Stochastic breakouts from instantons

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In this paper we study a simple model in which we simulate stochastic process which has controlled jumps (breakouts) on top of the standard Gaussian behaviour.

I. INTRODUCTION

In day-trading of derivatives [1] we see that a lot of the price fluctuations happen at support or resistance levels (S/R). The movement between these S/R levels is typically relatively abrupt. Such a behaviour is reminiscent of instantons (see e.g. [2] sec. 7) in physics where a tunneling of a particle through potential barrier happens almost instantaneously. In this paper we construct a model that has these properties and compare it to some price movement.

II. MODEL

The most naive way of modeling a price action of a derivative would be to assign a Gaussian transition probability to any subsequent step. Explicitly, what one could do is to define a stochastic process $\{X_t\}$ such that the transition probability is computed from the path-integral

$$\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) = \mathcal{N} \int_{x(t_0)=x_0}^{x(t_1)=x_1} \mathcal{D}x \exp\left(- \int_{t_0}^{t_1} dt \; \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 \right). \tag{1}$$

In physics the parameter m would be called a mass. Here $\mathcal{N}$ is a normalization constant to ensure

$$\int dx \; \mathbb{P}(X_{t_0} = x_0 \to X_{t_1} = x) = 1. \tag{2}$$

The integration over x is over all paths with boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$. We see here that the Gaussian measure in the path integral suppresses the choices of $x(t)$ that have large sudden changes as dx/dt is large at such point. Here X_t would represent the price of a derivative at time t. The paths in (1) allow negative values for x. We could restrict X_t to only have positive values by imposing that x can't be negative. One can evaluate the path-integral in (1) by discretizing the time-interval and evaluating the resulting Gaussian integrals (see Sec. III for the time-slicing) to obtain a Gaussian distribution

$$\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) \propto \exp\left(- \frac{m(x_1 - x_0)^2}{2(t_1 - t_0)} \right) \tag{3}$$

This is illustrated in FIG. 3.

The most obvious generalization to this model would be to include a potential $V(x)$ to the exponent:

$$\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) = \mathcal{N} \int_{x(t_0)=x_0}^{x(t_1)=x_1} \mathcal{D}x \exp\left(- \int_{t_0}^{t_1} dt \; \left[\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + V(x) \right] \right). \tag{4}$$

In the beginning we imposed that we want the system to mostly oscillate around a value, then jump almost instantaneously to another value and then continue oscillating around this new value. Such a behaviour is achieved in this model choosing the potential $V(x)$ in a way where there are local minima corresponding to these preset values around which we want the oscillation to happen. The jumps correspond to tunneling effects where the system tunnels from one local minima to another minima. From the path-integral perspective to leading order in m the choices of $x(t)$ that give these tunneling effects are called instantons and their profile is an instantaneous jump in x at some t^* (it looks like a step-function) [2] [3]. From the derivative pricing point of view these local minima correspond to predetermined S/R-levels of the particular derivative.

As we want to try to quantify a method used by traders, we should choose a derivative which is controlled mostly by traders. The chosen derivative should satisfy the following criteria:

- We want low market capitalization as traders do not have enough capital to move derivatives that have high market capitalization.
- We want to be able to leverage as traders want to have high return compared to the initial investment. The reason for this is that most traders do not have sufficiently large amount of capital to trade without leverage and get large enough profit.
- We want poor market sentiment because with good market sentiment there is money coming from other sources as well (for example retail investors). When the market sentiment is poor, only traders contribute to the volume.

Based on these we choose the pair CTSI/USDT trading on Binance around September 2023 at one hour time scale (i.e. one tick in the data corresponds to one hour) [4]. We want to track the price at relatively low time

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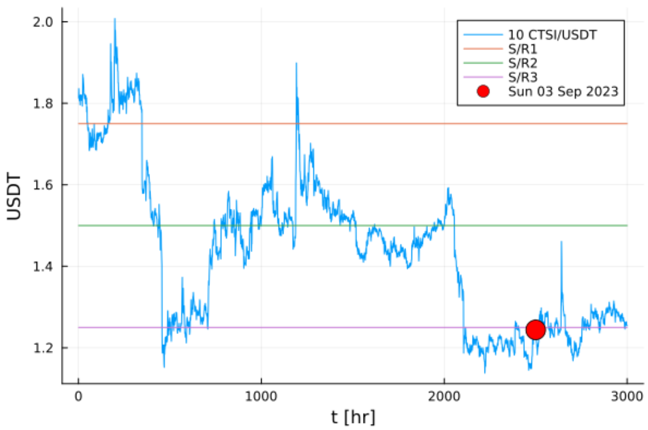

FIG. 1. The price action of CTSI/USDT before September 23 2023. We have scaled the original data by a factor of 10. The lines shown in the figure show the S/R-levels. One tick in the figure corresponds to an one hour. The breakouts (abrupt jumps) are clearly visible in the data.

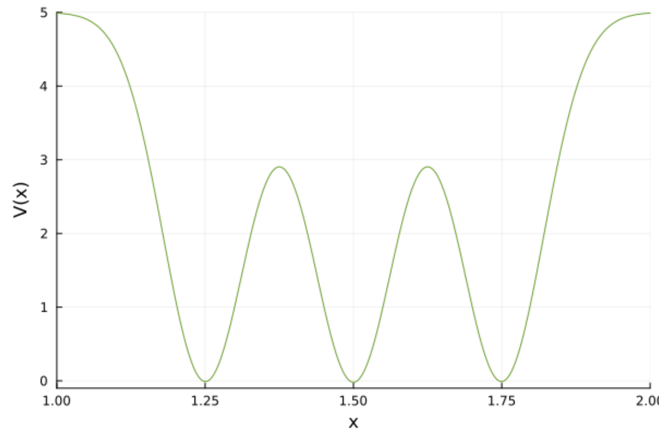

FIG. 2. The potential used in the path integral (4). The parameter values are $C = 5$, $\sigma_i = 0.01\ \forall i$.

scales as we assume that then the forces that move the price are day traders. At large time scales other phenomena come to play, like drift (positive/negative market sentiment). The potential we choose to use in this paper is

$$V(x) = C - C\sum_{i=1}^{n} \exp\left(-\frac{(x - x_i^{\mathrm{S/R}})^2}{\sigma_i}\right). \tag{5}$$

Here the parameters $x_i^{\mathrm{S/R}}$ correspond to the S/R-levels. For CTS/USDT we read from FIG. 1 that these are $x_1^{\mathrm{S/R}} = 1.25$, $x_2^{\mathrm{S/R}} = 1.5$ and $x_3^{\mathrm{S/R}} = 1.75$. The parameters σ_i control the width of the minima. The constant shift C does not matter, but we found it convenient to be ≈ 0. The potential is shown in the FIG. 2. The local minima for potential (5) are not necessarily strictly at $x_i^{\mathrm{S/R}}$ (it depends on the parameters). It is not a problem as the values $x_i^{\mathrm{S/R}}$ were determined visually.

III. NUMERICS

First we divide the time interval $t_1 - t_0 = \Delta t$ into n pieces where each piece has length $\delta t = (t_1 - t_0)/n$. After this we can write the path integral as

$$\begin{aligned}
&\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) \\
=&\mathcal{N} \int \prod_{i=1}^{i=n-1} dx^i \exp\left(-\sum_{j=0}^{j=n-1}\left[\frac{1}{2}m\left(\frac{x^{j+1} - x^j}{\delta t}\right)^2\right.\right. \\
&\left.\left.+ V\left(\frac{x^{j+1} + x^j}{2}\right)\right]\delta t\right).
\end{aligned} \tag{6}$$

Here $x^i = x(t_0 + i\delta t)$ so $x^0 = x_0$ and $x^n = x_1$. We chose to evaluate the potential at the middle point between two time steps. In the limit $n \to \infty$ this does not matter. However, we are simulating this system at finite n and the effect of this choice is that it suppresses the tunneling effects (if x^{j+1} and x^j are at two naighbouring minima, then the average is almost at the local maximum between these two minima). To perform the integrals $\int dx^i$ we use Monte Carlo integration. The idea in Monte Carlo integration is to approximate an integral by a discrete sum as

$$\int_D f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}. \tag{7}$$

Here the points $x_i \in D$ are sampled from the probability distribution $p : D \to [0, 1]$. The sum is guaranteed to converge to the integral when $N \to \infty$. However, we would like the sum to converge to the integral as fast as possible as in numerical computations we are restricted to finite values of N. For this reason some probability distributions p are preferred over others and this depends on the problem at hand. The simplest one is to sample the points $x_i \in D$ over a uniform distribution $p(x) = 1/\mathrm{Vol}_D$. The convergence of the sum is increased if we use a distribution $p(x)$ that has the same profile as $f(x)$. The technique of using different distribution than uniform is called importance sampling. We know that the end result is Gaussian for $V(x) = 0$. This motivates us to use a Gaussian distribution for our problem. See FIG. 3 for comparison between using uniform distribution and Gaussian distribution. In [5] another Monte Carlo technique for path integrals is discussed which is the so-called Metropolis algorithm. Next we describe how we implement the Gaussian sampling.

We generate N sample paths $\{x_0, x_k^1, x_k^2, ..., x_k^{n-1}\}$, where $k \in \{1, ..., N\}$, from Gaussian increments

$$p(x^i \to x^{i+1}) = \frac{1}{\sqrt{2\pi\delta t}} \exp\left(-\frac{m(x^{i+1} - x^i)^2}{2\delta t}\right). \tag{8}$$

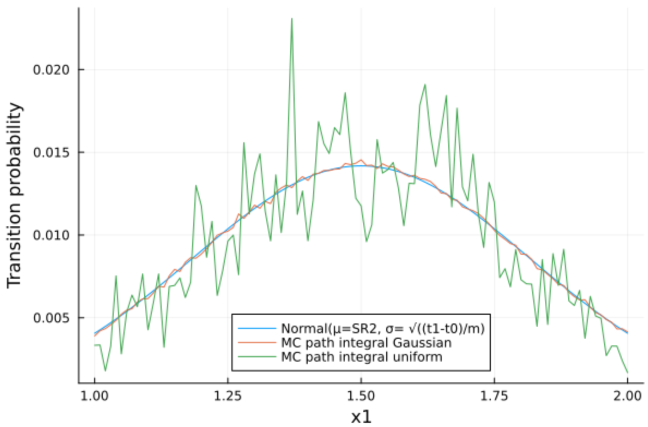

FIG. 3. Illustration on how much better the Gaussian distribution converges compared to using uniform distribution. Here we have set $V(x) = 0$ and the other parameters are $m = 100$, $x_0 = x_2^{\mathrm{S/R}}$, $n = 7$ and $N = 10000$. The convergence is even worse when we consider finer partitions of the time interval, i.e. n increases. This also shows how the path integral computes Gaussian transition probabilities (3) for vanishing potential.

The probability of a given path is

$$p(\{x_0, x_k^1, x_k^2, ..., x_k^{n-1}\}) = \frac{1}{\sqrt{2\pi(n-1)\delta t}} \exp\left(-\sum_{i=1}^{n-2} \frac{m(x_k^{i+1} - x_k^i)^2}{2\delta t}\right). \tag{9}$$

By denoting

$$f(\{x_0, x^1, x^2, ..., x^{n-1}, x_1\}) = \exp\left(-\sum_{j=0}^{j=n-1}\left[\frac{1}{2}m\left(\frac{x^{j+1}-x^j}{\delta t}\right)^2 + V\left(\frac{x^{j+1}+x^j}{2}\right)\right]\delta t\right) \tag{10}$$

the Monte Carlo integral can be written as ($\tilde{\mathcal{N}}$ is a new normalization constant) as

$$\begin{aligned}\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) &= \tilde{\mathcal{N}} \sum_{k=1}^{N} \frac{f(\{x_0, x^1, x^2, ..., x^{n-1}, x_1\})}{p(\{x_k^1, x_k^2, ..., x_k^{n-1}\})} \\ &= \tilde{\mathcal{N}} \sum_{k=1}^{N} \exp\left(-\left[\frac{1}{2}m\left(\frac{x_1 - x_k^{n-1}}{\delta t}\right)^2 + \sum_{j=0}^{j=n-1} V\left(\frac{x_k^{j+1}+x_k^j}{2}\right)\right]\delta t\right).\end{aligned} \tag{11}$$

The Gaussian factors nicely cancel when using using this distribution for the Monte Carlo integration. It is interesting to note that this expression resembles an integral of the form

$$\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) = \int \mathcal{D}(P(x)) \exp\left(-\int dt\, V(x(t))\right) \tag{12}$$

We can take care of the normalization by defining

$$\tilde{\mathbb{P}}(X_{t_1} = x_1 | X_{t_0} = x_0) = \frac{1}{\tilde{\mathcal{N}}} \mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0). \tag{13}$$

Then the transition probabilities are given by

$$\mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) = \frac{\tilde{\mathbb{P}}(X_{t_1} = x_1 | X_{t_0} = x_0)}{\sum_{x_1 \in S} \tilde{\mathbb{P}}(X_{t_1} = x_1 | X_{t_0} = x_0)} \tag{14}$$

where we assume that x_1 takes only discrete values in the set S. The possible values for x_1 we take to be $S = \{x_{\min} + i(x_{\max} - x_{\min})/n_1 \mid i \in \{0, 1, .., n_1\}\}$ for some n_1.

We also need to sample the transition probabilites we obtain from the Monte Carlo integration (11) to obtain simulated paths. To do this we use the inverse transform sampling technique. We need the cumulative distribution

$$\mathbb{C}(X_{t_1} \leq x | X_{t_0} = x_0) = \sum_{x_1 \leq x} \mathbb{P}(X_{t_1} = x_1 | X_{t_0} = x_0) \tag{15}$$

The cumulative distribution is a monotonic function so it is invertible as a function of x. Let the inverse be $\mathbb{C}^{-1} : S \to [0, 1]$. We draw a number $q \in [0, 1]$ from the uniform distribution. Now a sample drawn from our distribution (11) is given by $\mathbb{C}^{-1}(q)$.

Without exception we use parameter values $x_{\min} = 1$, $x_{\max} = 2$, $n_1 = 100$, $t_1 - t_0 = 10$, $\sigma_i = 0.01\ \forall i$ and $C = 5$ in our figures. We are basically enforcing the system to be contained in a finite interval $[x_{\min} = 1, x_{\max} = 2]$. This interval is split into $n_1 = 100$ possible positions that the system can occupy. We fix $t_1 - t_0 = 10$ throughout, because the relevant factor here is $m/(t_1 - t_0)$ so changing m essentially has the same effect as changing $t_1 - t_0$ (this is not exactly true as the potential term with our parametrization doesn't carry factor of m). The potential parameters are fine tuned to $\sigma_i = 0.01\ \forall i$ and $C = 5$ as with these parameters the potential has approximately the wanted form of having three minima with equal values around the S/R-levels. By changing σ_i:s arbitrarily might result to a potential where one minima is preferred over another, i.e the potential has an unique global minimum and is not degenerate.

IV. RESULTS

The system behaves as expected. From FIG. (4) we see that the system is prone to stay in the minima it started in. From FIG. (5) we see that as the mass m is increased, large jumps are less likely to happen. This comes from the factor $\exp(-\frac{m}{2}\left(\frac{dx}{dt}\right)^2)$ in the path integral which suppresses large changes. From FIG. (6) we

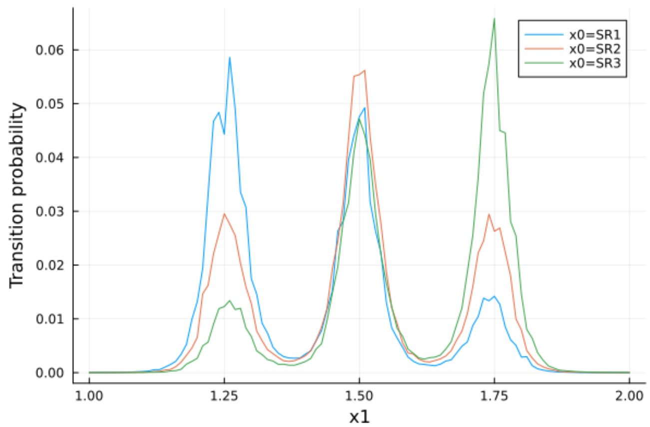

FIG. 4. The transition probabilities for various starting points x_0 for the potential (5). The other parameter values are $m = 100$, $n = 10$ and $N = 100000$

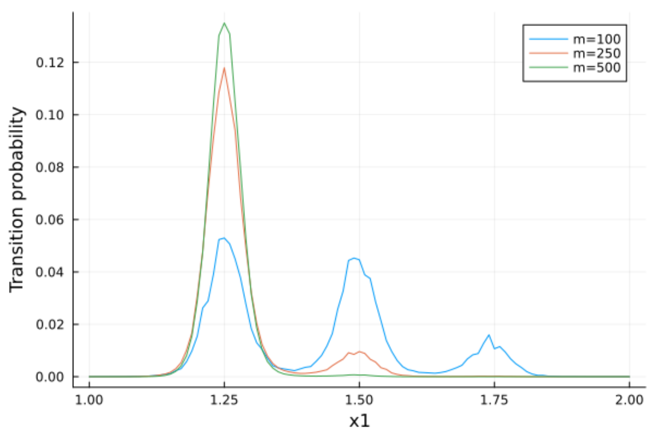

FIG. 5. The transition probabilities for different masses m using the potential (5). The other parameter values are $x_0 = x_1^{\mathrm{S/R}}, n = 10$ and $N = 100000$.

see that as the number of Monte Carlo iterations N is increased, the resulting transition probability converges to the correct distribution. From FIG. 7 we see that as the mesh size is increased (increasing n), the tunneling to another minima becomes more likely. We interpret this by emergence of instantons. As the mesh size is increased, there are more and more possibilities for the system to tunnel to another minima at intermediate steps.

In FIG. 8 and FIG. 9 we show some sample paths generated from the path integral (4). We see that the likelihood of tunneling is lowered as m is increased. In FIG. 8 too many tunneling events are happening. On the other hand, in FIG. 9 the tunneling events captures the breakouts in CTSI/USDT relatively well.

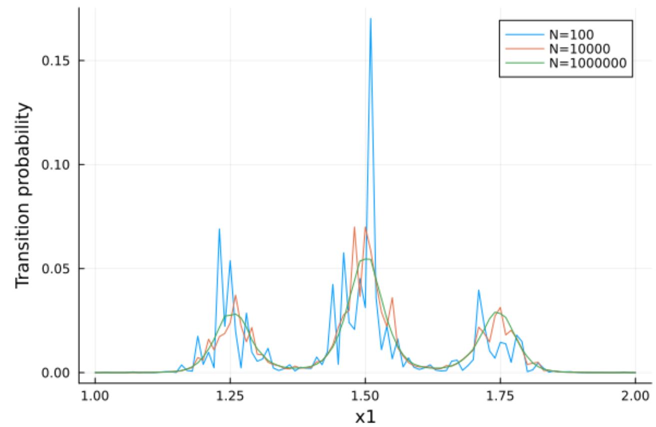

FIG. 6. The transition probabilities for different number of Monte Carlo iterations N using the potential (5). The other parameter values are $x_0 = x_2^{\mathrm{S/R}}, m = 100$ and $n = 10$

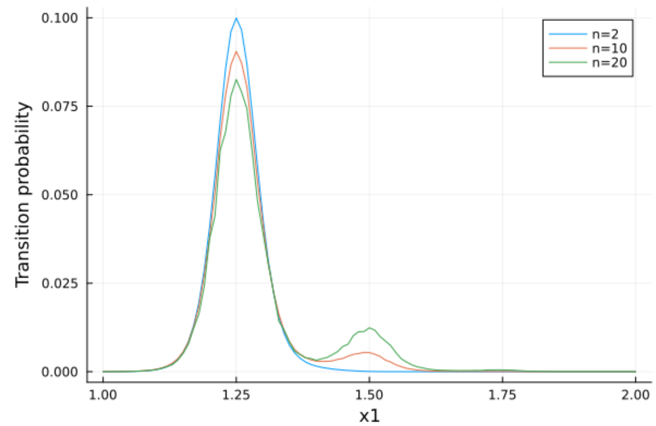

FIG. 7. The transition probabilities for different mesh sizes n using the potential (5). The other parameter values are $x_0 = x_1^{\mathrm{S/R}}$, $m = 200$ and $N = 100000$.

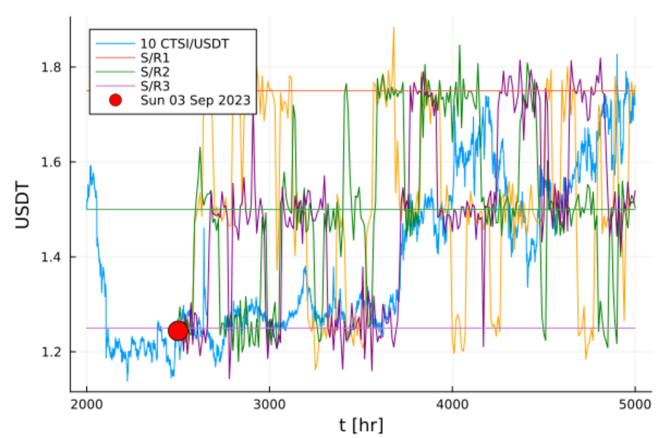

FIG. 8. The orange, green and purple curves are the sample paths generated by using the path integral. The parameter values here are $m = 250, n = 10$ and $N = 10000$.

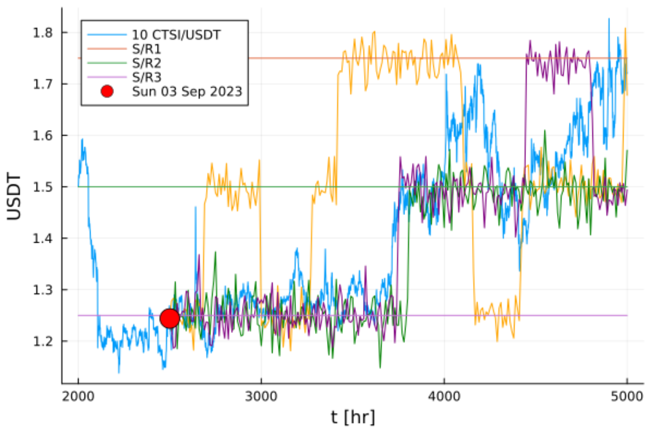

FIG. 9. The orange, green and purple curves are the sample paths generated by using the path integral. The parameter values here are $m = 450, n = 10$ and $N = 10000$.

V. CONCLUSIONS

The aim of this work was to build a quantitative model for a phenomenon observed by traders where a price of a derivative tends to jump abrubtly between the S/R-levels. We see from FIG. 8 and FIG. 9 that the model we have constructed has this property. The system oscillates around a minima for some time after which it tunnels to another minima, which most likely is a minimum next to the starting minimum (it is less likely for the system to instantaneously jump across a minimum). The idea of using only this principle to model a price action of a derivative seems quite far-fetched. Nevertheless, the quantitative model we have considered here can be fine-tuned to relatively high degree because of the potential $V(x)$. For example, for our potential choice (5) the minima are controlled by the parameters $x_i^{\mathrm{S/R}}$ and the variance around the minima is controlled by the parameters σ_i. One could possible determine these variances from the anterior data at even smaller time-scales where the system solely oscillates around a minimum. The amount of tunneling is affected by m. Maybe this parameter could also be determined from previous data by studying how many tunneling events happen during some period.

In trading there is a quote "bulls go up the stairs and bears go out the window". This means that it is easier to go down (bear) in price than to go up (bull). In our model this could be included by considering potentials whose value at the minima are not equal to each other. For example, a system with a potential that has three minima satisfying $V(x_1^{\mathrm{S/R}}) < V(x_2^{\mathrm{S/R}}) < V(x_3^{\mathrm{S/R}})$ is more likely to go through $x_3 \to x_2 \to x_1$ than $x_1 \to x_2 \to x_3$. Moreover at larger time-scales one might expect some sort of drift to happen where the price tends to either increase (positive market sentiment) or decrease (negative market sentiment) on average and not fluctuate around fixed levels. This can be modelled in our system by considering potentials that are time dependent $V(t, x)$. The techniques we have used here could be implemented for such potential ansatze as well.

Retrospectively a different choice for the potential (5) would have been better. The potential used here has the property that when the widths σ_i of the potential wells are changed, the value of the potential at these minima are also changed. It would have been nice to have total control over the values of the minima and the widths separate from each other. Maybe a better potential would be one where we do not have Gaussian wells but Gaussian potential barriers which separate the S/R-levels.

The program used in this project is open access and is available in `https://github.com/akapiisp/Stochastic-Breakouts.git`.

[1] A derivative is a contract or trade (or bet, depending on your prejudices) between two entities or counterparties whose value is a function of ‘‘derives from ‘‘the price of an underlying financial asset. This is from the book S. Blyth, An Introduction to Quantitative Finance (Oxford University Press, 2014).

[2] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, 1985).

[3] In the book by S. Coleman the role of mass m is played by $\hbar$. Also, the mass $\hbar$ in that book multiplies both the kinetic term and the potential. However, one can redefine the potential in such a way that it has a factor of mass.

[4] Binance is a cryptocurrency trading platform. Here USDT is a cryptocurrency that tries to mimic the US dollar, that is, one USDT is approximately one USD. CTSI is another cryptocurrency. Basically the pair CTSI/USDT tell us how much one unit of CTSI is in USD.

[5] M. J. E. Westbroek, P. R. King, D. D. Vvedensky, and S. D‘
rr, User‘‘s guide to monte carlo methods for evaluating path integrals, American Journal of Physics **86**, 293‘‘304 (2018).