Lecture 5: Image Transformations II: Sampling Theorem and Discrete Fourier Transform

Contents

Recently on IPCV ...

- Transformations: Represent image in different basis.
- Continuous Fourier transform analyses the frequency content of images.
- useful properties:
  - differentiation → multiplication with the frequency.
  - convolution in one domain → multiplication in the other
- The Fourier transform maps
  - box functions to sinc functions,
  - Gaussians to Gaussians with reciprocal variance,
  - delta combs to delta combs with reciprocal distance.

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Motivation

- We have seen that the Fourier transform is a useful tool for
  - representing images by their frequencies,
  - expressing convolutions and derivatives in terms of multiplications.
- Our considerations were based on the *continuous* Fourier transform. Its definition uses a continuous image with infinite extension.
- In practice, digital images are sampled and have a finite extent.
- Can Fourier theory help to analyse this sampling process and its difficulties?
- Is there a *discrete* Fourier transform that works on a finite domain?
- Can it be computed in an efficient way?

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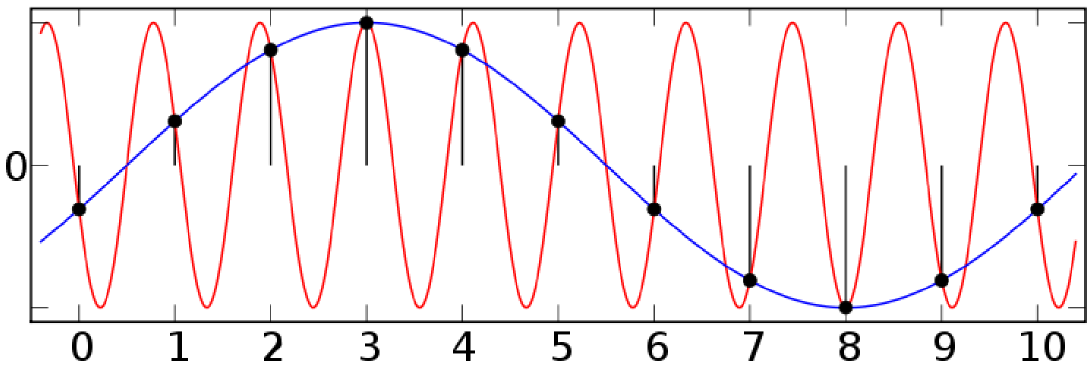
Towards the Discrete Setting: Sampling Theorem



Aliasing effect in 1-D. If the sampling rate is too low, high frequent signal components (red) are observed as low frequent artifacts (blue). Source: Wikipedia.

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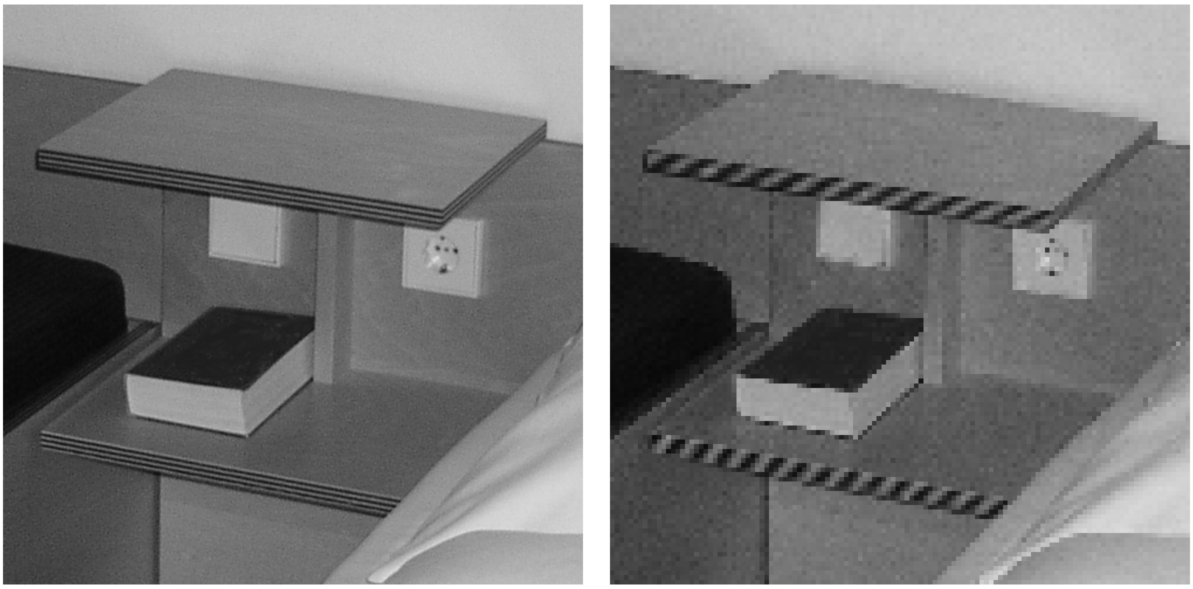
Towards the Discrete Setting: Sampling Theorem (2)



Aliasing effect in 2-D. **Left:** Original image, 496×496 pixels. **Right:** Downsampled with `xv` to 124×124 pixels. Author: J. Weickert.

Towards the Discrete Setting: Sampling Theorem (3)

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Left: Original image of a guitar player, 546×450 pixels. **Right:** Another version found on the internet, 345×282 pixels. Source: `http://www.johnlilley.com/aboutjohn.html`.

Aliasing Effect

- If a high-frequent signal is sampled too coarsely, low-frequent artifacts arise.
- This is called *aliasing,* for images sometimes also *Moiré effect.*
- can be observed quite often, e.g.
  - when using inappropriate programs for downsampling (such as `xv`),
  - when some internet browsers automatically scale down large images,
  - if the resolution of a scanner is too low,
  - as temporal aliasing in movies with rapidly rotating objects (e.g. propellors).
- Is there a theory that tells us how fine we have to sample?

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Sampling Theorem (Abtasttheorem)

(Whittaker 1915, Nyquist 1928, Kotelnikov 1933, Shannon 1949)

- Let a continuous signal f be *band-limited,* i.e. there exists a highest frequency W:
$$\hat{f}(u) \;=\; 0 \quad \text{for } |u| > W.$$
- In order to sample a band-limited signal without aliasing artifacts,
one must sample the highest frequency more than twice per period.

  For the sampling distance h this means
$$h \;<\; h_{max} \;:=\; \frac{1}{2W}.$$

Towards the Discrete Setting: Sampling Theorem (6)

Remarks

- Thus, there is a critical sampling frequency $2W$, below which aliasing starts. It is called *Nyquist frequency*.
- If the sampling theorem is obeyed, it is even possible to reconstruct the continuous signal $f(x)$ exactly (!) from its discrete samples $\{f(kh) \mid k \in \mathbb{Z}\}$:

  The *Whittaker–Shannon interpolation formula* states that in this case

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}\left(\frac{\pi}{h}(x-kh)\right).$$

  In Lecture 9, we will understand this formula better.
- For images, the sampling theorem must hold in x- and in y-direction.

Towards the Discrete Setting: Sampling Theorem (7)

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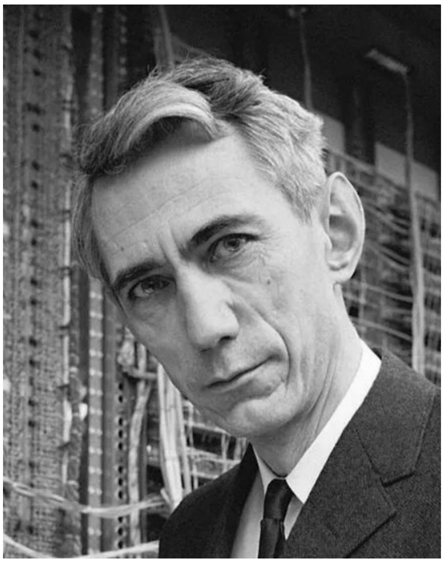
Edmund T. Whittaker


Harry Nyquist


Vladimir Kotelnikov

Three researchers who have contributed substantially to the discovery of the sampling theorem. **Left:** The British mathematician Sir Edmund Taylor Whittaker (1873–1956). **Middle:** The Swedish–American electrical engineer Harry Nyquist (1889–1976). **Right:** The Russian electrical engineer Vladimir Kotelnikov (1908–2005). Sources: `https://de.wikipedia.org/wiki/Edmund_Taylor_Whittaker`, `https://ethw.org/Harry_Nyquist`, and `https://www.mediastorehouse.com`.

Towards the Discrete Setting: Sampling Theorem (8)



The American mathematician, electrical engineer and cryptographer Claude E. Shannon (1916–2001) is regarded as the founder of information theory. He has devised the first chess-playing programmes, and he is also one of the pioneers of the sampling theorem. The right image shows his machine that simulates a labyrinth-solving electromechanical mouse. Source: `https://afflictor.com/tag/claude-shannon/`.

Towards the Discrete Setting: Sampling Theorem (9)

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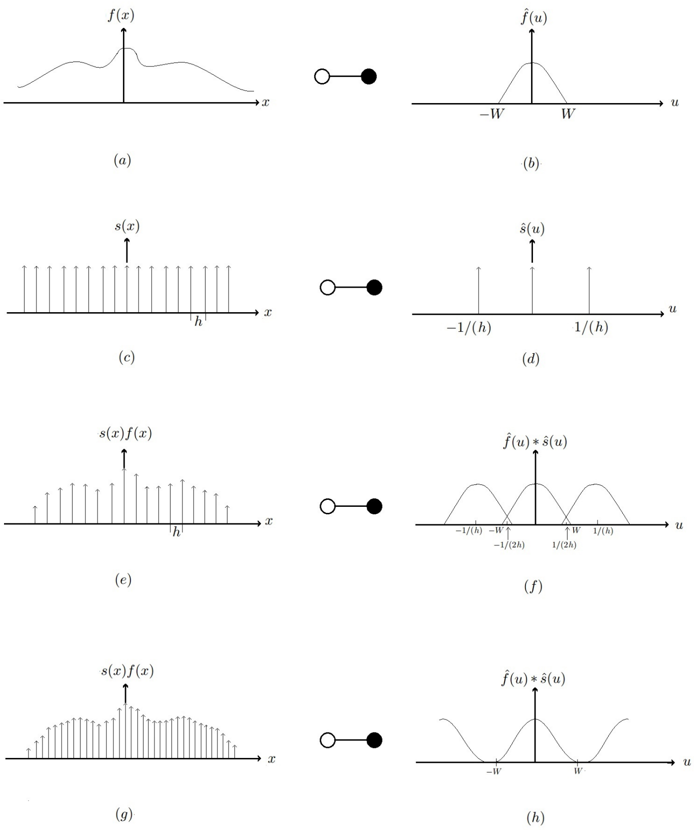

Illustration of the sampling theorem. **(a)** Band-limited signal. **(b)** Its Fourier transform. **(c)** Delta comb. **(d)** The Fourier transform of this delta comb is a delta comb with reciprocal grid distance. **(e)** Sampling a band-limited function by multiplication with a delta comb in the spatial domain. **(f)** In the Fourier domain this gives convolution of the Fourier transforms of (b) and (d). Overlapping frequency bands from different periods create aliasing: The high frequencies in $[\frac{1}{2h}, W]$ and $[-W, -\frac{1}{2h}]$ reappear as lower frequencies inside $[-\frac{1}{2h}, \frac{1}{2h}]$. **(g)** Reduction of the sampling distance. **(h)** In the Fourier domain the frequency bands do no longer overlap. No aliasing effects arise. Author: A. Goswami.

How Can One Avoid Aliasing ?

- Best solution, if possible:
  - Use a sufficiently high sampling rate.
  - This allows to represent also high frequencies adequately.
- Second best solution:
  - Suppress high frequencies by smoothing your image *before* downsampling.
  - Example of such a smoothing filter:
    Gaussian convolution with a sufficiently large σ:
    at least half the size of the coarser grid.
  - Formally this does not eliminate all high frequencies:
    However, it reduces them substantially such that artifacts are invisible.

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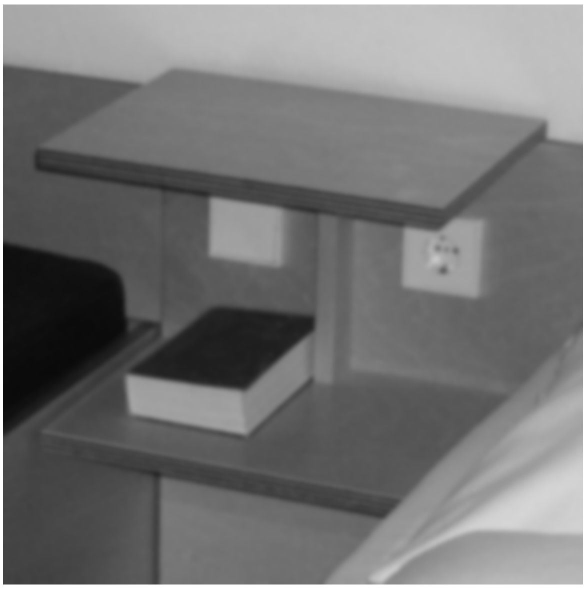

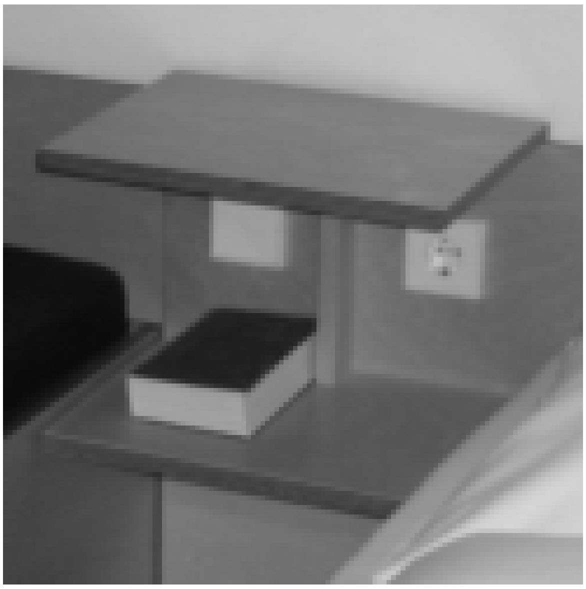

Avoidance of aliasing. **Left:** Original image (496×496 pixels) after suppression of high frequencies by Gaussian convolution with $\sigma = 2$. **Right:** Downsampling with xv to 124×124 pixels does not create visible aliasing effects in this case. Author: J. Weickert.

Lecture 5: Image Transformations II: Sampling Theorem and Discrete Fourier Transform

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Discrete Fourier Transform in 1-D

Goals

- discrete analogue to the continuous Fourier transform
- should deal with *sampled* signals of *finite* extent
- signal with M values is decomposed into M frequency components

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Discrete Fourier Transform in 1-D (2)

Reminder from Lecture 4

- Consider a continuous signal $f : \mathbb{R} \to \mathbb{R}$ with infinite extent.
  Its continuous Fourier transform is defined as

$$\hat{f}(u) \;=\; \int\limits_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx \qquad (u \in \mathbb{R})$$

  with $i^2 = -1$.

- The corresponding inverse continuous Fourier transform is given by

$$f(x) \;=\; \int\limits_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux}\, du \qquad (x \in \mathbb{R}).$$

Discrete Fourier Transform in 1-D (3)

Definition

- Consider a discrete signal $\boldsymbol{f} = (f_0, ..., f_{M-1})^\top$ with finite extent.
  Its *discrete Fourier transform (DFT)* is defined as

$$\hat{f}_p \;:=\; \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(-\frac{i2\pi pm}{M}\right) \qquad (p = 0, ..., M-1)$$

  with $i^2 = -1$.

- The corresponding *inverse discrete Fourier transform* is given by

$$f_m \;=\; \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \hat{f}_p \exp\left(\frac{i2\pi pm}{M}\right) \qquad (m = 0, ..., M-1)$$

Discrete Fourier Transform in 1-D (4)

Interpretation as Change of Basis

- Hermitian inner product of vectors $\boldsymbol{f} = (f_m)_{m=0}^{M-1}$ and $\boldsymbol{g} = (g_m)_{m=0}^{M-1}$ in $\mathbb{C}^M$:

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{m=0}^{M-1} f_m \, \bar{g}_m \,.$$

- One orthonormal basis of $(\mathbb{C}^M, \langle \cdot, \cdot \rangle)$ is given by the M vectors

$$\boldsymbol{b}_p \;:=\; \frac{1}{\sqrt{M}} \left(\exp\left(\frac{i2\pi p0}{M}\right), \exp\left(\frac{i2\pi p1}{M}\right), ..., \exp\left(\frac{i2\pi p(M-1)}{M}\right) \right)^{\top}$$

$$(p = 0, ..., M-1)\,.$$

- Representing a vector $\boldsymbol{f}$ in this *discrete Fourier basis* yields

$$\boldsymbol{f} \;=\; \sum_{p=0}^{M-1} \langle \boldsymbol{f}, \boldsymbol{b}_p \rangle \, \boldsymbol{b}_p.$$

- The DFT computes the Fourier coefficients $\hat{f}_p := \langle \boldsymbol{f}, \boldsymbol{b}_p \rangle$ for $p = 0, ..., M-1$. The inverse DFT reconstructs the signal from these coefficients via

$$\boldsymbol{f} = \sum_{p=0}^{M-1} \hat{f}_p \, \boldsymbol{b}_p.$$

Discrete Fourier Transform in 1-D (5)

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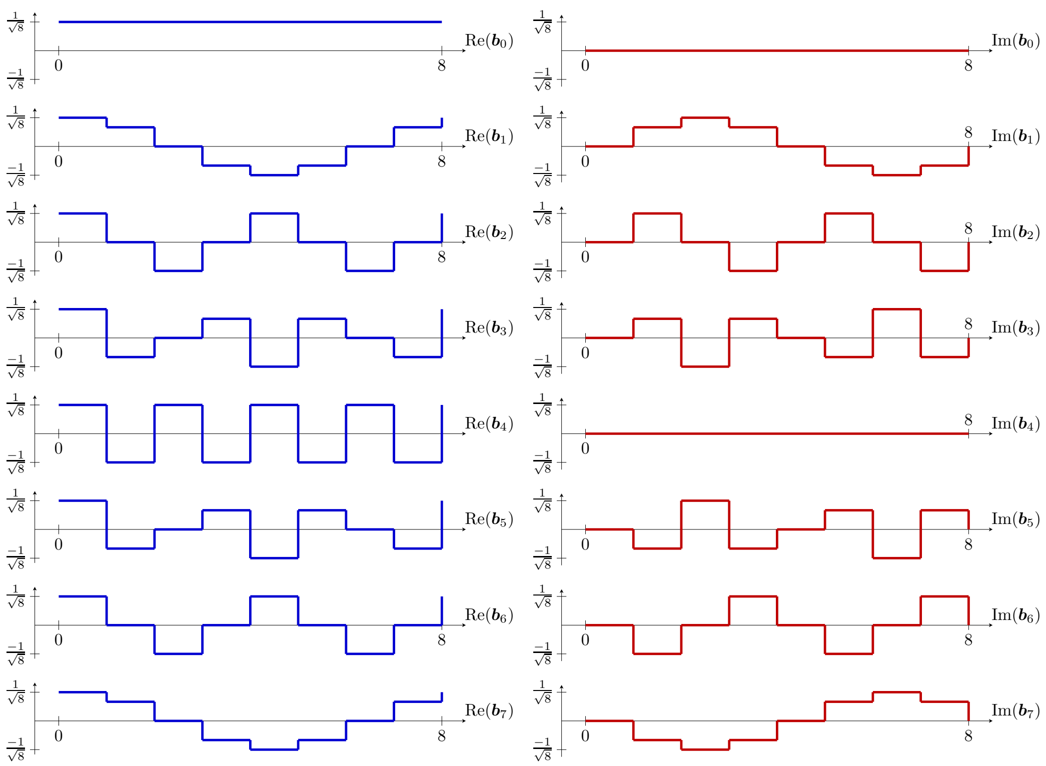

Signal representation of the basis vectors of the DFT for $M = 8$. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

Example: DFT of $\boldsymbol{f} = (6,4,5,1)^\top$

For $M=4$, the Fourier basis vectors are given by

$$\begin{aligned}
\boldsymbol{b}_0 &= \tfrac{1}{2}\left(e^{i0\frac{\pi}{2}0},\, e^{i0\frac{\pi}{2}1},\, e^{i0\frac{\pi}{2}2},\, e^{i0\frac{\pi}{2}3}\right)^\top = \tfrac{1}{2}\left(1,\,1,\,1,\,1\right)^\top,\\
\boldsymbol{b}_1 &= \tfrac{1}{2}\left(e^{i1\frac{\pi}{2}0},\, e^{i1\frac{\pi}{2}1},\, e^{i1\frac{\pi}{2}2},\, e^{i1\frac{\pi}{2}3}\right)^\top = \tfrac{1}{2}\left(1,\,i,\,-1,\,-i\right)^\top,\\
\boldsymbol{b}_2 &= \tfrac{1}{2}\left(e^{i2\frac{\pi}{2}0},\, e^{i2\frac{\pi}{2}1},\, e^{i2\frac{\pi}{2}2},\, e^{i2\frac{\pi}{2}3}\right)^\top = \tfrac{1}{2}\left(1,\,-1,\,1,\,-1\right)^\top,\\
\boldsymbol{b}_3 &= \tfrac{1}{2}\left(e^{i3\frac{\pi}{2}0},\, e^{i3\frac{\pi}{2}1},\, e^{i3\frac{\pi}{2}2},\, e^{i3\frac{\pi}{2}3}\right)^\top = \tfrac{1}{2}\left(1,\,-i,\,-1,\,i\right)^\top.
\end{aligned}$$

You can check that they are orthonormal w.r.t. the Hermitian inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle := \boldsymbol{f}^\top \bar{\boldsymbol{g}} = \sum_{m=0}^{3} f_m\, \bar{g}_m\,.$$

Do not forget to take the complex conjugate in the second argument!

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With this inner product, the Fourier coefficients of $\boldsymbol{f} = (6,4,5,1)^\top$ are given by

$$\begin{aligned}
\hat{f}_0 &= \langle \boldsymbol{f}, \boldsymbol{b}_0 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_0 = \tfrac{1}{2}\left(6\cdot 1 + 4\cdot 1 + 5\cdot 1 + 1\cdot 1\right) = 8\,,\\
\hat{f}_1 &= \langle \boldsymbol{f}, \boldsymbol{b}_1 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_1 = \tfrac{1}{2}\left(6\cdot 1 + 4\cdot(-i) + 5\cdot(-1) + 1\cdot i\right) = \tfrac{1}{2} - \tfrac{3}{2}\,i\,,\\
\hat{f}_2 &= \langle \boldsymbol{f}, \boldsymbol{b}_2 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_2 = \tfrac{1}{2}\left(6\cdot 1 + 4\cdot(-1) + 5\cdot 1 + 1\cdot(-1)\right) = 3\,,\\
\hat{f}_3 &= \langle \boldsymbol{f}, \boldsymbol{b}_3 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_3 = \tfrac{1}{2}\left(6\cdot 1 + 4\cdot i + 5\cdot(-1) + 1\cdot(-i)\right) = \tfrac{1}{2} + \tfrac{3}{2}\,i\,.
\end{aligned}$$

This is the discrete Fourier transform. It transforms the coefficients $(f_0, ..., f_3)^\top$ in the canonical basis to the coefficients $(\hat{f}_0, ..., \hat{f}_3)^\top$ in the Fourier basis.

By plugging in, one can check that

$$\boldsymbol{f} = \hat{f}_0\, \boldsymbol{b}_0 + \hat{f}_1\, \boldsymbol{b}_1 + \hat{f}_2\, \boldsymbol{b}_2 + \hat{f}_3\, \boldsymbol{b}_3\,.$$

This is the inverse discrete Fourier transform. It maps the coefficients $(\hat{f}_0, ..., \hat{f}_3)^\top$ in the Fourier basis to the coefficients $(f_0, ..., f_3)^\top$ in the canonical basis.

Discrete Fourier Transform in 1-D (8)

Remarks

The previous example illustrates some general properties of the DFT:

- $\hat{f}_0 = \frac{1}{\sqrt{M}} \sum\limits_{m=0}^{M-1} f_m$ is $\sqrt{M}$ times the average grey value $\frac{1}{M} \sum\limits_{m=0}^{M-1} f_m$.
- $\mathrm{Re}(\hat{f}_p)$ is an even function in p with respect to the index $M/2$:

$$\mathrm{Re}(\hat{f}_1) \;=\; \mathrm{Re}(\hat{f}_3)$$

- $\mathrm{Im}(\hat{f}_p)$ is an odd function in p with respect to the index $M/2$:

$$\mathrm{Im}(\hat{f}_1) \;=\; -\mathrm{Im}(\hat{f}_3).$$

  It vanishes for $p = M/2$:

$$\mathrm{Im}(\hat{f}_2) \;=\; 0.$$

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Outline

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Discrete Fourier Transform in 2-D

- Consider a discrete, image $\boldsymbol{f} = (f_{m,n})$ with $m = 0, ..., M-1$ and $n = 0, ..., N-1$. Its *discrete Fourier transform* is given by

$$\hat{f}_{p,q} \;:=\; \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{m,n} \exp\left(-\frac{i2\pi pm}{M}\right) \exp\left(-\frac{i2\pi qn}{N}\right)$$

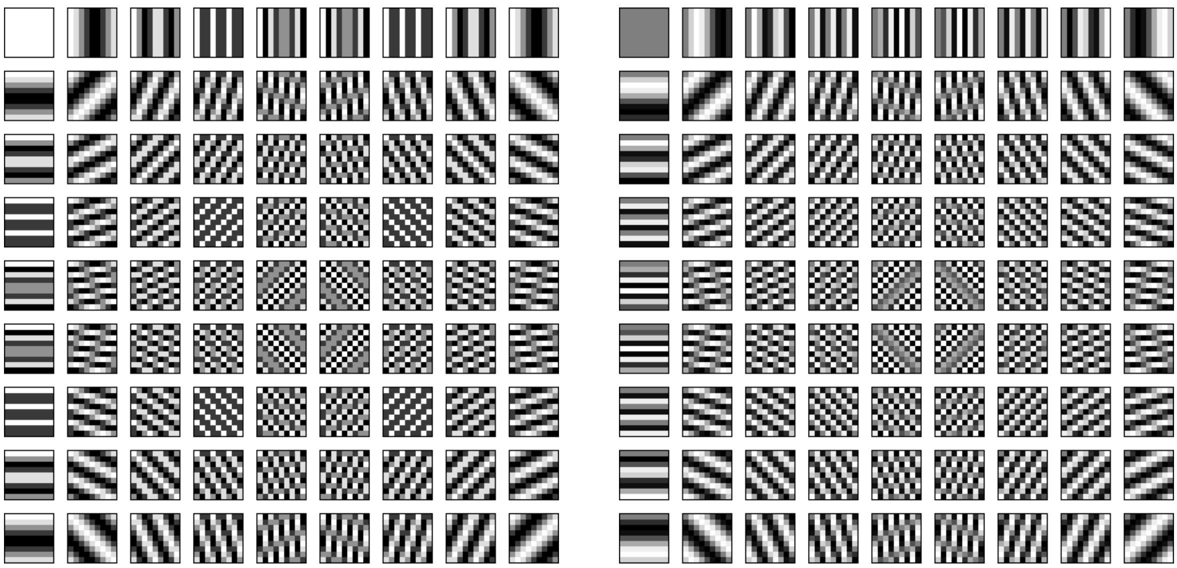
$$(p = 0, ..., M-1; \quad q = 0, ..., N-1)\,.$$

- The corresponding *discrete inverse Fourier transform* is defined as

$$f_{n,m} \;=\; \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \hat{f}_{p,q} \exp\left(\frac{i2\pi pm}{M}\right) \exp\left(\frac{i2\pi qn}{N}\right)$$

$$(n = 0, ..., M-1; \quad m = 0, ..., N-1)\,.$$

In higher dimensions, the DFT is defined in an analogue way.
Just like the continuous FT, the DFT is separable.



The 81 basis vectors of the DFT for $M = N = 9$. Here the y axis goes from the top to the bottom. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

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Properties of the Discrete Fourier Transform

Many important properties of the continuous FT carry over to the discrete FT:

- linearity
- shift theorem (when signal is extended periodically)
- convolution theorem

Some properties, however, can only be approximated on a discrete grid:

- scaling theorem
- rotation invariance

Often one uses the continuous FT for designing filters,
and the discrete FT for implementing them.

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Properties of the Discrete Fourier Transform (2)

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Shifting the Fourier Spectrum

- **Problem:** The DFT uses frequencies in $[0, M-1] \times [0, N-1]$.
  The origin (frequency 0) is in one corner of this rectangular domain.
  It would be nice to shift the origin of the spectrum to the centre $(\frac{M}{2}, \frac{N}{2})$.
  Then the DFT looks more similar to the continuous FT.
- The discrete shift theorem gives the transform pairs

$$f_{m-m_0,\, n-n_0} \quad \circ\!\!-\!\!\bullet \quad \hat{f}_{p,q} \exp\left(-\frac{i2\pi p m_0}{M}\right) \exp\left(-\frac{i2\pi q n_0}{N}\right)$$

$$f_{m,n} \exp\left(\frac{i2\pi p_0 m}{M}\right) \exp\left(\frac{i2\pi q_0 n}{N}\right) \quad \circ\!\!-\!\!\bullet \quad \hat{f}_{p-p_0,\, q-q_0}$$

- With $p_0 = \frac{M}{2}$ and $q_0 = \frac{N}{2}$, one replaces the image $f_{m,n}$ by

$$f_{m,n} \exp\left(\frac{i2\pi M m}{2M}\right) \exp\left(\frac{i2\pi N n}{2N}\right) = f_{m,n}\, e^{i\pi(m+n)} = f_{m,n}(-1)^{m+n}.$$

- Thus, all one has to do is to multiply $\boldsymbol{f}$ with a checkerboard-like sign pattern.

Properties of the Discrete Fourier Transform (3)

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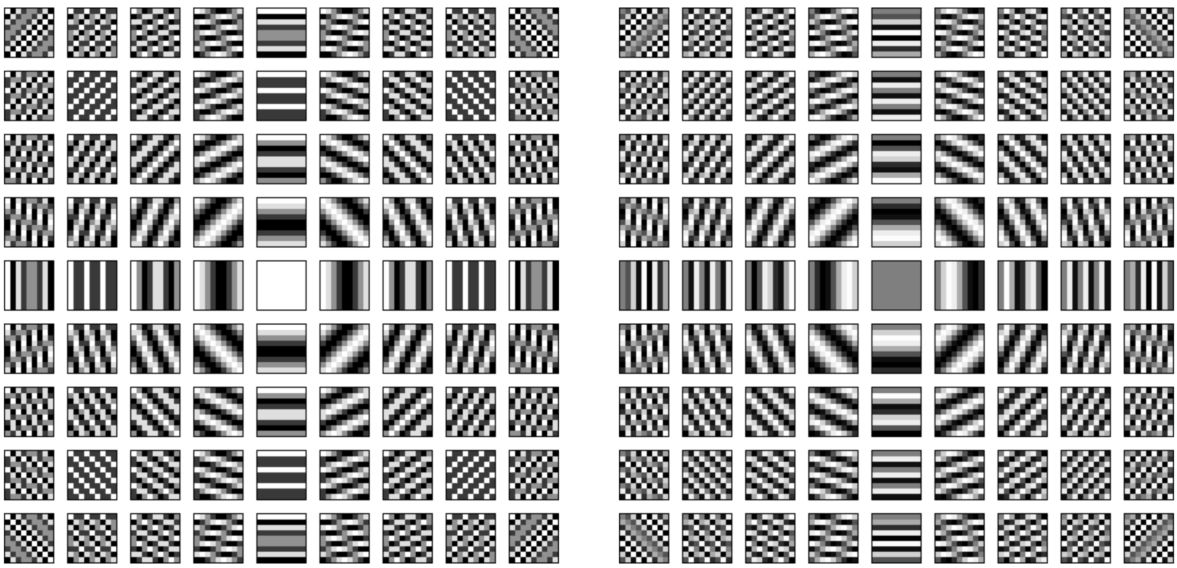

The 81 basis vectors of the shifted DFT for $M = N = 9$. The shift moves the low frequencies to the centre. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

Logarithmic Scaling of the Fourier Spectrum

- **Problem:** The range of the Fourier spectrum covers many orders of magnitude.
- Thus, for visualisation purposes one often uses a logarithmic transformation:

$$D_{p,q} \;=\; c \,\ln\left(1+|\hat{f}_{p,q}|\right).$$

  Adding 1 ensures that the result of the logarithm is nonnegative.
- Usually c is chosen such that

$$\max_{p,q} D_{p,q} = 255.$$

  This allows a convenient visualisation of the result, since its range is in $[0,255]$.
- *This logarithmic transformation is so common that often people simply forget to tell you that they have used it.*

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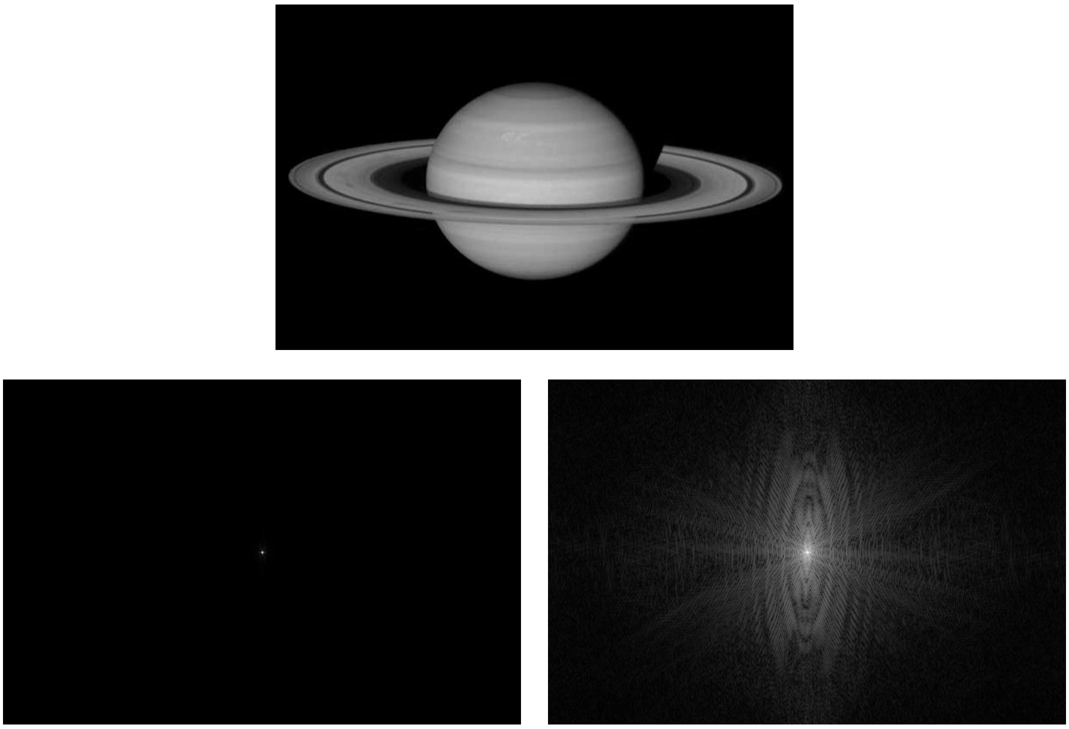

Top: Image of the planet Saturn, 600×400 pixels (`www.androidworld.com/Saturn.jpeg`). **Bottom left:** Fourier spectrum scaled to $[0,255]$. **Bottom right:** Fourier spectrum after logarithmic scaling with $\max D_{p,q} = 255$. Author: J. Weickert.

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Boundary Artifacts

- Fundamental difference between the continuous and the discrete FT:
  For the DFT, the signal $\boldsymbol{f}$ has *finite* extent: $f_0, ..., f_{M-1}$.
- The periodicity of the complex exponential function automatically creates a *periodic continuation* of the image in its Fourier and its spatial representation (see the definitions on Page 21):

$$\begin{aligned} \hat{f}_{p,q} &= \hat{f}_{p+kM,\, q+\ell N} \qquad (k, \ell \in \mathbb{Z}), \\ f_{n,m} &= f_{n+kM,\, m+\ell N} \qquad (k, \ell \in \mathbb{Z}). \end{aligned}$$

  This can create undesired boundary artifacts.
- **Example 1:**
  Periodically extended boundaries are usually discontinuous.
  This creates high-frequent Fourier components in x- and y-direction.
- **Example 2:**
  Wraparound errors in connection with convolutions:
  Grey values near the right boundary perturb grey values at the left boundary.

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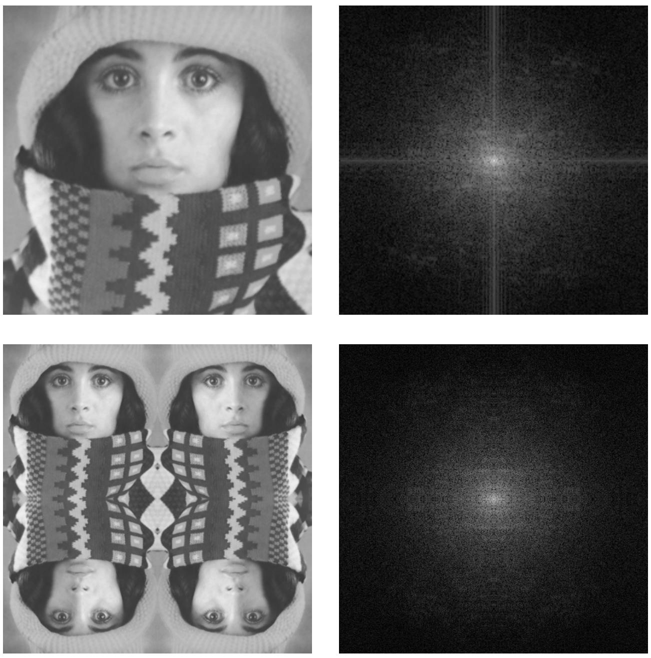
Boundary Artifacts (2)

How can These Artifacts be Handled ?

- **Fatalism**
  - Do nothing. Trust only convolution results far away from the boundaries.
  - Disadvantage: For large kernels, not many results are trustworthy.
  - Not recommended.
- **Zero Padding**
  - Supplement a layer of zeroes at the boundaries.
    Its thickness respects the size of the convolution kernel.
  - Disadvantage: Also zeroes can spoil your signal!
  - Not recommended.
- **Mirror Image at Boundaries**
  - Cleanest solution $\rightarrow$ Recommended.
  - Disadvantage: This doubles the signal length in each dimension.
    Thus, the computational load increases substantially.

Boundary Artifacts (3)



Top left: Original image, 256×256 pixels. **Top right:** Its logarithmic DFT spectrum, size 256×256. Note the horizontal and vertical artifacts due to discontinuities in the periodic extension. **Bottom left:** Mirrored image extension, 512×512 pixels. It does not have discontinuities at the image boundaries. **Bottom right:** Its logarithmic DFT spectrum has size 512×512 and does not show artifacts. Author: J. Weickert.

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The Fast Fourier Transform (FFT)

(Gauss 1805, Cooley/Tukey 1965)

- A literal implementation of 1-D DFT of a signal of length M is quite expensive: M^2 (complex-valued) multiplications and $M^2\!-\!M$ (complex-valued) additions
- Basic idea behind the *Fast Fourier Transform (FFT):* divide–and–conquer.
  - split problem of size M into two subproblems of size $\frac{M}{2}$
  - continue until size 1 is reached
- **Advantages**:
  - very efficient: $\mathcal{O}(M \log_2 M)$ operations
  - available in many numerical packages (see e.g. `www.fftw.org`)
- **Disadvantage**: standard FFT requires signals of size $M = 2^k$
- For 2-D images, one exploits the separability of the DFT:
  - hardly additional memory requirements
  - well-suited for parallel computing

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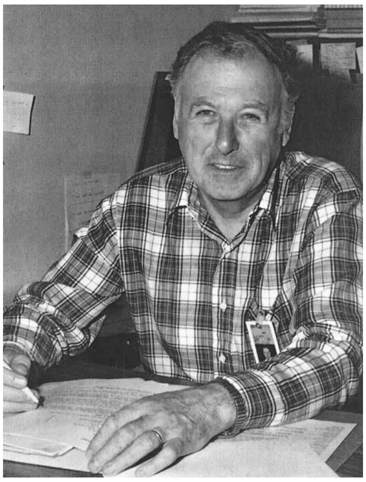
Fast Fourier Transform (2)

Carl Friedrich Gauss



James W. Cooley

John Wilder Tukey

Left: Carl Friedrich Gauss (1777–1855) was one of the most eminent mathematicians. Already in 1805, he introduced the FFT for some of his astronomical computations. Painting by Gottlieb Biermann (1887). **Middle:** James W. Cooley (1926–2016) was an applied mathematician who has popularised the FFT in 1965, in a joint publication with John Wilder Tukey. He perfomed the indexing and implementation work. Source: `https://www.nae.edu/219757/JAMES-W-COOLEY-19292017`. **Right:** The mathematician John Wilder Tukey (1915–2000) was one of the pioneers of robust statistics (leading also to median filtering), and he has coined the words *bit* and *software*. Source: `https://mathshistory.st-andrews.ac.uk/Biographies/Tukey/pictdisplay`.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

Fast Fourier Transform (3)

The Inverse DFT

- For computing the inverse DFT, no second algorithm is needed:
  - Just replace the Fourier coefficients $\hat{f}_p$ by their complex conjugates $\bar{\hat{f}}_p$.
  - Apply the DFT to these numbers.
- Explanation: Computing the inverse DFT

$$f_m = \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \hat{f}_p \exp\left(\frac{i2\pi pm}{M}\right)$$

and taking its complex conjugate gives

$$\bar{f}_m = \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \bar{\hat{f}}_p \exp\left(-\frac{i2\pi pm}{M}\right).$$

Note that $\bar{f}_m = f_m$ for real-valued images.

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

Summary

- A frequency must be sampled more than twice per period in order to avoid aliasing.
- The discrete Fourier transform (DFT) decomposes a discrete signal of size M into M frequency components.
- similar properties as continuous FT:
  complex-valued, linear, separable, shift theorem, convolution theorem
- main difference: finite signal size introduces periodic signal extension.
  This can create problems such as wraparound errors (remedy: mirroring).
- The Fast Fourier Transform (FFT) allows efficient computation of the DFT.
  In 1D its complexity is $\mathcal{O}(M \log_2 M)$.

References

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  (an entire book on aliasing!)
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  (further extensions on aliasing)
- Wikipedia: The Nyquist-Shannon Sampling Theorem.
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- R. C. Gonzalez, R. E. Woods: *Digital Image Processing*. Prentice Hall, Upper Saddle River, International Edition, 2017.
  (Chapter 4 gives an extended description of many technicalities.)
- K. R. Castleman: *Digital Image Processing*. Prentice Hall, Upper Saddle River, 1996.
  (One of the best textbooks on linear concepts such as the Fourier transform, convolutions and linear systems.)
- R. Bracewell: *The Fourier Transform and its Applications*. McGraw-Hill, New York, 2002.
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  (An entertaining article on the past, the presence and the future of the FFT.)

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