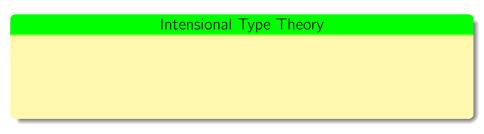
# Homotopy Type Theory and Hedberg's Theorem

Nicolai Kraus

16/11/12



### Intensional Type Theory

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e.g. 
$$\lambda f \to \lambda a \to f \ a \ a : (A \to A \to B) \to A \to B$$

## Reminder: Equality

#### **Definitional Equality**

Decidable equality for typechecking & computation; e.g.  $(\lambda a.b)x =_{\beta} b[x/a]$ 

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#### Propositional Equality

Equality needing a proof, i. e. a term of the identity type

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a, b : A

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#### Introduction

a : A

 $refl_a$ :  $a \equiv a$ 

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a, b : A  $a \equiv b : type$ 

### Elimination (J)

 $P: (a, b: A) \rightarrow a \equiv b \rightarrow Set$   $m: \forall a. P(a, a, refl_a)$  $J_{(a,b,q)}: P(a, b, q)$ 

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#### Introduction

 $\frac{a:A}{refl_a:a\equiv a}$ 

### Computation $(\beta)$

 $J_{(a,a,refl_a)} =_{\beta} ma$ 

Given a: A and  $p: a \equiv a$ , can we prove  $p \equiv refl_a$ ?

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#### Disadvantages

Intuitively wrong, impossible to express statements about equality, isomorphic sets cannot be equal

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Without UIP: new model of Type Theory (types as weak  $\omega$ -groupoids)

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- best expressible in Simplicial Sets SSets (the topos  $Sets^{\Delta^{op}}$ )
- realization functor  $R: SSets \rightarrow Top$
- R is a Quillen equivalence of model categories
- $\bullet$   $\Rightarrow$  (more or less) a model that uses topological spaces as types

Fix a type A.

### Decidable Equality

DecidableEquality :=  $\forall a b \rightarrow (a \equiv b + \neg a \equiv b)$ 

#### Hedberg's theorem

 $Decidable Equality \longrightarrow UIP$ 

#### **Constant Function**

$$const(f) := \forall ab \rightarrow fa \equiv fb$$

#### Constant Endofunction on Path Spaces

$$g: \forall a b \rightarrow a \equiv b \rightarrow a \equiv b$$
  
 $path\text{-}const(g):= \forall a b \rightarrow const g_{ab}$ 

#### Lemma 1

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- If decab = inlp, then  $g_{ab}(\underline{\ }) = p$

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- Proof with *J*: Just do it for (*a*, *a*, *refl<sub>a</sub>*). That's true!

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- Given any a, b : A and  $p, q : a \equiv b$ .
- Claim:  $p \equiv (g_{aa}refl_a)^{-1} \circ g_{ab}(p)$
- Proof with J: Just do it for (a, a, refl<sub>a</sub>). That's true!
- Same for q. But  $g_{aa}$  and  $g_{ab}$  are constant.

### Generalizations of Hedberg's theorem

We have seen

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#### Lemma 1

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DecidableEquality is a very strong property. How about something weaker? For example:

#### Separated

$$\forall ab \rightarrow \neg \neg (a \equiv b) \rightarrow a \equiv b$$

#### "general"

 $\forall a b \rightarrow [propositional \ evidence \ for \ a \equiv b] \rightarrow a \equiv b$ 

### **Propositions**

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Write **Prop** for this "subset" of **Type** 

### H-Propositional Reflection

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#### This means:

- A\* is in **Prop**
- $\eta:A\to A^*$
- if P is a proposition and  $A \rightarrow P$ , then  $A^* \rightarrow P$

### Generalizations of Hedberg's Theorem

"Propositional evidence for  $a \equiv b$ " is now just [an inhabitant of]  $(a \equiv b)^*$ .

#### H-Separated

$$\forall ab \rightarrow (a \equiv b)^* \rightarrow a \equiv b$$

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$$\forall ab \rightarrow (a \equiv b)^* \rightarrow a \equiv b$$

#### Theorem

h-separated<sub>A</sub>  $\longleftrightarrow \Sigma_g$  . path-const $(g) \longleftrightarrow UIP_A$ 

### Nearly uncountable many things to be done . . .

- Higher Inductive Types (see Mike Shulman's work)
- Model construction with modern abstract (not point-set) homotopy theory
- Constructive Simplicial Sets (the combinatorial version of what I have shown; see Thierry Coquand's / Simon Huber's work)
- Univalent foundations / Univalence ("alternative" to K) in general (see Voevodsky)
- ... and possible computational properties (Thorsten?)

#### THANK YOU!