Some constructions on ω-groupoids

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Outline

- Introduction to weak ω-groupoids
- Basic syntax of $\mathcal{T}_{\infty-\text{groupoid}}$ (describing the structure of weak ω-groupoids)
- Heterogeneous equality for syntactic terms
- Suspension and replacement
- Coherences constructions
- Semantic interpretation

Introduction to weak ω-groupoids I

- To internalize equality in types : setoids, groupoids
- A new interpreted of types : weak ω-groupoids (In Homotopy Type Theory)
  - weak ω-groupoid: A higher dimensional category (ω-category) where every morphism is an weak equivalence (equivalence for short)
  - Equivalence: invertible morphism up to all higher equivalence (generalization of isomorphism)
  - Weak: equality in coherence laws are up to equivalence (not strictly equal) e.g. $(f \circ g) \circ h \to f \circ (g \circ h)$
- How can we formalize weak ω-groupoids in type theory?
  - Warren's *strict* ω-groupoid model
  - Altenkirch and Rypacek's syntactic approach
  - Brunerie's syntactic approach: $\mathcal{T}_{\infty-\text{groupoid}}$

Basic syntax of $\mathcal{T}_{\infty-\text{groupoid}}$ I

- We use $\mathcal{T}_{\infty-\text{groupoid}}$ to describe the internal structure of a weak ω-groupoid

```
data Con          : Set
data Ty (Γ : Con) : Set
data Tm           : {Γ : Con}(A : Ty Γ) → Set
```

- The structure is inductively defined as

```
data Ty Γ where
  *     : Ty Γ
  _=h_  : {A : Ty Γ}(a b : Tm A) → Ty Γ
```

Structure of weak ω-groupoids I

- Equality on each levels:
- Setoid:

$$
\begin{aligned}
&\mathrm{id} : x = x \\
&\_^{-1} : x = y \to y = x \\
&\_ \circ \_ : y = z \to x = y \to x = z
\end{aligned}
$$

- Groupoid:

$$
\begin{aligned}
&\lambda : id \circ p = p \\
&\rho : p \circ id = p \\
&\alpha : p \circ (q \circ r) = (p \circ q) \circ r \\
&\kappa : p^{-1} \circ p = id \\
&\kappa' : p \circ p^{-1} = id
\end{aligned}
$$

Structure of weak ω-groupoids II

- In weak ω-groupoids, additionally we have more provable equalities on higher dimensions which are all weak (up to higher equivalences) e.g. horizontal composition, interchange law, coherence laws
- **Coherence laws** usually state that various compostions of elementary morphisms are equal. Example: There are two ways to show $(f \circ id) \circ g = f \circ g$

$$\begin{tikzcd}
(f \circ id) \circ g \arrow[r, "\alpha"] \arrow[dr, "\rho \cdot id"'] & f \circ (id \circ g) \arrow[d, "id \cdot \lambda"] \\
& f \circ g
\end{tikzcd}$$

- For higher dimensions things becomes much more complicated
- Infinitely many coherence constants

Contractible Contexts and Coherences I

- All coherence constants are derivable from J-eliminator of equality
- A analogous construction in $\mathcal{T}_{\infty-\text{groupoid}}$: **contractible contexts**
  - $\epsilon, *$
  - $\epsilon, x : *, y : *, \alpha : x = y$
  - ... $\Gamma, y : A, \alpha : x = y$ (Given $\Gamma \vdash A$ and $\Gamma \vdash x : A$)
- Anything in a contractible context is a coherence constant: intuitively the extension of contractible context internalise the J-eliminator

  Example: Assume $\epsilon, x : * \vdash x = x$ (weakening)

  $\Rightarrow \epsilon, x : *, x : *, \alpha : x = x \vdash x = x$ (J-eliminator)

  $\Rightarrow \epsilon, x : *, y : *, \alpha : x = y \vdash y = x$

Contractible Contexts and Coherences II

- Contexts for 0-level coherences in minimum contractible contexts:
  - $\epsilon, x : * \vdash x = x$ (id^*)
  - $\epsilon, x : *, y : *, \alpha : x = y \vdash y = x$ $(\_^{-1*})$
  - $\epsilon, x : *, y : *, \alpha : x = y, z : A, \beta : y = z \vdash x = z$ $(\_ \circ^* \_)$
- If we "replace" ϵ by arbitrary context Γ, and $*$ by arbitrary A, we obtain contexts for general 0-level coherences:
  - $\Gamma, x : A \vdash x = x$ (id)
  - $\Gamma, x : A, y : A, \alpha : x = y \vdash y = x$ $(\_^{-1})$
  - $\Gamma, x : A, y : A, \alpha : x = y, z : A, \beta : y = z \vdash x = z$ $(\_ \circ \_)$

Contractible Contexts and Coherences III

- Indeed any coherence constant exists in a context which we can substitute into a contractible context

```
data Tm where
   var   : ∀{Γ}{A : Ty Γ} → Var A → Tm A
   coh   : ∀{Γ Δ} → isContr Δ → (δ : Γ ⇒ Δ)
         → (A : Ty Δ) → Tm (A [ δ ]T)
```

Reasoning about syntactic terms

- Using homogeneous equality to reason about syntactic terms, we have to deal with **subst** e.g. subst $p\ x = y$, subst $p(\text{subst}\ p^{-1}\ x) = x$
- Heterogeneous equality for Tm

```
data _≅_ {Γ : Con}{A : Ty Γ}
      : {B : Ty Γ} → Tm A → Tm B → Set where
   refl : (b : Tm A) → b ≅ b
```

- Justification: The equality of inductively defined types are decidable, hence from Hedberg's Theorem they have UIP

Construction of coherences I

- To obtain a coherence term for arbitrary context Γ in two steps:
  1. a coherence term in a contractible context Δ
  2. a substitution $\Gamma \Rightarrow \Delta$
- To do the second step: *replacement* and suspension
- **Replacement**: Given an arbitrary type A in arbitrary context Γ, we can replace $*$ in a contractible context Δ by A and paste it onto A in Γ, such that we can obtain coherence B in Δ for type A

$$\frac{\Gamma \vdash A \quad \vdash \Delta \text{ contractible} \quad \Delta \vdash B}{\Gamma, \Delta^A \vdash B^A}$$

An example of Γ, Δ^A: $\Gamma, y : A, \alpha : x = y$

Construction of coherences II

- Intuitively, we can filter out variables in Γ which are unrelated to A. However it is very difficult to do that. Instead we build a new context using
- **Suspension**: build a minimum contractible context for type A of level n:
  - $(x_0 : *)$ (the one-variable context) for $n = 0$;
  - $(x_0 : *, x_1 : *, x_2 : x_0 =_{\mathsf{h}} x_1)$ for $n = 1$;
  - $(x_0 : *, x_1 : *, x_2 : x_0 =_{\mathsf{h}} x_1, x_3 : x_0 =_{\mathsf{h}} x_1, x_4 : x_2 =_{\mathsf{h}} x_3)$ for $n = 2$, etc.

  with Δ whose $*$ is replaced by A

$$\frac{\Gamma \vdash A \quad \vdash \Delta \text{ contractible} \quad \Delta \vdash B}{\Sigma A, \Delta^A \vdash B^A}$$

Construction of coherences III

- Thus we can define a substitution from a *replaced context* to a *suspended context* called **filter**

$$\Gamma, \Delta^A \Rightarrow \Sigma A, \Delta^A$$

Note: The suspended context is contractible because one-step suspension proves to preserve contractibility.

Construction of coherences IV

- Case: **Reflexivity**
  - 1st step: reflexivity in a minimum contractible context

```
refl*-Tm : Tm {x:*} (var v0 =h var v0)
refl*-Tm = Coh-Contr c*
```

  - 2nd step: reflexivity for arbitrary type A in arbitrary context Γ

```
refl-Tm     : {Γ : Con}(A : Ty Γ)
            → Tm (rpl-T {Δ = x:*} A (var v0 =h var v0))
refl-Tm A   = rpl-tm A refl*-Tm
```

Semantics

- A syntactic Grothendieck weak ω-groupoids is a globular set with an interpretation of syntactic coherence terms (coh)
- A globular set A consists coinductively of:
  - A set obj_A
  - For every $x, y : \mathrm{obj}_A$, a globular set $\mathrm{Hom}_A(x, y)$
- Example: the identity globular set $Id^\omega A$
  - $\mathrm{obj}_{Id^\omega A} = A$
  - $\mathrm{Hom}_{Id^\omega A}(a, b) = Id^\omega A(a = b)$
- The interpretation of contexts, types and terms

Conclusion

- Types bear the structure of weak ω-groupoids: the tower of iterated identity types
- An implementation of syntactic weak ω-groupoids in Agda
  - Basic syntax of the type theory $\mathcal{T}_{\infty-\text{groupoid}}$
  - Heterogeneous equality for terms
  - Constructions of coherences
  - Semantic interpretation with globular sets
- To complete a weak ω-groupoid model of type theory
- A computational interpretation of univalent axiom in Intensional Type Theory