# $X^2$ Model for Price Distributions

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## 1 Introduction

Financial markets exhibit complex dynamics, including heavy tails in price distributions, volatility clustering, and sudden jumps in asset prices. Traditional approaches often rely on agent-based models or stochastic processes to capture these phenomena.

We propose a stylized model inspired by statistical mechanics, where the effective Hamiltonian is

$$H = -\frac{1}{2}X^2, \quad X = \sum_i a_i s_i,$$

with $s_i$ representing either Ising, Heisenberg, or ternary spins $(-1, 0, 1)$, and $a_i$ sampled from Gaussian or Pareto distributions. The weighted magnetization

$$M(t) = \sum_i a_i s_i$$

represents market returns, or alternatively $M(t+1) - M(t)$ depending on the chosen interpretation. Furthermore, one can think of $M$ representing a form of market mode similar to $\sum \lambda_i \eta_i$ in portfolio optimization.

By sampling agent influences $a_i$ from heterogeneous distributions, we aim to study how collective agent interactions shape price dynamics, volatility, and emergent multimodality.

## 2 Model

### 2.1 Spin Variables

- Ising spins: $s_i = \pm 1$
- Heisenberg spins: continuous unit vectors
- Ternary spins: $s_i \in \{-1, 0, 1\}$

### 2.2 Agent Weights

The influence of each agent $a_i$ is drawn from either:

- Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$
- Pareto distribution, introducing heavy-tailed heterogeneity

## 2.3 Hamiltonian and Dynamics

The Hamiltonian is

$$H = -\frac{1}{2}X^2,$$

where $X = \sum_i a_i s_i$. Equilibrium behavior depends on $\beta$ in the Boltzmann distribution:

- $\beta > 0$: system tends to minimize $X$ (low-volatility, heterogeneous behavior)
- $\beta < 0$: system tends to maximize $X$ (high-volatility, herding behavior)

Time dynamics can be implemented using Glauber algorithm, where the choice affects volatility clustering. The weighted magnetization $M(t)$ directly maps to market returns, so all statistics on $M(t)$ correspond to financial observables.

## 2.4 Exact Distribution of Weighted Magnetization

In our approach, the weighted magnetization is defined as

$$M = \sum_{i=1}^{N} a_i s_i,$$

where the spins $s_i$ take values $\pm 1$ (or other discrete spin types). We assume that the spins are uncorrelated, which is valid e.g. in a high-temperature regime. This allows us to derive analytic formulas.

The uncorrelated nature enters explicitly when we write the characteristic function of $M$:

$$\phi(k) = \langle e^{ikM} \rangle = \prod_{i=1}^{N} \langle e^{ika_i s_i} \rangle = \prod_{i=1}^{N} \big[ \cos(ka_i) + i m_i \sin(ka_i) \big],$$

where $m_i = \langle s_i \rangle$ is the thermal average of spin $i$.

Here we need the self-consistency formula

$$m_i = \tanh(\beta a_i m_{\mathrm{SC}}),$$

with $m_{\mathrm{SC}}$ solving

$$m_{\mathrm{SC}} = \frac{1}{N} \sum_{i=1}^{N} a_i \tanh(\beta a_i m_{\mathrm{SC}}).$$

This formula is essential because it captures the collective influence of the spins on each other: the average of each spin depends on the effective "field" generated by the weighted sum of all spins.

The marginal probability distribution of $M$ is then obtained via an inverse Fourier transform:

$$P(M) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikM}\, \phi(k)\, dk.$$

Similarly, the distribution of returns $\Delta M = M(t+1) - M(t)$ can be computed using

$$\phi_{\Delta M}(k) = |\phi(k)|^2, \quad P(\Delta M) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\Delta M} |\phi(k)|^2\, dk.$$

In summary, the uncorrelated assumption enters at the level of the characteristic function, and the self-consistent $m_{\mathrm{SC}}$ formula is necessary to properly account for the collective effect of the spins in the distribution of $M$ and $\Delta M$.

# 3 Preliminary Results and Observations

- Qualitatively correct jump statistics are observed, but the underlying mechanism may differ from real markets. In reality, large jumps likely emerge from smaller agents learning about big players' behavior rather than from large agents acting alone.
- Multimodal price distributions appear for Pareto-distributed agents at low temperature (high $\beta$), whereas at high temperature (small $\beta$) distributions tend to be unimodal.
- Low-volatility periods arise from the system minimizing the market mode, reflecting heterogeneous behavior of agents.
- Noise-driven metastability, or possibly strange attractor dynamics, emerges at low temperatures: the system remains trapped in small neighborhoods before switching on longer timescales.
- Volatility clustering is influenced by the choice of dynamics (Glauber vs. other).
- Increasing the number of agents $N$ may reduce multimodality, suggesting multimodal distributions primarily occur when the system has few agents with heavy-tailed influence distributions.
- The weighted magnetization $M(t)$ captures the essential dynamics of returns, allowing analysis of volatility clustering, jump statistics, and multimodality.

# 4 Speculative Discussion

While the $X^2$ model reproduces some qualitative features of price dynamics, it does not fully capture real market mechanisms:

- Large jumps in reality may stem from the collective learning of small agents rather than the actions of big agents alone.
- Vanishing liquidity at certain price ranges suggests the need to explicitly model order flow and strategic interactions.
- Multimodal price distributions could be used as a diagnostic to infer the distribution of agent influences in real markets.

# 5 Future Work

1. Study the effect of system size $N$ on the persistence of multimodality. CLT might imply Gaussian.
2. Compare e.g. Glauber and other algorithms to incorporate different type of volatility clustering.
3. Map the weighted magnetization $M(t)$ more precisely to empirical financial returns.

4. Calibrate the model to real market data to infer the distribution of agent influences and validate multimodal features.

5. Investigate deviations from mean-field predictions at low temperatures and with correlated spins.

6. Explore agent interaction rules to model liquidity constraints and order flow mechanisms.

# 6 Conclusion

The $X^2$ model offers a promising simplified framework to study market price distributions and volatility dynamics through the lens of statistical mechanics. Preliminary results suggest the emergence of multimodal returns, volatility clustering, and metastability. While speculative, these features may provide insights into the behavior of heterogeneous agents in financial markets and guide future empirical investigations. Analytical solutions in the mean-field limit provide a useful reference. Code can be found from https://github.com/akapiisp/Spin-glass.

# References

[1] Taisei Kaizoji, Stefan Bornholdt, Yoshi Fujiwara, Dynamics of price and trading volume in a spin model of stock markets with heterogeneous agents, Physica A: Statistical Mechanics and its Applications, Volume 316, Issues 1–4, 2002, Pages 441-452, ISSN 0378-4371, https://doi.org/10.1016/S0378-4371(02)01216-5.

[2] Lattice $\phi^4$ field theory as a multi-agent system of financial markets,Dimitrios Bachtis, 2024,arXiv:2411.15813, https://arxiv.org/abs/2411.15813