Discrete scale invariance at strong coupling

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(Dated: April 26, 2025)

We study a holographic model in which we have broken a continuous scale invariance to it's discrete subgroup and compute two-point correlators.

I. INTRODUCTION

In the 21st century, holography has emerged as a powerful tool for studying strongly coupled systems. It has been applied to a wide range of phenomena, from condensed matter physics to quantum information theory, and has proven to be an invaluable framework. In this work, we use holography to explore strongly coupled systems that exhibit discrete scale invariance (DSI).

Many natural systems display discrete scale invariance, often signaled by log-periodicity in observables [1]. In holography, the bulk geometry must reflect the symmetries of the dual field theory. In our model, we introduce DSI by starting with a system that possesses continuous scale invariance and explicitly breaking it down to a discrete subgroup through the inclusion of a periodic term in the holographic coordinate. This approach allows for a controlled deformation, as the strength of this term can be tuned arbitrarily small, smoothly interpolating between continuous and discrete scale invariance.

By constructing such a model, we can qualitatively explore the effects of DSI on correlation functions in strongly coupled systems and contrast them with those of continuously scale-invariant theories. This framework provides new insights into the holographic realization of discrete scaling and its implications for field theory observables.

II. MODEL

We start from the warped metric

$$ds^2 = \frac{1}{z^2}dz^2 + \frac{1}{z^\beta}\delta_{ij}dx^i dx^i. \tag{1}$$

The Ricci scalar is $R = -\frac{d(d-1)\beta^2}{4}$. This has continuous scale invariance, which is given by $z \to \lambda z$ and $x \to \lambda^{\beta/2}x$, where $\lambda > 0$. If we choose $\beta = 2$ we obtain the Euclidean AdS (EAdS_d) metric. The boundary in these coordinates is at $z = 0$ where is also the UV of dual theory.

To controllably break the continuous symmetry into it's discrete subgroup, we introduce a log-periodic term in z. We'll consider the metric

$$ds^2 = \frac{1}{z^2}[1 + A\cos(k\log(z))]dz^2 + \frac{1}{z^\beta}\delta_{ij}dx^i dx^i. \tag{2}$$

The newly introduced term breaks the symmetry into $z \to \lambda_n z$ and $x \to \lambda_n^{\beta/2}x$, where now $\lambda_n = e^{2\pi n/k}$. The Ricci scalar is now

$$R = -\frac{\beta(d-1)}{2}\Bigg(\frac{d\beta}{2(1 + A\cos(k\log(z)))} \tag{3}$$

$$- \frac{Ak\sin(k\log(z))}{(1 + A\cos(k\log(z)))^2}\Bigg). \tag{4}$$

We must choose $|A| < 1$ to keep the zz-component of the metric non-zero.

It is convinient to instead consider coordinates $z \to z^{2/\beta}$. Now the metric can be written as

$$ds^2 = \frac{1}{z^2}([1 + A\cos(2k\log(z)/\beta)]dz^2 + \delta_{ij}dx^i dx^i). \tag{5}$$

However, to compare different metrics for different β at some point in the bulk, we need to do the inverse mapping for the radial coordinate as we identify the radial coordinate as the length scale.

Using the geodesic approximation, the two-point correlators are given by

$$\langle\phi(\mathbf{R})\phi(0)\rangle \sim \exp(-mL), \tag{6}$$

where L is the length of the geodesic that connects the points $(0, 0)$ and $(0, \mathbf{R})$ living at the boundary. This works for very heavy fields. Because of the assumed isotropicity, we can choose our coordinates such a way that the separation is only along one of the spatial directions. Essentially all the data we need about the correlators is in the geodesic length between the two points.

III. NUMERICS

So the goal is to evaluate geodesics in the background geometry (5). The geodesics minimize the length

$$\begin{aligned} L &= \int dx\mathcal{L}(z(x), z'(x)) \\ &= \int_0^R dx\frac{\sqrt{[1 + A\cos(2k\log(z)/\beta)]z'^2 + 1}}{z}. \end{aligned} \tag{7}$$

There is no explicit x dependence in $\mathcal{L}$, so $z'\frac{\partial\mathcal{L}}{\partial z'} - \mathcal{L} = C$ is constant along the path. This allows us to write

$$z' = \pm\sqrt{\frac{1/z^2 - C^2}{C^2[1 + A\cos(2k\log(z)/\beta)]}}. \tag{8}$$

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We can find C by assuming that along the path there is a tipping point $z_\star$ where $z' = 0$. We find $C^2 = 1/z_\star^2$. By substituting these findings back to (7), we get

$$L(z_\star) = 2\int_0^{z_\star} \frac{dz\ z_\star}{z^2}\sqrt{\frac{1 + A\cos(2k\log(z)/\beta)}{z_\star^2/z^2 - 1}}, \tag{9}$$

where we used (8) to express the integral in terms of z. Using the same expression we can write

$$R(z_\star) = \int_0^R dx = 2\int_0^{z_\star} \sqrt{\frac{1 + A\cos(2k\log(z)/\beta)}{z_\star^2/z^2 - 1}}. \tag{10}$$

There is a divergence in (7) arising from $z = 0$. We regularize this by subtracting a constant

$$L_{\rm ct} = 2\int_0^\infty dz \frac{\sqrt{1 + A\cos(2k\log(z)/\beta)}}{z}. \tag{11}$$

The regularized length $L_{\rm reg}(z_\star) = L(z_\star) - L_{\rm ct}$ is finite which we evaluate numerically by integrating over a finite interval $z \in [0, z_{\max}]$. We chose to work with $z_{\max} = 10^5$ which corresponds to roughly $R \sim 10^5$.

Comparing correlators between different theories can be subtle. We begin by matching

$$L^{A,k}_{\beta=2}(\bar{z}) = L^{0,0}_{\beta=2}(\bar{z}) \ \text{ and } \ \log(R^{A,k}_{\beta=2}(\bar{z})) = \log(R^{0,0}_{\beta=2}(\bar{z})) \tag{12}$$

at small $\bar{z}$, by shifting one side by a constant. A convenient choice for the matching point is $\bar{z} = e^{-2\pi}$. To compare correlators across different values of β, we rescale $\bar{z}$ and fix

$$L^{0,0}_{\beta}(\bar{z}^{\beta/2}) = L^{0,0}_{\beta=2}(\bar{z}) \ \text{ and } \ \log(R^{0,0}_{\beta}(\bar{z}^{\beta/2})) = \log(R^{0,0}_{\beta=2}(\bar{z})). \tag{13}$$

Finally, for arbitrary A, k, and β, we define the correlators by matching them to these expressions evaluated at $\bar{z}^{\beta/2}$. This procedure ensures that the logarithms of the correlators are matched at $\log(R^{0,0}_{\beta=2}(\bar{z}))$ to $L^{0,0}_{\beta=2}(\bar{z})$, assuming unit mass for the corresponding operators. For arbitrary β we match the correlators at $\bar{z}^{\beta/2}$, because this is the appropriate length scale that is comparable between different geometries.

IV. RESULTS

The results for the geodesic lengths are shown in Figs. 1 and 2. The linear behaviour of the solid line for $\beta = 2$ imply a power-law decay

$$\langle\phi(R)\phi(0)\rangle \sim 1/R^\alpha. \tag{14}$$

This is expected as this case corresponds to the standard scale-invariance. For different β the behaviour is reminiscent of

$$\langle\phi(R)\phi(0)\rangle_\beta \sim 1/(m^2 + R^\alpha). \tag{15}$$

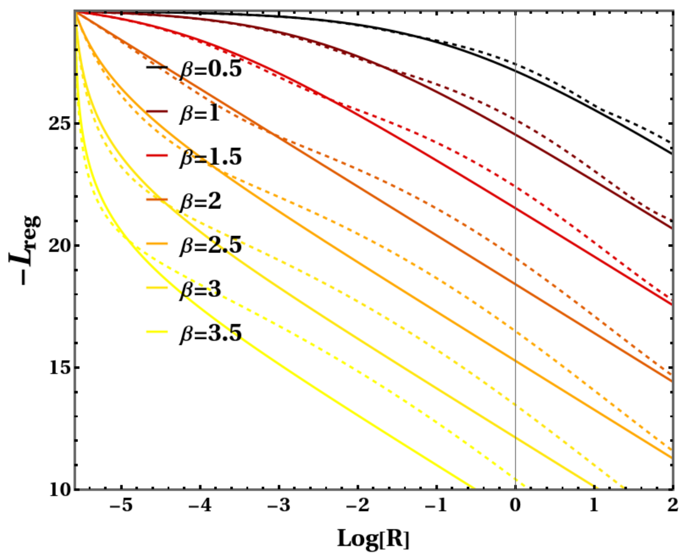

FIG. 1: Plots of the geodesics as a function of $\log R$ for various values of β. The dashed curves correspond to $A = 0.6$ and the solid curves to $A = 0$, which is the continuous limit. For values between we interpolate between these two curves. Here $k = 1$.

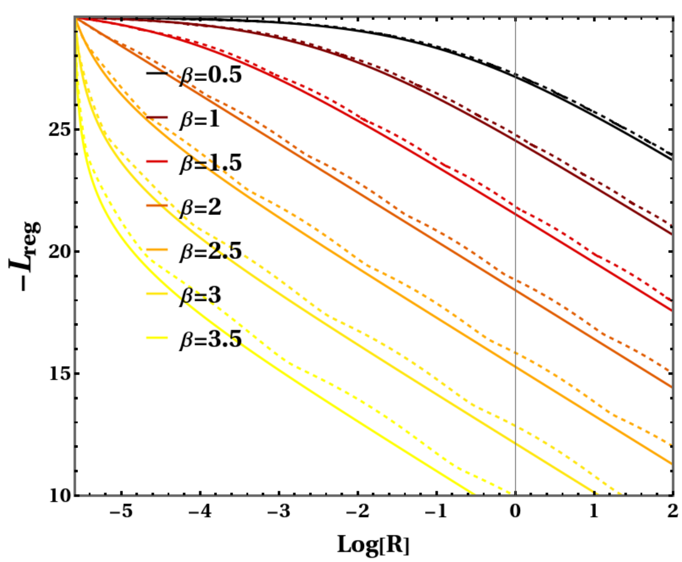

FIG. 2: Plots of the geodesics as a function of $\log R$ for various values of β. The dashed curves correspond to $A = 0.6$ and the solid curves to $A = 0$, which is the continuous limit. For values between we interpolate between these two curves. Here $k = 5$.

Different β introduce a "mass" scale m^2. The slope for large R is the same between different β so α seems to be independent of β. For $\beta > 2$ we have $m^2 < 0$ and for $\beta < 2$ we have $m^2 > 0$. After we introduce DSI, we see that the correlators get mostly enhanced. The anticipated log-periodicity in R can be seen from the figures.

V. DISCUSSION

In this work we have studied a holographic model that has DSI. We see that quite generally the correlation is slightly enhanced compared to having no DSI in these strongly coupled systems.

One should keep in mind that there could be some additional physics emerging at large scales which dissolve this effect. For example, effective cutoff arising for emergent disorder or system size limitations. On the same footing, it might also be possible that this effect is an artifact of having finite range for our coordinate z. However, it can be thought that there is an effective cutoff at large scales. For example, a finite size box can be modeled by considering finite range for z

We chose to present the model in arbitrary dimensions because the Ricci scalar might not be everywhere negative for arbitrary $|A| < 1$. However, for any such A we can find such d that makes (4) negative everywhere. If it's not negative everywhere, the method where the geodesics were taken to be symmetric with respect to the tipping point $z_\star$ might not be valid.

Due to the observed log-periodicity, one could possibly constraint on the functional form of the observed correlators. That is, there should be some deviation from the exponential form observed for continuous scale invariance and which displays log-periodicity. It would be interesting to understand more precisely about the functional form of these correlators.

It is interesting to note that DSI implies a form of discrete hierarchy in the system where different scales are related by discrete transformations. Thus it might be possible to think of DSI as a building block for hierarchical systems. In real world systems the explicit DSI can then be broken further.

There are also other ways to realize DSI in holography. Standard one is to consider metrics with conformal factors $\exp(2r/L + f(r))$, where $f(r)$ is a periodic function in r [2].

It is believed that financial markets show DSI before crashes [3]. This is captured by the LPPL-model. Furthermore, the markets show strong correlation across sectors and assets akin to the standard lore about holography being applicable at strong coupling at large number of degrees of freedom. Hence in the future with the above framework we might hope to capture some of this correlation via a holographic model.

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[2] M. Flory, Discrete scale invariance in holography revisited, Fortschritte der Physik **66**, 10.1002/prop.201700093 (2018).

[3] A. Johansen, D. Sornette, and O. Ledoit, Predicting financial crashes using discrete scale invariance (1999), arXiv:cond-mat/9903321 [cond-mat].