Lecture 12: Linear Filters II: Derivative Filters

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Recently on IPCV ...

- Linear shift invariant (LSI) filters are fully characterised by their impulse response.
- They are equivalent to convolutions.
- Fourier transform is useful for computing, analysing, and designing LSI filters.
- Lowpass filters allow to smooth the data
- Highpass filters eliminate low-frequent perturbations.
  The can also be used for sharpening image structures.
- Bandpass filters allow to extract features at certain scales.

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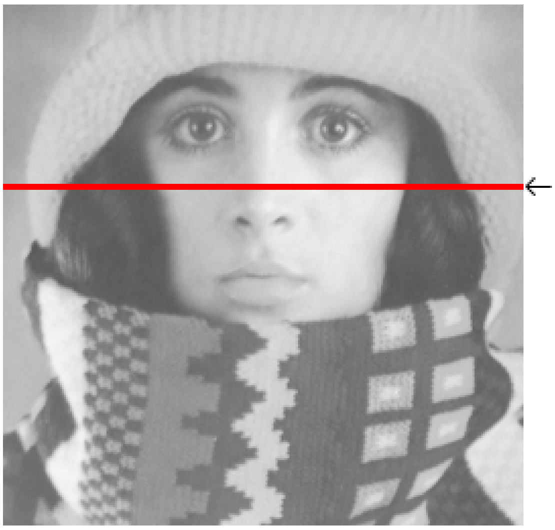
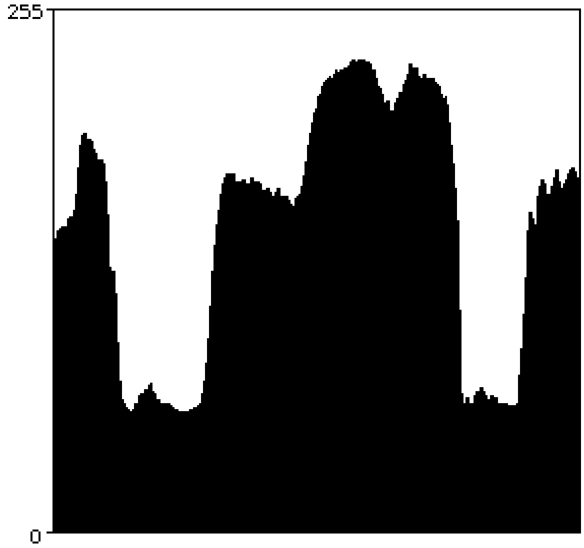
Introduction

Why Do We Need Derivative Filters ?

- Derivative filters are very important examples of linear shift invariant filters.
- Derivatives tell us something about local grey value changes in images.
- They can be used for detecting semantically important image features such as edges (next lecture).
- To compute derivatives in images, we need some mathematical insights:
  - What are the dangers when we want to compute derivatives?
  - What are useful derivative expressions for a continuous image?
  - How can we make these ideas applicable to discrete images?

Introduction (2)





Left: Image of size 256×256, from which a 1-D signal along a horizontal scanline has been extracted. **Right:** Intensity profile along this scanline. The largest intensity jumps mark the boundaries of the hair region. Author: T. Schneevoigt.

Outline

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Why is it Dangerous to Compute Derivatives?

Explanation in the Spatial Domain

- Small, but high-frequent fluctuations in the original signal can create very large perturbations in its derivatives.
- Example: The high-frequent 1-D perturbation

$$f(x) = \varepsilon \sin\left(\frac{x}{\varepsilon^2}\right)$$

becomes arbitrarily small in magnitude for $\varepsilon \to 0$. However, its derivative

$$f'(x) = \frac{1}{\varepsilon} \cos\left(\frac{x}{\varepsilon^2}\right)$$

exceeds all bounds !!!

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Explanation in the Fourier Domain

- Lecture 4:
  Differentiation in the spatial domain creates multiplication with the frequency in the Fourier domain:

$$\mathcal{F}\left[\frac{\partial^{n+m} f}{\partial x^n \, \partial y^m}\right](u,v) \;=\; (i2\pi u)^n \, (i2\pi v)^m \, \mathcal{F}[f](u,v)\,.$$

- Thus, high-frequent perturbations (e.g. noise) are massively amplified!

Remedy

- Perform lowpass filtering before computing derivatives.
- This damps the dangerous high frequent components.
- frequently used: Gaussian convolution (Lecture 11)
- Derivatives of Gaussian smoothed images are called *Gaussian derivatives.*

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Useful Concepts from Calculus in 2D

Partial Derivatives

- Consider a sufficiently smooth function $f(x, y)$ of the variables x and y.
  Then we can compute its *partial derivatives* w.r.t. x or y.
  To this end, one regards it as a function of a single variable.
  The other variable is treated like a constant.
- Example:

$$\begin{aligned} f(x,y) &= \sin(xy^2) + x^3, \\ \frac{\partial f}{\partial x}(x,y) &= y^2\cos(xy^2) + 3x^2, \\ \frac{\partial f}{\partial y}(x,y) &= 2xy\cos(xy^2)\,. \end{aligned}$$

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Useful Concepts from Calculus in 2D (2)

- Equivalent notations:

$$\frac{\partial f}{\partial x} = \partial_x f = f_x .$$

- Higher order partial derivatives can be computed consecutively:

$$\frac{\partial^2 f}{\partial x \, \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

- Under suitable smoothness assumptions (which we always assume to hold), we may exchange the order of partial differentiation:

$$f_{xy} = f_{yx} .$$

Useful Concepts from Calculus in 2D (3)

Nabla Operator

- The column vector of the partial derivatives is called *nabla operator* or *gradient:*

$$\boldsymbol{\nabla} := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

- Often it is possible to work with $\boldsymbol{\nabla}$ as if it were an ordinary vector.
  For instance, for a scalar-valued function $f(x, y)$, one gets

$$\boldsymbol{\nabla} f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}.$$

- $\boldsymbol{\nabla} f$ points in the direction of the steepest ascend of f.
- $|\boldsymbol{\nabla} f| = \sqrt{f_x^2 + f_y^2}$ is the slope in this direction.
  It is invariant under rotations.

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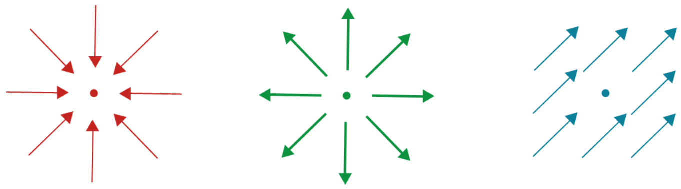
Divergence

- The inner product of the nabla operator and a vector-valued function $\boldsymbol{j}(x,y) = (j_1(x,y),\ j_2(x,y))^\top$ is called the *divergence (Divergenz)* of $\boldsymbol{j}$:

$$\operatorname{div} \boldsymbol{j} \;:=\; \boldsymbol{\nabla}^\top \boldsymbol{j} \;=\; (\partial_x, \partial_y) \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \;=\; \partial_x\, j_1 + \partial_y\, j_2\,.$$

- It measures the net outward flow through a small circle around the point.



Vector fields with negative, positive, and zero divergence.
From `https://www.khanacademy.org`.

- Thus, its sign allows to find sinks $(\operatorname{div} \boldsymbol{j} < 0)$ and sources $(\operatorname{div} \boldsymbol{j} > 0)$.
- We will need the divergence in Lecture 16 (Nonlinear Diffusion Filtering).

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Hessian

- The *Hessian (Hesse-Matrix)* of a scalar-valued function $f(x,y)$ is a 2×2 matrix that assembles all second-order partial derivatives:

$$\operatorname{Hess}(f) \;=\; \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

- If $\operatorname{Hess}(f)$ is positive definite everywhere, then f is strictly convex.

Laplacian

- The divergence of the gradient is called *Laplacian (Laplace-Operator):*

$$\Delta f \;:=\; \operatorname{div}(\boldsymbol{\nabla} f) \;=\; (\partial_x, \partial_y) \begin{pmatrix} f_x \\ f_y \end{pmatrix} \;=\; f_{xx} + f_{yy}\,.$$

- It can also be seen as the trace (sum of diagonal elements) of the Hessian.
- The Laplacian is invariant under rotations of f.
- It quantifies how much f differs from its average in a small neighbourhood.

Useful Concepts from Calculus in 2D (6)

Directional Derivatives

- Let a direction be characterised by a vector $\boldsymbol{v} = (c, s)^\top$ with norm 1.
- The *directional derivative (Richtungsableitung)* of f in direction $\boldsymbol{v}$ is given by

$$\partial_{\boldsymbol{v}} f \;=\; \boldsymbol{v}^\top \boldsymbol{\nabla} f \;=\; c\, f_x + s\, f_y\,.$$

  It measures the slope in the direction $\boldsymbol{v}$.

- The *second directional derivative* of f in direction $\boldsymbol{v}$ is computed as follows:

$$\begin{aligned} \partial_{\boldsymbol{vv}} f &= \boldsymbol{v}^\top \, \mathrm{Hess}(f)\, \boldsymbol{v} \\ &= (c, s) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \\ &= c^2 f_{xx} \,+\, 2cs\, f_{xy} \,+\, s^2 f_{yy}\,. \end{aligned}$$

  It tells us if f is curved upwards $(\partial_{\boldsymbol{vv}} f > 0)$ or downwards $(\partial_{\boldsymbol{vv}} f < 0)$ along $\boldsymbol{v}$.

Useful Concepts from Calculus in 2D (7)

Taylor Expansion

- **One-dimensional Taylor Expansion**

  A function $f : \mathbb{R} \to \mathbb{R}$ that is $n{+}1$ times continuously differentiable with bounded derivatives can be represented in $x{+}h$ by its Taylor expansion around x:

$$f(x+h) \;=\; \sum_{k=0}^{n} \frac{h^k}{k!}\, f^{(k)}(x) \;+\; \mathcal{O}(h^{n+1})\,.$$

- **Two-dimensional Taylor Expansion**

  A function $f : \mathbb{R}^2 \to \mathbb{R}$ that is $n{+}1$ times continuously differentiable with bounded derivatives can be represented in $\boldsymbol{x} + \boldsymbol{h}$ by

$$f(\boldsymbol{x} + \boldsymbol{h}) \;=\; \sum_{k=0}^{n} \frac{1}{k!} \langle \boldsymbol{h}, \boldsymbol{\nabla} \rangle^k f(\boldsymbol{x}) \;+\; \mathcal{O}(|\boldsymbol{h}|^{n+1})$$

  where we have e.g.

$$\begin{aligned} \langle \boldsymbol{h}, \boldsymbol{\nabla} \rangle^2 &= (h_1\, \partial_{x_1} + h_2\, \partial_{x_2})^2 \\ &= h_1^2\, \partial_{x_1 x_1} \,+\, 2\, h_1 h_2\, \partial_{x_1 x_2} \,+\, h_2^2\, \partial_{x_2 x_2}\,. \end{aligned}$$

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Numerical Approximation of Derivatives

For digital images, we must replace derivatives by discrete approximations.
These derivative approximations are called *finite differences.*
We find them with a Taylor expansion and a comparison of coefficients.

Example

Approximate the derivative f_i'' in pixel i with a stencil with pixels $i-1$, i, and $i+1$. The grid size is h. Compute the stencil weights.

- Let x_i denote the location of pixel i, and let $f_i := f(x_i)$ and $f_{i\pm1} := f(x_i \pm h)$.
- Then the Taylor expansions around pixel i are given by

$$
\begin{aligned}
f_{i-1} &= f_i - hf_i' + \tfrac{h^2}{2}f_i'' - \tfrac{h^3}{6}f_i''' + \tfrac{h^4}{24}f_i'''' - \tfrac{h^5}{120}f_i''''' + \mathcal{O}(h^6)\,, \\
f_i &= f_i\,, \\
f_{i+1} &= f_i + hf_i' + \tfrac{h^2}{2}f_i'' + \tfrac{h^3}{6}f_i''' + \tfrac{h^4}{24}f_i'''' + \tfrac{h^5}{120}f_i''''' + \mathcal{O}(h^6)\,.
\end{aligned}
$$

(We always assume that all required derivatives exist and are bounded.)

- We want to find the weights α_{-1}, α_0, and α_1 such that

$$f_i'' \;\approx\; \alpha_{-1}\, f_{i-1} \,+\, \alpha_0\, f_i \,+\, \alpha_1\, f_{i+1}\,.$$

- Using the Taylor expansions and comparing the coefficients in

$$\begin{aligned} 0\cdot f_i \,+\, 0\cdot f_i' \,+\, 1\cdot f_i'' &\overset{!}{=} \alpha_{-1}\, f_{i-1} \,+\, \alpha_0\, f_i \,+\, \alpha_1\, f_{i+1} \\ &= (\alpha_{-1}+\alpha_0+\alpha_1)\cdot f_i \\ &+ h\,(-\alpha_{-1}+\alpha_1)\cdot f_i' \\ &+ \tfrac{h^2}{2}\,(\alpha_{-1}+\alpha_1)\cdot f_i'' \,+\, \mathcal{O}(h^3) \end{aligned}$$

leads to a linear system in the unknown weights α_{-1}, α_0, and α_1:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{h^2} \end{pmatrix}.$$

- Its solution is given by $\alpha_{-1} = \frac{1}{h^2}$, $\alpha_0 = -\frac{2}{h^2}$, and $\alpha_1 = \frac{1}{h^2}$. This yields

$$f_i'' \;\approx\; \tfrac{1}{h^2}\, f_{i-1} \,-\, \tfrac{2}{h^2}\, f_i \,+\, \tfrac{1}{h^2}\, f_{i+1}.$$

How Accurate is This Approximation?

- Let us now study the error of our finite difference approximation of f_i''.
- Replacing f_{i-1} and f_{i+1} by their Taylor expansions gives

$$\begin{aligned} &\tfrac{1}{h^2}\, f_{i-1} \,-\, \tfrac{2}{h^2}\, f_i \,+\, \tfrac{1}{h^2}\, f_{i+1} \\ =\quad &\tfrac{1}{h^2}\left(f_i \,-\, hf_i' \,+\, \tfrac{h^2}{2} f_i'' \,-\, \tfrac{h^3}{6} f_i''' \,+\, \tfrac{h^4}{24} f_i'''' \,-\, \tfrac{h^5}{120} f_i''''' \,+\, \mathcal{O}(h^6)\right) \\ &-\tfrac{2}{h^2}\, f_i \\ &+\tfrac{1}{h^2}\left(f_i \,+\, hf_i' \,+\, \tfrac{h^2}{2} f_i'' \,+\, \tfrac{h^3}{6} f_i''' \,+\, \tfrac{h^4}{24} f_i'''' \,+\, \tfrac{h^5}{120} f_i''''' \,+\, \mathcal{O}(h^6)\right) \\ =\quad & f_i'' \,+\, \underbrace{\tfrac{h^2}{12}\, f_i'''' + \mathcal{O}(h^4)}_{\text{error}}. \end{aligned}$$

- Since f_i'''' is bounded, the error is quadratic in the grid size h. Thus, we have

$$f_i'' \;=\; \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \,+\, \mathcal{O}(h^2).$$

This is called an approximation with *consistency order (Konsistenzordnung)* 2.

Why is Consistency Important?

- The *consistency order* quantifies the error decay for finer sampling ($h \to 0$):
  - An order p corresponds to an $\mathcal{O}(h^p)$ error.
  - Higher consistency orders yield better accuracy.
- Discretisations must have at least consistency order 1, i.e. an error $\mathcal{O}(h)$:
  - Then the finite difference approximates the desired derivative for $h \to 0$.
  - Such an approximation is called *consistent.*
- *Inconsistent discretisations are unacceptable and useless:*
  - They do not approximate the continuous model at all.
  - An inconsistent algorithm does not solve the problem it was designed for.
- Note that consistency analysis describes the scaling behaviour for $h \to 0$.
  Thus, one should *always allow a variable grid size h* in all implementations.
  This is also important for problems beyond derivative approximations, e.g.
  - pyramids, where the grid size depends on the pyramid level.
  - 3D problems with different grid sizes in x-, y-, and z-direction.

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Derivative Filters in 1D

The Most Important Approximations

first derivative:

$$f_i' = \frac{f_{i+1} - f_i}{h} + \mathcal{O}(h) \qquad \text{forward difference}$$

$$f_i' = \frac{f_i - f_{i-1}}{h} + \mathcal{O}(h) \qquad \text{backward difference}$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + \mathcal{O}(h^2) \qquad \text{central difference}$$

second derivative:

$$f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + \mathcal{O}(h^2) \qquad \text{central difference}$$

Usually central differences have a higher order of consistency:
The symmetry causes cancellation effects of some Taylor coefficients.

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Filter Analysis in the Frequency Domain

- The continuous Fourier transform from Lecture 4 is also useful for analysing the frequency behaviour of linear shift invariant filters.
- Assume we want to analyse the frequency behaviour of the central difference approximation of f':

$$g(x) = \frac{1}{2h}\Big(f(x+h) - f(x-h)\Big)$$

  where h denotes the grid size. To this end, we express $\hat{g}$ in terms of $\hat{f}$.
- Using the linearity of the Fourier transform and the shift theorem we obtain

$$\begin{aligned}\hat{g}(u) &= \frac{1}{2h}\Big(\mathcal{F}[f(x+h)](u) - \mathcal{F}[f(x-h)](u)\Big)\\ &= \frac{1}{2h}\Big(e^{i2\pi hu} - e^{-i2\pi hu}\Big)\,\hat{f}(u).\end{aligned}$$

Derivative Filters in 1D (3)

- With $e^{i2\pi hu} - e^{-i2\pi hu} = 2i\,\sin(2\pi hu)$ we get the following frequency behaviour:

$$\hat{g}(u) \;=\; \frac{i}{h}\,\sin(2\pi hu)\,\hat{f}(u).$$

- With the Taylor expansion $\sin(2\pi hu) \approx 2\pi hu + \mathcal{O}(h^3u^3)$ we obtain

$$\hat{g}(u) \;=\; i2\pi u\,\hat{f}(u) \;+\; \mathcal{O}(h^2u^3).$$

- Remembering that $\mathcal{F}[f'](u) = i2\pi u\,\hat{f}(u)$ shows that

$$\hat{g}(u) \;=\; \mathcal{F}[f'](u) \;+\; \mathcal{O}(h^2u^3).$$

  Thus, the filter approximates the derivative f'.
- For a fixed frequency u, the approximation order is $\mathcal{O}(h^2)$.
  For a fixed grid size h, the approximation order is $\mathcal{O}(u^3)$.
  We see that the error increases cubically with the frequency!
- Thus, in contrast to a filter analysis in the spatial domain, the Fourier analysis gives also frequency-dependent results on the approximation quality.

Derivative Filters in 1D (4)

Improving the Order of Consistency with Larger Stencils

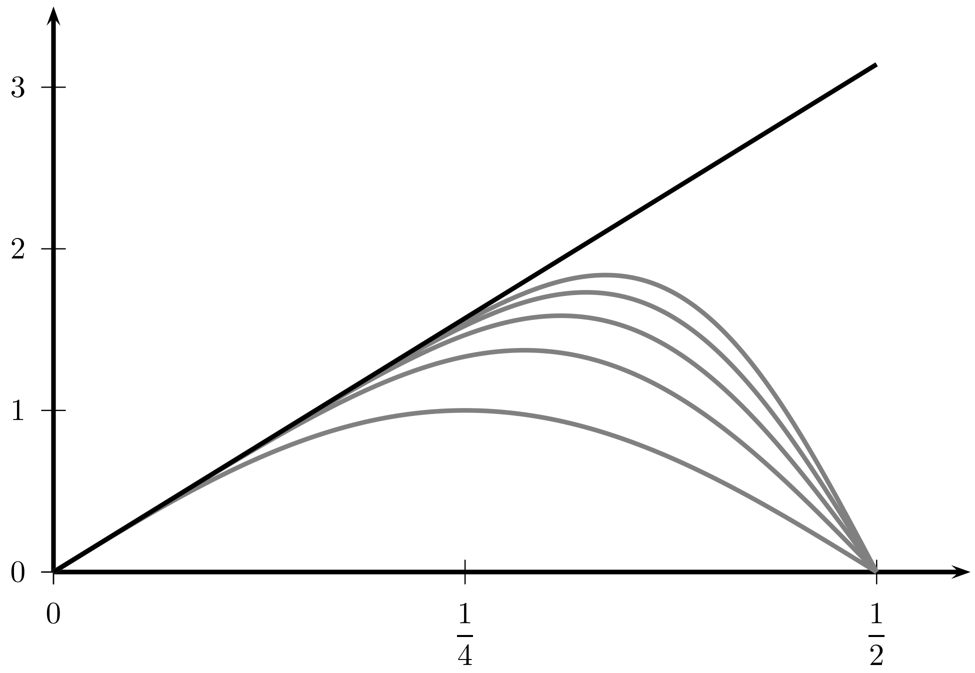
- By extending the stencil size, we can increase the order of consistency.
  However, this also increases the computational effort.
- Example: Central difference approximations of f' with stencil sizes 3, 5, and 7:

$$f_i' \;=\; \frac{f_{i+1} - f_{i-1}}{2h} \;+\; \mathcal{O}(h^2)\,,$$

$$f_i' \;=\; \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12h} \;+\; \mathcal{O}(h^4)\,,$$

$$f_i' \;=\; \frac{f_{i+3} - 9f_{i+2} + 45f_{i+1} - 45f_{i-1} + 9f_{i-2} - f_{i-3}}{60h} \;+\; \mathcal{O}(h^6)\,.$$

- The approximation quality can be visualised in the Fourier domain:
  - The ideal derivative operator gives $\hat{g}(u) \;=\; 2\pi iu\,\hat{f}(u) \;=:\; w(u)\,\hat{f}(u)$.
    Thus, the Fourier spectrum is amplified by $|w(u)| = |2\pi iu| = 2\pi|u|$.
  - The $\mathcal{O}(h^2)$ approximation amplifies the Fourier spectrum by $|w(u)| = \frac{1}{h}|\sin(2\pi hu)|$.



Amplification function $|w(u)|$ of the ideal derivative operator (black) and some of its central difference approximations (grey) for $h = 1$. The numerical approximations show a lowpass filter effect, i.e. there is some smoothing along the direction of the derivative. Increasing the mask size from 3 to 5, 7, 9, and 11 pixels one gets closer to the ideal derivative operator. Author: M. Mainberger.

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Derivative Filters in 2D

First-Order Derivatives

- In principle, the 1D masks can also be used in 2D.
- For the first order derivatives, the following stencils have consistency order 2:

$$\partial_x \approx \frac{1}{2h} \boxed{\begin{array}{c|c|c} -1 & 0 & 1 \end{array}}\,, \qquad \partial_y \approx \frac{1}{2h} \boxed{\begin{array}{c} 1 \\ \hline 0 \\ \hline -1 \end{array}}$$

(where the x axis goes from left to right, and the y axis from bottom to top)

- Problem:
  - Masks smooth in direction of derivative, but not orthogonal to it.
  - This suggests to introduce some smoothing perpendicular to the derivative direction, if one is interested in better isotropy.

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- The *Sobel operators* implement such a perpendicular smoothing.
  They convolve with the binomial kernel $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.
  This specific choice approximates the inherent smoothing of the 1D stencil:

$$\partial_x \approx \frac{1}{4} \boxed{\begin{array}{c} 1 \\ \hline 2 \\ \hline 1 \end{array}} * \frac{1}{2h} \boxed{\begin{array}{c|c|c} -1 & 0 & 1 \end{array}} = \frac{1}{8h} \boxed{\begin{array}{c|c|c} -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \end{array}}$$

$$\partial_y \approx \frac{1}{4} \boxed{\begin{array}{c|c|c} 1 & 2 & 1 \end{array}} * \frac{1}{2h} \boxed{\begin{array}{c} 1 \\ \hline 0 \\ \hline -1 \end{array}} = \frac{1}{8h} \boxed{\begin{array}{c|c|c} 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -2 & -1 \end{array}}$$

- One can show that this does not change the consistency order (still 2).
- However, Sobel operators approximate the rotation invariance of $|\boldsymbol{\nabla} f|$ better.

Derivative Filters in 2D (3)

Second Order Derivatives

- Standard approximation of the Laplacian $\Delta f = \partial_{xx} f + \partial_{yy} f$:

$$\Delta f_{i,j} \;=\; \frac{1}{h^2} \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \; f_{i,j}$$

$$-\;\frac{1}{12} h^2 (\partial_{xxxx} f_{i,j} + \partial_{yyyy} f_{i,j}) \;+\; O(h^4).$$

- Problem: Although the Laplacian is rotationally invariant, the derivative expression $\partial_{xxxx} f_{i,j} + \partial_{yyyy} f_{i,j}$ in its leading error term is not.

Derivative Filters in 2D (4)

- Better results are obtained with

$$\Delta f_{i,j} \;=\; \frac{1}{6h^2} \begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} \; f_{i,j}$$

$$-\;\frac{1}{12} h^2 (\partial_{xxxx} f_{i,j} + 2\,\partial_{xxyy} f_{i,j} + \partial_{yyyy} f_{i,j}) \;+\; O(h^4)$$

where the leading error term involves a rotationally invariant derivative expression, namely $\Delta(\Delta f)_{i,j}$ (cf. Assignment P6, Problem 2).

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Summary

- Image derivatives are useful for detecting features such as edges.
- Differentiation is dangerous. It can be stabilised with a lowpass filter.
- The weights of the discrete derivative approximations can be computed via a Taylor expansion with subsequent comparision of coefficients.
- The order of consistency can be increased by larger stencils.
- The continuous Fourier transform allows to analyse the frequency-dependent approximation quality.
- 2D derivative operators should have good rotation invariance.
  Example: Sobel operators.

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