1.e

Lemma 1 *Since,*

$$\frac{1}{1-z} = \ 1 + z + z^2 + z^3 + ...$$

By succesive differentiation,

$$\frac{1}{(1-z)^2} = \quad 1 + 2z + 3z^2 + 4z^3 + ...$$

$$\frac{2}{(1-z)^3} = \ 2(1) + (3)(2)z + (4)(3)z^2 + (5)(4)z^3 + ...$$

$$\frac{2z}{(1-z)^3} = \ 2(1)z + (3)(2)z^2 + (4)(3)z^3 + (5)(4)z^4 + ...$$

$$\frac{2z^2}{(1-z)^3} = \ 2(1)z^2 + (3)(2)z^3 + (4)(3)z^4 + (5)(4)z^5 + ...$$

Lemma 2 *If* $T(z) \ = \ \{t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ ....\} \ = \ t_n$*, then,*

$$\begin{aligned} T^{'}(z) \ &= \ \{t_1,\ 2t_2,\ 3t_3,\ 4t_4,\ ....\} \ = \ (n+1)t_{n+1} \\ zT^{'}(z) \ &= \ \{0,\ t_1,\ 2t_2,\ 3t_3,\ 4t_4,\ ....\} \ = \ (n)t_n \end{aligned}$$

$$\begin{aligned} T^{''}(z) \ &= \ \{(2)(1)t_2,\ (3)(2)t_3,\ (4)(3)t_4,\ ....\} \ = \ (n+2)(n+1)t_{n+2} \\ zT^{''}(z) \ &= \ \{0,\ (2)(1)t_2,\ (3)(2)t_3,\ (4)(3)t_4,\ ....\} \ = \ (n+1)(n)t_{n+1} \end{aligned}$$

$$\begin{aligned} T^{'''}(z) \ &= \ \{(3)(2)(1)t_3,\ (4)(3)(2)t_4,\ ....\} \ = \ (n+3)(n+2)(n+1)t_{n+3} \\ zT^{'''}(z) \ &= \ \{0,\ (3)(2)(1)t_3,\ (4)(3)(2)t_4,\ ....\} \ = \ (n+2)(n+1)(n)t_{n+2} \\ z^2T^{'''}(z) \ &= \ \{0,\ 0,\ (3)(2)(1)t_3,\ (4)(3)(2)t_4,\ ....\} \ = \ (n+1)(n)(n-1)t_{n+1} \\ z^3T^{'''}(z) \ &= \ \{0,\ 0,\ 0,\ (3)(2)(1)t_3,\ (4)(3)(2)t_4,\ ....\} \ = \ (n)(n-1)(n-2)t_n \end{aligned}$$

Given from 1.d)

$$\boxed{z(1-z)^2T^{(3)}(z) \;+\; 2(1-z)T^{(2)}(z) \;+\; 2(1-6z)T^{(1)}(z) \;=\; \frac{8\left(z^2+3z+1\right)}{(1-z)^3}}$$

Let, $LHS \;=\; z(1-z)^2T^{(3)}(z) \;+\; 2(1-z)T^{(2)}(z) \;+\; 2(1-6z)T^{(1)}(z),\; RHS \;=\; \dfrac{8\left(z^2+3z+1\right)}{(1-z)^3}$

$$\boxed{LHS = \;\left[z+z^3-2z^2\right]T^{(3)}(z) \;+\; [2 \;-\; 2z]T^{(2)}(z) \;+\; [2-12z]T^{(1)}(z)} \tag{1}$$

$$\begin{aligned}\left[z+z^3-2z^2\right]T^{(3)}(z) \;&=\; zT^{(3)}(z) \;+\; z^3T^{(3)}(z) \;-2z^2T^{(3)}(z)\\ &= (n+2)(n+1)(n)t_{n+2} \;+\; (n)(n-1)(n-2)t_n \;-\; 2(n+1)(n)(n-1)t_{n+1}\end{aligned} \tag{2}$$

$$[2 \;-\; 2z]T^{(2)}(z) \;=\; 2T^{(2)}(z) \;-\; 2zT^{(2)}(z) = \; 2(n+2)(n+1)t_{n+2} \;-\; 2(n+1)(n)t_{n+1} \tag{3}$$

$$[2-12z]T^{(1)}(z) \;=\; 2T^{(1)}(z) \;-\; 12zT^{(1)}(z) = \; 2(n+1)t_{n+1} \;-\; 12(n)t_n \tag{4}$$

(1) = (2)+(3)+(4) $\Longrightarrow$

$$LHS \;=\; t_n[n(n-1)(n-2)-12n] \;-\; 2(n+1)t_{n+1}[n(n-1)+n-1] \;+\; (n+2)(n+1)t_{n+2}[n+2]$$

$$\boxed{LHS = \; t_n[n(n-1)(n-2)-12n] \;-\; 2(n+1)\left(n^2-1\right)t_{n+1} \;+\; (n+2)^2(n+1)t_{n+2}}$$

$$\boxed{RHS \;=\; \frac{8\left(z^2+3z+1\right)}{(1-z)^3}}$$

$$RHS \;=\; \left[\frac{8z^2}{(1-z)^3} \;+\; \frac{24z}{(1-z)^3} \;+\; \frac{8}{(1-z)^3}\right] \tag{5}$$

From Lemma 1 and writing in vector form,

$$\frac{2}{(1-z)^3} \;=\; \{2(1),\; (3)(2),\; (4)(3),\; (5)(4),\; ....\} \;=\; (n+1)(n+2)$$

$$\frac{2z}{(1-z)^3} = \; \{0,\; 2(1),\; (3)(2),\; (4)(3),\; (5)(4),\; ....\} \;=\; n(n+1)$$

$$\frac{2z^2}{(1-z)^3} = \; \{0,\; 0,\; 2(1),\; (3)(2),\; (4)(3),\; (5)(4),\; ....\} \;=\; n(n-1)$$

$$\therefore RHS \;=\left[\frac{8z^2}{(1-z)^3} \;+\; \frac{24z}{(1-z)^3} \;+\; \frac{8}{(1-z)^3}\right] \;=\; 4\left[\frac{2z^2}{(1-z)^3}\right] \;+\; 12\left[\frac{2z}{(1-z)^3}\right] \;+\; 4\left[\frac{2}{(1-z)^3}\right]$$

$$\therefore RHS \;=\left[\frac{8z^2}{(1-z)^3} \;+\; \frac{24z}{(1-z)^3} \;+\; \frac{8}{(1-z)^3}\right] = \; 4[n(n-1)] \;+\; 12[n(n+1)] \;+\; 4[(n+1)(n+2)]$$

$$\boxed{RHS \;=\; 4[n(n-1)] \;+\; 12[n(n+1)] \;+\; 4[(n+1)(n+2)]}$$

We know $LHS \; = \; RHS$,

$$\therefore \; t_n[n(n-1)(n-2)-12n] \; - \; 2(n+1)\left(n^2-1\right)t_{n+1} \; + \; (n+2)^2(n+1)t_{n+2}$$
$$= \; 4[n(n-1)] \; + \; 12[n(n+1)] \; + \; 4[(n+1)(n+2)]$$

Replacing n with $n-1$,

$$\begin{aligned} \equiv t_{n-1}[(n-1)(n-2)(n-3)-12(n-1)] \; - \; 2(n)\left((n-1)^2-1\right)t_{n+1} \; + \; (n+1)^2(n)t_{n+1} \\ = \; 4[(n-1)(n-2)] \; + \; 12[(n-1)(n)] \; + \; 4[(n)(n+1)] \end{aligned} \tag{6}$$

$$\begin{aligned} t_{n-1}[(n-1)(n-2)(n-3)-12(n-1)] \; - \; 2(n)\left((n-1)^2-1\right)t_{n+1} \; + \; (n+1)^2(n)t_{n+1} \\ = \; t_{n-1}[(n+1)(n-6)(n-1)] \; - \; 2n^2(n-2)t_n \; + \; (n+1)^2(n)t_{n+1} \end{aligned} \tag{7}$$

$$4[(n-1)(n-2)] \; + \; 12[(n-1)(n)] \; + \; 4[(n)(n+1)] \; = \; 4\left[5n^2 \; - \; 5n \; + \; 2\;\right] \; = \; 20n(n-1) \; + \; 8$$
$$\therefore 4[(n-1)(n-2)] \; + \; 12[(n-1)(n)] \; + \; 4[(n)(n+1)] \; = \; 20n(n-1) \; + \; 8 \tag{8}$$

Substituiting (7), (8) in (6)

$$t_{n-1}[(n+1)(n-6)(n-1)] \; - \; 2n^2(n-2)t_n \; + \; (n+1)^2(n)t_{n+1} \; = \; 20n(n-1) \; + \; 8$$

Re arranging,

$$n^2(n-2)t_n \; = \; \frac{(n+1)[(n-1)(n-6)t_{n-1} \; + \; (n+1)(n)t_{n+1}] \; - \; 20n(n-1) \; - \; 8}{2}$$

$$\boxed{\therefore n^2(n-2)t_n \; = \; \frac{1}{2}(n+1)[(n-1)(n-6)t_{n-1} \; + \; (n+1)(n)t_{n+1}] \; - \; 10n(n-1) \; - \; 4}$$