List of Corrections

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Quotient Types in Type Thoery

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Abstract

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Quotient Types in Type Thoery

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The Thesis Abstract is written here (and usually kept to just this page). This thesis mainly covered the quotient types in type theory

Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include your project advisor. . .

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mention the definable quotient is very useful

1.5 Overview

compact, comprehensive Overview:Add overview of each part, as much as you can, and compact

In chapter 2, we will discuss the backgroud of this research. Type theory is a popular topic in theoretical computer science. It is quite powerful not only a a theory but also as a programming language. We use a dependent functional programming language called Agda which is design based on Martin-Löf type theory. The related work will also be discussed in this chapter.

In chapter 3, we will discuss quotient types which is the topic of this thesis in detail. Quotient types can be understood as a interpretation of quotient set in set theory. It is an extensional concept which is also related to other extensional concepts. It can be encoded in different ways. Categorically speaking it is a coequalizer, and a split quotient is a just a split coequalizer.

In chapter 4, we start introducing one of our achievements, the definable quotients. It is usually very unreadable, unorganised and complicated to write some programs without abstracting. It is also applied to quotient types. If we have some types that can be abstract as a quotient type of some common types, then it will be easily encoded and manipulated. As a example, integers can be encoded as the quotients types of paired natural numbers over the equivalence relation that two pairs are equal if they represent the same subtraction.

In chapter 5,

In chapter 6, we will talk about the setoid model approach to encode extensional concepts. The work is mainly extending the setoid model done by Altenkirch in [1] to quotient types.

In chapter 7, we will discuss the new area between mathematics and computer science – Homotopy Type Theory. We will talk about the higher inductive types and also the weak ω-groupoids-model which is used to interpret homotopy types in intensional type theory. Quotient types can be encoded Homotopy Type Theory simply.

Chapter 2

Background

Regular mathematics is based on classical logic and ZFC set theory. Set theory is a language to describe definitions of most mathematical objects, in other words, it serves as a foundational system for mathematics.

George Cantor and Richard Dedekind invented set theory in 1870s. However, in the 1900s, Bertrand Russell discovered a paradox in their system. In this naive set theory, there was no distinction between small sets like the set of natural numbers or the set of real numbers and "larger" sets like the set of all sets. This lead to the famous contradiction named after Russell. To avoid this paradox, he proposed the theories theory? of types [2] as an alternative to naïve set theory. Each mathematical object is assigned a type. This is done in a hierarchical structure such that "larger" sets and small sets reside in different levels. The set of all sets is no longer on the same level as its elements and the paradox disappears.

2.1 Type Theory

The elementary notion of type theory is *type* which plays a similar role to set in set theory, but differs fundamentally. Every object in type theory comes with its unique type, while an object in set theory can appear in multiple sets and we can talk about an object without knowing which sets it belongs to. To explain the difference, we use the the number 2 as an example. In set theory, 2 is not an element of only one specific set, it belongs to the set of natural numbers $\mathbb{N}$ and also the set of integers $\mathbb{Z}$. While in type theory, it is impossible to avoid mentioning the type of object 2. Usually the term suc (suc zero) stands for 2 of type $\mathbb{N}$ and we have a different term of type $\mathbb{Z}$ constructed by constructors of $\mathbb{Z}$ which is a different object to the one of $\mathbb{N}$.

Since Russell's type theory, a variety of type theories have been developed by mathematicians and computer scientists, for example Gödel's System T [3]. There are two families of famous type theories building the bridges between mathematics and computer science, *lambda calculus* and *Martin-Löf type theory*.

2.1.1 Lambda Calculus

Alonzo Church introduced lambda calculus in the 1930s. He first introduced an untyped lambda calculus which turned out to be inconsistent due to the Kleene-Rosser paradox ?cite? . Then he refined it with adding types. This theory is also called Church's theory of types or simply typed lambda calculus is introduced as a foundation of mathematics. An important change is that functions become primitive objects which means functions are types defined inductively using the $\rightarrow$ type former. It is widely applied to various fields especially computer science. Some languages are extentions of lambda calculus, for example Haskell. Haskell belongs to one of the variants of lambda calculus called System F, although it has evovled into System FC recently. There are also other refinements of lambda calculus which is illustrated by the λ-cube [4].

2.1.2 Per Martin-Löf's Type Theories

In 1970s, Per Martin-Löf [5, 6] developed his profound intuitionistic type theory. His 1971's formulation which is impredicative was proved to be inconsistent with the Girard's paradox [7]. The impredicativty means that for a universe U, there is an axiom $\mathsf{U} \in \mathsf{U}$. The later version is predicative and is more widely used.

Like set theory, it can also serve as also a foundation of constructive mathematics [8]. Different to set theory whose axioms are based on first-order logic or intuitionistic logic, Martin-Löf type theory provides a means of implementing intuitionistic logic. This is achieved by the Curry-Howard isomorphism:

"propositions can be interpreted as types and their proofs are inhabitants of that types"

more explanation

Different to simply lambda calculus, dependent types are introduced.

Definition 2.1. *Dependent type.* Dependent types are types that depends on values of other types [9].

With dependent types, the quantifiers like $\forall$ and $\exists$ can be encoded. The Curry-Howard isomorphism is then extended to predicate logic. A predicate on X can be written as a dependent type Px where $x : X$.

There are also two variants of Martin-Löf type theory based on the treatment of equality. The notion of equality is one of the most profound topic in type theory. we have two kinds of equality, one is definitional equality, the other is propositional equality.

Definition 2.2. *definitional equality* definitional equality is a judgement-level equality, which holds when two objects have the same normal forms[10].

With dependent types, it is possible to write a type to encode the equality of objects.

Definition 2.3. *Propositional equality* Propositional equality is a type which represents a propostion that two objects of the same type are equal.

Intuitively if two objects are definitionally equal, they must be propositionally equal.

$$\frac{a \equiv b}{a = b}\ Id - intro$$

But how about the other way around? Are two propositional equal objects definitional equal?

In intensional type theory, the answer is no. Propositional equality (also called intensional equality [10]) is different to definitional equality. The definitional equality is always decidable hence type checking that depends on definitional equality is decidable as well [1]. Therefore intensional type theory has better computational behaviors. Types like $\mathsf{a} = \mathsf{b}$ which stands for a propostion that a equals b are propositional equalities. They are some types which we need to prove or disprove by construction. Each of them has an unique element refl which only exists if a and b are definitionally equal in all cases. However it is not enough for other extensional equalities, for example the equality of functions.

In extensional type theory, the propositional equality is extensional and is undistinguished with definitional equality, in other words, two propositional equal objects are judgementally equal. This is called reflecition rule.

$$\frac{a = b}{a \equiv b}\ Reflection \tag{2.1}$$

Objects of different normal forms, for example point-wise equal functions or different proofs of the same proposition, may be definitionally equal. This is called functional extensionality.

$$\frac{f g : A \to B, \forall a : A, f\ a = g\ a}{f = g}\ functional - extensionality \tag{2.2}$$

It is provable in extensional type theory.

Lemma 2.4. *We can prove 2.2 in extensional type theory where we have 2.1.*

Proof. Suppose $\Gamma \vdash f\ a = g\ a$, with reflection rule we have $\Gamma \vdash f\ a \equiv g\ a$. Then using ξ-rule, we know that $\Gamma \vdash \lambda a.f\ a \equiv \lambda a.g\ a$. From η-equivalence, we know that $\Gamma \vdash f \equiv g$. We can conclude that $\Gamma \vdash f = g$. $\square$

In intensional type theory, this is not provable. If we add it as an axiom, we will lose the canonicity since we can construct a natural number with this extensionality and substitution for propositional equality. It means that the definitional equality is not decidable and type checking becomes undecidable as well. The type checker will not always terminate.

Usually we have to make a choice of them, either the better adpation of extensional equality or better computational behaviors. Agda chooses the intensional one while NuPRL chooses the extensional one. However Altenkirch and McBride introduced a variant of extensional type theory called *Observational Type Theory* [11] in which definitional equality is decidable and propositional equality is extensional.

Martin-Löf type theory can be encoded as programming languages in which the evaluation of a well-typed program always terminates [10]. There are various implementations based on different variants of it, such as NuPRL, LEGO, Coq, Epigram and Agda. Since we usually discuss in intensional type theory, we will use Agda throughout this thesis.

2.2 Agda

Agda is a dependently typed functional programming language which is designed based on intensional version of Martin-Löf type theory [12].

As we have seen, Martin-Löf type theory is based on the Curry-Howard isomorphism: types are identified with propositions and terms (or programs) are identified with proofs. It turns Agda is into a proof assistant like Coq, which allows users to do mathematical reasoning. We can also reason about Agda programs inside itself. Usually to prove the correctness of programs, we need to state some theorems of programming languages on the meta-level, but in Agda we can prove and use these theorems alongwith writing programs.

There are more features of Agda as follows:

- *Dependent type*. As mentioned in 2.1, dependent types are types that depends on values of other types [9]. They enable us to write more expresive types as program speficication or propositions in order to reduce bugs. In Haskell and other Hindley-Milner style languages, types and values are clearly distinct [13], In Agda, we can define types depending on values which means the border between types and values is vague. To illustrate what this means, the most common example is VectorAn where we can length-explicit lists called vectors. It is a data type which represents a vector containing elements of type **A** and depends on a natural number **n** which is the length of the list. We can specify types with more constraints such that the we can express what programs we can better and leave the checking work to the type chekcer. For instance, to use the length-explicit vector, we will not encounter exceptions like out of bounds in Java, since it is impossible to define such functions before compiling.

- *Functional programming language*. As the name indicates that, functional programming languages emphasizes the application of functions rather than changing data in the imperative style like C++ and Java. The base of functional programming is lambda calculus. The key motivation to develop functional programmming language is to eliminating the side effects which means we can ensure the result will be the same no matter how many times we input the same data. There are several generations of functional programming languages, for example Lisp, Erlang, Haskell etc. Most of the applications of them are currently in the academic fields, however as the functional programming developed, more applications will be explored.

- *A proof assistant* Based on the Curry-Howard isomorphism, we have predicate logic available in Agda and we can prove mathematical theorems and theorems about the programs encoded in Agda itself.

  The general approach to do theorem proving in Agda is as follows: First we give the name of the proposition and encode it as the type. Then we can gradually refine the goal to formalise a type-correct program namely the proof. As long as we have the proof, it can be used as a lemma in other proofs or programs. Usually, there are no tactics like in Coq (it may be implemented in the future). But with the gradually refinement mechanism, the process of building proofs is very similar to conceiving proofs in regular mathematics.

As a functional programming languages, Agda has some nice features for theorem proving,

- *Pattern matching.* The mechanism for dependently typed pattern matching is very powerful [14]. We could prove propositions case by case. In fact it is similar to the approach to prove propositions case by case in regular mathematics. Pattern match is a more intuitive way to use the eliminators of types.

- *Inductive & Recursive definition.* In Agda, types are often defined inductively, for example, natural numbers is defined as

```
data ℕ : Set where
  zero : ℕ
  suc  : (n : ℕ) → ℕ
```

  The function for inductive types are usually written in recursive style, for example, the double function for natural numbers,

```
double : ℕ → ℕ
double zero = zero
double (suc n) = suc (suc (double n))
```

  The availability of recursive definition enables programmers to prove propositions in the same manner of mathematical induction.

- *Construction of functions.* One of the advantage of using a functional programming language as a theorem prover is the construction of functions which makes the proving more flexible.

  In functional programming languages, complicated programs are commonly built gradually using aunxiliary functions and frequently used functions in the library.

  Described as a proof assistant, complicated theorems are commonly proved gradually using lemmas and other theorems we have proved.

  This decreases the difficulty of interpreting proofs in mathematics into Agda.

- *Lazy evaluation.* Lazy evaluation could eliminate unecessary operation because Agda is lazy to delay a computation until we need its result. It is often used to handle infinite data structures. [15]

Agda also has some special functions in its interactive emacs interface beyond simple functional programming languages.

- *Type Checker.* Type checker is an essential part of Agda. You can use to to type check a file without compiling it. It is the type checker that detect type mismatch problem and for theorem proving, it means the proof is incorrect. It interactively shows the goals, assumptions and variables when buiding a proof.

  The coverage checker makes sure that the patterns cover all possible cases [16].

  The termination checker will warn possiblily non-terminated error. The missing cases error will be reported by type checker. The suspected non-terminated definition can not be used by other ones. All programs must terminate in Agda so that it will not crash [13]. The type checker then ensures that the proof is complete and not been proved by itself.

  In Agda, type signatures are essential due to the presence of type checker.

- *Interactive interface.* It has a Emacs-based interface for interactively writing and verifying proofs. With type checker we can refine our proofs step by step [16]. It also has some convenient functions and emacs means the potential to be extended.

- *Unicode support.* In Haskell and Coq, unicode support is not an essential part. However in Agda, to be a better theorem prover, it reads unicode symbols like: β, $\forall$ and $\exists$ and supports mixfix operators like: $+$ and $-$, which are very common for mathematics. It provides more meaningful names for types and lemmas and more flexible way to define operators. This also improve the readablity of the Agda proofs. For example, the commutativity of plus for natural numbers can be encoded as follows

  $$\mathsf{comm} : \forall\ (a\ b : \mathbb{N}) \to a + b \equiv b + a$$

  We can use symbols we are familiar in regular mathematics.

  Secondly we could use symbols to replace some common-used properties to simply the proofs a lot. The following code was simplied using several symbols,

  Finally, we could use some other languages characters to define functions such as Chinese characters.

- *Code navigation.* As long as a program is loaded, it provides shortcut keys to move to the original definitions of certain object and move back. In real life programming it alleviates a great deal of work of programmers to look up the library.

- *Implicit arguments.* Sometime it is unnessary to write an argument since it can be inferred from other arguments by the type checker. It can simplify the application of functions and make the programs more concise. For example, to define a polymorphic function id,

$$\mathsf{id} : \{A : \mathsf{Set}\} \to A \to A$$
$$\mathsf{id}\ a = a$$

  Whenever we give an argument a, its type A must be inferable.

- *Module system.* The mechanism of parametrised modules makes it possible to define generic operations and prove a whole set of generic properties.

- *Coinduction.* We can define coinductive types like streams in Agda which are typically infinite data structures. Coinductive occurences must be labelled with ∞ and coninductive types do not need to terminate but has to be productive. It is often used in conjunction with lazy evaluation. [17]

With these helpful features, Agda is a very powerful proof assisstant. It does not magically prove theorems for people, but it really helps mathematicians and computer scientists to do formalised reasoning with verification by high-performance computers.

2.2.1 basic syntax

to help the reader understand the Agda code in text

To understand the code in this thesis, I will introduce some basic types and syntax of Agda.

First of all, like other languages, we use "=" for function definition rather than propositional equality type former which is "$\equiv$". This is inconsistent with our conventional choices of symbols in articles, but it follows the conventions in Haskell and other programming languages that "=" is used for definition.

Different to Haskell, we use single colon : for typing judgement, for example a:A means that a is of type A.

We have universe levels parameters in a lot of definitions which makes code looks unnecessarily cumbersome. We will follow the **typical ambiguity** in this thesis which says that we write A:Set for A:Set$a$ and Set:Set which stands for Set$i$:Set$(i+1)$.

The underscore marks the spaces for the explicit arguments in non-prefix operators.

2.2.2 Identity Type

Identity type is the type introduced by Martin-Löf to encode the propositional equality for definitionally equal terms [10]. For any two terms of $\mathsf{a}\,\mathsf{b} : \mathsf{A}$, we have the type $\mathsf{Id}(\mathsf{A}, \mathsf{a}, \mathsf{b})$ which is inhabitted when a and b are definitionally equal. Here we use an alternative equivalent version named after Paulin-Mohring which is parameterized with the left side of the identity.

```
data _≡_ {A : Set} (x : A) : A → Set where
  refl : x ≡ x
```

In Agda eliminators are not automatically derived for the types defined. Instead we have pattern matching generally which is sometimes stronger than eliminators. As long as we pattern match on a variable of an identity type with the unique inhabitant refl, all occurences of both variables become the same. It is stronger and it provides the eliminator J.

```
J : (A : Set)(a : A) → (P : (b : A) → a ≡ b → Set)
  → P a refl
  → (b : A)(p : a ≡ b) → P b p
J A .b P m b refl = m
```

2.2.3 Extensionality

In regular mathematics, equality does not only exists between intensionally equal terms. Pointwise equal functions are usually identified and it is called functional extensionality as we mentioned 2.2. Therefore the identity type in intensional type theory is not powerful enough, we need extensionality in intensional type theory.

As Martin Hofmann summarises in [18], there are several related extensional concepts, *Functional extensionality*, *Uniqueness of identity*, *Proof-irrelevance*, *Subset types*, *Propositional extensionality*, *Quotient types*. These notions are not available in currect setting of intensional type theory, but they are worth interpreting to help both Mathematics and programs constructions. However, it only makes sense if the type-checking decidability and terms canonicity are not sacrificed.

Quotient types is one of the most interesting extensional concepts in type theory. It generally enables us to redefine equality on a type with a given equivalence relation so that we have more tools to formalise mathematical objects, like the real numbers.

2.3 Homotopy Type Theory

Cosider moving this part to chapter 7

Homotopy Type Theory is a variant of intensional Martin-Löf type theory which is a new branch developed between theoretical computer science and mathematics. Vladimir Voevodsky found a surprising connection between homotopy theory and type theory [19]. He proposed the univalence axiom, which identifies isomorphic structures, as a univalent foundation for mathematics.

In Homotopy Type Theory, there is an observation that notions in type theory can be interpreted by homotopy-theoretical terms. A type is regarded as a *space* and a term of this type is a *point* of this space. Functions between types are *continous maps* and identity types are usually considered as *paths*. Identity types of identity types are *homotopies*. Although these notions are originally defined with topological bases, we only employ them as homotopical notions on a higher level.

As univalence axiom states, equality is equivalent to equivalence. Acutally it can be seen as an formal acceptance of the "common sense" in Mathematics that isomorphic structures can be identitfied. The higher structures of the equivalence also allows us to study the different ways of identification. Therefore it is more appropriate to interpret types as higher groupoids. People is trying to implement Homotopy Type Theory in intensional type theory and one possible way is to interpret weak ω-groupoids first in Agda. The author has done some work in this direction which can be found in Chapter 8.

Higher inductive types are another important ingredient of Homotopy Type Theory. It provides us an enriched way to define types, with the paths as constructors of the types as well. A circle $\mathsf{base} : \mathbb{S}^1$ can be *inductively* defined with two constructors,

- A point $\mathsf{base} : \mathbb{S}^1$, and

- A path $\mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$.

The eliminator for this type has to take into account the path as well.

Homotopy Type Theory does not only help us model type theory with a focus on the equality, but also provides mathematicians type theoretical tools to study homotopy theory.

I will not explain this topic in detail here but in chapter 7.

For further reference, a well-written text book on Homotopy Type Theory which is written by many brilliant mathematicians and computer scientists is available [20].

2.4 Some conventions

What conventions we will follow in this thesis and some commonly used symbols and function

The universe of small types is encoded as $\mathbf{Set}_0$ or **Set** rather than **Type**, even though it is not a set in set-theoretical sense.

2.5 a very short introduction to category theory

Category theory is a very useful tools to formalise mathematical notions, particularly focussing on morphisms which can be functions, relations and trasformations. A category has a collection of objects and one collection of arrows for each pair of objects. A simple finite category can be visualised as a directed diagram but there are also a set of conditions which are callled categorical laws to obey.

The most accessible example is the category of sets. Objects are sets, arrows are functions and all categorical conditions fulfilled. It is also very helpful to formulate type theoretical concepts in a categorical way. Type theory and category theory are closely related, especially Homotopy type theory [1].

Category theory abstracts a lot of similar concepts in different fields and provides a concise language for mathematics for mathematicians.

[1] http://ncatlab.org/nlab/show/relation+between+type+theory+and+category+theory

2.6 Extensional concepts in intensional type theory

2.6.1 Propositional extensionality

2.6.2 Functional extensionality

2.6.3 Hedberg Theorem

2.6.4 Quotient types

2.6.5 Why do we use type theory?

one page explanation of why type theory is useful

It is always debatable to choose set theory or type theory. First of all, it depends on whether do you think in a constructive way: does any proposition can be claimed true only if we have witness? It the answer is yes, then type theory is a better choice. From a computer scientist's point of view, it is more natural to think in a constructive way. Something without a type makes no sense to us because we are not sure what it stands for and how do we use it. The type definition describe the syntax so that some symbol makes sense and the semantic meaning may be revealed from the construction.

There is another question, whether mathematics is a collection of patterns and laws which is observed, or it is a system created and built by people to explain the patterns and laws in the world. I think people prefer the second answer usually accept the type theory more easily, although most people (probably 99.9 percent) prefer the first one. When we learn what is natural numbers, we learn it as "numbers like 1, 2, 3 ,4 and perhaps 0", the commutative law, associate law are axioms because there is no way to prove it if we introduce it in this manner. We are convinced by some examples like "$2 + 3 = 3 + 2$" and we find it works for most of the cases then we accept it by observations. It is some methods physicians used a lot – to conclude some laws from a number of facts. It is a proper method for physicians because what they research on is world can only be observed. However for mathematics, even though it is applied to the real world, it is a system completely created by people. People used their fingers to count, wrote symbols for results, even though it was very shallow it is obviously a aritificial system. People extend

Type theory is strongly connected with computation theory. Set

Type theory has fewer axioms, simpler model than set theory which has mutual foudations: logic and axioms.

Chapter 3

Quotient Types

should say something of computer science first rather than mathematics maybe, because it is a compu

Quotient types is an important part in dependently-typed programming languages. It makes some type much easier to define and for some other types possibly to define. Also there are a set of quotient properties such that we can lift functions from the base types. I will start to introduce the quotients from mathematical quotients to quotients in programmming languages to give you an overview what quotient types are.

3.1 Quotients in Mathematics

moving the categorical intuition earlier, because it essentially generalises the quotients in different top

Quotient is a basic notion in mathematics. Usually the first quotient we learn is the result of division. $8 \div 4$ (or $8/4$) gives the result 2 which is called quotient.

At first most mathematicians are only interested in numbers. As long as they start working on other mathematical objects, some more abstract structures, many notions are extended. As a simple case, the product of numbers is extended to the product of vectors and the product of sets.

Similarly, quotient is also extended to other objects, for example the quotient in set theory.

3.1.1 Example : quotient sets

In set theory, we have a similar operation which turns some set into another set but the divisor is not the same kind of object as the dividend. We use equivalence relation to divide a set,

Definition 3.1. Equivalence relation. An equivalence relation is a binary relation which is refleixive, symmetric and transitive.

Intuitively, given any equivalence relation a set is partitioned into some cells, so that the elements equivalent to each other are in the same cell. The cells are called equivalence classes.

Definition 3.2. Equivalence class.

$$[a] = \{x : A \mid a \sim x\} \tag{3.1}$$

The set of these equivalence classes is called the quotient set.

Definition 3.3. Quotient set. Given a set A equipped with an equivalence relation $\sim$, a quotient set is denoted as $A/\sim$ which contains the set of equivalence classes.

$$A/\sim = \{[a] \mid a : A\} \tag{3.2}$$

Not only in set theory, the quotient of some algebraic structures is a common notions in other branches of mathematics. The notion of equivalence relation is extended to spaces, groups, categories and so does the quotient derived using the same construction.

Generally speaking, it describes the collection of equivalent classes of some equivalent relation on sets, spaces or other abstract structures. In type theory, following similar procedure, quotient type is also a conceivable notion.

3.1.2 Quotient types in extensional type theory

In extensional type theory like NuPrl, it is possible to redefine equality type of some types. However there is also some problems about it:

3.1.3 Quotient types in intensional type theory

Quotient types are not available in original intensional type theory. Alternatively we have *setoids* to simulate quotients.

Definition 3.4. *Setoid.* A setoid $(A, \sim)$: $\mathbf{Set}_1$ is a set [1] A : $\mathbf{Set}$ equipped with an equivalence relation $\sim : A \longrightarrow A \longrightarrow \mathbf{Prop}$.

It contains Carrier for an underlying set, _≈_ for a binary relation on Carrier and a proof that it is an equivalence relation.

In Agda, we define a setoid as

```
record Setoid : Set₁ where
  field
    Carrier       : Set
    _≈_           : Carrier → Carrier → Set
    isEquivalence : IsEquivalence _≈_
```

We can use setoids to represent quotients, just like the quotient 4 can be represented as the pair $(8, 2)$. However several problems arise from this approach.

First of all, originally many operations and types are defined based on sets, which means that we have to redefine all these types and operations for setoids because they are different sorts. It is easy to see if we consider a question: how to represent a quotient if its base type A is already represented by a setoid.

Secondly, from a programming perspective, setoids are unsafe because we have access to the underlying sets [21]. The operations on setoids may not respect the equivalence relation and make no sense.

Therefore, ideally it is better that the object represent a quotient should also be of type **Set**, just as if we divide 8 by 2 we prefer 4 than $(8, 2)$ which makes more sense and can be manipulated uniformly. It is also the case in the other mathematical theories, the base object and the quotient object are of the same sort. So how quotient type should look like in intensional type theory?

Given a setoid $(A, \sim)$, a type Q : $\mathbf{Set}$ can represent the quotient type of this setoid, if it has several laws:

[1] Setoid could be universe polymorphic.

$$\frac{A : \mathbf{Set} \quad \sim : A \to A \to \mathbf{Prop}}{A/{\sim} : \mathbf{Set}}\; Q - \mathbf{Form}$$

$$\frac{a : A}{[a] : A/{\sim}}\; Q - \mathbf{Intro}$$

$$\frac{\begin{array}{l} B : A/{\sim} \to \mathbf{Set} \\ f : (a : A) \to B\,[a] \\ (a, b : A) \to (p : a \sim b) \to f a \overset{p}{=} f b \end{array}}{\hat{f} : (q : Q) \to B\, q}\; Q - \mathbf{elim}$$

3.2 Categorical intuition

3.2.1 split quotient/coequalizer

Categorically speaking, a quotient is a coequalizer.

Definition 3.5. Coequalizer. Given two objects X and Y and two parallel morphisms $f, g : X \to Y$, a coequalizer is an object Q with a morphism $q : Y \to Q$ such that $q \circ f = q \circ g$. It has to be universal as well. Any pair (Q' , q') $q' \circ f = q' \circ g$ has a unique factorisation u such that $q' = u \circ q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "u"] \\ & & Q' \end{tikzcd}$$

$$\begin{tikzcd} A \times A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "{[\cdot]}"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "u"] \\ & & Q' \end{tikzcd}$$

a split quotient is just a split coequalizer, with an embedding function which finds a representative in each equivalence class.

3.2.2 Adjunction between Sets and Setoids

From a higher point of view, Quotient is a Functor which is left-adjoint to ∇ which is the trivial embedding functor from **Sets** to **Setoids**.

Definition 3.6. $\nabla A = (A, \equiv)$, $Q(B, \sim) = B/\sim$

We have following isomorphism for the adjunction of the Quotient Functor and ∇ functor.

$$\frac{B/\sim \;\to A}{\overline{(B, \sim) \to \nabla A}}$$

3.3 Impredicative encoding of quotient types

In set theory, we have the subset relation such that we can construct equivalence classes and then quotient set. However in intensional type theory, the subset types are also unvailable and equivalence classes cannot be implemented.

The introduction of the univalence axiom for propositions, which is also called propositional extensionality changes this situation.

Definition 3.7. *propositional extensionality.*

$\forall P, Q : \textbf{Prop}, (P \iff Q) \iff P = Q.$

Voevodsky firsrtly constructs quotients using the following impredicative approach in Homotopy Type Theory using Coq [22].

suppose we have $A : Set$, $\sim : A \to A \to Set$.

A quotient type $A/\sim$ is defined as a predicate and a proof that it gives rise to an equivalence class and it is non-empty.

$A/\sim = \Sigma P : A \to \textbf{Prop}, EqClass(P) \times (\exists x : A, P(x))$

where the equivalence class proof is encoded as

$EqClass(P) = \forall a, b : A, P(a) \to (P(b) \iff a \sim b)$

The proof part on the right is the $(-1) - truncation$ such that different proofs will not gives different terms for the same equivalence class.

question of where shall we apply pe with the proposional extensionality we can prove that

Theorem 3.8. *Given* $(P, prf) : A/\!\sim$ *, all proofs of* $\exists x : A, P(x)$ *are equal*

Proof. Given any two proofs of $\exists x : A, P(x)$ written as (x, px) and (y, py), apply the $EqClass(P)$ to (x, y, px, py) we know that $x \sim y$. Hence the truncation of □

Moreover we have another suprising theorem:

Theorem 3.9. *if we have propositional univalence, we can prove that all quotients are effective.*

Proof. Suppose we have a set A with equivalence relation $\sim$.

Given $a : A$, and a predicate $P : A \to Prop$ defined as

$$Px = a \sim x$$

With the lifting operator for quotient types, we have a lifted version of P such that

$$\hat{P}[x] = Px$$

Suppose we have a premise $[a] = [b]$, it is true that

$$[a] = [b]$$

and then

$$Pa = \hat{P}[a] = \hat{P}[b] = Pb$$

which is just

$$a \sim a = a \sim b$$

with propositional univalence, we know that they are logically equivalent

$$a \sim a \iff a \sim b$$

since $a \sim a$ is the reflexivity which is true trivially,

$$a \sim b$$

is also true.

Therefore we have a proof that $[a] = [b]- > a \sim b$ which means that the quotient is effective. □

3.3.1 Functional extensionality and quotient types

As we have mentioned before, in intensional type theory propositional equality $Id(A, a, b)$ is inhabited if and only if a and b are definitionally equal terms. The Agda definition could be written as

However the equality of functions are not only judged by definitions. Functions are usually viewed extensionally as black boxes. If two functions pointwise generate the same outputs for the same inputs, they are equivalent even though their definitions may differ. This is called functional extensionality which is not inhabited [1] in original intensional type theory and can be expressed as following,

given two types A and B, and two functions f, $g : A \longrightarrow B$,

$$Ext = \forall\, x \colon A, fx = gx \longrightarrow f = g$$

The problem seems easy to solve by just adding a constant $ext : Ext$ to intensional type theory as following codes in Agda

However, postulating something could lead to inconsistence. If we postulate Ext, then theory is no longer adequate, which means it is possible to define irreducible terms. It can be easily verified in Agda through formalising a non-canonical term for a natural number by an eliminator of intensional equality.

Using the eliminator |J| [2] of the |Id A a b| :

we can construct an irreducible term of natural number as

With this term, we can construct irreducible terms of any type A by a mapping $f : \mathbb{N} \to A$. This will destroy some good features of intensional type theory since it could leads to nonterminating programs.

[2] It is originally used by Martin-Löf [10] and a good explanation could be found in [23]

Altenkirch investigates this issue and gives a solution in [1]. He proposes an extension of intensional type theory by a universe of propositions **Prop** in which all proofs of same propositions are definitionally equal, namely the theory is proof irrelevant. At the same time, a setoid model where types are interpreted by a type and an equivalence relation acts as the metatheory and η-rules for Π-types and Σ-types hold in the metatheory. The extended type theory generated from the metatheory is decidable and adequate, Ext is inhabited and it permits large elimination (defining a dependent type by recursion). Within this type theory, introduction of quotient types is straightforward. The set of functions are naturally quotient types, the hidden information is the definition of the functions and the equivalence relation is the functional extensionality.

There are more problems concerning quotient types and most of them are related to equality. One of the main problems is how to lift the functions for base types to the ones for quotient types. Only functions respecting the equivalence relation can be lifted. Even in extensional type theory, the implementation of quotient types does not stop at replacing equality of the types. We will discuss these in next section.

3.4 Example of Quotients

The introduction of quotient types is very helpful. Many types can be defined using quotient types, some of them can only be defined with quotient types, such as real numbers (the reason will be covered in here).

quotient groups, quotient space, partiality monad.

3.5 literature review

In [24], Mendler et al. have firstly considered building new types from a given type using a quotient operator //. Their work is done in an implementation of extensional type theory, NuPRL.

In NuPRL, every type comes with its own equality relation, so the quotient operator can be seen as a way of redefining equality in a type. But it is not all about building new types. They also discuss problems that arise from defining functions on the new type which can be illustrated using a simple example.

Assume the base type is A and the new equivalence relation is E, the new type can be formed as $A//E$.

When we want to define a function $f : A//E \longrightarrow Bool$, $f\,a \neq f\,b$ may exists for $a, b : A$ such that $E\,a\,b$. This will lead to inconsistency since $E\,a\,b$ implies a converts to b in extensional type theory, hence the left hand side $f\,a$ can be converted to $f\,b$, namely we get $f\,b \neq f\,b$ which is contradicted with the equality reflection rule.

Therefore a function is said to be well-defined [24] on the new type only if it respects the equivalence relation E, namely

$$\forall\, a\, b : A, E\,a\,b \longrightarrow f\,a = f\,b$$

We call this *soundness* property in [21].

After the introduction of quotient types, Mendler further investigates this topic from a categorical perspective in [25]. He uses the correspondence between quotient types in Martin-Löf type theory and coequalizers in a category of types to define a notion called *squash types*, which is further discussed by Nogin [26].

To add quotient types to Martin-Löf type theory, Hofmann proposes three models for quotient types in his PhD thesis [18]. The first one is a setoid model for quotient types. In this model all types are attached with partial equivalence relations, namely all types are setoids rather than sets. Types without a specific equivalence relation can be seen as setoids with the basic intensional equality. This is similar to extensional type theory in some sense. The second one is groupoid model which solves some problems but it is not definable in intensional type theory. He also proposes a third model to combine the advantages of the first two models, but it also has some disadvantages. Later in [27] he gives a simple model in which we have type dependency only at the propositional level, he also shows that extensional Type Theory is conservative over intensional type theory extended with quotient types and a universe [28].

Nogin [26] considers a modular approach to axiomatizing the same quotient types in NuPRL as well. Despite the ease of constructing new types from base types, he also discusses some problems about quotient types. For example, since the equality is extensional, we cannot recover the witness of the equality. He suggests including more axioms to conceptualise quotients. He decomposes the formalisation of quotient type into several smaller primitives such that they can be handled much simpler.

Homeier [29] axiomatises quotient types in Higher Order Logic (HOL), which is also a theorem prover. He creates a tool package to construct quotient types as a conservative extension of HOL such that users are able to define new types in HOL. Next he defines the normalisation functions and proves several properties of these. Finally he discussed the issues when quotienting on the aggregate types such as lists and pairs.

Courtieu [30] shows an extension of Calculus of Inductive Constructions with *Normalised Types* which are similar to quotient types, but equivalence relations are replaced by normalisation functions. However not all quotient types have normal forms. Normalised types are proper subsets of quotient types, because we can easily recover a quotient type from a normalised type as below

Barthe and Geuvers [31] also propose a new notion called *congruence types*, which is also a special class of quotient types, in which the base type are inductively defined and with a set of reduction rules called the term-rewriting system. The idea behind it is the β-equivalence is replaced by a set of β-conversion rules. Congruence types can be treated as an alternative to the pattern matching introduced in [32]. The main purpose of introducing congruence types is to solve problems in term rewriting systems rather than to implement quotient types.

Barthe and Capretta [33] compare different ways to setoids in type theory. The setoid is classified as partial setoid or total setoid depending on whether the equality relation is reflexive or not. They also consider obtain quotients with different kinds of setoids, especially the ones from partial setoids are difficult to define because the lack of reflexivity.

Abbott, Altenkirch et al. [34] provides the basis for programming with quotient datatypes polymorphically based on their works on containers which are datatypes whose instances are collections of objects, such as arrays, trees and so on. Generalising the notion of container, they define quotient containers as the containers quotiented by a collection of isomorphisms on the positions within the containers.

Voevodsky [22] implements quotients in Coq based on a set of axioms of Homotopy Type Theory. It is based on the groupoid model for intensional type theory where isomorphisms are equalities. He firstly implement equivalence class and use it to implement quotients which is an analogy to the construction of quotient sets in set theory.

Chapter 4

Definable Quotients

1. Distributivity 2. Rational numbers 3. other definable quotients

Quotient types may be necessary for defining some types in intensional type theory, but it is not always the case. The set of integers and the set of rational numbers can be defined without quotients types, but there are more properties revealed if we view them as quotients. However there are some good properties if we relate them with the base types and equivalence relation, for example we can lift functions and their properties from base types to quotient types. Moreover, if the base types are simpler to manipulate, it is worthwhile using the base type to define functions and reasoning and then lifting them. We can achieve more convenience by manipulating base types and then lifting the operators and propositions according to the relation between quotient types and base types. The main things we are going to discuss in this chapter is called definable quotient structures which does not require quotient types to be added into the intensional type theory.

In this Chapter we will show this using one of the examples, the set of integers. Some of the work is is conducted by Thorsten Altenkirch, Thomas Anberrée and the author together, and summarised in [21] .

4.1 Integers

From the usual symbols to represent integers, we can easily figure out one inductive definition for integers,

```
data ℤ : Set where
```

```
+_ : ℕ → ℤ
zero : ℤ
-_ : ℕ → ℤ
```

However we face a trade-off: three different representation for zero or to use code +0 for number +1. Usually the principle is to not losing canonicity because it requires unnecessary checking for whether some functions respect the equivalence or not. Therefore, the second choice makes more sense and we refine it a bit as:

```
data ℤ : Set where
  +suc_ : ℕ → ℤ
  zero  : ℤ
  -suc_ : ℕ → ℤ
```

This is better, but in practice it is expected to have more cases if we use pattern matching. Every time we use pattern matching, a case will be split into three. This becomes worse and worse when we have mutiple integer arguments and we have to do case analysis on all of them. A simple refinement is combining the first two constructors:

```
data ℤ : Set where
  +_    : ℕ → ℤ
  -suc_ : ℕ → ℤ
```

This is the most proper version we decided to use for the set of integers. It is inductively defined and is readable because it is just an intepretation of the usual symbols for integers in regular mathematics.

Usually we believe that the reason of inventing integers is the lack of symbols to represent the results of subtraction between two natural numbers. Integers are used to represent these results, and vice versa, every integer can be represented as a pair of natural numbers and the choice is not unique. For example, from the equation $1 - 4 = -3$, it is clear that the integer -3 can be represented the pair $(1, 4)$. Therefore we can use the paired natural numbers as an alternative definition for integers.

$$\mathbb{Z}_0 = \mathbb{N} \times \mathbb{N}$$

However since there are different pairs for one integer, we have to quotient it with an equivalence relation. For any two pairs of natural numbers (n_1, n_2) and (n_3, n_4), we know they represent the same integer if

$$n_1 - n_2 = n_3 - n_4$$

Technically, this does not work because the subtraction defined for natural numbers only returns zero if the pair is for negative number. We only need to do some small modifications:

$$n_1 + n_4 = n_3 + n_2$$

This helps us define a relation but it is not enough. This is an equation in mathamtics, but in Type Theory we have to prove that it is an equivalence relation, namely, it is reflexive, symmetric and transitive.

Combining the carrier (the pair of natural numbers), the equivalence relation and its proof, we have a setoid.

```
ℤ-Setoid : Setoid
ℤ-Setoid = record
  { Carrier       = ℤ₀
  ; _≈_           = _∼_
  ; isEquivalence = _∼_isEquivalence
  }
```

Since the set of integers is definable as we discussed before, they can be seen as the normal forms of the equivalent classes. The normalisation function can be defined as follows:

```
[_]               : ℤ₀ → ℤ
[ m , 0 ]         = + m
[ 0 , suc n ]     = -suc n
[ suc m , suc n ] = [ m , n ]
```

The function should be proved well-defined on the setoid, namely it has to respect the equivalence relation. We call it soundness here. It is not trivial but easy to observe that the function is sound[1].

A setoid and a function respects this equivalence (not necessary to be a normalisation function) constitute a prequotient.

Definition 4.1. Prequotient.

Given a setoid $(A, \sim)$, a *prequotient* $(Q, [\_], \text{sound})$ over that setoid consists in

1. a set Q,
2. a function $[\_] : A \longrightarrow Q$,
3. a proof *sound* that the function $[\_]$ is compatible with the relation $\sim$, that is

$$\text{sound}\colon (a, b : A) \longrightarrow a \sim b \longrightarrow [a] = [b],$$

Prequotient only includes the formalisation rules and introduction rules. To complete a *quotient*, we also need the elimination rule added into such a prequotient

4. for any $B : Q \longrightarrow \mathbf{Set}$, an eliminator

$$\begin{aligned} \text{qelim}_B \ &: \ (f\colon (a : A) \longrightarrow B\,[a]) \\ &\longrightarrow ((p : a \sim b) \longrightarrow f\;a \simeq_{\text{sound }p} f\;b) \\ &\longrightarrow ((q : Q) \longrightarrow B\;q) \end{aligned}$$

such that $\text{qelim-}\beta : \text{qelim}_B\ f\,p\,[a] \equiv f\;a$.

This eliminator is also called dependent lifting function because it actually lifts a function which is well-defined on the setoid to a function defined on the quotient type. The result type is also dependent on the quotient type. There is an equivalent definition given by Martin Hofmann which has a non-dependent eliminator with an induction principle instead.

$$\text{lift}\colon (f\colon A \longrightarrow B) \longrightarrow (\forall a, b \cdot a \sim b \longrightarrow f\;a \equiv f\;b) \longrightarrow (Q \longrightarrow B)$$

[1]the formal proof can be found in appendix (we cheat a bit by defining embedding function to make it simpler)

Suppose B is a predicate,

$$\text{qind}\colon ((a : A) \longrightarrow B\,[a]) \longrightarrow ((q : Q) \longrightarrow B\,q)$$

However, it is oberservable that given any non-empty set Q, all constant functions fit in this definition. Any element of Q has to be mapped from at most one equivalence class of the setoid $(A, \sim)$. This property is called *exact* here

5. *exact* : $(\forall a, b : A) \longrightarrow [a] \equiv [b] \longrightarrow a \sim b.$

The quotient is exact if exactly one equivalence class corresponds to an element of Q.

We already know that the integer is definable and it is plausible to find a representative in each equivalence classes. Since we treat elements of Q as the name for the equivalence classes, the selection function can be defined as an embedding function from the quotient type Q to base type A. This is an alternative and more flexible way to eliminate the quotient type Q and if a prequotient $(Q, [\,\cdot\,], \text{sound})$ on a setoid $(A, \sim)$ has an embedding function which is specified as

$$\begin{aligned}
\text{emb} &: Q \longrightarrow A \\
\text{complete} &: (a : A) \longrightarrow \text{emb}\,[a] \sim a \\
\text{stable} &: (q : Q) \longrightarrow [\text{emb}\ q] \equiv q.
\end{aligned}$$

Then it is a *definable quotient*. Composing the "normalisation" function $[\,\cdot\,]$ with the embedding function, we obtain the real normalisation function. A definable quotient is an *exact* quotient which is proved in [21].

Operations For a definable quotient, we can lift an operation by mixing the normalisation and embedding functions. For example, given an unary operator

$$\begin{aligned}
&\mathsf{lift}_1 : (op : \mathbb{Z}_0 \to \mathbb{Z}_0) \to \mathbb{Z} \to \mathbb{Z} \\
&\mathsf{lift}_1\ op = [_] \circ op \circ \ulcorner_\urcorner
\end{aligned}$$

To lift binary or n-ary operators, we only need to apply the operator to the representative for each "equivalence class" and "normalise" the result so that it becomes a function defined on the set of "equivalence classes".

But there is a unavoidable problem: not all operations defined on $\mathbb{Z}_0$ is defind on the setoid, namely respect the equivalence relation. Therefore it is reasonable to verify if the function is well-defined on the setoid:

$$a \sim b \to op\, a \sim op\, b$$

We will show how to define the addition for the quotient integers. Given two numbers (a_1, b_1) and (a_2, b_2) We only need to add them pair-wisely together, and it can be verified easily because we know that

$(a_1 - b_1) + (a_2 - b_2) = (a_1 + a_2) - (b_1) - (b_2)$

The verification is not necessary here but should be important in other cases.

```
_+_ : ℤ₀ → ℤ₀ → ℤ₀
(x+ , x-) + (y+ , y-) = (x+ ℕ+ y+) , (x- ℕ+ y-)
```

Properties We can also define the ring of $\mathbb{Z}$. It contains a lot of properties to prove which are seemed as axioms in classic mathematics. In constructive mathematics, the only axioms for integers are the constructors and the elimination rules.

As what we have menetioned, even though the definition of integers only has two constructors, it gradually increase the difficulty of proving when doing case analyses on more and more integers. One example is the proving of distributivity.

An efficient utilization of quotient structure: the proving of distributivity One of the most important motivations of using setoid integers is that the setoid definition reduces the complexity of programs involving integers. We have shown it is simpler to define some operators, but it will be much more evident when proving properties of the ring of integers.

Most simple laws of the ring of integers are not as unbearably complicated as the distributivity laws. An attempt of the right distributivity for $\mathbb{Z}$ ($(y+z) \times x = y \times x + z \times x$) is shown as below. The multiplication is not defined with pattern matching, but in the arithmetic approach.

```
_ℤ*_ : ℤ → ℤ → ℤ
```

```
i ℤ* j = sign i S* sign j ◃ ∣ i ∣ ℕ* ∣ j ∣
```

The first idea which I came up with is to use the right distributivity law of natural numbers. It is observable that if all three variables have the same signs, it is easy to apply the right distributivity law of natural numbers. We can relax the constraint a bit, only y and z have the same symbol, it is still plausible. To prove this part we need three lemmas,

```
lem1 : ∀ x y → sign x ≡ sign y → ∣ x ℤ+ y ∣ ≡ ∣ x ∣ ℕ+ ∣ y ∣
lem1 (-suc x) (-suc y) e = cong suc (sym (m+1+n≡1+m+n x y))
lem1 (-suc x) (+ y) ()
lem1 (+ x) (-suc y) ()
lem1 (+ x) (+ y) e = refl

lem2 : ∀ x y → sign x ≡ sign y → sign (x ℤ+ y) ≡ sign y
lem2 (-suc x) ( -suc y) e = refl
lem2 (-suc x) (+ y) ()
lem2 (+ x) (-suc y) ()
lem2 (+ x) (+ y) e = refl


lem3 : ∀ x y s → s ◃ (x ℕ+ y) ≡ (s ◃ x) ℤ+ (s ◃ y)
lem3 0 0 s = refl
lem3 0 (suc y) s = sym (ℤ-id-l _)
lem3 (suc x) y s = trans (h s (x ℕ+ y)) (
    trans (cong (λ n → (s ◃ suc 0) ℤ+ n) (lem3 x y s)) (
    trans (sym (ℤ-+-assoc (s ◃ suc 0) (s ◃ x) (s ◃ y))) (
    cong (λ n → n ℤ+ (s ◃ y)) (sym (h s x)))))
  where
  h : ∀ s y → s ◃ suc y ≡ (s ◃ (suc 0)) ℤ+ (s ◃ y)
  h s 0 = sym (ℤ-id-r _)
  h Sign.- (suc y) = refl
  h Sign.+ (suc y) = refl
```

The following is a partial definition with the first case that

```
distʳ   : _ℤ*_ DistributesOverʳ _ℤ+_
distʳ x y z with sign y S≟ sign z
distʳ x y z    | yes p
   rewrite lem1 y z p | p
               | lem2 y z p =
   trans (cong (λ n → sign z S* sign x ◃ n)
   (distribʳ | x | | y | (| z |)))
   (lem3 (| y | ℕ* | x |) (| z | ℕ* | x |)
   (sign z S* sign x))
distʳ x y z | no ¬p = ...
```

If y and z have different signs, it is impossible to apply the right distributivity law for natural numbers. We have to prove it from scratch. Even though it seems that there are only two cases to prove, it is conceivable that how many lemmas we need as prerequisites. The proving is as complicated as using pattern matching on each variable. This is not the best solution we want. We should benefit more from the observation that any equation of integers can be turned into equation of natural numbers**??**.

However, if we prove the laws for quotient integers, it is much simpler since there is only one case to prove.

However, if we prove it for the quotient integers, it is much easier. In fact, it is in generally automatically provable. Since the equality of any two quotient integers is essentially the equality of two natural numbers after normalising. To prove the distributivity, the simpliest way is to use semiring solver for natural numbers. DistributesOver^l means that the distributivity of the first operators over the second one.

Remark 4.2. A ring solver is an automatic equation checker for rings e.g. the ring of integers. It is implemented based on the theory described in "Proving Equalities in a Commutative Ring Done Right in Coq" by Grégoire and Mahboubi [35].

```
distˡ :   _*_ DistributesOverˡ _+_
distˡ (a , b) (c , d) (e , f) = solve 6
   (λ a b c d e f → a :* (c :+ e) :+ b :* (d :+ f) :+
      (a :* d :+ b :* c :+ (a :* f :+ b :* e))
      :=
      a :* c :+ b :* d :+ (a :* e :+ b :* f) :+
```

$(a :* (d :+ f) :+ b :* (c :+ e)))$ refl $a\ b\ c\ d\ e\ f$

The utilization of ring solver can be simplified even further by adopting "reflection". It helps us quote the type of the goal so that we can define a function that automatically do it without explicitly providing the equations. There is already some work done by van der Walt [36]. It can be seen as an analogy of the "ring" tactic from Coq.

The main drawback of this method is the type verification of the terms automatically generated requires more computations than the terms we manually construct. The optimization of Agda already shows a big improvement in this technical efficiency issue. However it will affect other functions that use this proof. It may heavily slow the type checking. Therefore although it is still very convenient to use the ring solver to prove any proposition for the quotient integers, we decide to prove commonly used properties for the commutative ring of the integers by hand. Luckily it is still much simpler than the ones for the set of integers $\mathbb{Z}$.

This is a good example of how the definable quotient structure helps simplifying definiting and proving about new types based on existing functions and theorems.

4.2 Rational numbers

add this paragraph: From a discussion on Agda mailing list, a better way to manipulate the rational numbers is required. The set definition of rational numbers contains a proof of the coprime property which means that whenther you want to define a function, you have to reduce the fraction until the enumerator and denominator are comprime. However from the experience of using fracitons in arithematics, we can leave them without reducing and it is still correct. Therefore, it is better for us to have the flexibility of using the un-reduced fractions or reduced ones in operations. With the help of definable quotient structures, it is feasible. We can define operations and prove theorems about rational numbers using the setoid definition and lift them to the set version. It makes the definition much easier and whenever we want the reduced fraction, we just need to apply the normalisation function within the structure. It also improves the efficiency of programs using rational numbers, even some people claims that it will be too large such that it is less efficient.

The quotient of rational numbers is better known than the previous quotient of integers. We usually write two integers m and n (n is not zero) in fractional form $\frac{m}{n}$ to represent a rational number. Alternatively we can use an integer and a positive natural number such that it is simpler to exclude 0 in the denominator. Two fractions are equal if they are reduced to the same irreducible term. If the numerator and denominator of a fraction are coprime, it is said to be an irreducible fraction. Based on this observation, it is naturally to form a definable quotient, where the base type is

$$\mathbb{Q}_0 = \mathbb{Z} \times \mathbb{N}$$

The integer stands for the *numerator* and the natural number is *denominator-1* (We use N for N^+ to avoid invalid fractions from construction rather than from zero test)

In Agda, to make the terms more meaningful we define it as

```
data ℚ₀ : Set where
  _/suc_ : (n : ℤ) → (d : ℕ) → ℚ₀
```

In mathematics, to judge the equality of two fractions, it is easier to conduct the following conversion,

$$\frac{a}{b} = \frac{c}{d} \iff a \times d = c \times b$$

Therefore the equivalence relation can be defined as,

```
_*_ : ℤ → ℕ → ℤ
(+ x) * d = + (x ℕ* d)
(-suc x) * 0 = + 0
(-suc x) * suc d = -suc (x ℕ+ suc x ℕ* d)

_∼_   : Rel ℚ₀ _
(n1 /suc d1) ∼ (n2 /suc d2) =   n1 * suc d2 ≡ n2 * suc d1
```

The normal form of rational numbers, namely the quotient type in this quotient is the set of irreducible fractions. We only need to add a restriction that the numerator and denominator is coprime,

$$\mathbb{Q} = \Sigma(n\colon \mathbb{Z}).\Sigma(d\colon \mathbb{N}).\mathsf{coprime}\, n\, (d+1)$$

It can be defined as follows which is available in standard library,

```
record ℚ : Set where
  field
    numerator     : ℤ
    denominator-1 : ℕ
    isCoprime     : True (C.coprime? ∣ numerator ∣ (suc denominator-1))

  denominator : ℤ
  denominator = + suc denominator-1

  coprime : Coprime numerator denominator
  coprime = toWitness isCoprime
```

The normalisation function is an implementation of the reducing process. But first we need to the |gcd| function which calculates the greatest common divisor can help us reduce the fraction and give us the proof of coprime. First we need to define the conversion from the results of GCD to normal rational numbers (the full definition can be found in Appendix Appendix A),

```
GCD′→ℚ : ∀ x y di → y ≢ 0 → C.GCD′ x y di → ℚ
```

To normalise a fractional, we split it into 3 cases with respect to the numerator. The idea is to calculate the "gcd” and then use the above function to get the normalised rational number.

```
[_] : ℚ₀ → ℚ
[ (+ 0) /suc d ] = ℤ.+_ 0 ÷ 1
[ (+ (suc n)) /suc d ] with gcd (suc n) (suc d)
[ (+ suc n) /suc d ] | di , g = GCD′→ℚ (suc n) (suc d) di (λ ()) (C.gcd-gcd′ g)
```

The embedding function is simple. We only need to forget the coprime proof in the normal form,

```
⌜_⌝ : ℚ → ℚ₀
⌜ x ⌝ = (ℤcon (ℚ.numerator x)) /suc (ℚ.denominator-1 x)
```

Similarly, we are able to construct the setoid, the prequotient and then the definable quotient of rational numbers. We can benefit from the ease of defining operators and proving theorems on setoids while still using the normal form of rational numbers, the lifted operators and properties which are safer.

The same approach works here as well. Since we can easily embed the natural numbers into integers, the equations of the quotient rational numbers are degraded to equations of the integers. The commutative ring of integers also enable us to prove all properties of rational numbers automatically.

Chapter 5

Real numbers and other undefinable quotients

If you mean something specific, always write "the"

In the previous chapter, only definable quotient types are investigated. But some other types are undefinable in intensional type theory without quotients. In this chapter, the real numbers, the multisets and the partiality monad and also the proofs that the undefinablity of them will be disccussed in detail.

5.1 Real numbers

We have several choices to represent real numbers. We choose Cauchy sequences of rational numbers to represent real numbers [37].

$$\mathbb{R}_0 = \{s : \mathbb{N} \longrightarrow \mathbb{Q} \mid \forall \varepsilon : \mathbb{Q}, \varepsilon > 0 \longrightarrow \exists m : \mathbb{N}, \forall i : \mathbb{N}, i > m \longrightarrow |s_i - s_m| < \varepsilon\}$$

It is implementable in Type Theory, but there is a problem of the choice of type of the second part of the cauchy sequence, namely the property that it is a cauchy sequence. Do we distinguish the same sequences with different proofs? Logically speaking we should not. It means that we need to truncate it to proposition but we will lose the important tool to guess what the real number is. To avoid this problem, we could use an alternative equivalent definition which is a subset of $\mathbb{R}_0$:

$$\mathbb{R}_0' = \left\{ f : \mathbb{N} \longrightarrow \mathbb{Q} \mid \forall k : \mathbb{N}, \forall m, n > k, \longrightarrow |f_m - f_n| < \frac{1}{k} \right\}$$

With the definition, the condition part is propositional and we can guess the number by applying any number k to the sequence and we know the interval where it should be located.

Different cauchy sequences can represent the same number. Therefore an equivalence relation[1] is expected. In mathematics two Cauchy sequences $\mathbb{R}_0$ are said to be equal if their pointwise difference converges to zero,

$$r \sim s = \forall \varepsilon : \mathbb{Q}, \varepsilon > 0 \longrightarrow \exists m : \mathbb{N}, \forall i : \mathbb{N}, i > m \longrightarrow |r_i - s_i| < \varepsilon$$

5.1.1 Non-normalizability of Cauchy Sequences

To prove that it is impossible to give a full definable quotient structure of real numbers with the setoid of cauchy sequences, we could show it by proving that it is impossible to define a normalisation function for the cauchy sequences.

Definition 5.1. We say that a quotient structure $A/\sim$ is definable via a normalisation, if we have a normalisation function

$$nf : A \longrightarrow A \tag{5.1}$$

with the property that it respects $\sim$

$$p : \Pi_{c_1, c_2 : A} c_1 \sim c_2 \longrightarrow nf(c_1) = nf(c_2). \tag{5.2}$$

such that

$$q : \Pi_{c:A} nf(c) \sim c. \tag{5.3}$$

It is equivalent to say that we have a definable quotient structure in the sense of [21], because we can form the set of equivalence classes as

$$Q :\equiv \Sigma_{c:\mathbb{R}_0} nf(c) = c$$

where the second part is propositional, and the "normalisation" function can be defined as

$$[c] :\equiv nf(c), p(q)$$

[1] The Agda version is in Appendix

and the embedding function is just the first projection. The properties can be verified easily.

In the other way around, the true normalisation function is just

$$n := emb \circ [\_]$$

and the properties hold as well.

We have made an attempt in the original version of our [21] draft, but there is something important problem pointed out by Martin Escardo. Laterly, Nicolai Kraus suggests to fix the proof by proving it as a meta-theoretical property. We will show an adaption of his proof here.

Some preliminaries In fact the proof is mainly conducted using topological tools. The following definitions are helpful for someone who are not so familiar with topological concepts.

Definition 5.2. Metric space. In mathematics, a metric space is a set where a notion of distance (called a metric) between elements of the set is defined.

A metric space is an ordered pair (M, d) where M is a set and d is a metric on M:

1. M is a set,
2. and $d : M \times M \to \mathbb{R}$ s.t.
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) = d(y, x)$
5. $d(x, y) + d(y, z) \geq d(x, z)$

We usually define a standard topological structure for discrete types.

- $(\mathbf{2}, h)$ where $h(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$
- $(\mathbb{N}, d)$ where $d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$
- $(\mathbb{Q}, e)$ where $e(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$

We use a slightly different definition of cauchy sequences of rational numbers here:

Definition 5.3. We call a function $f : \mathbb{N}^+ \longrightarrow \mathbb{Q}$[2] a *Cauchy Sequence* if it satisfies

$$\mathsf{isCauchy}(f) := \forall (n : \mathbb{N}^+).\, \forall (m : \mathbb{N}^+).\, m > n \longrightarrow |f(n) - f(m)| < \frac{1}{n}. \tag{5.4}$$

The type of Cauchy Sequences is thus

$$\mathbb{R}_0 := \Sigma_{f:\mathbb{N}^+ \longrightarrow \mathbb{Q}} \mathsf{isCauchy}(f).$$

And the standard metric space for the sequences $\mathbb{N}^+ \to \mathbb{Q}$ is defined by the distance function

$$g(f_1, f_2) = 2^{-\inf\{k \in \mathbb{N} \mid f_1(k) \neq f_2(k)\}} \tag{5.5}$$

Among all these standard metric spaces, It is a folklore that all definable functions are continous.

Theorem 5.4. *All definable functions are continous.*

Let us introduce the following auxiliary definition:

Definition 5.5. For a sequence $f : \mathbb{N} \longrightarrow \mathbb{Q}$, we say that f is *Cauchy with factor* k, written as $\mathsf{isCauchy}_k$, for some $k \in \mathbb{Q}$, $k > 0$, if

$$\mathsf{isCauchy}_k(f) \;:=\; \forall (n\, m : \mathbb{N}).\, m > n \longrightarrow |f(n) - f(m)| < \frac{1}{k \cdot n}. \tag{5.6}$$

The usual Cauchy condition $\mathsf{isCauchy}$ is therefore "Cauchy with factor 1".

Remark 5.6. If we claim a function f is defined on $\mathbb{R}_0$ that respects $\sim$, it means that we have a proof

$$p : \Pi_{c_1, c_2 : \mathbb{R}_0} c_1 \sim c_2 \longrightarrow f(c_1) = f(c_2). \tag{5.7}$$

Now we have enough tools to prove the following proposition.

Proposition 5.7. *"$\mathbb{R}_0/\sim$" **is connected.** It means that any continuous function f*

$$f : \mathbb{R}_0 \longrightarrow \mathbf{2} \tag{5.8}$$

that respects $\sim$ is constant. We prove that it is impossible to find $c_1, c_2 : \mathbb{R}_0$ such that $f(c_1) \neq f(c_2)$ meta-theoretically, instead of deriving a proof term of this in type theory.

[2] we use $\mathbb{N}^+$ instead of $\mathbb{N}$ because n must be positive in $\frac{1}{n}$

The definability of function implies that it is a continous function 5.4 between the standard metric spaces for $\mathbb{R}_0$ and $\mathbf{2}$[3].

Proof. The general idea is to interpret our definition in classical mathematics, assume we have a non-constant function and deduce a contradiction.

Consider the "naive" set model (with "classical standard mathematics" as meta-theory). This clearly works if we are in a minimalisitic type theory with Π, Σ, W, $=$, $\mathbb{N}$; however, if we restrict ourselves to the types in the lowest universe of homotopy type theory (which is enough), it also works for HoTT. We use $[\![\cdot]\!]$ as an interpreation function; for example, we write $\mathbb{R}$ for the field of real numbers which can be defined as $[\![\mathbb{R}_0]\!]/[\![\sim]\!]$. By abuse of notation, we write $[\![\mathbb{R}_0]\!]$ for the set of Cauchy sequences in the model that fulfill the Cauchy condition, without the actual proof thereof. This is justified as this property is propositional.

For readability, we use the symbol $=$ for equality in the theory as well in the model, and we do not use the semantic brackets for natural numbers such as 2 or 4. In the model, we use $\bar{\cdot} : [\![\mathbb{R}_0]\!] \longrightarrow \mathbb{R}$ as the function that gives us the limit of a Cauchy sequence (Not all functions in the "naive" set model have to be continuous). Thus, for $r : \mathbb{R}_0$, we write $\overline{[\![r]\!]} \in \mathbb{R}$ for the real number it represents.

Assume f, p are given. We prove that $[\![f]\!] : [\![\mathbb{R}_0]\!] \longrightarrow [\![\mathbf{2}]\!]$ is constant in the model, which implies the statement of Proposition 5.7.

Thus, assume $[\![f]\!]$ is non-constant, there are $c_1, c_2 : [\![\mathbb{R}_0]\!]$ with $[\![f]\!](c_1) \neq [\![f]\!](c_2)$.

Define

$$m_1 :\equiv \mathsf{sup}\{\overline{d} \in \mathbb{R} \mid d \in [\![\mathbb{R}_0]\!], \overline{d} \leq \mathsf{max}(\overline{c_1}, \overline{c_2}), [\![f]\!](d) = [\![1_{\mathbf{2}}]\!]\} \tag{5.9}$$

$$m_2 :\equiv \mathsf{sup}\{\overline{d} \in \mathbb{R} \mid d \in [\![\mathbb{R}_0]\!], \overline{d} \leq \mathsf{max}(\overline{c_1}, \overline{c_2}), [\![f]\!](d) = [\![0_{\mathbf{2}}]\!]\} \tag{5.10}$$

(note that one of these two necessarily has to be $\overline{c_1}$ or $\overline{c_2}$, whichever is bigger). Set $m :\equiv \mathsf{min}(m_1, m_2)$, and we can observe that in *every* neighborhood U of m, given any t, we can always find another point $x \in U$ such that $x = \overline{e}$ (for some e) with $[\![f]\!](e) \neq [\![f]\!](t)$.

Let $c \in [\![\mathbb{R}_0]\!]$ be a Cauchy sequence such that $\overline{c}$ is equal to m. We may assume that c satisfies the condition $[\![\mathsf{isCauchy}_5]\!]$.[4]

As f (and thereby $[\![f]\!]$) is continuous (remember the metric spaces), there exists $n_0 \in [\![\mathbb{N}]\!]$ such that for any Cauchy sequence $c' \in [\![\mathbb{R}_0]\!]$, if the first n_0 sequence elements of c'

[3]The metric of $\mathbb{R}$ comes from the first component. Technically, if $\mathbb{R}_0$ is defined by **??**, this would not make it a metric space (as the distance between non-equal elements could be 0); however, this would not matter, and for our definition, there is no problem anyway.

[4]the factor 5 is chosen due to the need of a later proof.

coincide with those of c (namely the distance $g(c,c') = 2^{-\mathsf{inf}\{k\in\mathbb{N} \mid c(k)\neq c'(k)\}} \leq 2^{-n_0}$), then $[\![f]\!](c') = [\![f]\!](c)$. Write $U \subset [\![\mathbb{R}_0]\!]$ for the set of Cauchy sequences which fulfill this property, and $\overline{U} := \{\overline{d} \mid d \in U\}$ for the set of reals that U corresponds to.

We claim that $\overline{U}$ is a neighborhood of m. More precisely, we prove: The interval $I := (m - \frac{1}{2n_0}, m + \frac{1}{2n_0})$ is contained in $\overline{U}$. Let $x \in \mathbb{R}$ be in I. There is a sequence $t : [\![\mathbb{N} \longrightarrow \mathbb{Q}]\!]$ such that $[\![\mathsf{isCauchy}_{5n_0}]\!](t)$ and $\overline{t} = x$. Let us now "concatenate" the first n_0 elements of the sequence c with t, that is, define

$$g : [\![\mathbb{N} \longrightarrow \mathbb{Q}]\!] \tag{5.11}$$

$$g(n) = \begin{cases} c(n) & \text{if } n \leq n_0 \\ t(n - n_0) & \text{else.} \end{cases} \tag{5.12}$$

Observe that g is also a Cauchy sequence, i.e. $[\![\mathsf{isCauchy}]\!](g)$: The only thing that needs to be checked is whether the two "parts" of g work well together. Let $0 < n \leq n_0$ and $m > n_0$ be two natural numbers. We need to show that

$$|g(n) - g(m)| < \frac{1}{n}. \tag{5.13}$$

Calculate

$$\begin{aligned}
& |g(n) - g(m)| && (5.14)\\
= \;& |c(n) - t(m - n_0)| && (5.15)\\
= \;& |c(n) - \overline{c} + \overline{c} - \overline{t} + \overline{t} - t(m - n_0)| && (5.16)\\
\leq \;& |c(n) - \overline{c}| + |\overline{c} - \overline{t}| + |\overline{t} - t(m - n_0)| && (5.17)\\
\leq \;& \frac{1}{5n} + \frac{1}{2n_0} + \frac{1}{5n_0 \cdot (m - n_0)} && (5.18)\\
\leq \;& \frac{1}{5n} + \frac{1}{2n} + \frac{1}{5n_0} && (5.19)\\
< \;& \frac{1}{n}. && (5.20)
\end{aligned}$$

From the continuity property of f and the definition of g we know that $[\![f]\!](g) = [\![f]\!](c)$. Clearly, $\overline{g} = \overline{t} = x \in I$. Therefore *all* $s \in [\![\mathbb{R}_0]\!]$ with $\overline{s} \in I$, we can use the "concatenation" approach to find a g satisfies $s\,[\![\sim]\!]\,g$ (namely $\overline{s} = \overline{g}$), and by the condition that f (and thereby $[\![f]\!]$) repects $\sim$, we can conclude that $[\![f]\!](s) = [\![f]\!](g) = [\![f]\!](c)$.

However, as we have seen, in *every* neighborhood of m, and thus in particular in $(m - \frac{1}{2n_0}, m + \frac{1}{2n_0})$, there is an x such that $x = \overline{e}$ (for some e) with $[\![f]\!](e) \neq [\![f]\!](c)$, in contradiction to the just established statement. $\square$

The proposition that "$\mathbb{R}_0/\sim$ is connected" implies the following corollary:

Corollary 5.8. *Any continous function from $\mathbb{R}_0$ to any discrete type that respects $\sim$ is constant.*

Theorem 5.9. *Any continuous endofunction f on $\mathbb{R}_0$ that respects $\sim$ which means*

$$p : \Pi_{c_1,c_2:\mathbb{R}_0} c_1 \sim c_2 \longrightarrow f(c_1) = f(c_2). \tag{5.21}$$

is constant (in the sense of corollary 5.8).

Proof. Assume we have f, p as required.

To prove f is constant, it is enough to show that the sequence part is constant because the proof part is propositional. Again, by slight abuse of notation, we write $[\![f]\!] : [\![\mathbb{R}_0]\!] \longrightarrow [\![\mathbb{R}_0]\!]$, omitting the proof part of f.

Given a postive natural number $n : [\![\mathbb{N}^+]\!]$, we have a projection function $\pi_n : [\![\mathbb{R}_0]\!] \longrightarrow [\![\mathbb{Q}]\!]$. Define a function $h_n : [\![\mathbb{R}_0]\!] \longrightarrow [\![\mathbb{Q}]\!]$ as

$$h_n(c) :\equiv \pi_n \circ f$$

By corollary 5.8 we know that h_n is constant, hence f is constant everywhere, it is enough to show that f is constant.

□

Corollary 5.10. *There is no definable normalisation function on $\mathbb{R}_0$ in the sense of 5.1*

Corollary 5.11. $\mathbb{R}_0/\sim$ *is not definable in the sense of [21].*

However, it doesn't imply that we cannot define the set of real numbers in minimalisitic type theory with $\Pi, \Sigma, \mathsf{W}, =, \mathbb{N}$. The meaning of definability of real numbers is still not clear enough. To make it more precise, we define it as whether there is a type A in TT (minimalisitic type theory) such that its embedding $[\![A]\!]$ in $\mathsf{TT}+\mathsf{Q}$ (type theory extended with quotient types) is isomorphic to $[\![\mathbb{R}]\!]_0/[\![\sim]\!]$ (where it is a valid type). We conjecture that it is still not definable.

Proof. Assume the set of real numbers is definable, we have a type A and its embedding in $\mathsf{TT}+\mathsf{Q}$ is $[\![A]\!]$. It also gives us a normalisation function and a representative function between $[\![\mathbb{R}]\!]_0$ and $[\![A]\!]$.

P(T) -> Connected (T) -> Contractible(T), $\mathbb{R}/\sim$ is connected but not contractible?

□

5.1.2 Cauchy completeness of the Cauchy reals

expand Cauchy completeness of Cauchy reals: it should rely on the axiom of countable choice

Whether our definition of Cauchy sequence is Cauchy complete? In other words, is there a representative Cauchy sequence as a limit for every equivalence class? The answer is no.

In the HoTT book [20], an alternative definition is used instead which is called cauchy approximation. Because every approximation has been proven to have a limit, it is Cauchy complete. The definition uses the higher inductive types which will be discussed in later Chapter 7.

5.2 Multisets

Definition 5.12. Multiset. A multiset (or bag) is a set without the constraint that there is no repetitive elements.

A set is just a special case of multiset when the *multiplicity*(the number of the occurences) of every element is one. Multisets (or bags) are believed to be used in ancient times, but it is only studied by mathematicans from 20th century.

In set theory, a multiset is defined as a pair of a set A and a occurences counting function $m : A \to \mathbb{N}$.

However in type theory, the set-theoretical *"set"* is not a primitive notion and we need to define multisets in a type-theoretic way. In type theoretical language, a multiset can be seen as a unordered list. It can be encoded as lists which identifies permutations. To make it simpler we use length-explict list *Vector* ("Vector A 3" stands for a list of type A of length 3).

The equivalence relation required is as follows:

Given two lists of type A and has length n, $p\, q : Vector\, A\, n$, they are equivalent if we have an isomorphism between them.

$$p \sim q = \Sigma \phi : Fin\, n \to Fin\, n, \phi\, is\, a\, bijection \land \forall x, p_x = q_{\phi(x)}$$

decidable order -> of course definable, A-> N definable, give a explicit proof

definable -> split quotient because definable is too general

5.3 Partiality monad

The partiality monad is a conindutive type which is available in the standard library of Agda. It is used to encode delayed computation.

```
data Delay (A : Set) : Set where
  now : A → Delay A
  later : ∞ (Delay A) → Delay A
```

A non-terminating program can be defined in a coinductive way.

```
never : {A : Set} → Delay A
never = later (♯ never)
```

We have two equality for the Delay type: strong bisimilarity and weak bisimilarity. Two computation are strongly bisimilar if they are the same after the same number of steps delay (there can be infinite steps).

```
data _∼_ {A : Set} : Delay A → Delay A → Set where
  now   : ∀ {x} → (now x) ∼ (now x)
  later : ∀ {x y} (x∼y : ∞ ((♭ x) ∼ (♭ y))) → (later x) ∼ (later y)
```

We inductively define an operator which states that "x terminates with y" if we write $x))\downarrow y$.

```
infix 4 _↓_

data _↓_ {A : Set} : Delay A → A → Set where
  nowT   : ∀{a} → (now a) ↓ a
```

```
laterT : ∀{d a} → d ↓ a → (later (♯ d)) ↓ a
```

And two computation are weakly bisimilar if they terminates with the same value.

```
data _≈_ {A : Set} : Delay A → Delay A → Set where
  now   : ∀ {x y a} → x ↓ a → y ↓ a → x ≈ y
  later : ∀ {x y} (x∼y : ∞ ((♭ x) ≈ (♭ y))) → (later x) ≈ (later y)
```

The quotient derived from the equivalent relation (weak bisimilarity) which represent the set of all computations can also be a good example of undefinable quotient.

It is equivalent to another definition inductively on either left side or right side.

5.4 Not all connected type is contractible

Martin's example is that we can define a type which is proved different but we cannot find a function from it to *2*.

Martin Escardo's http://www.cs.bham.ac.uk/~mhe/agda/FailureOfTotalSeparatedness.html should be c

conjecture: every quotient definable in pure type theory without quotient is split

one way to prove it is any connected type is contractible.

the question is still open.

This will implies that reals are not only non-split but also undefianble in general.

1. why quotients simplifies reasoning. (writened up) 2. Undefinablity 3. omega-groupoids model

Chapter 6

The Setoid Model

the Setoid Model

Quotient types are one of the extensional concepts in Type Theory [18]. There are several existing intensional models for extensional concepts. The first one we are going to work with is Altenkirch's setoid model. To introduce an extensional propositional equality in intensional type theory, Altenkirch [1] proposes an intensional setoid model with a proof-irrelevant universe of propositions **Prop**.

$$[proof - irr]\frac{\Gamma \vdash P : \mathbf{Prop} \qquad \Gamma \vdash p, q : P}{\Gamma \vdash p = q : P} \tag{6.1}$$

It only contains "propositional" sets which has at most one inhabitant. Notice that it is not a definition of types, which means that we cannot conclude a type is of type **Prop** if we have a proof that all inhabitants are definitionally equal.

The propositional universe is closed under "Π" and "Σ", namely dependent functions and dependent products.

$$[\Pi - Prop]\frac{\Gamma \vdash A : \mathbf{Set} \qquad \Gamma, x : A \vdash P \in \mathbf{Prop}}{\Gamma \vdash \Pi\, x : A.P} \tag{6.2}$$

$$[\Sigma - Prop]\frac{\Gamma \vdash P : \mathbf{Prop} \qquad \Gamma, x : P \vdash Q \in \mathbf{Prop}}{\Gamma \vdash \Sigma\, x : P.Q} \tag{6.3}$$

It is called a setoid model since types are interpreted as setoids. The solution to introduce the extensional equality is an object type theory defined inside the setoid model which serves as the metatheory. He also proved that the extended type theory generated from the metatheory is decidable and adequate, functional extensionality is inhabited and it permits large elimination (defining a dependent type by recursion). Within this type theory, introduction of quotient types is straightforward.

This model is different to a setoid model as an E-category, for instance the one introduced by Hofmann [38] . An E-category is a category equipped with an equivalence relation for homsets. To distinguish them, we call this category **E-setoids**. All morphisms of **E-setoids** gives rise to types and they are cartesian closed, namely it is a a locally cartesian closed category (LCCC). Not all morphisms in our category of setoids give rise to types and it is not an LCCC. Every LCCC can serve as a model for categories with families but not every category with families has to be an LCCC.

write why this model is not lccc explicitly. refer to Nicolais's result

The category of setoids is not a LCCC The pullback functor.

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f^*(a)"'] & Y' \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Observe that $X \to 1 \cong X$, therefore the pullback of y which is $X/1 \to X \times Y/Y$ can be seen as a pullback of X of type $X \to X/Y$.

The left adjoint to the pullback functor $f*$ is just the post composition of f written as $f \circ \_$ or Σ_f.

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d, "a"'] & X' \arrow[d, "\Sigma_f a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

However this setoid model is still a model for Type Theory just like the groupoid model which is a generalisation of it. To develop this model of type theory in Agda, we have implemented the categories with families of setoids. We build a category with families of setoids to accommodate the types theory described in [1] so that it is possible to define quotient types following Martin Hofmann's Paper [27]. Only necessary part for the Setoid model will be present here.

6.1 An implementation of categories with Families in Agda

Following the work in [?], we first define a proof-irrelevant universe of propositions. We name it as **hProp** since **Prop** is a reserved word which can't be used and **hProp** is a notion from Homotopy Type Theory which we will introduce later.

6.2 hProp

A proof-irrelvant universe only contains sets with at most one inhabitant.

```
record hProp : Set₁ where
  constructor hp
  field
    prf : Set
    Uni : {p q : prf} → p ≡ q

open hProp public renaming (prf to <_>)
```

We can extract the proof of any propostion $A : hProp$ by using <> and there is always a proof that all inhabitants of it are the same, in other words, if there is any proof of it, the proof is unique. This is not exactly the same as the *Prop* universe in Altenkirch's approach which is judgemental. It is just a judgement whether a set behaves like a *Proposition*. The *hProp* we define above is propositional since we can extract the proof of uniqueness.

We would like to have some basic propositions $\top$ and $\bot$. To distinguish them with the ones for non-proof irrelevant propositions which are already available in Agda library, we add a prime to all similar symbols.

```
⊤' : hProp
⊤' = hp ⊤ refl

⊥' : hProp
```

$$\bot' = \mathsf{hp}\ \bot\ (\lambda\ \{p\} \to \bot\text{-}\mathsf{elim}\ p)$$

We also want the universal and existential quantifier for $hProp$, namely it is closed under Π-types and Σ-types. The universal quantifier of $hProp$ can be axiomitised but we decide to explicitly state that we require the functional extensionality to use this module. The reason is that functional extensionality is actually equivalent to the closure under Π-types.

$$\forall' : (A : \mathsf{Set})(P : A \to \mathsf{hProp}) \to \mathsf{hProp}$$
$$\forall'\ A\ P = \mathsf{hp}\ ((x :\ A) \to <\ P\ x >)\ (ext\ (\lambda\ x \to \mathsf{Uni}\ (P\ x)))$$

$$\Sigma' : (P : \mathsf{hProp})(Q : <\ P > \to \mathsf{hProp}) \to \mathsf{hProp}$$
$$\Sigma'\ P\ Q = \mathsf{hp}\ (\Sigma <\ P > (\lambda\ x \to <\ Q\ x >))$$
$$(\lambda\ \{p\}\ \{q\} \to$$
$$\mathsf{sig\text{-}eq}\ (\mathsf{Uni}\ P)\ (\mathsf{Uni}\ (Q\ (\mathsf{proj}_1\ q))))$$

Implication and conjuction which are independent ones of them follow simply.

$$\_\Rightarrow\_ : (P\ Q : \mathsf{hProp}) \to \mathsf{hProp}$$
$$P \Rightarrow Q = \quad \forall' <\ P > (\lambda\ \_ \to Q)$$

$$\_\wedge\_ : (P\ Q : \mathsf{hProp}) \to \mathsf{hProp}$$
$$P \wedge Q = \Sigma'\ P\ (\lambda\ \_ \to Q)$$

As long as we have implication and conjuction, more operators on proposition can be defined, for instances negation and logical equivalence.

```
¬ : hProp → hProp
¬ P = P ⇒ ⊥'

_↔_  : (P Q : hProp) → hProp
P ↔ Q = (P ⇒ Q) ∧ (Q ⇒ P)
```

6.3 Category

To define category of setoids we should define category first.

```
record Category : Set where
  constructor CatC
  field
    obj          : Set

    hom          : obj → obj → Set

    id           : ∀ a
                 → hom a a

    [_⇒_]_∘_     : ∀ α {β} γ
                 → hom β γ
                 → hom α β
                 → hom α γ

    isCategory : IsCategory obj hom id [_⇒_]_∘_
```

isCategory contains all the laws for this structure to be a category, for instance the associativity laws for composition.

6.4 Category of setoids

Then we could define setoids using **hProp**. An equivalence relation has three properties reflexivity, symmetry and transitivity. Since we have $refl$ here, we call the reflexivity for propositional equality from the library with prefix as $PE.refl$.

```
record ishEquivalence {A : Set}(_≈h_ : A → A → hProp) : Set₁ where
  constructor _,_,_
  field
    refl  : {x : A} → < x ≈h x >
    sym   : {x y : A} → < x ≈h y > → < y ≈h x >
    trans : {x y z : A} → < x ≈h y > → < y ≈h z > → < x ≈h z >
```

Here we use **hSetoid** as the name because **Setoid** is already used for non-proof-irrelvant setoids in the library. For each setoid, we have a carrier type and an equivalence relation.

```
record hSetoid : Set₁ where
  constructor _,_,_
  infix 4 _≈h_ _≈_
  field
    Carrier : Set
    _≈h_    : Carrier → Carrier → hProp
    isEquiv : ishEquivalence _≈h_
```

A morphism in this category is a function of the underlying sets which respects the equivalence relation. We don't identify the extensional equal functions in the homsets as in **E-setoids**.

```
record _⇉_ (A B : hSetoid) : Set₁ where
  constructor fn:_resp:_
  field
    fn   : | A | → | B |
    resp : {x y : | A |} →
      [ A ] x ≈ y →
      [ B ] fn x ≈ fn y
```

The definitions of identity morphism and composition are straightforward and the categorical laws hold trivially as follows.

```
id' : {Γ : hSetoid} → Γ ⇉ Γ
id' = record { fn = id; resp = id}

_∘c_ : ∀{Γ Δ Z} → Δ ⇉ Z → Γ ⇉ Δ → Γ ⇉ Z
yz ∘c xy = record
  { fn = [ yz ]fn ∘ [ xy ]fn
  ; resp = [ yz ]resp ∘ [ xy ]resp
  }

id₁ : ∀ Γ Δ (ch : Γ ⇉ Δ) → ch ∘c id' ≡ ch
id₁ _ _ ch = PE.refl

id₂ : ∀ Γ Δ (ch : Γ ⇉ Δ) → id' ∘c ch ≡ ch
id₂ _ _ ch = PE.refl

comp : ∀ Γ {Δ Φ} Ψ
  (f : Γ ⇉ Δ)
  (g : Δ ⇉ Φ)
  (h : Φ ⇉ Ψ)
  → h ∘c g ∘c f ≡ h ∘c (g ∘c f)
comp _ _ f g h = PE.refl
```

Combined all components we obtain the category of setoids.

```
setoid-Cat : Category
setoid-Cat = CatC hSetoid _⇉_ (λ _ → id') (λ _ _ → _∘c_)
  (IsCatC id₁ id₂ comp)
```

This category has a terminal object which is just the unit set with trivial equality. As a terminal object there is precisely one morphism from every object to it.

```
⊤-setoid : hSetoid
⊤-setoid = record {
  Carrier = ⊤;
  _≈h_    = λ _ _ → ⊤';
  isEquiv = record {
    refl    = tt;
    sym     = λ _ → tt;
    trans   = λ _ _ → tt } }

⋆ : {Δ : hSetoid} → Δ ⇉ ⊤-setoid
⋆ = record
  { fn = λ _ → tt
  ; resp = λ _ → tt }

unique⋆ : {Δ : hSetoid} → (f : Δ ⇉ ⊤-setoid) → f ≡ ⋆
unique⋆ f = PE.refl
```

6.5 categories with families of setoids

A Category with families consists of a base category and a functor [39]. We firstly define the categories with families of sets in Agda as a guidance for the one for setoids. We would present the setoid one here since it is relevant.

We would like to show two formalisation of category with families for setoids here. The first one is simple and short but not comprehensive. We have to extract all complicated components from the simple definition. However the second one gives these components one by one so that it more understandable and convenient.

The category with families works as a model for type theory. So we will introduce them from a type theoretical point of view.

The base category is the category for contexts. In the setoid version we interpret a context as a setoid as well.

To define the second component, namely the presheaf functor, it is necessary to construct the target category first. The objects of this category are families of setoids.The index setoids are the semantic types and the indexed families of setoids are terms. The morphisms are component-wise morphisms between setoids. All the categorical laws hold trivially.

```
inxSetoids : Set₁
inxSetoids = Σ[ I : hSetoid ] (| I | → hSetoid)

_⇉setoid_ : inxSetoids → inxSetoids → Set₁
(I , f) ⇉setoid (J , g) =
  Σ[ i-map : I ⇉ J ]
    ((i : | I |) → f i ⇉ g ( [ i-map ]fn i))

Fam-setoid : Category
Fam-setoid = CatC
        inxSetoids
        _⇉setoid_
        (λ _ → id' , (λ _ → id'))
        (λ { _ _ (fty , ftm) (gty , gtm) → fty ∘c gty ,
          (λ i → ftm ([ gty ]fn i) ∘c gtm i)})
        (IsCatC
```

```
        (λ α β f → PE.refl)
        (λ α β f → PE.refl)
        (λ α δ f g h → PE.refl))
```

Since we already specify the category of contexts, we only need the presheaf which is a contravariant functor from the category of contexts to the category we defined above. The definition of category with families of setoids could be as simple as follows.

```
record CWF-setoid : Set₁ where
  field
  T : Functor (Op setoid-Cat) Fam-setoid
```

All details of this definition are hidden including the functor laws. Therefore we will show the details as the second version.

The semantic contexts are setoids and the terminal object is just the empty context.

```
Con = hSetoid

emptyCon = ⊤-setoid

emptysub = ⋆
```

A semantic type has following components. fm is a setoid of all types. $substT$ is the substitution between types within the context. It should be a morphism between setoids so it has to preserve the equivalence relation. We also need to specify the computation rules for substitution.

```
record Ty (Γ : Con) : Set₁ where
  field
  fm     : | Γ | → hSetoid

  substT : {x y : | Γ |} →
    [ Γ ] x ≈ y →
    | fm x | →
    | fm y |
  subst* : ∀{x y : | Γ |}
    (p : [ Γ ] x ≈ y)
    {a b : | fm x |} →
    [ fm x ] a ≈ b →
    [ fm y ] substT p a ≈ substT p b

  refl*  : ∀(x : | Γ |)
    (a : | fm x |) →
    [ fm x ] substT [ Γ ]refl a ≈ a
  trans* : ∀{x y z : | Γ |}
    (p : [ Γ ] x ≈ y)
    (q : [ Γ ] y ≈ z)
    (a : | fm x |)
    → [ fm z ] substT q (substT p a)
      ≈ substT ([ Γ ]trans p q) a
```

Some other lemmas on the proof irrelevance derived from these fields are not shown here since they are just auxiliary functions.

Then we have to define the substituting in a type given a context morphism and verify it preserves equivalence relation as well.

```
_[_]T : ∀ {Γ Δ : Con} → Ty Δ → Γ ⇉ Δ → Ty Γ
A [ f ]T
  = record
  { fm     = fm ∘ fn
  ; substT = substT ∘ resp
```

```
    ; subst* = subst* ∘ resp
    ; refl*  = λ _ _ → subst-pi'
    ; trans* = λ _ _ _ →
      [ fm (fn _) ]trans (trans* _ _ _) subst-pi
    }
    where
      open Ty A
      open _⇉_ f
```

The semantic terms are simpler. It should also preserve the equivalence relation on the elements of contexts.

```
  record Tm {Γ : Con}(A : Ty Γ) : Set where
    constructor tm:_resp:_
    field
      tm    : (x : | Γ |) → | [ A ]fm x |
      respt : ∀ {x y : | Γ |} →
        (p : [ Γ ] x ≈ y) →
        [ [ A ]fm y ] [ A ]subst p (tm x) ≈ tm y
```

Substitution for terms can be defined as

```
  _[_]m : ∀ {Γ Δ : Con}{A : Ty Δ} →
    Tm A →
    (f : Γ ⇉ Δ)
    → Tm (A [ f ]T)
  _[_]m t f = record
    { tm = [ t ]tm ∘ [ f ]fn
    ; respt = [ t ]respt ∘ [ f ]resp
    }
```

Syntactically we can form a new context by using a context Γ and a type $A : Ty\,\Gamma$. To introduce a term of it, we need a term of the semantic context Γ and a term of semantic type A. It is called context comprehension.

```
_&_ : (Γ : Con) → Ty Γ → Con
Γ & A = record
  { Carrier = Σ[ x : | Γ | ] | fm x |
  ; _≈h_    = λ{(x , a) (y , b) →
                 Σ'[ p : x ≈h y ] [ fm y ] substT p a ≈h b}
  ; isEquiv =
  record
  { refl    = refl , (refl* _ _)
  ; sym     = λ {(p , q) → (sym p) ,
     [ fm _ ]trans
     (subst* _ ([ fm _ ]sym q))
     trans-refl }
  ; trans = λ {(p , q) (m , n) →
     trans p m ,
     [ fm _ ]trans
     ([ fm _ ]trans
     ([ fm _ ]sym (trans* _ _ _)) (subst* _ q)) n }
  }
  }
```

There are also some other morphisms come with it. Any morphism from a context Γ to a context $\Delta \& A$ consists of a morphism from Γ to Δ and a term of type A substituted. In other words, There is an isomorphism between $Hom(\Gamma, \Delta\&A)$ and $\Sigma\gamma : Hom(\Gamma, \Delta)A[\gamma]$.

fst projects the morphism and snd projects the term. Indeed the fst operation provides weakening for types, and the snd projection enables us to interpret variables. $fst\&$ defines a morphism for each type A which is a canonical projection of A. We need to use id' which are identity context morphisms to achieve these.

$$
\begin{array}{l}
\mathsf{fst} : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Delta) \to \Gamma \rightrightarrows (\Delta\ \&\ A) \to \Gamma \rightrightarrows \Delta \\
\mathsf{fst}\ A\ f = \mathsf{record} \\
\qquad \{\ \mathsf{fn} = \mathsf{proj}_1 \circ [\ f\ ]\mathsf{fn} \\
\qquad ;\ \mathsf{resp} = \mathsf{proj}_1 \circ [\ f\ ]\mathsf{resp} \\
\qquad \}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{fst\&} : \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \Gamma\ \&\ A \rightrightarrows \Gamma \\
\mathsf{fst\&}\ A = \mathsf{fst}\ A\ \mathsf{id'}
\end{array}
$$

$$
\begin{array}{l}
\_\mathsf{+T}\_ : \{\Gamma : \mathsf{Con}\} \to \mathsf{Ty}\ \Gamma \to (A : \mathsf{Ty}\ \Gamma) \to \mathsf{Ty}\ (\Gamma\ \&\ A) \\
B\ \mathsf{+T}\ A = B\ [\ \mathsf{fst\&}\ A\ ]\mathsf{T}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{snd} : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Delta) \to \\
\quad (f : \Gamma \rightrightarrows (\Delta\ \&\ A)) \\
\quad \to \mathsf{Tm}\ (A\ [\ \mathsf{fst}\ A\ f\ ]\mathsf{T}) \\
\mathsf{snd}\ A\ f = \mathsf{record} \\
\qquad \{\ \mathsf{tm} = \mathsf{proj}_2 \circ [\ f\ ]\mathsf{fn} \\
\qquad ;\ \mathsf{respt} = \mathsf{proj}_2 \circ [\ f\ ]\mathsf{resp} \\
\qquad \}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{v0} : \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Tm}\ (A\ \mathsf{+T}\ A) \\
\mathsf{v0}\ A = \mathsf{snd}\ A\ \mathsf{id'}
\end{array}
$$

Inversely we could define a pairing operation to combine a context morphism with a term. The η-law for the projection and pairing holds trivially.

$$
\begin{array}{l}
\_,,\_ : \{\Gamma\ \Delta : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Delta\}(f : \Gamma \rightrightarrows \Delta) \to \\
\quad (\mathsf{Tm}\ (A\ [\ f\ ]\mathsf{T})) \\
\quad \to \Gamma \rightrightarrows (\Delta\ \&\ A) \\
f\ ,,\ t = \mathsf{record} \\
\qquad \{\ \mathsf{fn} = \langle\ [\ f\ ]\mathsf{fn}\ ,\ [\ t\ ]\mathsf{tm}\ \rangle \\
\qquad ;\ \mathsf{resp} = \langle\ [\ f\ ]\mathsf{resp}\ ,\ [\ t\ ]\mathsf{respt}\ \rangle \\
\qquad \}
\end{array}
$$

$$
\mathsf{\&\text{-}eta} : \{\Gamma\ \Delta : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Delta\}(f : \Gamma \rightrightarrows (\Delta\ \&\ A))
$$

```
    → _,_ {A = A} (fst A f) (snd A f) ≡ f
  &-eta f = PE.refl
```

Then a lifting operation could help us define Π-types.

```
  lift : {Γ Δ : Con}(f : Γ ⇉ Δ)(A : Ty Δ) → Γ & A [ f ]T ⇉ Δ & A
  lift f A = record
    { fn = ⟨ [ f ]fn ∘ proj₁ , proj₂ ⟩
    ; resp = ⟨ [ f ]resp ∘ proj₁ , proj₂ ⟩
    }

  lift-eta : {Γ Δ : Con}
      (f : Γ ⇉ Δ)(A : Ty Δ)(x : | Γ |)
      (a : | [ A ]fm ([ f ]fn x) |)
      → [ lift f A ]fn (x , a) ≡ ( [ f ]fn x , a)
  lift-eta f A x a = PE.refl
```

One of the most complicated part of this definition is the Π-types. Π-types is also called dependent function types. Semantically it is a function type on the underlying semantic types with a proof that the the functions respect the equivalence relation.

```
  Π : {Γ : Con}(A : Ty Γ)(B : Ty (Γ & A)) → Ty Γ
```

It also comes with two necessary operation on the terms of Pi-types, λ-abstraction and application. There are $\beta-\eta$ laws to verfify for them so that we could form an isomorphism with these two operations. however technically it causes stack overflow. We may simplify these definition in the future so that we could verify them in Agda.

$$\mathsf{lam} : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\}\{B : \mathsf{Ty}\ (\Gamma\ \&\ A)\} \to \mathsf{Tm}\ B \to \mathsf{Tm}\ (\Pi\ A\ B)$$

$$\mathsf{app} : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\}\{B : \mathsf{Ty}\ (\Gamma\ \&\ A)\} \to \mathsf{Tm}\ (\Pi\ A\ B) \to \mathsf{Tm}\ B$$

Non-dependent version of Π-types namely function types can be defined with type weakening. Since the dependence disappears, it is possible to define it straightforwardly without using Π-types.

```
_⇒'_ : {Γ : Con}(A B : Ty Γ) → Ty Γ
A ⇒' B = Π A (B +T A)
```

6.6 What we can do in this model

6.7 Examples of types

We also implement some common types within the syntactic structure of this type theory. For example, natural numbers and the simply typed universe. I will only present the natural numbers here.

The semantic of natural numbers is just natural numbers and the equivalence relation is defined recursively with respect to case analysis on natural numbers. Other operations and properties of this type can be proved easily.

```
⟦Nat⟧ : {Γ : Con} → Ty Γ
⟦Nat⟧ = record
  { fm = λ γ → record
    { Carrier = ℕ
```

```
    ; _≈h_   = _≈nat_
    ; isEquiv = record
      { refl = λ {n} → reflNat {n}
      ; sym = λ {x} {y} → symNat {x} {y}
      ; trans = λ {x} {y} {z} → transNat {x} {y} {z}
      }
    }
  ; substT = λ _ → id
  ; subst* = λ _ → id
  ; refl* = λ x a → reflNat {a}
  ; trans* = λ p q a → reflNat {a}
  }
```

Zero and successor operator can be defined as follows.

```
⟦0⟧ : {Γ : Con} → Tm {Γ} ⟦Nat⟧
⟦0⟧ = record
  { tm = λ _ → 0
  ; respt = λ p → tt
  }

⟦s⟧ : {Γ : Con} → Tm {Γ}⟦Nat⟧ → Tm {Γ} ⟦Nat⟧
⟦s⟧ (tm: t resp: respt)
  = record
  { tm = suc ∘ t
  ; respt = respt
  }
```

The equality type is an essential part of a type theory. We could define it by using the equivalence relation from the setoid representation of type A. The equivalence relation is trivial since it is proof-irrelevant.

```
Rel : {Γ : Con} → Ty Γ → Set₁
Rel {Γ} A = Ty (Γ & A & A +T A)

⟦Id⟧ : {Γ : Con}(A : Ty Γ) → Rel A
⟦Id⟧ A
  = record
    { fm = λ {((x , a) , b) →   record
      { Carrier = [ [ A ]fm x ] a ≈ b
      ; _≈h_ = λ x₁ x₂ → ⊤'
      ; isEquiv = record
        { refl = λ {x₁} → tt
        ; sym = λ x₂ → tt
        ; trans = λ x₂ x₃ → tt
        }
      } }
    ; substT = λ {((x , a) , b) x0 →
      [ [ A ]fm _ ]trans
      ([ [ A ]fm _ ]sym a)
      ([ [ A ]fm _ ]trans
      ([ A ]subst* _ x0) b)
      }
    ; subst* = λ p x₁ → tt
    ; refl* = λ x a → tt
    ; trans* = λ p q a → tt }
```

The unique inhabitant $refl$ is defined as

```
cm-refl : {Γ : Con}(A : Ty Γ) → Γ & A ⇉ (Γ & A & A +T A)
cm-refl A = record { fn = λ x' → x' , proj₂ x'
    ; resp = λ x' → x' , proj₂ x' }

⟦refl⟧⁰ : {Γ : Con}(A : Ty Γ)
  → Tm {Γ & A} (⟦Id⟧ A
  [ cm-refl A ]T)
```

```
⟦refl⟧⁰ A = record
      { tm = λ {(x , a) → [ [ A ]fm x ]refl {a} }
      ; respt = λ p → tt
      }

⟦refl⟧ : {Γ : Con}(A : Ty Γ)
   → Tm {Γ} (Π A (⟦Id⟧ A
   [ cm-refl A ]T) )
⟦refl⟧ {Γ} A =   lam {Γ} {A} (⟦refl⟧⁰ A)
```

We have an abstracted $refl$ term as well. Using Π-types we could define the eliminator for Id, but it is more involved.

We have done the basics for category of families of setoids. There are more types can be interpreted in this model so that we could show that it is a valid model for Type Theory. We would like to interpret quotient types in this model by following Hofmann's method in [27] or by ourselves.

6.8 Quotient types in setoid model

6.9 Observational equality

Later in in [40], Altenkirch and McBride further simplifies the setoid model by adopting McBride's heterogeneous approach to equality. They identifies values up to observation rather than construction which is called observational equality. It is the propositional equality induced by the Setoid model. In general we have a heterogeneous equality which compares terms of types which are different in construction. It only make sense when we can prove the types are the same. It helps us avoids the heavy use of *subst* which makes formalisation and reasoning involved. We could simplify the setoid model by adapting this approach and the implementation could be easier.

Chapter 7

Homotopy Type Theory and higher inductive types

Homotopy Type Theory (HoTT) is a new "*hot*" field developed as a branch between theoretical computer science and mathematics. Indeed it arises from a new interpretation of intensional Martin-Löf type theory into homotopy theory where identity types are interpreted as paths and the identity types of "paths" as homotopies.

It is extended with Vladimir Voevodsky's *univalence axiom* which identifies isomorphic structures. Formally speaking, it identifies isomorphism with equality.

In category theory, we usually use isomorphism instead of "equality". The univalent axiom seems like a formalisation of this vague notion.

In Homotopy Type Theory, we interpret types as higher groupoids which is a "multi-sorted object" which contains points for terms, paths for identity types and higher paths for higher level iterated identity types[**?**]. To accomodate this interpretation, inductive types are not enough, then a more general schema for defining types is invented —*higher inductive types*. Briefly speaking, in a definition of a higher inductive type, there can be constructors not only for points but also for paths. With higher inductive types, it is much easier to define quotient types. However the implementation of Homotopy Type Theory in intensional type theory is still an open problem. We work on defining semi-simplicial types and weak ω-groupoids to solve it. There is also the very new model using cubical sets proposed by Bezem, Coquand and Huber in [41].

In this chapter, we will discuss some basic notions in Homotopy Type Theory like higher inductive types, quotient types in Homotopy Type Theory and a discussion of the implementation of Homotopy Type Theory in intensional type theory. It can be seen as an extended introduction to Homotopy Type Theory and serves as an essential prerequiste for next chapter.

7.1 Basic Homotopy Type theoretical notions

7.1.1 Univalent axiom

Vladimir Voevodsky proposed to add the following *univalence* to type theory, as the fundamental axiom of the Homotopy Type Theory.

Definition 7.1. *univalence axiom.* for any two types X and Y,

$$X = Y \simeq X \simeq Y$$

holds.

7.1.2 Homotopy types

The Homotopy Type Theory first provides an interpretation of type theory into homotopy theory. Notice we mainly focus on homotopical notions rather than lying down the topological basis first.

- Types are interpreted as spaces. $a : A$ can be stated as a is a point of space A.
- Terms are continous functions, for example, $f : A \to B$ is a continous function between spaces and it is equivalent to say a is a point of space or $a : 1 \to A$ is a continous function.
- Identity types are path spaces,
- Identity terms are
- Identity types of identity types are called homotopies (if we represent a path as a continous function $p : [0, 1] \to X$).
- There are also 3-homotopies and 4-homotopies and even higher levels which forms an infinite structure called ω-groupoids (or ∞-groupoids).

From the interpretation, it is natural to define the types as some generalisation of setoids or groupoids.

It is all about adding more structures on types.

A setoid is a set with an equivalence relation. Categorically, terms are objects and in setoid models, there is at most one morphism between terms which stands for the equivalence relation. The reflexivity is the identity morphism, the symmetry provides an inverse of each morphism and the transitivity is just the composition of morphism. One important deduction is that for any $p : a \sim b, p\dot{p}^{-1} = id$. However it is not true for groupoid models.

In groupoid models, Instead of the definitional equality $p : a \sim b, p.p^{-1} = id$, we have a 2-dimensional isomorphism $G3 : p.p^{-1} \sim id$ (It is one of the groupoid laws [42]) which stands for the propositional equality. Setoids are just a special cases of groupoids. The morphism is not unique in each homset, i.e. the identity types are non-trivial.

The second level is non-trivial in a groupoid, what if the third level and higher levels are non-trivial? It we turn the equalities, namely groupoid laws of each level, into explicit morphisms (which are called equivalence), we get ω-groupoids.

By building the infinite tower of identity types, every type has a weak ω-groupoids [43]. Similar to the interpretation of types as setoids, weak ω-groupoids is more general choice. It has more expressive power than setoid models.

7.2 Higher inductive types

Inductive types are types which can be freely generated by finite constructors. Constructors are functions of arbitrary number of arguments which can be the type itself being defined. Examples like natural numbers can be defined as,

- $0 : \mathbb{N}$
- $suc : \mathbb{N} \to \mathbb{N}$

In homotopy type theories, inductive types are not enough because we should take into account the *higher* structures of types, namely the iterated identity types on each levels. Therefore, we need a new and more general approach to describe types: the higher inductive types: a type does not only has constructors for terms but also constructors for paths. One common example is a circle $\mathsf{base} : \mathbb{S}^1$ (1-sphere) can be *inductively* defined with two constructors,

- A point $\mathsf{base} : \mathbb{S}^1$, and

- A path $\mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$.

Of course, this also implies that the eliminator should include the scheme to elimate the paths as well. It means that if we want to define some function $f : \mathsf{base} : \mathbb{S}^1 \to B$, assume $f(base) = b$ we have to map loop to an identity path $l : b = b$ as well $ap_f(\mathsf{loop}) = l$.

7.3 Quotient types with help of higher inductive types

By utilizing higher inductive types, it is simpler to encode the equivalence relation into the definition of certain types. Therefore it is possible to define quotient types in this model.

Assume A is a set and $\_ \sim \_ : A \to A \to \mathbf{Prop}$ (**Prop** stands for mere propositions in Homotopy Type Theory). The quotient set $A/\sim$ can be defined as a higher inductive type:

- $[\_] : A \to A/\sim$
- $eqv : (a, b : A) \to A \sim B \to [a] = [b]$

  which is a set as well, so
- $isSet : (x.y : A/\sim) \to (p1, p2 : x = y) \to p1 = p2$

Notice that the first one is the constructor of objects, the second one builds the morphism and the third one states that all path spaces are contractible so that the resulting types are h-sets. It is not a generic quotient construction of any types but specially for sets, so we call it set-quotient.

7.4 Quotient inductive inductive types

Altenkirch points out that we could apply higher inductive types in a specific way. We can construct a *set* by using a a mere relation. So in this sense, each non-trivial morphism is the unique one. This looks like a setoid, but indeed there is no difference between sets and setoids categorically because a skeleton of "setoids" is just a discrete category which is equivalent to sets (?review).

discuss with Thorsten

7.5 Various implementations of Homotopy Type Theory

7.5.1 Simplicial sets

7.5.2 Cubical sets

7.5.3 Brunerie's syntactic approach

7.6 Summary

In this chapter we introduce the basic notions in Homotopy Type Theory, discuss the various implementations of Homotopy Type Theory. In the next chapter we will focus on thesyntactic implementation of weak ω-groupoids following Brunerie's approach. We attempt to formalise the groupoid model of Homotopy Type Theory in intensional type theory, specially in Agda.

Maybe I should combine HIT chapter with weak ω-groupoids model chapter?

Chapter 8

the ω-groupoids Model

WORK ON THIS FIRST SO THAT WE CAN REVISE OUR PAPER

It is possible to define the quotient types in Homotopy Type Theory but to implement the Homotopy Type Theory in intensional type theory, it is still a difficult problem. We work on defining semi-simplicial types and weak ω-groupoids to solve it.

In Type Theory, a type can be interpreted as a setoid which is a set equipped with an equivalence relation [1]. The equivalence proof of the relation consists of reflexivity, symmetry and transitivity whose proofs are unique. However in Homotopy Type Theory, we reject the principle of uniqueness of identity proofs (UIP). Instead we accept the univalence axiom which says that equality of types is weakly equivalent to weak equivalence. Weak equivalence can been seen as a refinement of isomorphism without UIP [44]. For example, a weak equivalence between two objects A and B in a 2-category is a morphism $f : A \longrightarrow B$ which has a corresponding inverse morphism $g : B \longrightarrow A$, but instead of the proofs of isomorphism $f \circ g = 1_B$ and $g \circ f = 1_A$ we have two 2-cell isomorphisms $f \circ g \cong 1_B$ and $g \circ f \cong 1_A$.

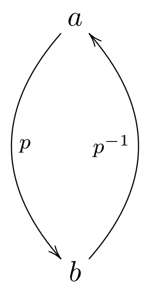
Voevodsky proposed the univalence axiom which basically says that isomorphic types are equal. This can be viewed as a strong extensionality axiom and it does imply functional extensionality (a coq proof of this can be found in [45]). However, adding univalence as an axiom destroys canonicity, i.e. that every closed term of type $\mathbb{N}$ is reducible to a numeral. In the special case of extensionality and assuming a strong version of UIP we were able to eliminate this issue [1, 40] using setoids. However, it is not clear how to generalize this in the absence of UIP univalence which is incompatible with UIP. To solve the problem we should generalise the notion of setoids, namely to enrich the structure of the identity proofs.

The generalised notion is called weak ω-groupoids and was proposed by Grothendieck 1983 in a famous manuscript *Pursuing Stacks* [46]. Maltsiniotis continued his work and suggested a simplification of the original definition which can be found in [47]. Later Ara also presents a slight variation of the simplication of weak ω-groupoids in [48]. Categorically speaking an ω-groupoid is an ω-category in which morphisms on all levels are equivalences. As we know that a set can be seen as a discrete category, a setoid is a category where every morphism is unique between two objects.

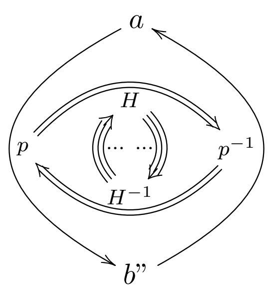
$$a$$

$$b$$

A groupoid is more generalised, every morphism is isomorphism but the proof of isomorphism is unique, namely the composition of a morphism with its inverse is equal to an identity morphism.



Similarly, an n-groupoid is an n-category in which morphisms on all levels are equivalence.



ω-groupoids which are also called ∞-groupoids is an infinite version of n-groupoids. To model Type Theory without UIP we also require the equalities to be non-strict, in other words, they are not definitionally equalities. Finally we should use weak ω-groupoids to interpret types and eliminate the univalence axiom.

There are several approaches to formalise weak ω-groupoids in Type Theory. For instance, Altenkirch and Rypáček [44], and Brunerie's notes [49]. This paper mainly explains an implementation of weak ω-groupoids following Brunerie's approach in Agda which is a well-known theorem prover and also a variant of intensional Martin-Löf type theory. The approach is to specify when a globular set is a weak ω-groupoids by first defining a type theory called $\mathcal{T}_{\infty-groupoid}$ to describe the internal language of Grothendieck weak ω-groupoids, then interpret it with a globular set and a dependent function. All coherence laws of the weak ω-groupoids should be derivable from the syntax, we will present some basic ones, for example reflexivity. One of the main contribution of this paper is to use the heterogeneous equality for terms to overcome some very difficult problems when we used the normal homogeneous one. In this paper, we omit some complicated and less important programs, namely the proofs of some lemmas or the definitions of some auxiliary functions. it is still possible for the reader who is interested in the details to check the code online, in which there are only some minor differences.

In this chapter we present an syntactic implementation of weak ω-groupoids following Brunerie's approach. The main contributions are adapting using heterogeneous equality, some syntactic construction of coherence laws and suspensions.

8.1 An implementation of weak ω-groupoids

It is very interesting to investigate the approach to define quotient types in Homotopy Type Theory which is a variant of Martin-Löf type theory. In Homotopy Type Theory, we reject the principle of uniqueness of identity proofs (UIP) but instead we accept the univalence axiom which says that equality of types is weakly equivalent to weak equivalence. Weak equivalence can been seen as a refinement of isomorphism without UIP [44]. To make it more precise, a weak equivalence between two objects A and B in a 2-category is a morphism $f : A \longrightarrow B$ which has a corresponding inverse morphism $g : B \longrightarrow A$, but instead of the proofs of isomorphism $f \circ g = 1_B$ and $g \circ f = 1_A$ we have two 2-cell isomorphisms $f \circ g \cong 1_B$ and $g \circ f \cong 1_A$. Since the setoid interpretation of types in setoid model, as we mentioned before, relies on UIP, it has to be generalised so that we could formalise it in intensional type theory.

The generalised notion is called Grothendieck ω-groupoids. Grothendieck introduced the notion of ω-groupoids in 1983 in a famous Manuscript *Pursuing Stacks* [46]. Maltsiniotis continued his work and suggested a simplification of the original definition which can be found in [47]. Later Ara also present a slight variation of the simplification of weak ω-groupoids in [48]. Categorically speaking an ω-groupoid is an ω-category in which morphisms on all levels are equivalences. As we know that a set can be seen as a discrete category, a setoid is a category where every morphism is unique between two objects. A groupoid is more generalised, every morphism is isomorphism but the proof of isomorphism is unique, namely the composition of a morphism with its inverse is equal to an identity morphism. Similarly, an n-groupoid is an n-category in which morphisms on all levels are equivalence. ω-groupoids which are also called ∞-groupoids is an infinite version of n-groupoids. To model Type Theory without UIP we also require the equalities to be non-strict, in other words, they are not definitionally equalities. Finally we should use weak ω-groupoids to interpret types and eliminate the univalence axiom.

There are several approaches to formalise weak ω-groupoids in Type Theory. For instance, Altenkirch and Rypacek's paper [44], and Brunerie's notes [49]. We work on an implementation of weak ω-groupoids following Brunerie's approach in Agda. The approach is to specify when a globular set is a weak ω-groupoid by first defining a type theory called $\mathcal{T}_{\infty-groupoid}$ to describe the internal language of Grothendieck weak ω-groupoids, then interpret it with a globular set and a dependent function. All coherence laws of the weak ω-groupoids should be derivable from the syntax, we will present some basic ones, for example reflexivity. One of the main contribution of our work is to use the heterogeneous equality for terms to overcome some very difficult problems when we used the normal homogeneous one. When introducing our implementation, we omit some complicated but less important programs, namely the proofs of some lemmas or the definitions of some auxiliary functions. it is still possible for the reader who is interested in the details to check the code online, in which there are only some minor differences.

8.1.1 Syntax

Since the definitions of contexts, types and terms involve each others, we adopt a more liberal way to do mutual definition in Agda which is a feature available since version 2.2.10. Something declared is free to use even it has not been completely defined.

8.2 Syntax

We develop the type theory of ω-groupoids formally, following [49]. This is a Type Theory with only one type former which we can view as equality types and interpret as the homsets of the ω groupoid. There are no definitional equalities which correspond to the fact that we consider weak ω-groupoids. That is no laws are strict (i.e. definitional) but all are witnessed by terms. Compared to [44] the definition is very much simplified by the observation that all laws of a weak ω-groupoid follow from the existence of coherence constants for any contractible context.

In our formalisation we exploit the more liberal way to do mutual definitions in Agda, which was implemented recently following up a suggestion by the first author. It allows us to first introduce a type former but give its definition later.

Since we are avoiding definitional equalities we have to define a syntactic substitution operation which we need for the general statement of the coherence constants. However, defining this constant requires us to prove a number of substitution laws at the same time. We address this issue by using a heterogeneous equality which exploits uniqueness of identity proofs (UIP). Note that UIP holds because all components defined here are sets in the sense of Homotopy Type Theory.

8.2.1 Basic Objects

We first declare the syntax of our type theory which is called $\mathcal{T}_{\infty-groupoid}$ namely the internal language of weak ω-groupoids. The following declarations in order are contexts as sets, types are sets dependent on contexts, terms and variables are sets dependent on types, Contexts morphisms and the contractible contexts.

```
data Con             : Set
data Ty (Γ : Con)    : Set
data Tm              : {Γ : Con}(A : Ty Γ) → Set
data Var             : {Γ : Con}(A : Ty Γ) → Set
data _⇒_             : Con → Con → Set
data isContr         : Con → Set
```

It is possible to complete the definition of contexts and types first. Contexts are inductively defined as either an empty context or a context with a type of it. Types are defined as either $*$ which we call it 0-cell, or a morphism between two terms of some type A. If the type A is n-cell then we call the morphism $(n+1)$-cell.

```
data Con where
  ε     : Con
  _,_   : (Γ : Con)(A : Ty Γ) → Con
```

```
data Ty Γ where
  *      : Ty Γ
  _=h_   : {A : Ty Γ}(a b : Tm A) → Ty Γ
```

8.2.2 Heterogeneous Equality for Terms

One of the big challenge we encountered at first is the difficulty to formalise and to reason about the equalities of terms, which is essential when defining substitution. When we used the common identity types which are homogeneous, we had to use *subst* function in Agda to unify the types on both sides of the equation. It created a lot of technical issues that made the encoding too involved to proceed. However we found that the syntactic equality of types of given context which will be introduced later, is decidable which means that it is an h-set. In other words, the equalities of types is unique, so that it is safe to use the JM equality (heterogeneous equality) for terms of different types. The equality is inhabited only when they are definitionally equal.

```
data _≅_ {Γ : Con}{A : Ty Γ} : {B : Ty Γ} → Tm A → Tm B → Set where
  refl : (b : Tm A) → b ≅ b
```

Once we have the heterogeneous equality for terms, we can define a proof-irrelevant substitution which we call coercion here since it gives us a term of type A if we have a term of type B and the two types are equal. We can also prove that the coerced term is heterogeneously equal to the original term. Combined these definitions, it is much more convenient to formalise and to reason about term equations.

```
_⟦_⟫        : {Γ : Con}{A B : Ty Γ}(a : Tm B) → A ≡ B → Tm A
a ⟦ refl ⟫  = a

cohOp       : {Γ : Con}{A B : Ty Γ}{a : Tm B}(p : A ≡ B)
              → a ⟦ p ⟫ ≅ a
cohOp refl  = refl _
```

8.2.3 Substitutions

With context morphism, we can define substitutions for types variables and terms. Indeed the composition of contexts can be understood as substitution for context morphisms as well.

$$
\begin{array}{llllll}
\_[\_]\mathsf{T} & : \{\Gamma\ \Delta : \mathsf{Con}\} & \to \mathsf{Ty}\ \Delta & \to (\delta : \Gamma \Rightarrow \Delta) & \to \mathsf{Ty}\ \Gamma \\
\_[\_]\mathsf{V} & : \{\Gamma\ \Delta : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Delta\} & \to \mathsf{Var}\ A & \to (\delta : \Gamma \Rightarrow \Delta) & \to \mathsf{Tm}\ (A\ [\ \delta\ ]\mathsf{T}) \\
\_[\_]\mathsf{tm} & : \{\Gamma\ \Delta : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Delta\} & \to \mathsf{Tm}\ A & \to (\delta : \Gamma \Rightarrow \Delta) & \to \mathsf{Tm}\ (A\ [\ \delta\ ]\mathsf{T}) \\
\_\circledcirc\_ & : \{\Gamma\ \Delta\ \Theta : \mathsf{Con}\} & \to \Delta \Rightarrow \Theta & \to (\delta : \Gamma \Rightarrow \Delta) & \to \Gamma \Rightarrow \Theta
\end{array}
$$

8.2.4 Weakening Rules

We can freely add types to the contexts of any given type judgments, term judgments or context morphisms. These are weakening rules.

$$
\begin{array}{lllll}
\_+\mathsf{T}\_ & : \{\Gamma : \mathsf{Con}\} & (A : \mathsf{Ty}\ \Gamma) & \to (B : \mathsf{Ty}\ \Gamma) & \to \mathsf{Ty}\ (\Gamma\ ,\ B) \\
\_+\mathsf{tm}\_ & : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\} & (a : \mathsf{Tm}\ A) & \to (B : \mathsf{Ty}\ \Gamma) & \to \mathsf{Tm}\ (A +\mathsf{T}\ B) \\
\_+\mathsf{S}\_ & : \{\Gamma : \mathsf{Con}\}\{\Delta : \mathsf{Con}\} & (\delta : \Gamma \Rightarrow \Delta) & \to (B : \mathsf{Ty}\ \Gamma) & \to (\Gamma\ ,\ B) \Rightarrow \Delta
\end{array}
$$

To define the variables and terms we have to use the weakening rules. A Term can be either a variable or a coherence constant (coh). We use typed de Bruijn indices to define variables as either the rightmost variable of the context, or some variable in the context which can be found by cancelling the rightmost variable along with each vS. The coherence constants are one of the major part of this syntax, which are primitive terms of the primitive types in contractible contexts which will be introduced later. Since contexts, types, variables and terms are all mutually defined, most of their properties have to be proved simultaneously.

```
data Var where
  v0 : {Γ : Con}{A : Ty Γ}    → Var (A +T A)
  vS : {Γ : Con}{A B : Ty Γ}(x : Var A) → Var (A +T B)

data Tm where
  var : {Γ : Con}{A : Ty Γ} → Var A → Tm A
```

```
coh : {Γ Δ : Con} → isContr Δ → (δ : Γ ⇒ Δ) → (A : Ty Δ) → Tm (A [ δ ]T)
```

Another core part of the syntactic framework is contractible contexts. Intuitively speaking, a context is contractible if its geometric realization is contractible to a point. It either contains one variable of the 0-cell $*$ which is the base case, or we can extend a contractible context with a variable of an existing type and an n-cell, namely a morphism, between the new variable and some existing variable.

```
data isContr where
  c*   : isContr (ε , *)
  ext  : {Γ : Con} → isContr Γ → {A : Ty Γ}(x : Var A)
       → isContr (Γ , A , (var (vS x) =h var v0))
```

Context morphisms are defined inductively similarly to contexts. A context morphism is a list of terms corresponding to the list of types in the context on the right hand side of the morphism.

```
data _⇒_ where
  •     : {Γ : Con} → Γ ⇒ ε
  _,_   : {Γ Δ : Con}(δ : Γ ⇒ Δ){A : Ty Δ}(a : Tm (A [ δ ]T))
        → Γ ⇒ (Δ , A)
```

8.2.5 Lemmas

The following four lemmas state that to substitute a type, a variable, a term, or a context morphism with two context morphisms consecutively, is equivalent to substitute with the composition of substitution.

```
[⊚]T    : {Γ Δ Θ : Con}{A : Ty Θ}{ϑ : Δ ⇒ Θ}{δ : Γ ⇒ Δ}
        → A [ ϑ ⊚ δ ]T ≡ (A [ ϑ ]T)[ δ ]T

[⊚]v    : {Γ Δ Θ : Con}{A : Ty Θ}(x : Var A){ϑ : Δ ⇒ Θ}{δ : Γ ⇒ Δ}
        → x [ ϑ ⊚ δ ]V ≅ (x [ ϑ ]V) [ δ ]tm
```

$$\begin{aligned}
\textsf{[⊚]tm} \quad & : \{\Gamma\ \Delta\ \Theta : \textsf{Con}\}\{A : \textsf{Ty}\ \Theta\}(a : \textsf{Tm}\ A)\{\vartheta : \Delta \Rightarrow \Theta\}\{\delta : \Gamma \Rightarrow \Delta\} \\
& \rightarrow a\ [\ \vartheta \circledcirc \delta\ ]\textsf{tm} \cong (a\ [\ \vartheta\ ]\textsf{tm})\ [\ \delta\ ]\textsf{tm}
\end{aligned}$$

$$\begin{aligned}
\textsf{⊚assoc} \quad & : \{\Gamma\ \Delta\ \Theta\ \Omega : \textsf{Con}\}(\gamma : \Theta \Rightarrow \Omega)\{\vartheta : \Delta \Rightarrow \Theta\}\{\delta : \Gamma \Rightarrow \Delta\} \\
& \rightarrow (\gamma \circledcirc \vartheta) \circledcirc \delta \equiv \gamma \circledcirc (\vartheta \circledcirc \delta)
\end{aligned}$$

Weakening inside substitution is equivalent to weakening outside.

$$\begin{aligned}
\textsf{[+S]T} \quad & : \{\Gamma\ \Delta : \textsf{Con}\}\{A : \textsf{Ty}\ \Delta\}\{\delta : \Gamma \Rightarrow \Delta\}\{B : \textsf{Ty}\ \Gamma\} \\
& \rightarrow A\ [\ \delta \ \textsf{+S}\ B\ ]\textsf{T} \equiv (A\ [\ \delta\ ]\textsf{T})\ \textsf{+T}\ B
\end{aligned}$$

$$\begin{aligned}
\textsf{[+S]tm} \quad & : \{\Gamma\ \Delta : \textsf{Con}\}\{A : \textsf{Ty}\ \Delta\}(a : \textsf{Tm}\ A)\{\delta : \Gamma \Rightarrow \Delta\}\{B : \textsf{Ty}\ \Gamma\} \\
& \rightarrow a\ [\ \delta \ \textsf{+S}\ B\ ]\textsf{tm} \cong (a\ [\ \delta\ ]\textsf{tm})\ \textsf{+tm}\ B
\end{aligned}$$

$$\begin{aligned}
\textsf{[+S]S} \quad & : \forall\{\Gamma\ \Delta\ \Delta_1 : \textsf{Con}\}\{\delta : \Delta \Rightarrow \Delta_1\}\{\gamma : \Gamma \Rightarrow \Delta\}\{B : \textsf{Ty}\ \Gamma\} \\
& \rightarrow \delta \circledcirc (\gamma \ \textsf{+S}\ B) \equiv (\delta \circledcirc \gamma)\ \textsf{+S}\ B
\end{aligned}$$

They are useful to derive some auxiliary functions. The following is one of them which is used a lot in proofs.

$$\begin{aligned}
\textsf{wk-tm+} \quad & : \{\Gamma\ \Delta : \textsf{Con}\}\{A : \textsf{Ty}\ \Delta\}\{\delta : \Gamma \Rightarrow \Delta\}(B : \textsf{Ty}\ \Gamma) \\
& \rightarrow \textsf{Tm}\ (A\ [\ \delta\ ]\textsf{T}\ \textsf{+T}\ B) \rightarrow \textsf{Tm}\ (A\ [\ \delta \ \textsf{+S}\ B\ ]\textsf{T}) \\
\textsf{wk-tm+}\ B\ t \quad & = t\ [\![\ \textsf{[+S]T}\ \rangle\!\rangle
\end{aligned}$$

We can cancel the last term in the substitution for weakened objects since weakening doesn't introduce new variables in types and terms.

$$\begin{aligned}
\textsf{+T[,]T} \quad & : \{\Gamma\ \Delta : \textsf{Con}\}\{A : \textsf{Ty}\ \Delta\}\{\delta : \Gamma \Rightarrow \Delta\}\{B : \textsf{Ty}\ \Delta\}\{b : \textsf{Tm}\ (B\ [\ \delta\ ]\textsf{T})\} \\
& \rightarrow (A\ \textsf{+T}\ B)\ [\ \delta\ ,\ b\ ]\textsf{T} \equiv A\ [\ \delta\ ]\textsf{T}
\end{aligned}$$

$$\textsf{+tm[,]tm} \quad : \{\Gamma\ \Delta : \textsf{Con}\}\{A : \textsf{Ty}\ \Delta\}(a : \textsf{Tm}\ A)\{\delta : \Gamma \Rightarrow \Delta\}\{B : \textsf{Ty}\ \Delta\}\{c : \textsf{Tm}\ (B\ [\ \delta\ ]\textsf{T})\}$$

$$\rightarrow (a \; \mathsf{+tm} \; B) \; [\; \delta \; , \; c \;]\mathsf{tm} \cong a \; [\; \delta \;]\mathsf{tm}$$

Most of the substitutions are defined as usual, except the one for coherence constants. We do substitution in the context morphism part of the coherence constants.

$$\begin{aligned}
&\mathsf{var}\; x && [\; \delta \;]\mathsf{tm} = x\; [\; \delta \;]\mathsf{V} \\
&\mathsf{coh}\; c\Delta\; \gamma\; A && [\; \delta \;]\mathsf{tm} = \mathsf{coh}\; c\Delta\; (\gamma \circledcirc \delta)\; A \; [\![\; \mathsf{sym}\; [\circledcirc]\mathsf{T} \;\rangle\!\rangle
\end{aligned}$$

8.3 Some Important Derivable Constructions

In this section we show that it is possible to reconstruct the structure of a (weak) ω-groupoid from the syntactical framework presented in Section 8.2 in the style of [44]. To this end, let us call a term $a : \mathsf{Tm}\; A$ an n-cell if $\mathsf{level}\; A \equiv n$, where

$$\begin{aligned}
&\mathsf{level} && : \forall\; \{\Gamma\} \rightarrow \mathsf{Ty}\; \Gamma \rightarrow \mathbb{N} \\
&\mathsf{level}\; * && = 0 \\
&\mathsf{level}\; (_\mathsf{=h}_\; \{A\}\; _\; _) && = \mathsf{suc}\; (\mathsf{level}\; A)
\end{aligned}$$

In any ω-category, any n-cell a has a domain (source), $s^n_m\, a$, and a codomain (target), $s^n_m\, a$, for each $m \leq n$. These are, of course, (n-m)-cells. For each pair of n-cells such that for some m $s^n_m a \equiv t^n_m b$, there must exist their composition $a \circ^n_m b$ which is an n-cell. Composition is (weakly) associative. Moreover for any (n-m)-cell x there exists an n-cell $\mathsf{id}^n_m\, x$ which behaves like a (weak) identity with respect to $\circ^n_m$. For the time being we discuss only the construction of cells and omit the question of coherence.

For instance, in the simple case of bicategories, each 2-cell a has a horizontal source $s^1_1\, a$ and target $t^1_1\, a$, and also a vertical source $s^2_1\, a$ and target $t^2_1 a$, which is also the source and target, of the horizontal source and target, respectively, of a. There is horizontal composition of 1-cells $\circ^1_1$: $x \xrightarrow{f} y \xrightarrow{g} z$, and also horizontal composition of 2-cells $\circ^2_1$, and vertical composition of 2-cells $\circ^2_2$. There is a horizontal identity on a, $\mathsf{id}^1_1\, a$, and vertical identity on a, $\mathsf{id}^2_1\, a = \mathsf{id}^2_2 \mathsf{id}^1_1\, a$.

Thus each ω-groupoid construction is defined with respect to a *level*, m, and depth n-m and the structure of an ω-groupoid is repeated on each level. As we are working purely syntactically we may make use of this fact and define all groupoid structure only at level

$m = 1$ and provide a so-called *replacement operation* which allows us to lift any cell to an arbitrary type A. It is called 'replacement' because we are syntactically replacing the base type $*$ with an arbitrary type, A.

An important general mechanism we rely on throughout the development follows directly from the type of the only nontrivial constructor of Tm, coh, which tells us that to construct a new term of type $\Gamma \vdash A$, we need a contractible context, Δ, a type $\Delta \vdash T$ and a context morphism $\delta : \Gamma \Rightarrow \Delta$ such that

$$T[\,\delta\,]\mathsf{T} \equiv A$$

Because in a contractible context all types are inhabited we may in a way work freely in Δ and then pull back all terms to A using δ. To show this formally, we must first define identity context morphisms which complete the definition of a *category* of contexts and context morphisms:

$$\mathsf{IdCm} : \forall\ \{\Gamma\} \rightarrow \Gamma \Rightarrow \Gamma$$

It satisfies the following property:

$$\mathsf{IC\text{-}T} \quad : \forall\{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\} \rightarrow A\ [\ \mathsf{IdCm}\ ]\mathsf{T} \equiv A$$

The definition proceeds by structural recursion and therefore extends to terms, variables and context morphisms with analogous properties. It allows us to define at once:

$$\begin{aligned}
&\mathsf{Coh\text{-}Contr} && : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\} \rightarrow \mathsf{isContr}\ \Gamma \rightarrow \mathsf{Tm}\ A \\
&\mathsf{Coh\text{-}Contr}\ isC && = \mathsf{coh}\ isC\ \mathsf{IdCm}\ _\ [\![\ \mathsf{sym}\ \mathsf{IC\text{-}T}\ \rangle\!\rangle
\end{aligned}$$

We use Coh-Contr as follows: for each kind of cell we want to define, we construct a minimal contractible context built out of variables together with a context morphism that populates the context with terms and a lemma that states a definitional equality between the substitution and the original type.

8.3.1 Suspension and Replacement

For an arbitrary type A in Γ of level n one can define a context with $2n$ variables, called the *stalk* of A. Moreover one can define a morphism from Γ to the stalk of A such that its substitution into the maximal type in the stalk of A gives back A. The stalk of A depends only on the level of A, the terms in A define the substitution. Here is an example of stalks of small levels: ε (the empty context) for $n = 0$; $(x_0 : *, x_1 : *)$ for $n = 1$; $(x_0 : *, x_1 : *, x_2 : x_0 =_{\mathsf{h}} x_1, x_3 : x_0 =_{\mathsf{h}} x_1)$ for $n = 2$, etc.

This is the $\Delta = \varepsilon$ case of a more general construction where in we *suspend* an arbitrary context Δ by adding $2n$ variables to the beginning of it, and weakening the rest of the variables appropriately so that type $*$ becomes $x_{2n-2} =_{\mathsf{h}} x_{2n-1}$. A crucial property of suspension is that it preserves contractibility.

8.3.1.1 Suspension

Suspension is defined by iteration level-A-times the following operation of one-level suspension. $\Sigma\mathsf{C}$ takes a context and gives a context with two new variables of type $*$ added at the beginning, and with all remaining types in the context suspended by one level.

$$\begin{aligned}
&\Sigma\mathsf{C} : \mathsf{Con} \to \mathsf{Con} \\
&\Sigma\mathsf{T} : \{\Gamma : \mathsf{Con}\} \to \mathsf{Ty}\ \Gamma \to \mathsf{Ty}\ (\Sigma\mathsf{C}\ \Gamma)
\end{aligned}$$

$$\begin{aligned}
&\Sigma\mathsf{C}\ \varepsilon && = \varepsilon\ ,\ *\ ,\ * \\
&\Sigma\mathsf{C}\ (\Gamma\ ,\ A) && = \Sigma\mathsf{C}\ \Gamma\ ,\ \Sigma\mathsf{T}\ A
\end{aligned}$$

The rest of the definitions is straightforward by structural recursion. In particular we suspend variables, terms and context morphisms:

$$\begin{aligned}
&\Sigma\mathsf{v} && : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\} \to \mathsf{Var}\ A \to \mathsf{Var}\ (\Sigma\mathsf{T}\ A) \\
&\Sigma\mathsf{tm} && : \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\} \to \mathsf{Tm}\ A \to \mathsf{Tm}\ (\Sigma\mathsf{T}\ A) \\
&\Sigma\mathsf{s} && : \{\Gamma\ \Delta : \mathsf{Con}\} \to \Gamma \Rightarrow \Delta \to \Sigma\mathsf{C}\ \Gamma \Rightarrow \Sigma\mathsf{C}\ \Delta
\end{aligned}$$

The following lemma establishes preservation of contractibility by one-step suspension:

$$\mathsf{\Sigma C\text{-}Contr} : (\Delta : \mathsf{Con}) \to \mathsf{isContr}\ \Delta \to \mathsf{isContr}\ (\mathsf{\Sigma C}\ \Delta)$$

It is also essential that suspension respects weakening and substitution:

$$\begin{aligned}
&\mathsf{\Sigma T[+T]} &&: \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma)(B : \mathsf{Ty}\ \Gamma)\\
& &&\to \mathsf{\Sigma T}\ (A\ \mathsf{+T}\ B) \equiv \mathsf{\Sigma T}\ A\ \mathsf{+T}\ \mathsf{\Sigma T}\ B\\
&\mathsf{\Sigma tm[+tm]} &&: \{\Gamma : \mathsf{Con}\}\{A : \mathsf{Ty}\ \Gamma\}(a : \mathsf{Tm}\ A)(B : \mathsf{Ty}\ \Gamma)\\
& &&\to \mathsf{\Sigma tm}\ (a\ \mathsf{+tm}\ B) \cong \mathsf{\Sigma tm}\ a\ \mathsf{+tm}\ \mathsf{\Sigma T}\ B
\end{aligned}$$

$$\begin{aligned}
&\mathsf{\Sigma T[\Sigma s]T} &&: \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Delta)(\delta : \Gamma \Rightarrow \Delta)\\
& &&\to (\mathsf{\Sigma T}\ A)\ [\ \mathsf{\Sigma s}\ \delta\ ]\mathsf{T} \equiv \mathsf{\Sigma T}\ (A\ [\ \delta\ ]\mathsf{T})
\end{aligned}$$

General suspension to the level of a type A is defined by iteration of one-level suspension. For symmetry and ease of reading the following suspension functions take as a parameter a type A in Γ, while they depend only on its level.

$$\mathsf{\Sigma C\text{-}it} : \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Con} \to \mathsf{Con}$$

$$\mathsf{\Sigma T\text{-}it} : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Ty}\ \Delta \to \mathsf{Ty}\ (\mathsf{\Sigma C\text{-}it}\ A\ \Delta)$$

$$\mathsf{\Sigma tm\text{-}it} : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma)\{B : \mathsf{Ty}\ \Delta\} \to \mathsf{Tm}\ B \to \mathsf{Tm}\ (\mathsf{\Sigma T\text{-}it}\ A\ B)$$

Finally, it is clear that iterated suspension preserves contractibility.

$$\mathsf{\Sigma C\text{-}it\text{-}Contr} : \forall\ \{\Gamma\ \Delta\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{isContr}\ \Delta \to \mathsf{isContr}\ (\mathsf{\Sigma C\text{-}it}\ A\ \Delta)$$

By suspending the minimal contractible context, *, we obtain a so-called span. They are also stalks with a top variable added. For example $(x_0 : *)$ (the one-variable context) for $n = 0$; $(x_0 : *, x_1 : *, x_2 : x_0 =_{\mathsf{h}} x_1)$ for $n = 1$; $(x_0 : *, x_1 : *, x_2 : x_0 =_{\mathsf{h}} x_1, x_3 : x_0 =_{\mathsf{h}} x_1, x_4 : x_2 =_{\mathsf{h}} x_3)$ for $n = 2$, etc. Spans play an important role later in the definition of composition.

8.3.1.2 Replacement

After we have suspended a context by inserting an appropriate number of variables, we may proceed to a substitution which fills the stalk for A with A. The context morphism representing this substitution is called filter. In the final step we combine it with Γ, the context of A. The new context contains two parts, the first is the same as Γ, and the second is the suspended Δ substituted by filter. However, we also have to drop the stalk of A becuse it already exists in Γ.

Geometrically speaking, the context resulting from replacing $*$ in Δ by A is a new context which corresponds to the pasting of Δ to Γ to A.

As always, we define replacement for contexts, types and terms:

$$\begin{array}{ll}
\mathsf{rpl\text{-}C} & : \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Con} \to \mathsf{Con} \\
\mathsf{rpl\text{-}T} & : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Ty}\ \Delta \to \mathsf{Ty}\ (\mathsf{rpl\text{-}C}\ A\ \Delta) \\
\mathsf{rpl\text{-}tm} & : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma)\{B : \mathsf{Ty}\ \Delta\} \to \mathsf{Tm}\ B \to \mathsf{Tm}\ (\mathsf{rpl\text{-}T}\ A\ B)
\end{array}$$

Replacement for contexts, $\mathsf{rpl}-\mathsf{C}$, defines for a type A in Γ and another context Δ a context which begins as Γ and follows by each type of Δ with $*$ replaced with (pasted onto) A. To this end we must define the substitution filter which pulls back each type from suspended Δ to the new context.

$$\mathsf{filter} : \{\Gamma : \mathsf{Con}\}(\Delta : \mathsf{Con})(A : \mathsf{Ty}\ \Gamma) \to \mathsf{rpl\text{-}C}\ A\ \Delta \Rightarrow \Sigma\mathsf{C\text{-}it}\ A\ \Delta$$

$$\begin{array}{ll}
\mathsf{rpl\text{-}C}\ \{\Gamma\}\ A\ \varepsilon & = \Gamma \\
\mathsf{rpl\text{-}C}\ A\ (\Delta\ ,\ B) & = \mathsf{rpl\text{-}C}\ A\ \Delta\ ,\ \mathsf{rpl\text{-}T}\ A\ B
\end{array}$$

$$\mathsf{rpl\text{-}T}\ A\ B = \Sigma\mathsf{T\text{-}it}\ A\ B\ [\ \mathsf{filter}\ \_\ A\ ]\mathsf{T}$$

8.3.2 First-level Groupoid Structure

We can proceed to the definition of the groupoid structure of the syntax. We start with the base case: 1-cells. Replacement defined above allows us to lift this structure to an arbitrary level n (we leave most of the routine details out). This shows that the syntax is a 1-groupoid on each level. In the next section we show how also the higher-groupoid structure can be defined.

We start by an essential lemma which formalises the discussion at the beginning of this Section: to construct a term in a type A in an arbitrary context, we first restrict attention to a suitable contractible context Δ and use lifting and substitution – replacement – to pull the term built by coh in Δ back. This relies on the fact that a lifted contractible context is also contractible, and therefore any type lifted from a contractible context is also inhabited.

$$\begin{array}{l}
\mathsf{Coh\text{-}rpl} : \{\Gamma\ \Delta : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma)(B : \mathsf{Ty}\ \Delta) \to \mathsf{isContr}\ \Delta \to \mathsf{Tm}\ \{\mathsf{rpl\text{-}C}\ A\ \Delta\}\ (\mathsf{rpl\text{-}T}\ A\ B) \\
\mathsf{Coh\text{-}rpl}\ \{\Delta = \Delta\}\ A\ B\ isc = \mathsf{coh}\ (\mathsf{\Sigma C\text{-}it\text{-}\varepsilon\text{-}Contr}\ A\ isc)\ (\mathsf{filter}\ \Delta\ A)\ (\mathsf{\Sigma T\text{-}it}\ A\ B)
\end{array}$$

Next we define the reflexivity, symmetry and transitivity terms of any type . Let's start from the basic case as for the base type *. It is trivially inhabited because the context is the basic case of a contractible context.

$$\begin{array}{l}
\mathsf{refl*} : \mathsf{Tm}\ \{\mathsf{x{:}*}\}\ (\mathsf{var\ v0 =h\ var\ v0}) \\
\mathsf{refl*} = \mathsf{Coh\text{-}Contr\ c*}
\end{array}$$

To obtain the reflexivity term for any given type, we just use replacement.

$$\begin{array}{lll}
\mathsf{refl\text{-}Tm} & : & \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \\
 & & \to \mathsf{Tm}\ (\mathsf{rpl\text{-}T}\ \{\Delta = \mathsf{x{:}*}\}\ A\ (\mathsf{var\ v0 =h\ var\ v0})) \\
\mathsf{refl\text{-}Tm}\ A & = & \mathsf{rpl\text{-}tm}\ A\ \mathsf{refl*}
\end{array}$$

Symmetry (inverse) is defined similarly. Note that the intricate names of contexts, as in Ty x:*,y:*,α:x=y indicate their definitions which have been hidden. For instance we are assuming the definition: x:*,y:*,α:x=y = ε ,* , * , (var (vS v0) =h var v0)

$$\begin{array}{l}
\mathsf{sym*\text{-}Ty} : \mathsf{Ty\ x{:}*,y{:}*,\alpha{:}x{=}y} \\
\mathsf{sym*\text{-}Ty} = \mathsf{vY =h\ vX}
\end{array}$$

$$\begin{array}{l}
\mathsf{sym*\text{-}Tm} : \mathsf{Tm}\ \{\mathsf{x{:}*,y{:}*,\alpha{:}x{=}y}\}\ \mathsf{sym*\text{-}Ty} \\
\mathsf{sym*\text{-}Tm} = \mathsf{Coh\text{-}Contr}\ (\mathsf{ext\ c*\ v0})
\end{array}$$

$$\mathsf{sym\text{-}Tm} \quad : \{\Gamma : \mathsf{Con}\}(A : \mathsf{Ty}\ \Gamma) \to \mathsf{Tm}\ (\mathsf{rpl\text{-}T}\ A\ \mathsf{sym*\text{-}Ty})$$

```
sym-Tm A    = rpl-tm A sym*-Tm
```

Finally, transitivity (composition). Note that each of these cells is defined by a different choice of the contractible context Δ.

```
trans*-Ty : Ty x:*,y:*,α:x=y,z:*,β:y=z
trans*-Ty = (vX +tm _ +tm _) =h vZ

trans*-Tm : Tm trans*-Ty
trans*-Tm = Coh-Contr (ext (ext c* v0) (vS v0))

trans-Tm     : {Γ : Con}(A : Ty Γ) → Tm (rpl-T A trans*-Ty)
trans-Tm A   = rpl-tm A trans*-Tm
```

For each of reflexivity, symmetry and transitivity we can construct appropriate coherence 2-cells witnessing the groupoid axioms. The base case for variable contexts is proved simply using contractibility. We use substitution to define the application of the three basic terms we have defined above.

```
Tm-right-identity* : Tm {x:*,y:*,α:x=y}
  (trans*-Tm [ IdCm , vY , reflY ]tm =h vα)
Tm-right-identity* = Coh-Contr (ext c* v0)

Tm-left-identity* : Tm {x:*,y:*,α:x=y}
  (trans*-Tm [ ((IdCm ⊚ pr1 ⊚ pr1) , vX) , reflX , vY , vα ]tm =h vα)
Tm-left-identity* = Coh-Contr (ext c* v0)

Tm-right-inverse* : Tm {x:*,y:*,α:x=y}
  (trans*-Tm [ (IdCm , vX) , sym*-Tm ]tm =h reflX)
Tm-right-inverse* = Coh-Contr (ext c* v0)

Tm-left-inverse* : Tm {x:*,y:*,α:x=y}
  (trans*-Tm [ ((• , vY) , vX , sym*-Tm , vY) , vα ]tm =h reflY)
Tm-left-inverse* = Coh-Contr (ext c* v0)

Tm-G-assoc* : Tm Ty-G-assoc*
```

$$\text{Tm-G-assoc*} = \text{Coh-Contr } (\text{ext } (\text{ext } (\text{ext } c* \text{ v0}) \text{ (vS v0)}) \text{ (vS v0)})$$

Their general versions are defined using replacement. For instance, for associativity, we define:

$$\begin{aligned}
&\text{Tm-G-assoc} &&: \{\Gamma : \text{Con}\}(A : \text{Ty } \Gamma) \to \text{Tm } (\text{rpl-T } A \text{ Ty-G-assoc*})\\
&\text{Tm-G-assoc } A &&= \text{rpl-tm } A \text{ Tm-G-assoc*}
\end{aligned}$$

8.3.3 Higher Structure

In this section we propose how also higher groupoid structure can be introduced in the syntactical framework. We use the more abstract language of category theory to communicate the gist of the construction leaving the tedious formalisation for future work. To this end note that contexts and context morphisms form a category up to definitional quality. Because equality of contexts is decidable we may assume UIP on context morphisms and we are therefore working in a honest 1-category where equality of arrows is definitional equality of context morphisms. This category will be denoted Con.

8.3.3.1 Identities

For each type of level $n \in \mathbf{N}$, we have defined in Section 8.3.1.2 a context called *span* which has $2n + 1$ variables which except for the top level, n, there are two variables on each level whose type is the equality type of the two variables on the level below, except for the bottom-level variables which are of type $*$. We call denote a span of any type of level n, S_n. Note that all such spans are isomorphic.

In each case we call the last variable the *peak*. Note that each S_n is contractible because it is a suspension of a contractible context. We call the proof of contractibility of S_n is-contr S_n.

In each S_n define the type σ_n as $x_{2n-2} =_{\mathsf{h}} x_{2n-1}$. It is the type of the top variable. We are going to show that the following

$$\begin{tikzcd} S_0 \arrow[r, "i_0"] & S_1 \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "t_0"] \arrow[r, "i_1"] & S_2 \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "t_1"] \cdots S_n \arrow[r, "i_n"] & S_{n+1} \cdots \arrow[l, shift right=2, "s_n"'] \arrow[l, shift left=2, "t_n"] \end{tikzcd}$$

is a *reflexive globular object* in Con. I.e. we define morphisms s_n, t_n, i_n between spans that it satisfy the following usual *globular identities*:

$$\begin{aligned} s_n t_{n+1} &= s_n s_{n+1} \\ t_n t_{n+1} &= t_n s_{n+1} \end{aligned} \tag{8.1}$$

together with

$$s_n i_n = \mathsf{id} = t_n i_n \tag{8.2}$$

To this end, for each n, define context morphisms $s_n, t_n : S_{n+1} \Rightarrow S_n$ by the substitutions

$$\begin{array}{rlrcll} s_n & = & x_k & \leftarrow\!\dashv & x_k & k < 2n \\ & & x_{2n} & \leftarrow\!\dashv & x_{2n} & \\ t_n & = & x_k & \leftarrow\!\dashv & x_k & k < 2n \\ & & x_{2n} & \leftarrow\!\dashv & 2n+1 & \end{array}$$

In words, s_n forgets the last two variables of S_{n+1} and t_n forgets the peak and its domain. It's easy to check that s and t indeed satisfy (8.1).

In order to define $i_n : S_n \Rightarrow S_{n+1}$, we must consider stalks (see Section 8.3.1), which are contexts, hereby denoted $\overline{S_n}$, which are like S_n without the last variable, together with types $\overline{\sigma}_n$ which are like σ_n but considered in the smaller context.

For each n there is a context morphism $\overline{i_n} : S_n \Rightarrow \overline{S}_{n+1}$ defined by

$$\begin{array}{rlrcll} \overline{i_n} & = & x_k & \leftarrow\!\dashv & x_k & k \leq 2n \\ & & x_{2n+1} & \leftarrow\!\dashv & x_{2n} & \end{array}$$

The substitution of $\overline{\sigma}_{n+1}$ along $\overline{i_n}$, $\overline{i_n}[\overline{\sigma}_{n+1}]_{\mathsf{T}}$, is equal to $x_{2n} =_{\mathsf{h}} x_{2n}$ in S_n. So in order to extend $\overline{i_n}$ to $i_n : S_n \Rightarrow S_{n+1}$ we must define a term in $\overline{i_n}[\overline{\sigma}_{n+1}]_{\mathsf{T}}$. We can readily do that by coh:

$$i_n \;=\; \overline{i_n}\,, \mathsf{coh}\ (\mathsf{IdCm}\ \mathsf{S_n})\ (x_{2n} =_{\mathsf{h}}\ x_{2n})\ (\mathsf{is\text{-}contr}\ S_n)$$

It's easy to check that i_n satisfies (8.2).

For each n consider the chain

$$i_0^n \;\equiv\; S_0 \xrightarrow{i_0} S_1 \xrightarrow{i_1} S_2 \longrightarrow \cdots \xrightarrow{i_{n-1}} S_n$$

The substitution $\sigma_n[i_0]_{\mathsf{T}} \cdots [i_{n-2}]_{\mathsf{T}}[i_{n-1}]_{\mathsf{T}} \equiv \sigma_n[i_0^n]_{\mathsf{T}}$ is a type, λ_n, in S_0. We call λ_n the *n-iterated loop type* on x_0. The term $S_0 \vdash (\mathsf{var}\ x_{2n})[i_0^n]_{\mathsf{tm}} : \lambda_n$ is the iterated identity term on x_0.

8.3.3.2 Composition

For $m > n$ write s_n^m, t_n^m for $m - n$ iteration of s and t, respectively. Explicitly:

$$\begin{aligned} s_n^m &= s_{m-1} \cdots s_n & &: S_m \Rightarrow S_n \\ t_n^m &= t_{m-1} \cdots t_n & &: S_m \Rightarrow S_n \end{aligned}$$

For each $n \in \mathbb{N}$ define context Y_n by the pullback in:

$$\begin{tikzcd} S_0 & S_n \arrow[l, "t_0^n"'] \\ S_n \arrow[u, "s_0^n"] & Y_n \arrow[l, "r_n"] \arrow[u, "l_n"'] \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

By definition of pullbacks, Y_n looks like a pair of spans S_n together with the proviso that the variable 1 of one is always equal to variable 0 of the other. I.e. Y_n has the shape of two spans pasted target-to-source at level 0. It is easy to check that this is indeed a pullback.

By the globular identities the two outer squares in the diagram below commute and by the universal property of the pullback imply a pair of mediating arrows as indicated.

$$\begin{tikzcd} S_0 & S_n \arrow[l, "t_0^n"'] & S_{n+1} \arrow[l, "s_n"'] \\ S_n \arrow[u, "s_0^n"] & Y_n \arrow[l, "r_n"] \arrow[u, "l_n"'] \arrow[ul, phantom, very near start, "\ulcorner"] & \\ S_{n+1} \arrow[u, "s_n"] & & Y_{n+1} \arrow[ll, "r_{n+1}"] \arrow[uu, "l_{n+1}"'] \arrow[ul, dashed, "\langle s,s \rangle_n"'] \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

Similarly we obtain an arrow $Rr_n : Y_{n+1} \longrightarrow Y_n$. The morphisms l_n and r_n provide projections onto the left and right span of Y_n respectively. The mediating arrows $\langle s, s \rangle_n$ and $\langle t, t \rangle_n$ provide projections out of Y_{n+1} onto the join of the sources and targets of the left and right parts respectively.

In order to define composition we define for each n a third morphism $c_n : Y_n \Rightarrow S_n$ with the property that both the s-squares and t-squares below commute.

$$\begin{tikzcd}
S_n & S_{n+1} \arrow[l, shift right, "s_n"'] \arrow[l, shift left, "t_n"] \\
Y_n \arrow[u, "c_n"] & Y_{n+1} \arrow[l, shift right, "\langle s,s \rangle_n"'] \arrow[l, shift left, "\langle t,t \rangle_n"] \arrow[u, "c_{n+1}"']
\end{tikzcd} \tag{8.3}$$

The commutativity of (8.3) expresses the fact that the source of a composition is a composition of sources and the target of a composition is a composition of target.

It follows from all of this that for a context Γ and a pair of morphisms $a, b : \Gamma \Rightarrow S_n$, there is a context morphism $c\langle a, b\rangle : \Gamma \Rightarrow S_n$ from $s_0^n a$ to $t_0^n b$ which is the composition of a and b.

8.4 Semantics

8.4.1 Globular Types

To interpret the syntax, we need globular types [1] . Globular types are defined coinductively as follows.

```
record Glob : Set₁ where
  constructor _||_
  field
    obj   : Set
    hom   : obj → obj → ∞ Glob
open Glob public renaming (obj to |_|)
```

If all the object types are indeed sets, i.e. uniqueness of identity types holds, we call this a globular set.

As an example, we could construct the identity globular type called Idw.

```
Idω     : (A : Set) → Glob
```

[1]Even though we use the Agda |Set|, this isn't necessarily a set in the sense of Homotopy Type Theory.

```
Idω A   = A || (λ a b → ♯ Idω (a ≡ b))
```

Note that this is usually not a globular set.

Given a globular set G, we can interpret the syntactic objects.

The record definition also require some semantic weakening and semantic substitution laws. The semantic weakening rules tell us how to deal with the weakening inside interpretation.

```
record Semantic (G : Glob) : Set₁ where
  field
    [[_]]C   : Con → Set
    [[_]]T   : ∀{Γ} → Ty Γ → [[ Γ ]]C → Glob
    [[_]]tm  : ∀{Γ A} → Tm A → (γ : [[ Γ ]]C) → | [[ A ]]T γ |
    [[_]]cm  : ∀{Γ Δ} → Γ ⇒ Δ → [[ Γ ]]C → [[ Δ ]]C
```

π provides the projection of the semantic variable out of a semantic context.

```
    π   : ∀{Γ A}(x : Var A)(γ : [[ Γ ]]C) → | [[ A ]]T γ |
```

Following are the computation laws for the interpretations of contexts and types.

```
  [[_]]C-β1   : [[ ε ]]C ≡ ⊤
  [[_]]C-β2   : ∀ {Γ A} → [[ Γ , A ]]C ≡ Σ [[ Γ ]]C (λ γ   → | [[ A ]]T γ |)

  [[_]]T-β1   : ∀{Γ}{γ : [[ Γ ]]C} → [[ * ]]T γ ≡ G
  [[_]]T-β2   : ∀{Γ A u v}{γ : [[ Γ ]]C}
              → [[ u =h v ]]T γ ≡
                ♭ (hom ([[ A ]]T γ) ([[ u ]]tm γ) ([[ v ]]tm γ))
```

The semantic substitution properties are essential,

```
  semSb-T   : ∀ {Γ Δ}(A : Ty Δ)(δ : Γ ⇒ Δ)(γ : [[ Γ ]]C)
```

$$\rightarrow [\![A [\delta]\mathsf{T}]\!] T\ \gamma \equiv [\![A]\!] T\ ([\![\delta]\!] cm\ \gamma)$$

$$\begin{aligned}\mathsf{semSb\text{-}tm} : & \forall\{\Gamma\ \Delta\}\{A : \mathsf{Ty}\ \Delta\}(a : \mathsf{Tm}\ A)(\delta : \Gamma \Rightarrow \Delta)(\gamma : [\![\Gamma]\!] C) \\ & \rightarrow \mathsf{subst}\ |\_|\ (semSb\text{-}T\ A\ \delta\ \gamma)\ ([\![a [\delta]\mathsf{tm}]\!] tm\ \gamma) \\ & \quad \equiv [\![a]\!] tm\ ([\![\delta]\!] cm\ \gamma)\end{aligned}$$

$$\begin{aligned}\mathsf{semSb\text{-}cm} : & \forall\ \{\Gamma\ \Delta\ \Theta\}(\gamma : [\![\Gamma]\!] C)(\delta : \Gamma \Rightarrow \Delta)(\vartheta : \Delta \Rightarrow \Theta) \\ & \rightarrow [\![\vartheta \circledcirc \delta]\!] cm\ \gamma \equiv [\![\vartheta]\!] cm\ ([\![\delta]\!] cm\ \gamma)\end{aligned}$$

Since the computation laws for the interpretations of terms and context morphisms are well typed up to these properties.

$$\begin{aligned}[\![\_]\!]\mathsf{tm}\text{-}\beta 1\ \ & : \forall\{\Gamma\ A\}\{x : \mathsf{Var}\ A\}\{\gamma : [\![\Gamma]\!] C\} \\ & \rightarrow [\![\mathsf{var}\ x]\!] tm\ \gamma \equiv \pi\ x\ \gamma\end{aligned}$$

$$[\![\_]\!]\mathsf{cm}\text{-}\beta 1\ \ : \forall\{\Gamma\}\{\gamma : [\![\Gamma]\!] C\} \rightarrow [\![\bullet]\!] cm\ \gamma \equiv \mathsf{coerce}\ [\![\_]\!] C\text{-}\beta 1\ \mathsf{tt}$$

$$\begin{aligned}[\![\_]\!]\mathsf{cm}\text{-}\beta 2\ \ & : \forall\{\Gamma\ \Delta\}\{A : \mathsf{Ty}\ \Delta\}\{\delta : \Gamma \Rightarrow \Delta\}\{\gamma : [\![\Gamma]\!] C\}\{a : \mathsf{Tm}\ (A [\delta]\mathsf{T})\} \\ & \rightarrow [\![\delta , a]\!] cm\ \gamma \equiv \mathsf{coerce}\ [\![\_]\!] C\text{-}\beta 2\ (([\![\delta]\!] cm\ \gamma)\ , \\ & \quad \mathsf{subst}\ |\_|\ (semSb\text{-}T\ A\ \delta\ \gamma)\ ([\![a]\!] tm\ \gamma))\end{aligned}$$

The semantic weakening properties should actually be deriavable since weakening is equivalent to projection substitution.

$$\begin{aligned}\mathsf{semWk\text{-}T}\ \ & : \forall\ \{\Gamma\ A\ B\}(\gamma : [\![\Gamma]\!] C)(v : |\ [\![B]\!] T\ \gamma\ |) \\ & \rightarrow [\![A +\mathsf{T}\ B]\!] T\ (\mathsf{coerce}\ [\![\_]\!] C\text{-}\beta 2\ (\gamma\ , v)) \equiv [\![A]\!] T\ \gamma\end{aligned}$$

$$\begin{aligned}\mathsf{semWk\text{-}cm}\ \ & : \forall\ \{\Gamma\ \Delta\ B\}\{\gamma : [\![\Gamma]\!] C\}\{v : |\ [\![B]\!] T\ \gamma\ |\}(\delta : \Gamma \Rightarrow \Delta) \\ & \rightarrow [\![\delta +\mathsf{S}\ B]\!] cm\ (\mathsf{coerce}\ [\![\_]\!] C\text{-}\beta 2\ (\gamma\ , v)) \equiv [\![\delta]\!] cm\ \gamma\end{aligned}$$

$$\begin{aligned}\mathsf{semWk\text{-}tm} : & \forall\ \{\Gamma\ A\ B\}(\gamma : [\![\Gamma]\!] C)(v : |\ [\![B]\!] T\ \gamma\ |)(a : \mathsf{Tm}\ A) \\ & \rightarrow \mathsf{subst}\ |\_|\ (semWk\text{-}T\ \gamma\ v)\ ([\![a +\mathsf{tm}\ B]\!] tm\ (\mathsf{coerce}\ [\![\_]\!] C\text{-}\beta 2\ (\gamma\ , v))) \\ & \quad \equiv ([\![a]\!] tm\ \gamma)\end{aligned}$$

Here we declare them as properties because they are essential for the computation laws of function π.

$$
\begin{aligned}
\text{π-}\beta 1 \quad & : \forall\{\Gamma\ A\}(\gamma : [\![\ \Gamma\ ]\!]C)(v : |\ [\![\ A\ ]\!]T\ \gamma\ |) \\
& \to \mathsf{subst}\ |\_|\ (semWk\text{-}T\ \gamma\ v)\ (\pi\ \mathsf{v0}\ (\mathsf{coerce}\ [\![\_]\!]C\text{-}\beta 2\ (\gamma\ ,\ v))) \equiv v \\
\text{π-}\beta 2 \quad & : \forall\{\Gamma\ A\ B\}(x : \mathsf{Var}\ A)(\gamma : [\![\ \Gamma\ ]\!]C)(v : |\ [\![\ B\ ]\!]T\ \gamma\ |) \\
& \to \mathsf{subst}\ |\_|\ (semWk\text{-}T\ \gamma\ v)\ (\pi\ (\mathsf{vS}\ \{\Gamma\}\ \{A\}\ \{B\}\ x)\ (\mathsf{coerce}\ [\![\_]\!]C\text{-}\beta 2\ (\gamma\ ,\ v))) \\
& \equiv \pi\ x\ \gamma
\end{aligned}
$$

The only part of the semantics where we have any freedom is the interpretation of the coherence constants:

$$
[\![\mathsf{coh}]\!] : \forall\{\Theta\} \to \mathsf{isContr}\ \Theta \to (A : \mathsf{Ty}\ \Theta) \to (\vartheta : [\![\ \Theta\ ]\!]C) \to |\ [\![\ A\ ]\!]T\ \vartheta\ |
$$

However, we also need to require that the coherence constants are well behaved wrt to substitution which in turn relies on the interpretation of all terms. To address this we state the required properties in a redundant form because the correctness for any other part of the syntax follows from the defining equations we have already stated. However, there seems to be no way to avoid this.

If the underlying globular type is not a globular set we need to add coherence laws, which is not very well understood. On the other hand, restricting ourselves to globular sets means that our prime examle Idω is not an instance anymore. We should still be able to construct non-trivial globular sets, e.g. by encoding basic topological notions and defining higher homotopies as in a classical framework. However, we don't currently know a simple definition of a globular set which is a weak ω-groupoid. One possibility would be to use the syntax of type theory with equality types. Indeed, we believe that this would be an alternative way to formalize weak ω-groupoids.

8.5 Summary

In this chapter, we present an implementation of weak ω-groupoids following Brunerie's work. Briefly speaking, we define the syntax of the type theory $\mathcal{T}_{\infty-groupoid}$, then a weak ω-groupoid is a globular set with the interpretation of the syntax. To overcome some technical problems, we use heterogeneous equality for terms, some auxiliary functions and loop context in all implementation. We construct the identity morphisms and verify some groupoid laws in the syntactic framework. The suspensions for all sorts of objects are also defined for other later constructions.

There are still a lot of work to do within the syntactic framework. For instance, we would like to investigate the relation between the $\mathcal{T}_{\infty-groupoid}$ and a Type Theory with equality types and J eliminator which is called $\mathcal{T}_{eq}$. One direction is to simulate the J eliminator syntactically in $\mathcal{T}_{\infty-groupoid}$ as we mentioned before, the other direction is to derive J using *coh* if we can prove that the $\mathcal{T}_{eq}$ is a weak ω-groupoid. The syntax could be simplified by adopting categories with families. An alternative may be to use higher inductive types directly to formalize the syntax of type theory.

We would like to formalise a proof of that Idω is a weak ω-groupoid, but the base set in a globular set is an h-set which is incompatible with Idω. Perhaps we could solve the problem by instead proving a syntactic result, namely that the theory we have presented here and the theory of equality types with J are equivalence. Finally, to model the Type Theory with weak ω-groupoids and to eliminate the univalence axiom would be the most challenging task in the future.

Chapter 9

Summary

9.0.1 Future work

Appendix A

Appendix A

```
GCD′→ℚ : ∀ x y di → y ≢ 0 → C.GCD′ x y di → ℚ
GCD′→ℚ .(q₁ ℕ* di) .(q₂ ℕ* di) di neo (C.gcd-* q₁ q₂ c) = record { numerator = numr
  ; denominator-1 = pred q₂
  ; isCoprime = iscoprime }
  where
  numr = ℤ.+_ q₁
  deno = suc (pred q₂)

  numr≡q1 : ∣ numr ∣ ≡ q₁
  numr≡q1 = refl

  lzero : ∀ x y → x ≡ 0 → x ℕ* y ≡ 0
  lzero .0 y refl = refl

  q2≢0 : q₂ ≢ 0
  q2≢0 qe = neo (lzero q₂ di qe)

  invsuc : ∀ n → n ≢ 0 → suc (pred n) ≡ n
  invsuc zero nz with nz refl
  ... | ()
  invsuc (suc n) nz = refl

  deno≡q2 : deno ≡ q₂
  deno≡q2 = invsuc q₂ q2≢0
```

```
transCop : ∀ {a b c d} → c ≡ a → d ≡ b → C.Coprime a b → C.Coprime c d
transCop refl refl c = c

copnd : C.Coprime ∣ numr ∣ deno
copnd = transCop numr≡q1 deno≡q2 c

witProp : ∀ a b → GCD a b 1 → True (C.coprime? a b)
witProp a b gcd1 with gcd a b
witProp a b gcd1 | zero , y with GCD.unique gcd1 y
witProp a b gcd1 | zero , y | ()
witProp a b gcd1 | suc zero , y = tt
witProp a b gcd1 | suc (suc n) , y with GCD.unique gcd1 y
witProp a b gcd1 | suc (suc n) , y | ()

iscoprime : True (C.coprime? ∣ numr ∣ deno)
iscoprime = witProp ∣ numr ∣ deno (C.coprime-gcd copnd)
```

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