t)^\top \nabla f_i(\mathbf{x}_t^{(i)}) = -\nabla f(\bar{\mathbf{x}}_t)^\top \nabla f(\bar{\mathbf{x}}_t) + \frac{1}{n} \sum_{i=1}^n \nabla f(\bar{\mathbf{x}}_t)^\top \left(\nabla f(\bar{\mathbf{x}}_t) - \nabla f_i(\mathbf{x}_t^{(i)}) \right)$$

$$\leq -\frac{1}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \frac{1}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f_i(\mathbf{x}_t^{(i)}) \right\|^2$$

$$\leq -\frac{1}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \frac{L^2}{2n} \sum_{i=1}^n \left\| \bar{\mathbf{x}}_t - \mathbf{x}_t^{(i)} \right\|^2$$

Note: $-\mathbf{a}^\top \mathbf{b} \leq \frac{1}{2} \left\| \mathbf{a} \right\|^2 + \frac{1}{2} \left\| \mathbf{b} \right\|^2$.

Continued

And by adding and subtracting $\nabla f(\bar{\mathbf{x}}_t)$ in the last term: $\|\mathbf{a}+\mathbf{b}\|^2 \leq 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$

$$\begin{aligned}
\frac{1}{2}\left\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\mathbf{x}_t^{(i)})\right\|^2 &\leq \left\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\mathbf{x}_t^{(i)}) - \nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \|\nabla f(\bar{\mathbf{x}}_t)\|^2 \\
&\leq \frac{L^2}{n}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t - \mathbf{x}_t^{(i)}\right\|^2 + \|\nabla f(\bar{\mathbf{x}}_t)\|^2
\end{aligned}$$

Now we plug everything together, and use $\gamma \leq \frac{1}{4L}$.

$$\mathbb{E}f(\bar{\mathbf{x}}_{t+1}) \leq \mathbb{E}f(\bar{\mathbf{x}}_t) + \left(\gamma^2 L - \frac{\gamma}{2}\right)\|\nabla f(\bar{\mathbf{x}}_t)\|^2 + \frac{\gamma^2 L\sigma^2}{2n} + \left(\frac{\gamma L^2}{2} + \gamma^2 L^3\right)\frac{1}{n}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t - \mathbf{x}_t^{(i)}\right\|^2$$

Proof of Lemma (Difference)

Note that if t is a multiple of τ, then $R_t = 0$ and there is nothing to prove. Otherwise note that

$$\begin{aligned}\mathbb{E}R_{t+1} &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\bar{\mathbf{x}}_{t+1} - \mathbf{x}_{t+1}^{i}\right\|^2 = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\bar{\mathbf{x}}_{t} - \mathbf{x}_{t}^{i} + \gamma\mathbf{g}_t^i - \gamma\bar{\mathbf{g}}_t\right\|^2 \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\bar{\mathbf{x}}_{t} - \mathbf{x}_{t}^{i} + \gamma\nabla f_i(\mathbf{x}_t^i) - \gamma\bar{\mathbf{v}}_t\right\|^2 + \gamma^2\sigma^2\,,\end{aligned}$$

where $\bar{\mathbf{g}}_t := \frac{1}{n}\sum_{i=1}^{n}\mathbf{g}_t^i$ and $\bar{\mathbf{v}}_t := \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\mathbf{x}_t^{(i)})$ denote the average of the client gradients. With the inequality $\|\mathbf{a}+\mathbf{b}\|^2 \leq (1+\tau^{-1})\|\mathbf{a}\|^2 + 2\tau\|\mathbf{b}\|^2$ for $\tau \geq 1$, $\mathbf{a},\mathbf{b}\in\mathbb{R}^d$, we continue:

$$\begin{aligned}\mathbb{E}R_{t+1} &\leq \left(1+\frac{1}{\tau}\right)\mathbb{E}R_t + \frac{2\tau\gamma^2}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f(\mathbf{x}_t^{(i)}) - \bar{\mathbf{v}}_t\right\|^2 + \gamma^2\sigma^2 \\ &\leq \left(1+\frac{1}{\tau}\right)\mathbb{E}R_t + \frac{2\tau\gamma^2}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(\mathbf{x}_t^{(i)})\right\|^2 + \gamma^2\sigma^2 \\ &\leq \left(1+\frac{1}{\tau}\right)\mathbb{E}R_t + \frac{2\tau\gamma^2}{n}\sum_{i=1}^{n}\mathbb{E}\left(2\left\|\nabla f_i(\bar{\mathbf{x}}_t)\right\|^2 + 2\left\|\nabla f_i(\mathbf{x}_t^{(i)}) - \nabla f_i(\bar{\mathbf{x}}_t)\right\|^2\right) + \gamma^2\sigma^2\end{aligned}$$

$(*)$

Continued

We now use $f_1 = f_2 = \cdots = f_n$ and $\gamma \leq \frac{1}{20L\tau}$ to simplify:

$$\begin{aligned}
\mathbb{E}R_{t+1} &\leq \left(1 + \frac{1}{\tau}\right)\mathbb{E}R_t + \frac{1}{100L^2\tau}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \frac{1}{100\tau}\mathbb{E}R_t + \gamma^2\sigma^2 \\
&\leq \left(1 + \frac{3}{2\tau}\right)\mathbb{E}R_t + \frac{1}{100L^2\tau}\mathbb{E}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \gamma^2\sigma^2
\end{aligned}$$

The lemma now follows by unrolling, and noting that $\left(1 + \frac{3}{2\tau}\right)^j \leq 5$ for all $0 \leq j \leq \tau$.

Lecture 7 Recap

- Federated Learning
  - studied the convergence properties of local SGD
  - in practice: FedAvg

- Homogeneous/IID optimization setting
  - might not be realistic for real-world applications!

Bibliography I

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