Lecture 4:

Image Transformations I:
Continuous Fourier Transform

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Recently on IPCV ...

- visible light: **electromagnetic waves** that illuminate objects
- Human **colour perception** is based on three cone types.
- They correspond to the three primary colours red, green, and blue.
  Their additive blend creates other colour impressions.
- technical realisation: **RGB** colour space.
- for printers and copiers: substractive **CMY** and **CMYK** colour spaces
- **HSV** colour space: cylindrical representation, useful for illumination changes
- **YCbCr**: separates luma from chroma information, useful later for compression

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Motivation (1)

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This lecture is the first of a series of four lectures that take you to a desert ride. At first glance, the desert looks dry and dusty, but you will see structures much clearer. Your math knowledge is your survival kit. Source: `https://morocco-places.com`.

Motivation (2)

Motivation

- Transformations are useful for analysing the properties of images and processing them in an efficient way.
- For analysing audio signals, a decomposition into their frequencies is very natural.
- Similar things can be done with images, which can be seen as 2-D signals:
  - They can be decomposed into their frequency content in x- and y-direction.
  - useful for many things, e.g. filter design and fast convolution algorithms
- **Fourier transform**: Represents signals in terms of their frequencies.
  - It can be regarded as a change of basis.
  - Trigonometric basis functions represent different frequencies.
- To understand this, we first have to remember two concepts:
  - complex numbers for writing trigonometric functions more compactly
  - representations of vectors in orthonormal bases

Motivation (3)

Joseph Fourier (1768–1830) did not only discover the so-called Fourier transform, he also introduced the diffusion equation, made archeological discoveries in Egypt, and acted as a prefect in Grenoble. Source: `http://de.wikipedia.org/wiki/Joseph_Fourier`.

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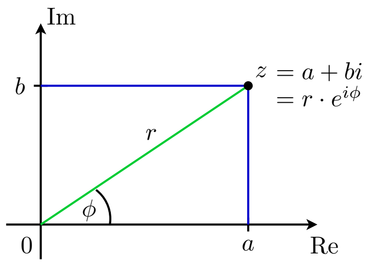
Complex Numbers

- The set $\mathbb{C}$ of complex numbers extends the set $\mathbb{R}$ of real numbers.
  It allows to compute the square root of a negative number.
- A complex number $z = a + bi \in \mathbb{C}$ consists of two parts:
  a *real part* $\mathrm{Re}(z) := a \in \mathbb{R}$ and an *imaginary part* $\mathrm{Im}(z) := b \in \mathbb{R}$.
  The number i denotes the *imaginary unit*. It satisfies $i^2 = -1$.
- The *conjugate complex number* of $z = a + bi$ is defined as $\bar{z} := a - bi$.
- $a + bi$ may also be identified with a vector $(a, b)^\top \in \mathbb{R}^2$.
  This is its so-called *Cartesian representation*.



A complex number in Cartesian and polar form (see later). Later on we will need that $\cos\phi = \frac{a}{r}$.
Author: M. Mainberger.

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Complex Numbers (2)

- Addition and subtraction of complex numbers is done componentwise:

$$(a+bi) \;+\; (c+di) \;=\; (a+c) \;+\; (b+d)\,i\,,$$
$$(a+bi) \;-\; (c+di) \;=\; (a-c) \;+\; (b-d)\,i\,.$$

- Multiplication uses $i^2 = -1$:

$$\begin{aligned}(a+bi)\cdot(c+di) &= ac+adi+bci+bdi^2\\ &= (ac-bd) \;+\; (ad+bc)\,i\,.\end{aligned}$$

- Division expands the fraction by the complex conjugate of the denominator:

$$\frac{a+bi}{c+di} = \frac{(a+bi)\cdot(c-di)}{(c+di)\cdot(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i\,.$$

- The *norm (modulus, magnitude)* of a complex number $z = a+ib$ is given by

$$|z| \;:=\; \sqrt{z\bar{z}} \;=\; \sqrt{(a+bi)\,(a-bi)} \;=\; \sqrt{a^2-b^2\,i^2} \;=\; \sqrt{a^2+b^2}\,.$$

It coincides with the Euclidean norm of the vector $(a,b)^\top$ in Cartesian form.
Note that $\sqrt{zz} = \sqrt{a^2+2abi-b^2}$ would not give the desired result $\sqrt{a^2+b^2}$.

Complex Numbers (3)

- With the power series representations

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}\,, \qquad \cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\,, \qquad \sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

one obtains the very important *Euler's formula*

$$e^{i\phi} \;=\; \cos\phi \;+\; i\sin\phi\,.$$

- Euler's formula implies that $e^{i\phi}$ lies on the unit circle,

$$|e^{i\phi}| \;=\; \sqrt{\cos^2\phi+\sin^2\phi} \;=\; 1 \qquad \forall\,\phi\,,$$

and is 2π-periodic:

$$e^{i(\phi+2k\pi)} \;=\; e^{i\phi} \qquad \forall\,k\in\mathbb{Z}\,.$$

- Writing down Euler's formula for ϕ and $-\phi$ and remembering that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$ yields two useful results:

$$\cos\phi \;=\; \frac{e^{i\phi}+e^{-i\phi}}{2}\,, \qquad \sin\phi \;=\; \frac{e^{i\phi}-e^{-i\phi}}{2i}\,.$$

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Complex Numbers (4)

- One can express any nonzero complex number $z = a + bi$ in its *polar form*

$$z \;=\; r\, e^{i\phi}$$

with radius $r := |z| = \sqrt{a^2+b^2}$ and *argument* (angle between z and real axis)

$$\phi \;:=\; \arg(z) \;:=\; \begin{cases} \arccos\left(\frac{a}{r}\right) & \text{if } b \geq 0, \\ -\arccos\left(\frac{a}{r}\right) & \text{if } b < 0. \end{cases}$$

This representation gives a polar angle $\phi \in (-\pi, \pi]$.
(Note that the $\arccos$ function always yields values in $[0, \pi]$.)

- The polar form is convenient for multiplications,

$$z_1 z_2 \;=\; |z_1|\,|z_2|\, e^{i(\phi_1+\phi_2)},$$

and for raising a complex number to some power p:

$$z^p \;=\; |z|^p\, e^{ip\phi}.$$

In polar form, the complex conjugate of $z \;=\; r\, e^{i\phi}$ is given by $\bar{z} \;=\; r\, e^{-i\phi}$.

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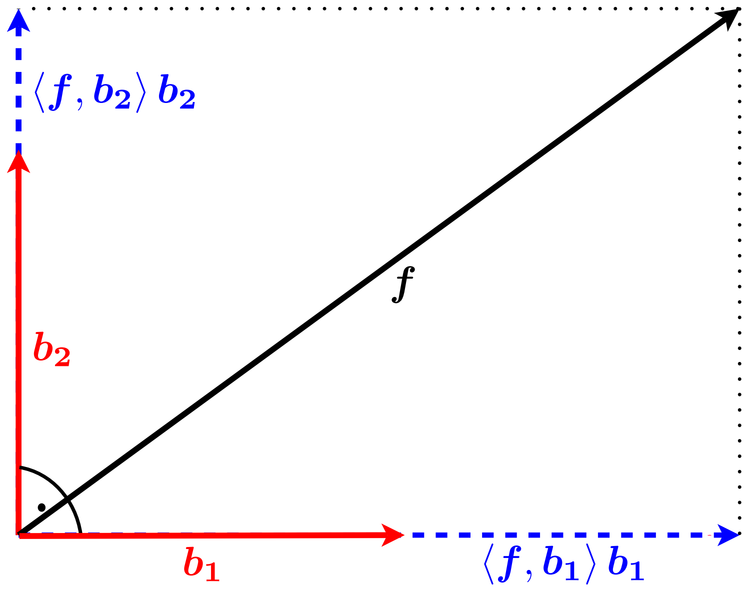
Prerequisites from Linear Algebra



A vector $\boldsymbol{f} \in \mathbb{R}^2$ can be represented in an arbitrary orthonormal (orthogonal with norm 1) basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{R}^2$ by the formula $\boldsymbol{f} = \langle \boldsymbol{f}, \boldsymbol{b_1} \rangle \, \boldsymbol{b_1} + \langle \boldsymbol{f}, \boldsymbol{b_2} \rangle \, \boldsymbol{b_2}$. This insight can be generalised. It will be very useful for us in this and the next three lectures. Author: P. Peter.

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- Expressing a vector $\boldsymbol{f} \in \mathbb{R}^N$ in an orthonormal basis $\{\boldsymbol{b_1}, ..., \boldsymbol{b_N}\}$ of $\mathbb{R}^N$ gives

$$\boldsymbol{f} \;=\; \sum_{k=1}^{N} \langle \boldsymbol{f}, \boldsymbol{b_k} \rangle \, \boldsymbol{b_k}$$

  where $\langle ., . \rangle$ denotes the *Euclidean inner product (euklidisches Skalarprodukt):*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{j=1}^{N} f_j \, g_j .$$

  The coefficient $\langle \boldsymbol{f}, \boldsymbol{b_k} \rangle$ quantifies the projection of $\boldsymbol{f}$ onto the basis vector $\boldsymbol{b_k}$: It tells us how much of the vector $\boldsymbol{b_k}$ is contained in $\boldsymbol{f}$.

- For a complex-valued vector $\boldsymbol{f} \in \mathbb{C}^N$, one must use the *Hermitian inner product*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{j=1}^{N} f_j \, \bar{g}_j$$

  where $\bar{g}_j$ is the complex conjugate of g_j.
  Complex conjugation allows to define the norm of $\boldsymbol{f}$ via $|\boldsymbol{f}| := \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle}$.

Prerequisites from Linear Algebra (3)

- An N-dimensional vector $\boldsymbol{f} \in \mathbb{R}^N$ has N components $\{f_j \mid j = 1, ..., N\}$.
  A function $f : \mathbb{R} \to \mathbb{R}$ has infinitely many values $\{f(x) \mid x \in \mathbb{R}\}$.
  In this sense it can be interpreted as an "infinite-dimensional vector".
- Representing $f : \mathbb{R} \to \mathbb{R}$ with an infinite set of orthonormal basis functions $\{b_u \,|\, u \in \mathbb{R}\}$ yields

$$f \;=\; \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$$

  with the inner product

$$\langle f, g \rangle \;:=\; \int_{\mathbb{R}} f(x) \, g(x) \, dx.$$

  The coefficient $\langle f, b_u \rangle$ quantifies the projection of f onto the basis function b_u.
  It measures how much of the function b_u is contained in f.
- For complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ one uses the inner product

$$\langle f, g \rangle \;:=\; \int_{\mathbb{R}} f(x) \, \bar{g}(x) \, dx.$$

  Note again the complex conjugation.

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Continuous Fourier Transform in 1-D

Goals

- decompose a signal into its frequency components
- compute convolutions in a highly efficient way
- have an ideal tool for analysing and designing linear shift-invariant filters (later)

Basic Intuition

- express a 1-D signal $f : \mathbb{R} \to \mathbb{R}$ in a specific basis $\{b_u \mid u \in \mathbb{R}\}$:

$$f \;=\; \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$$

- choose basis functions $\{b_u \mid u \in \mathbb{R}\}$ such that they represent all frequencies u
- coefficient $\langle f, b_u \rangle$ measures the content of the frequency u within f

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How to Choose the Basis Functions

- To represent a function f in terms of its frequencies, a natural idea would be to use trigonometric functions of cosine and sine type:

$$\begin{aligned} c_u(x) \;&=\; \cos(2\pi u x), \\ s_u(x) \;&=\; \sin(2\pi u x), \end{aligned}$$

  where u denotes the frequency (number of oscillations within x-interval $[0, 1]$).
- Since we are lazy, we combine $c_u(x)$ and $s_u(x)$ with Euler's formula:
  We use them as real and imaginary part of the complex-valued function

$$b_u(x) \;=\; c_u(x) + i\, s_u(x) \;=\; e^{2\pi i u x}.$$

  This creates our set $\{b_u \,|\, u \in \mathbb{R}\}$ of basis functions.

Continuous Fourier Transform in 1-D (3)

The Fourier Transform

- The Fourier transform yields the coefficient $\hat{f}(u) := \langle f, b_u \rangle$ for each frequency u. This coefficient measures the contribution of a frequency u to the signal f.
- The *Fourier transform (FT)* of a 1-D function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\boxed{\hat{f}(u) \;:=\; \mathcal{F}[f](u) \;:=\; \langle f, b_u \rangle \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx\,.}$$

- Note that the minus sign comes from the complex conjugation of $b_u(x) = e^{2\pi iux}$.
- $\{\hat{f}(u) \,|\, u \in \mathbb{R}\}$ is the desired representation of the function $\{f(x) \,|\, x \in \mathbb{R}\}$ in the *frequency domain (Fourier domain).*

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Continuous Fourier Transform in 1-D (4)

Remarks on the Complex-Valuedness

The Fourier transform is complex-valued:

- $\mathrm{Re}(\hat{f}(u))$ measures the contribution of the function $c_u(x) = \cos(2\pi ux)$ to $f(x)$. It vanishes for odd signals $f(x)$ ($f(x) = -f(-x)$), since cosine is even.
- $\mathrm{Im}(\hat{f}(u))$ measures the contribution of the function $s_u(x) = \sin(2\pi ux)$ to $f(x)$. It vanishes for even signals $f(x)$, i.e. $f(x) = f(-x)$, since the sine is odd.

The Polar Form and Fourier Spectra

- The polar form of $\hat{f}(u)$ gives useful insights.
- The magnitude $\left|\hat{f}(u)\right|$ is called *Fourier spectrum (Fourierspektrum)*.
  It expresses the total importance of the frequency u within the signal f.
- The angle $\phi(u) = \arg\left(\hat{f}(u)\right)$ is called the *phase angle (Phasenwinkel)*.
  It characterises the phase shift relative to a cosine function:
  - $\phi(u) = 0$ corresponds to the pure cosine function $c_u(x) = \cos(2\pi ux)$.
  - $\phi(u) = \frac{\pi}{2}$ corresponds to the pure sine function $s_u(x) = \sin(2\pi ux)$.
  - An arbitrary angle $\phi(u)$ corresponds to the function $t_{u,\phi}(x) = \cos(2\pi ux - \phi)$.
- Often one is only interested in the Fourier spectrum $\left|\hat{f}(u)\right|$.
  Also the so-called *power spectrum (Powerspektrum)* $\left|\hat{f}(u)\right|^2$ is popular.

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Continuous Fourier Transform in 1-D (5)

The Inverse Fourier Transform

- The goal of the inverse Fourier transform is to synthesise the signal f from its Fourier coefficients $\hat{f}(u) = \langle f, b_u \rangle$.
- Thus, we have to use the formula $f = \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$ with $b_u = e^{2\pi iux}$.
- The *inverse 1-D Fourier transform* of $\hat{f}(u)$ is defined as

$$\boxed{f(x) \;=\; \mathcal{F}^{-1}[\hat{f}](x) \;:=\; \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux}\, du\,.}$$

- Note that there is no minus sign here.

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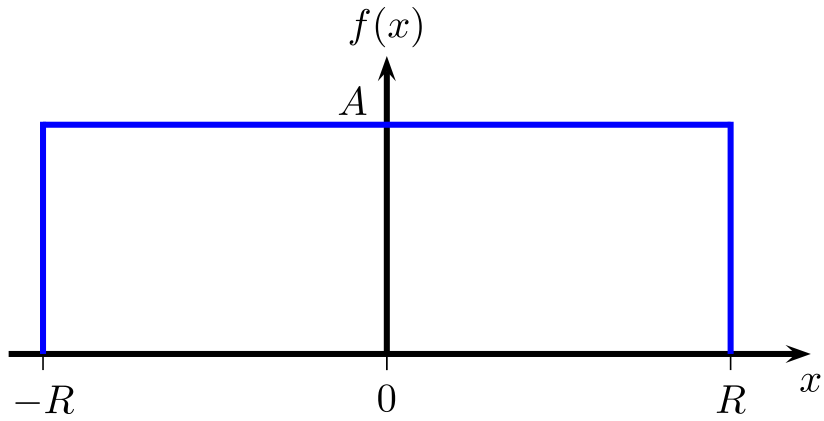
Example: Fourier Transform of a Box Function



A box function. Author: M. Mainberger.

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- For $u \neq 0$, the Fourier transform of this box function is given by

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx = \int_{-R}^{R} A\, e^{-i2\pi ux}\, dx \\
&= A \left[\frac{-1}{i2\pi u}\, e^{-i2\pi ux} \right]_{-R}^{R} \\
&= \frac{-A}{i2\pi u} \left(e^{-i2\pi uR} - e^{i2\pi uR} \right) = \frac{A}{i2\pi u} \left(e^{i2\pi uR} - e^{-i2\pi uR} \right) \\
&= \frac{A}{i2\pi u}\, 2i\, \sin(2\pi uR) = \frac{A}{\pi u}\, \sin(2\pi uR)\,.
\end{aligned}$$

- For $u = 0$, we directly get

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi 0x}\, dx = A\, [x]_{-R}^{R} = 2RA\,.$$

Continuous Fourier Transform in 1-D (8)

- Note that $\hat{f}(u)$ is real-valued, since $f(x)$ is an even function.
- For $u \neq 0$, the Fourier spectrum is given by

$$\left|\hat{f}(u)\right| = \left|\frac{A}{\pi u}\right| \, |\sin(2\pi u R)| = 2RA \left|\frac{\sin(2\pi u R)}{2\pi u R}\right|$$

- Furthermore we have $\left|\hat{f}(0)\right| = 2RA$.
- With the so-called *sinc function*

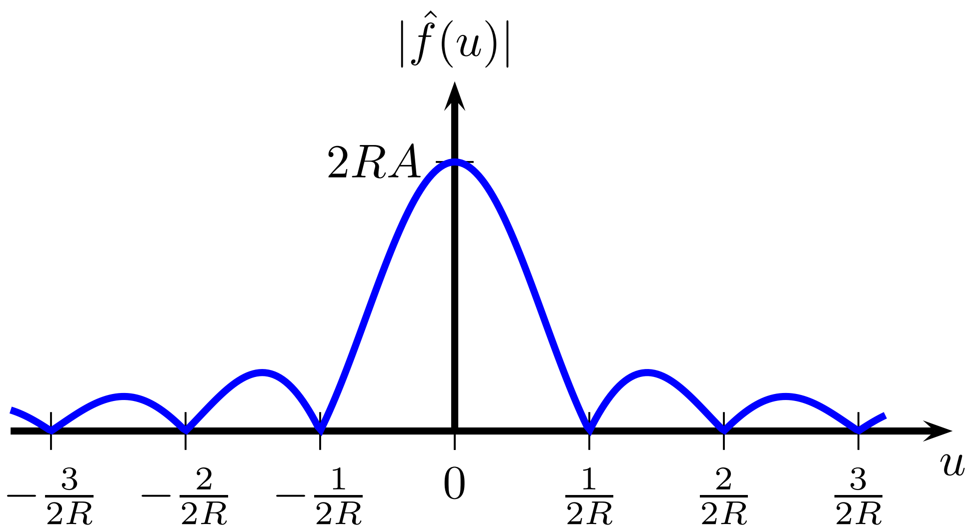
$$\operatorname{sinc}(x) \ := \begin{cases} \frac{\sin(x)}{x} & x \neq 0\,, \\ 1 & x = 0\,. \end{cases}$$

we obtain

$$\left|\hat{f}(u)\right| = 2RA\,|\operatorname{sinc}(2\pi u R)|$$

- The sinc function satisfies $\operatorname{sinc}(0) = 1$ and $\operatorname{sinc}(k\pi) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$.
- Thus, $\left|\hat{f}(u)\right|$ satisfies $\left|\hat{f}(0)\right| = 2RA$ and $\left|\hat{f}(\frac{k}{2R})\right| = 0$ with $k \in \mathbb{Z} \setminus \{0\}$.

Continuous Fourier Transform in 1-D (9)



Fourier spectrum $\left|\hat{f}(u)\right|$ of the box function. Author: M. Mainberger.

Remark:

- While the box function $f(x)$ has finite extent in the spatial domain, its Fourier transform $\hat{f}(u)$ has infinite extent in the frequency domain.

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Continuous Fourier Transform in 2-D

Definition

- The *Fourier transform (FT)* of a 2-D function $f(x,y)$ is defined as

$$\hat{f}(u,v) \;:=\; \mathcal{F}[f](u,v) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

  where u and v are the frequencies in x- and y-direction.

- The *inverse 2-D Fourier transform* is given by

$$f(x,y) \;=\; \mathcal{F}^{-1}[\hat{f}](x,y) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u,v)\, e^{i2\pi(ux+vy)}\, du\, dv.$$

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Don’t Fear High Dimensions!

- In higher dimensions the definition proceeds in the same way.
- Because of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi ux}\, dx\right) e^{-i2\pi vy}\, dy$$

it follows that the Fourier transform is *separable:*

- First compute the Fourier transform in x-direction.
  Then apply the Fourier transform in y-direction to this result.
- An m-dimensional Fourier transform is computed via a sequence of m one-dimensional transforms.
- This is computationally very nice and greatly reduces the workload.

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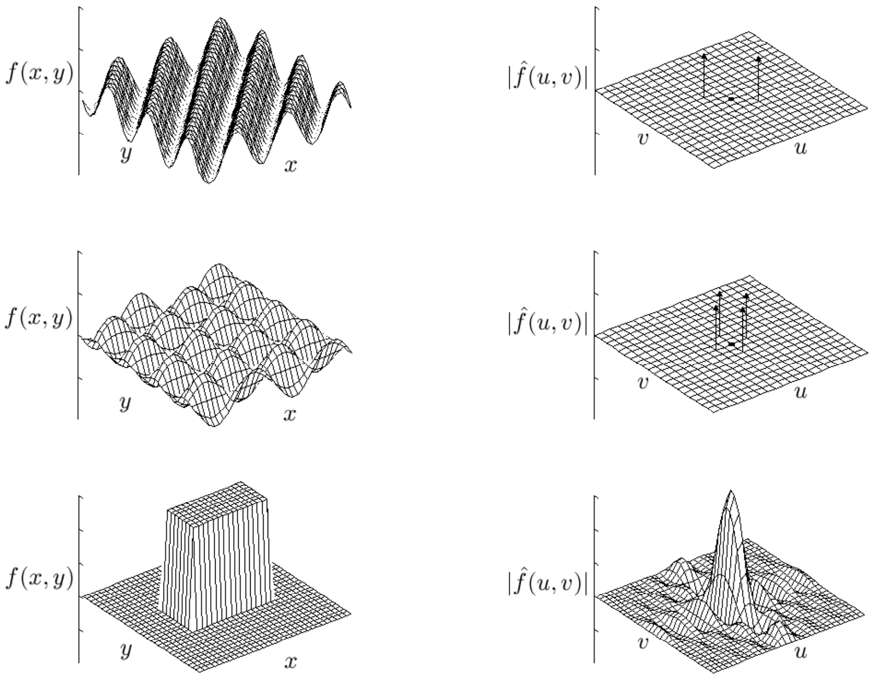
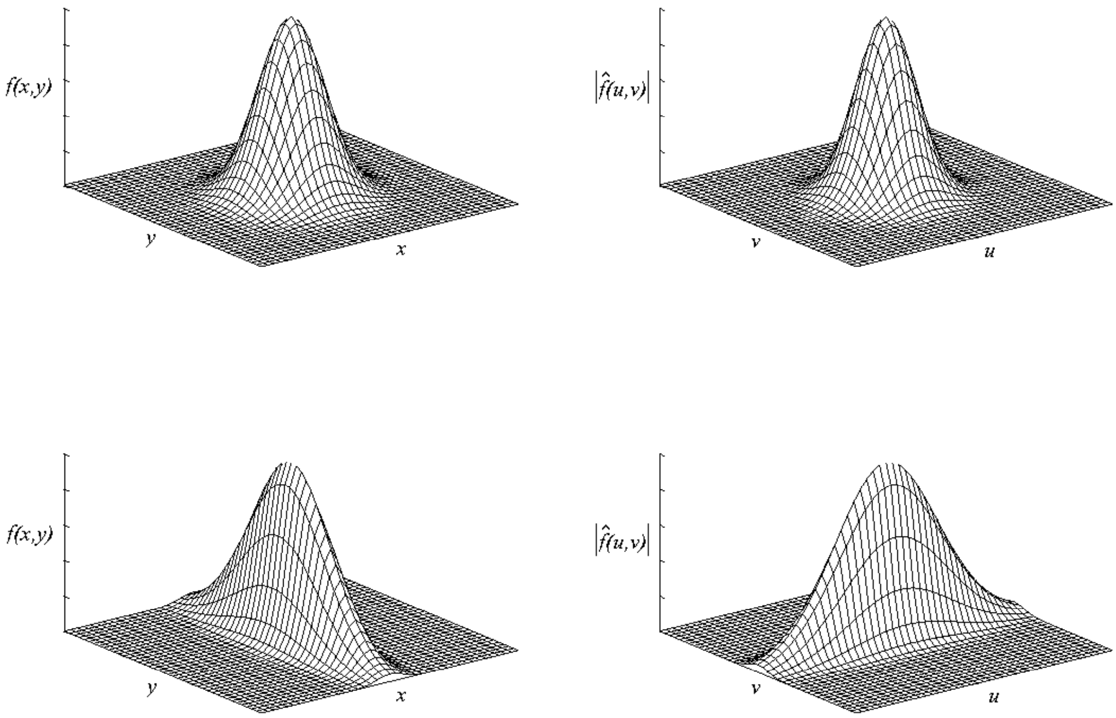


Fourier spectra of some 2-D functions. Author: N. Khan.

Continuous Fourier Transform in 2-D (4)



Further 2-D Fourier spectra. Author: N. Khan.

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Image Transformations I: Continuous Fourier Transform

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Properties of the Continuous Fourier Transform

- **Linearity**

  Let f and g be functions and $a, b \in \mathbb{R}$.
  Then the Fourier transform satisfies the *superposition principle:*

$$\mathcal{F}[af + bg] \;=\; a\mathcal{F}[f] + b\mathcal{F}[g]\,.$$

- **Similarity Theorem**

$$\mathcal{F}[f(ax, by)](u, v) \;=\; \frac{1}{|ab|}\,\mathcal{F}[f]\left(\frac{u}{a}, \frac{v}{b}\right) \qquad \forall\, a, b \in \mathbb{R} \setminus \{0\}.$$

  Elongation in the spatial domain gives shortening in the Fourier domain:
  Both domains are reciprocal.

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- **Differentiation**

$$\mathcal{F}\left[\frac{\partial^{n+m} f}{\partial x^n\, \partial y^m}\right](u, v) \;=\; (i2\pi u)^n\,(i2\pi v)^m\,\mathcal{F}[f](u, v).$$

  Differentiation gives multiplication with the frequency in the Fourier domain.
  Thus, high frequent components (e.g. noise) are amplified!

- **Shift Theorem**

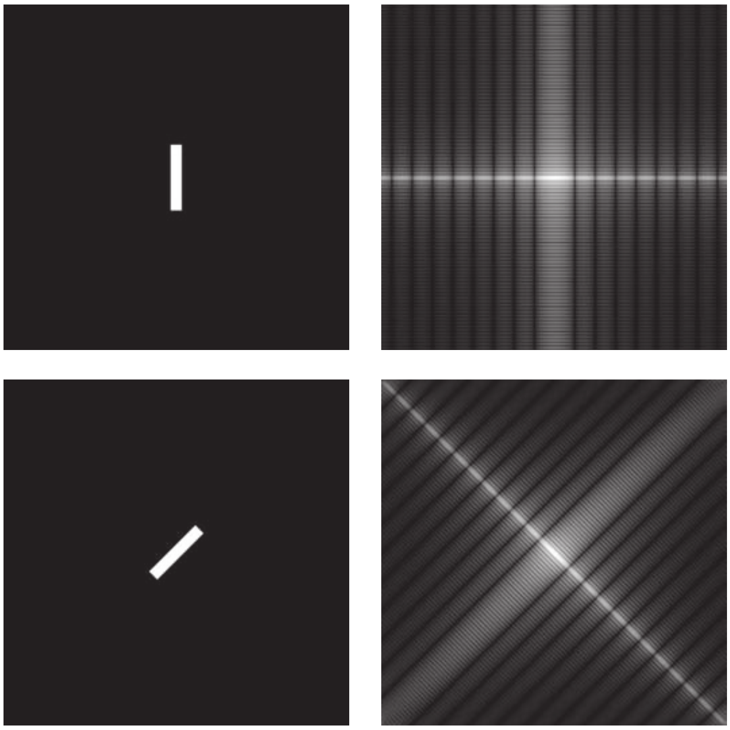
$$\mathcal{F}[f(x - x_0,\, y - y_0)](u, v) \;=\; e^{-i2\pi(ux_0 + vy_0)}\,\mathcal{F}[f](u, v)$$

  Shift in the spatial domain rotates the phase angle in the Fourier domain.
  The Fourier *spectrum*, however, is not affected, since $\left|e^{-i2\pi(ux_0+vy_0)}\right| \;=\; 1.$
  Thus, the Fourier spectrum is shift-invariant.

- **Rotation Invariance**

  If the image is rotated, its FT is rotated by the same angle.

Properties of the Continuous Fourier Transform (3)



Rotation invariance of the Fourier transform. **(a) Top left:** Original image. **(b) Top right:** Its Fourier spectrum. **(c) Bottom left:** Rotated image. **(d) Bottom right:** Its Fourier spectrum. Authors: R. C. Gonzalez, R. E. Woods.

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Properties of the Continuous Fourier Transform (4)

- **Convolution Theorem**

  The convolution of two functions $f(x,y)$ and $g(x,y)$ is given by (cf. Lecture 2)

  $$(f * g)(x,y) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x', y-y')\, g(x',y')\, dx'\, dy'.$$

  It will be fundamental for all shift-invariant linear filters (Lecture 11). Take e.g.

  $$g(x,y) \;:=\; \begin{cases} \frac{1}{\pi r^2} & \text{for } x^2+y^2 \le r^2, \\ 0 & \text{else.} \end{cases}$$

  Then $f * g$ smoothes the image f by averaging all grey values within a neighbourhood of radius r. Computing this integral is expensive if r is large.

  However, convolution is easily computed as multiplication in the Fourier domain:

  $$\boxed{\mathcal{F}[f * g] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[g]}$$

  Afterwards the results must be transformed back to the spatial domain.

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Properties of the Continuous Fourier Transform (5)

- **Fourier Transform of the Product of Two Functions**

  Multiplication of two functions in the spatial domain becomes convolution in the Fourier domain:

$$\mathcal{F}[f \cdot g] \;=\; \mathcal{F}[f] * \mathcal{F}[g].$$

  This is the reciprocal convolution theorem.
  Computationally this is not advantageous.
  However, it will help us to understand sampling (more details in Lecture 5).

- **Fourier Transform of a Gaussian**

  gives a Gaussian-like function with reciprocal variance. In 2D:

$$f(x,y) \;:=\; \frac{1}{2\pi\sigma^2} \exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right) \;\Longrightarrow\; \hat{f}(u,v) \;=\; \exp\left(\frac{-(2\pi)^2(u^2+v^2)}{2\sigma^{-2}}\right).$$

  However, the Gaussian is not the only function that is invariant under the FT:

Properties of the Continuous Fourier Transform (6)

- **Fourier Transform of a Delta Comb**

  A (continuous) *delta pulse* δ is a model for an infinitely sharp peak
  (think e.g. of a Gaussian with standard deviation $\sigma \to 0$).
  It is centred in 0, and its area (integral) is 1:

$$\int_{-\infty}^{\infty} \delta(x)\, dx \;=\; 1.$$

  The integral of a function f times a delta pulse δ evaluates f in 0,

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx \;=\; f(0),$$

  while a shifted delta pulse $\delta(. - x_0)$ yields $f(x_0)$:

$$\int_{-\infty}^{\infty} f(x)\, \delta(\underbrace{x - x_0}_{x'})\, dx \;=\; \int_{-\infty}^{\infty} f(x' + x_0)\, \delta(x')\, dx' \;=\; f(x_0).$$

Properties of the Continuous Fourier Transform (7)

Consider an infinitely extended comb of delta pulses with peak distance λ. Then its FT is a delta comb with reciprocal peak distance $1/\lambda$:

$$g(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\lambda) \quad \Longrightarrow \quad \hat{g}(u) = \sum_{k=-\infty}^{\infty} \delta\left(u - \frac{k}{\lambda}\right).$$

This formula is important for the sampling a continuous signal f: Sampling f means computing the product of f with a delta comb g.

Summary

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- The continuous Fourier transform analyses the frequency content of images.
- It is complex-valued, linear, separable, and invariant under rotations.
- Spatial and Fourier domain are reciprocal w.r.t. localisation and orientation.
- Spatial shifts become phase shifts.
  The Fourier spectrum remains unchanged.
- Differentiation becomes multiplication with the frequency.
- Convolution in one domain becomes multiplication in the other.
- The Fourier transform maps
  - box functions to sinc functions,
  - Gaussians to Gaussians with reciprocal variance,
  - delta combs to delta combs with reciprocal distance.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33

References

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  (the classical reference when you want to learn the full story about the FT)