

Numbers, quotients, and definability of reals

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November 20, 2012

1 Numbers

Since Agda is known as a proof assistant, the library of numbers is crucial. In such kind of proof assistants which are based on Martin-Löf type theory, we need to construct the type of numbers and the usual properties of them should be verifiable rather than axiomatic.

There are different ways of defining numbers, even though they are mathematically equivalent, they are technically different, which means the proving of theorems about the numbers varies. For example, integers can be defined by exploiting the isomorphism between \mathbb{Z} and $\mathbb{N} + \mathbb{N}$:

```
data  $\mathbb{Z}$  : Set where
  -[1+_] : (n :  $\mathbb{N}$ ) →  $\mathbb{Z}$   - -[1+ n ] stands for - (1 + n).
  +_ : (n :  $\mathbb{N}$ ) →  $\mathbb{Z}$   - + n stands for n.
```

And this is exactly the definition in Agda standard library version 0.6. This definition is easy to use since it has two cases and for each integer you have one unique representation. However intuitively we lose the "special position" held by 0. Of course we can define three cases definition with distinct 0 constructor but too many cases are not ideal for proving.

Alternatively we have another isomorphism between \mathbb{Z} and $\mathbb{N} \times \mathbb{N} / \sim$, namely constructing the set of integers from quotienting the set of $\mathbb{N} \times \mathbb{N}$ by the following equivalence relation:

$$(n_1, n_2) \sim (n_3, n_4) \iff n_1 + n_4 = n_3 + n_2 \quad (1)$$

From this observation we can define a setoid implementation for integers.

```
data  $\mathbb{Z}_0$  : Set where
  _,_ :  $\mathbb{N}$  →  $\mathbb{N}$  →  $\mathbb{Z}_0$ 
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$$\begin{array}{c} _ \sim _ : \text{Rel } \mathbb{Z}_0 _ \\ (x1, x2) \sim (y1, y2) = (x1 \text{ N+ } y2) \equiv (y1 \text{ N+ } x2) \end{array}$$

The common operations defined on $\mathbb{Z}_0(\mathbb{N} \times \mathbb{N} / \sim)$ has only one case which are much simpler than the one for the previous definition,

$$\begin{array}{c} _ + _ : \mathbb{Z}_0 \rightarrow \mathbb{Z}_0 \rightarrow \mathbb{Z}_0 \\ (x1, x2) + (y1, y2) = (x1 \text{ N+ } y1), (x2 \text{ N+ } y2) \end{array}$$

and we only need to prove that it respects the equivalence relation,

$$\begin{array}{c} _ \text{ respects } _ : \{A : \text{Set}\} \rightarrow \text{Op}_2 A \rightarrow \text{Rel } A _ \rightarrow \text{Set} \\ _ \circledast _ \text{ respects } _ \approx _ = \forall a b c d \rightarrow a \approx b \rightarrow c \approx d \rightarrow (a \circledast c) \approx (b \circledast d) \end{array}$$

Given two pairs equal setoid integers $(x, x_1) \sim (x_2, x_3)$ and $(x_4, x_5) \sim (x_6, x_7)$, the goal just turns into some simple expression of natural numbers

$$x\mathbb{N} + x_4\mathbb{N} + (x_3\mathbb{N} + x_7) \equiv x_2\mathbb{N} + x_6\mathbb{N} + (x_1\mathbb{N} + x_5)$$

which can be automatically solved in Agda (the detail looks cumbersome).

Given any operation respects the equivalence relation, we can easily turns it into the corresponding operation for the set definition \mathbb{Z} in a general way.

First, we need a normalization function to find the corresponding $z : \mathbb{Z}$ for any given $z_0 : \mathbb{Z}_0$.

$$\begin{array}{c} _ : \mathbb{Z}_0 \rightarrow \mathbb{Z} \\ [m, 0] = + m \\ [0, \text{suc } n] = -[1+ n] \\ [\text{suc } m, \text{suc } n] = [m, n] \end{array}$$

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$$\begin{array}{c} _ : \mathbb{Z} \rightarrow \mathbb{Z}_0 \\ \lceil + n \rceil = n, 0 \\ \lceil -[1+ n] \rceil = 0, \text{suc } n \end{array}$$

Then the general lifting function is

$$\begin{aligned}
\text{lift}_2 & : (_ \circledast _ : \mathbf{Op}_2 \mathbb{Z}_0) \\
& \rightarrow \mathbf{Op}_2 \mathbb{Z} \\
\text{lift}_2 _ \circledast _ a b & = [\ulcorner a \urcorner \circledast \ulcorner b \urcorner]
\end{aligned}$$

Since we can prove the operation respects the equivalence relation, the lifted operation should have the same properties, which means we can also lift the proofs of the theorems on setoid integers.

In fact, this kind of relationship between setoids and sets can be generalized as quotients. Given a setoid (A, \sim) , some set $Q : \mathbf{Set}$ can be seen as the corresponding quotient type over this setoid, if we have a function $[\cdot] : A \rightarrow Q$ such that it fulfils certain set of properties (details in [1]).

For the set rational numbers, we could also define it using setoids from the construction of fractions and the equivalence relation on fractions.

For real numbers, we could use cauchy sequences of rational numbers to represent them. However, we still can not find a way to define the set of reals in current setting of Intensional Type Theory.

From the observations above, there is a pattern between different kinds of numbers, namely we can create a setoid using already defined number sets to represent a new number set. This kind of relation better exploits the relation between the number sets such that we can prove theorems much simpler.

2 quotients

The quotient type is unavailable in Intensional Type Theory, since it is extensional.

3 definability

References

- [1] Thorsten Altenkirch, Thomas Anberrée, and Nuo Li. Definable Quotients in Type Theory. 2011.