Equations over Groups

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An Elementary Problem

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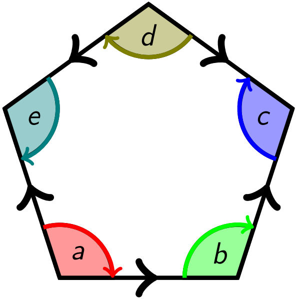

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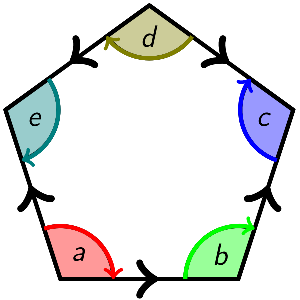

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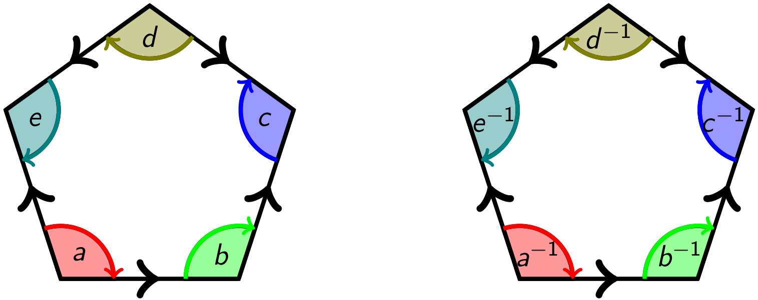

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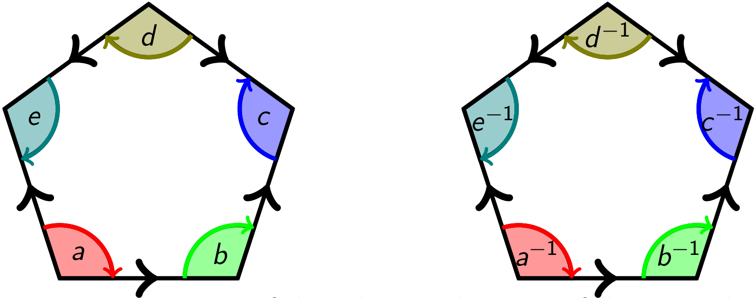

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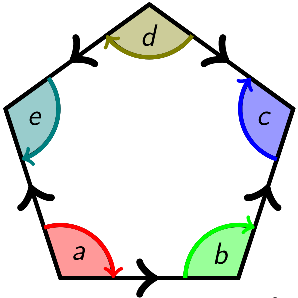
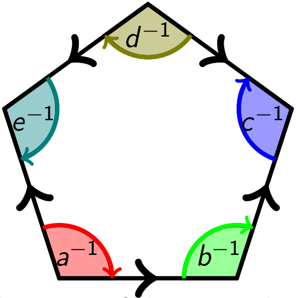
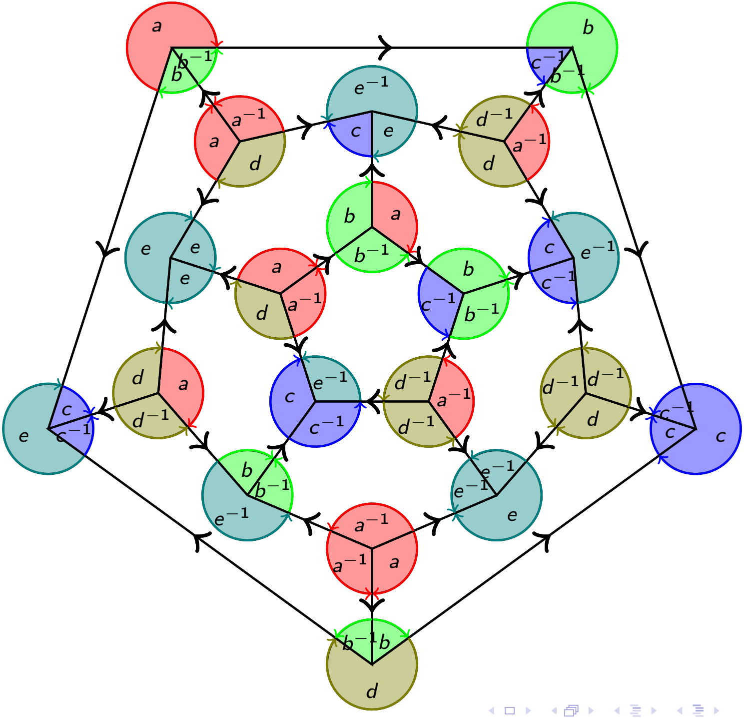




Let a topological tiling of the sphere with copies of these tiles be given.
Edge orientations must match!



a b^{-1} b
b c^{-1} b^{-1}
e^{-1} c e
a^{-1} a d
d^{-1} a^{-1} d
b a b^{-1}
e e e
a d a^{-1}
b c^{-1} b^{-1}
c e^{-1} c^{-1}
d a d^{-1}
c e^{-1} c^{-1}
d^{-1} a^{-1} d^{-1}
d^{-1} d^{-1} d
c e c^{-1}
c^{-1} c c
b b^{-1} e^{-1}
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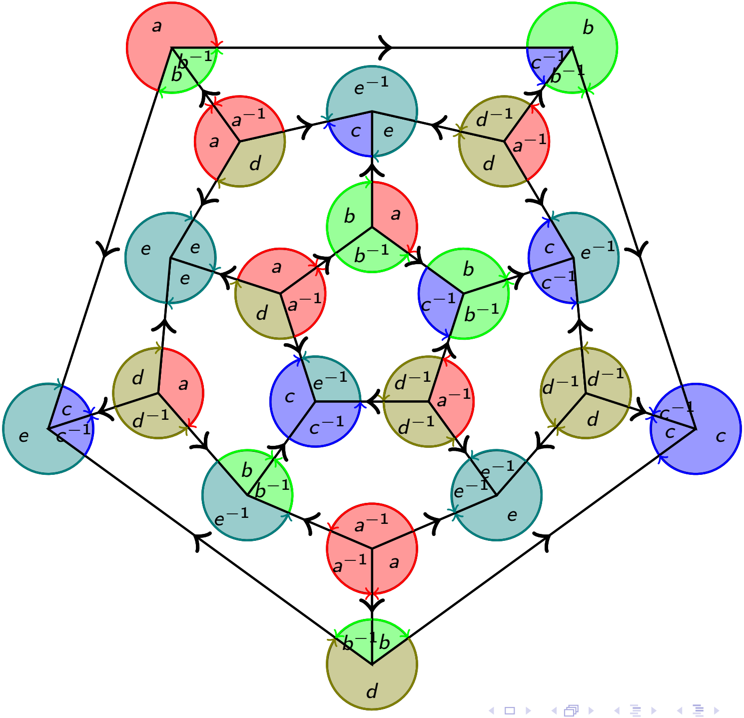
Conjecture: If positively oriented edges equal negatively oriented edges in number, this will always be possible.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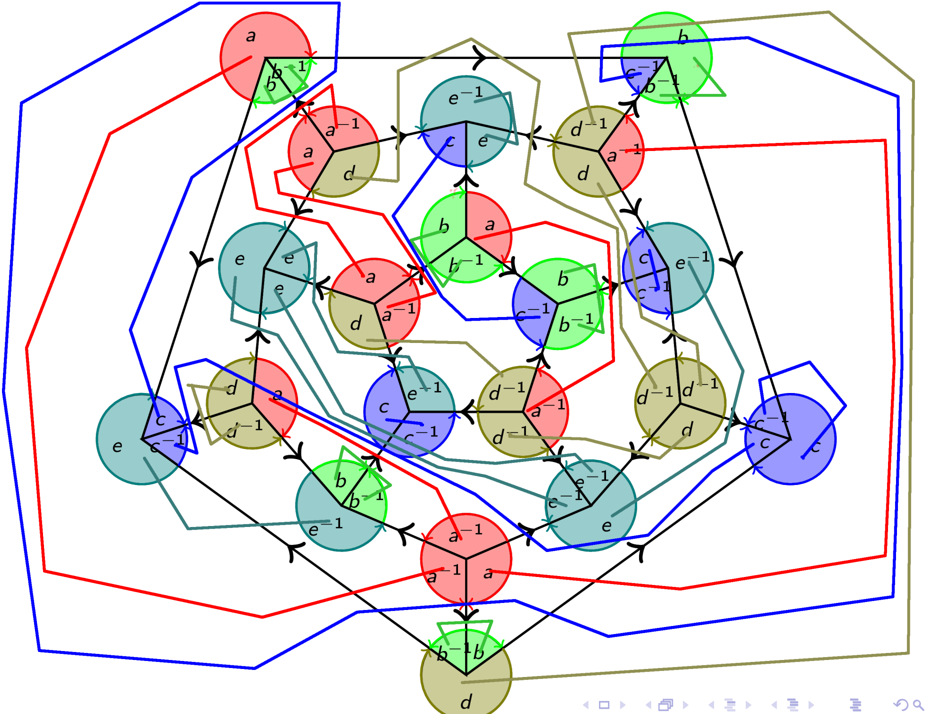

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(Recall: S^H denotes normal closure of set S in H.)

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Conjecture: Define $s : G * \mathbb{Z} \to \mathbb{Z}$ by $s = [\text{const}\, 0, \text{id}]$.
If $s(r) \neq 0$, the above holds.

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Group G, "unknown" t, equation $a_1 t^{d_1} \ldots a_n t^{d_n} = e$ over G

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Groups are perverse objects.

Revelations

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All these problems are actually the same!

An Important Problem: Progress

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$|d_1| + \ldots + |d_n| \leq 5$ (Evangelidou, 2003)

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$|d_1| + \ldots + |d_n| \leq 6 + x$ (Agda and Coq, 2014)?