Constructing inductive-inductive types using a domain-specific type theory[1]

Ambrus Kaposi

j.w.w. Thorsten Altenkirch, Péter Diviánszky and András Kovács

TYPES 2018
Braga
21 June 2018

SZÉCHENYI 2020
Magyarország Kormánya
Európai Unió
Európai Szociális Alap
BEFEKTETÉS A JÖVŐBE

[1] This work was supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002) and COST Action EUTypes CA15123.

Contents

1. Specifying inductive-inductive types

2. Constructing inductive-inductive types

Specifying inductive-inductive types

How to specify inductive types?

By listing their constructors.

How to specify inductive types?

By listing their constructors.

E.g. natural numbers are specified by

$$
\begin{aligned}
\mathit{Nat} &: \mathrm{Type} \\
\mathit{zero} &: \mathit{Nat} \\
\mathit{suc} &: \mathit{Nat} \to \mathit{Nat}
\end{aligned}
$$

How to specify inductive types?

By listing their constructors.
E.g. natural numbers are specified by

$$\begin{aligned} \mathit{Nat} &: \mathrm{Type} \\ \mathit{zero} &: \mathit{Nat} \\ \mathit{suc} &: \mathit{Nat} \to \mathit{Nat} \end{aligned}$$

Inductive-inductive types allow multiple sorts indexed over each other, e.g.

$$\begin{aligned} \mathit{Con} &: \mathrm{Type} \\ \mathit{Ty} &: \mathit{Con} \to \mathrm{Type} \\ \bullet &: \mathit{Con} \\ - \triangleright - &: (\Gamma : \mathit{Con}) \to \mathit{Ty}\,\Gamma \to \mathit{Con} \\ U &: (\Gamma : \mathit{Con}) \to \mathit{Ty}\,\Gamma \\ \mathit{Pi} &: (\Gamma : \mathit{Con})(A : \mathit{Ty}\,\Gamma) \to \mathit{Ty}\,(\Gamma \triangleright A) \to \mathit{Ty}\,\Gamma \end{aligned}$$

What does it mean to list the constructors?

What does it mean to list the constructors?

A signature for an inductive-inductive type is a context

What does it mean to list the constructors?

A signature for an inductive-inductive type is a context in a domain-specific type theory.

What does it mean to list the constructors?

A signature for an inductive-inductive type is a context in a domain-specific type theory.

- Variables
- Empty universe U with underline for El:

$$\frac{}{\Gamma \vdash \mathrm{U}} \qquad \frac{\Gamma \vdash a : \mathrm{U}}{\Gamma \vdash \underline{a}}$$

- Restricted function space:

$$\frac{\Gamma \vdash a : \mathrm{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t\, u : B[x \mapsto u]}$$

What does it mean to list the constructors?

A signature for an inductive-inductive type is a context in a domain-specific type theory.

- Variables
- Empty universe U with underline for El:

$$\frac{}{\Gamma \vdash \mathrm{U}} \qquad \frac{\Gamma \vdash a : \mathrm{U}}{\Gamma \vdash \underline{a}}$$

- Restricted function space:

$$\frac{\Gamma \vdash a : \mathrm{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t\, u : B[x \mapsto u]}$$

Signature for natural numbers:
$\Theta := (\cdot, \mathit{Nat} : \mathrm{U}, \mathit{zero} : \underline{\mathit{Nat}}, \mathit{suc} : \mathit{Nat} \Rightarrow \underline{\mathit{Nat}})$

What does it mean to list the constructors?

A signature for an inductive-inductive type is a context in a domain-specific type theory.

- Variables
- Empty universe U with underline for El:

$$\frac{}{\Gamma \vdash \mathrm{U}} \qquad \frac{\Gamma \vdash a : \mathrm{U}}{\Gamma \vdash \underline{a}}$$

- Restricted function space:

$$\frac{\Gamma \vdash a : \mathrm{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t\, u : B[x \mapsto u]}$$

Signature for natural numbers:
$\Theta := (\cdot,\ Nat : \mathrm{U},\ zero : \underline{Nat},\ suc : Nat \Rightarrow \underline{Nat})$

Not possible: $(\cdot,\ T : \mathrm{U},\ evil : (T \Rightarrow \underline{T}) \Rightarrow \underline{T})$

Standard interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{A}} \in \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{A}} \in \Gamma^{\mathsf{A}} \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{A}} &:= (\gamma \in \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}(\gamma) \\
\mathsf{U}^{\mathsf{A}}(\gamma) &:= \mathsf{Set} \\
((x : a) \Rightarrow B)^{\mathsf{A}}(\gamma) &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to B^{\mathsf{A}}(\gamma, \alpha)
\end{aligned}$$

Standard interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{A}} \in \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{A}} \in \Gamma^{\mathsf{A}} \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{A}} &:= (\gamma \in \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}(\gamma) \\
\mathsf{U}^{\mathsf{A}}(\gamma) &:= \mathsf{Set} \\
((x : a) \Rightarrow B)^{\mathsf{A}}(\gamma) &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to B^{\mathsf{A}}(\gamma, \alpha)
\end{aligned}$$

$-^{\mathsf{A}}$ on a context gives algebras for that signature.
E.g. $\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$

Logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\,\gamma\,\gamma' \to A^{\mathsf{A}}\,\gamma \to A^{\mathsf{A}}\,\gamma' \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{M}}((\gamma,\alpha),(\gamma',\alpha')) &:= (\gamma_M : \Gamma^{\mathsf{M}}(\gamma,\gamma')) \times A^{\mathsf{M}}(\gamma_M,\alpha,\alpha') \\
\mathsf{U}^{\mathsf{M}}(\gamma_M, a, a') &:= a \to a' \to \mathsf{Set} \\
(\underline{a})^{\mathsf{M}}(\gamma_M,\alpha,\alpha') &:= a^{\mathsf{M}}(\gamma_M,\alpha,\alpha') \\
((x : a) \Rightarrow B)^{\mathsf{M}}(\gamma_M, f, f') &:= (\alpha_M \in a^{\mathsf{M}}(\gamma,\alpha,\alpha')) \to \\
&\qquad B^{\mathsf{M}}\Big((\gamma_M,\alpha_M), f(\alpha), f'(\alpha')\Big)
\end{aligned}$$

Tweaked logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\,\gamma\,\gamma' \to A^{\mathsf{A}}\,\gamma \to A^{\mathsf{A}}\,\gamma' \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{M}}((\gamma,\alpha),(\gamma',\alpha')) &:= (\gamma_M : \Gamma^{\mathsf{M}}(\gamma,\gamma')) \times A^{\mathsf{M}}(\gamma_M,\alpha,\alpha') \\
\mathsf{U}^{\mathsf{M}}(\gamma_M, a, a') &:= a \to a' \\
(\underline{a})^{\mathsf{M}}(\gamma_M,\alpha,\alpha') &:= (a^{\mathsf{M}}(\gamma_M)(\alpha) = \alpha') \\
((x : a) \Rightarrow B)^{\mathsf{M}}(\gamma_M, f, f') &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to \\
&\qquad B^{\mathsf{M}}\Big((\gamma_M, \mathsf{refl}), f(\alpha), f'\big(a^{\mathsf{M}}(\gamma_M)(\alpha)\big)\Big)
\end{aligned}$$

Tweaked logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\,\gamma\,\gamma' \to A^{\mathsf{A}}\,\gamma \to A^{\mathsf{A}}\,\gamma' \to \mathsf{Set}}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{M}}((\gamma,\alpha),(\gamma',\alpha')) &:= (\gamma_M : \Gamma^{\mathsf{M}}(\gamma,\gamma')) \times A^{\mathsf{M}}(\gamma_M,\alpha,\alpha') \\
\mathsf{U}^{\mathsf{M}}(\gamma_M, a, a') &:= a \to a' \\
(\underline{a})^{\mathsf{M}}(\gamma_M,\alpha,\alpha') &:= (a^{\mathsf{M}}(\gamma_M)(\alpha) = \alpha') \\
((x : a) \Rightarrow B)^{\mathsf{M}}(\gamma_M, f, f') &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to \\
&\qquad B^{\mathsf{M}}\Big((\gamma_M, \mathsf{refl}), f(\alpha), f'(a^{\mathsf{M}}(\gamma_M)(\alpha))\Big)
\end{aligned}$$

$-^{\mathsf{M}}$ on a context gives homomorphisms of algebras. E.g.

$$\begin{aligned}
\Theta^{\mathsf{M}}((N,z,s),(N',z',s')) = (N_M : N \to N') \times (N_M(z) = z') \times \\
((\alpha \in N) \to N_M(s(\alpha)) = s'(N_M(\alpha)))
\end{aligned}$$

Constructing inductive-inductive types

Constructing natural numbers

We use the domain-specific type theory:

$$\mathbb{N} := \{t \mid \Theta \vdash t : \underline{Nat}\}$$

Constructing natural numbers

We use the domain-specific type theory:

$$\mathbb{N} := \{t \mid \Theta \vdash t : \underline{\mathit{Nat}}\}$$

N.B.

$$\Theta = (\cdot,\ \mathit{Nat} : \mathsf{U},\ \mathit{zero} : \underline{\mathit{Nat}},\ \mathit{suc} : \mathit{Nat} \Rightarrow \underline{\mathit{Nat}})$$

Constructing natural numbers

We use the domain-specific type theory:

$$\mathbb{N} := \{t \mid \Theta \vdash t : \underline{Nat}\}$$

N.B.

$$\Theta = (\cdot,\ Nat : \mathsf{U},\ zero : \underline{Nat},\ suc : Nat \Rightarrow \underline{Nat})$$

Constructors:

$$\mathsf{zero} := zero \in \mathbb{N}$$

$$\mathsf{suc}(n \in \mathbb{N}) := suc\ n \in \mathbb{N}$$

Recursion

We need:

$$\mathsf{rec}_{\mathbb{N}} : \mathbb{N} \to (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)$$

Recursion

We need:

$$\mathsf{rec}_{\mathbb{N}} : \mathbb{N} \to (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)$$

The standard interpretation of $t \in \mathbb{N}$, i.e. $\Theta \vdash t : \underline{Nat}$:

$$t^{\mathsf{A}} \in (x \in \Theta^{\mathsf{A}}) \to \underbrace{(\underline{Nat})^{\mathsf{A}}(x)}_{=\mathsf{proj}_1(x)}$$

Recursion

We need:

$$\mathsf{rec}_{\mathbb{N}} : \mathbb{N} \to (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)$$

The standard interpretation of $t \in \mathbb{N}$, i.e. $\Theta \vdash t : \underline{Nat}$:

$$t^{\mathsf{A}} \in (x \in \Theta^{\mathsf{A}}) \to \underbrace{(\underline{Nat})^{\mathsf{A}}(x)}_{=\mathsf{proj}_1(x)}$$

$$\mathsf{rec}_{\mathbb{N}}(t) := t^{\mathsf{A}}$$

Model for the initial algebra

Specification (fix a Θ):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{C}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{A}}}$$

On the universe:

$$\mathsf{U}^{\mathsf{C}}(\nu, a) := \{t \,|\, \Theta \vdash t : \underline{a}\}$$

Model for the initial algebra

Specification (fix a Θ):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{C}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{A}}}$$

On the universe:

$$\mathsf{U}^{\mathsf{C}}(\nu, a) := \{t \mid \Theta \vdash t : \underline{a}\}$$

Initial algebra for Θ is $\Theta^{\mathsf{C}}(\mathrm{id}_\Theta) \in \Theta^{\mathsf{A}}$.

Model for the recursor

Specification (fix a Θ and a $\theta \in \Theta^{\mathsf{A}}$):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{R}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{M}}(\Gamma^{\mathsf{C}}(\nu), \nu^{\mathsf{A}}(\theta))}$$

On the universe:

$$\mathsf{U}^{\mathsf{R}}(\nu, a)(t) := t^{\mathsf{A}}(\theta)$$

Model for the recursor

Specification (fix a Θ and a $\theta \in \Theta^{\mathsf{A}}$):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{R}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{M}}(\Gamma^{\mathsf{C}}(\nu), \nu^{\mathsf{A}}(\theta))}$$

On the universe:

$$\mathsf{U}^{\mathsf{R}}(\nu, a)(t) := t^{\mathsf{A}}(\theta)$$

Recursor is given by $\Theta^{\mathsf{R}}(\mathsf{id}_{\Theta}) \in \Theta^{\mathsf{M}}(\text{initial algebra for } \Theta, \theta)$.

Summary

Domain-specific type theory for signatures.

We do universal algebra by defining models of this type theory.

Standard model:	algebras
Tweaked logical relations:	algebra homomorphisms
Model where U is terms:	initial algebra
Model where U is the standard interpretation:	recursor
Logical predicates:	families
Tweaked dependent logical relations:	sections
Model where U is logical predicate translation:	eliminator

Summary

Domain-specific type theory for signatures.

We do universal algebra by defining models of this type theory.

Standard model:	algebras
Tweaked logical relations:	algebra homomorphisms
Model where U is terms:	initial algebra
Model where U is the standard interpretation:	recursor
Logical predicates:	families
Tweaked dependent logical relations:	sections
Model where U is logical predicate translation:	eliminator

All of this extends to quotient inductive-inductive types.
Challenge: what about higher inductive-inductive types?

Constructing inductive types à la Awodey-Frey-Speight

System F impredicative encoding:

$$\mathbb{N} := ((x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x))$$

Constructing inductive types à la Awodey-Frey-Speight

System F impredicative encoding:

$$\mathbb{N} := ((x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x))$$

N.B.

$$\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$$

Constructing inductive types à la Awodey-Frey-Speight

System F impredicative encoding:

$$\mathbb{N} := ((x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x))$$

N.B.

$$\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$$

Problem: given an $f \in \Theta^{\mathsf{M}}(x, x')$ and an $n \in \mathbb{N}$, $f(n(x)) \stackrel{?}{=} n(x')$

Constructing inductive types à la Awodey-Frey-Speight

System F impredicative encoding:

$$\mathbb{N} := ((x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x))$$

N.B.

$$\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$$

Problem: given an $f \in \Theta^{\mathsf{M}}(x, x')$ and an $n \in \mathbb{N}$, $f(n(x)) \stackrel{?}{=} n(x')$

Solution: let's build this in the definition!

$$\mathbb{N} := (n \in (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)) \times (\forall x, x', f. f(n(x)) = n(x'))$$