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Doctoral Thesis

Thesis Title

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Abstract

Faculty Name
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by Nuo Li

The Thesis Abstract is written here (and usually kept to just this page). This thesis mainly covered the quotient types in type theory

Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include your project advisor…

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Background: Quotient Types

Quotient is a very common notion is mathematics. It refers to the result of division at first and is extended to other abstract branches of mathematics. More generally, it describes the collection of equivalent classes of some equivalent relation on sets, spaces or other abstract structures. In type theory, following similar procedure, quotient type is also a conceivable notion.

2.1 Introduction

Quotient sets In set theory, the old brother of type theory, given a set A equipped with an equivalence relation $\sim$, a quotient set is denoted as $A/\sim$ which contains the set of equivalence classes.

$$[a] = \{x : A \mid a \sim x\} \tag{2.1}$$

$$A/\sim = \{[a] \mid a : A\} \tag{2.2}$$

Quotient types In type theory, quotient type can be formalised as following:

$$\frac{A \quad \sim : A \to A \to \mathbf{Prop}}{A/\sim}\ Q - \mathbf{Form}$$

$$\frac{a : A}{[a] : A/\sim}\ Q - \mathbf{Intro}$$

$$\frac{\begin{array}{l} B : A/\!\sim \to \mathbf{Set} \\ f : (a : A) \to B\,[a] \\ (a, b : A) \to (p : a \sim b) \to f\,a \overset{p}{=} f\,b \end{array}}{\hat{f} : (q : Q) \to B\,q}\; Q - \mathbf{elim}$$

In general, quotient types are unavailable in intensional type theory. Quotients are everywhere, for example, rational numbers, real numbers, multi-sets. The introduction of quotient types is very helpful. Some of them can be defined more effectively, such as the set of integers. Some of them can only be defined with quotient types, such as real numbers (the reason will be covered in O).

2.2 Categorical intuition

Categorically speaking, a quotient is a coequalizer.

Definition 2.1. Given two objects X and Y and two parallel morphisms $f, g : X \to Y$, a coequalizer is an object Q with a morphism $q : Y \to Q$ such that $q \circ f = q \circ g$. It has to be universal as well. Any pair (Q' , q') $q' \circ f = q' \circ g$ has a unique factorisation u such that $q' = u \circ q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "u"] \\ & & Q' \end{tikzcd}$$

$$\begin{tikzcd} A \times A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "{[\cdot]}"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "u"] \\ & & Q' \end{tikzcd}$$

2.2.1 Adjuction between Sets and Setoids

From a higher point of view, Quotient is a Functor which is left-adjoint to ∇ which is the trivial embedding functor from **Sets** to **Setoids**.

Definition 2.2. $\nabla A = (A, =)$

$$B/\!\sim\, \to A$$

$$(B, \sim) \to \nabla A$$

2.3 Definable Quotients

Some types can be defined without quotient, however it does not possess good properties from being quotient types. Examples like integers, which can be defined as either negative or non-negative,

Sometimes, quotient types are more difficult to reason about than their base types. We can achieve more convenience by manipulating base types and then lifting the operators and propositions according to the relation between quotient types and base types. Therefore it is worthwhile for us to conduct a research project on the implementation of quotients in intensional type theory.

The work of this project will be divided into several phases. This report introduces the basic notions in my project on implementing quotients in type theory, such as type theory setoids, and quotient types, reviews some work related to this topic and concludes with some results of the first phase. The results done by Altenkirch, Anberrée and I in [1] will be explained with a few instances of quotients.

2.3.1 Natural numbers as the basis

2.3.2 Integers

2.3.3 Rational numbers

$\frac{3}{2}$

2.3.4 Real numbers and Complex numbers

2.3.5 Multisets(bags)

Bibliography

[1] Thorsten Altenkirch, Thomas Anberrée, and Nuo Li. Definable Quotients in Type Theory. 2011.