Internal parametricity, without an interval

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Identity type in type theory

$$? : \mathsf{Id}_{\mathbb{N}}\,(1 + 1)\,2$$

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In general:

$$\frac{A : \mathsf{Type}}{\mathsf{Id}_A : A \to A \to \mathsf{Type}} \qquad \frac{a : A}{\mathsf{refl}_a : \mathsf{Id}_A\,a\,a} \qquad \dots$$

What kind of type is Id?

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$$\begin{aligned}
\mathsf{Id}_{\mathbb{N}}\,0 \quad & 0 && := \top \\
\mathsf{Id}_{\mathbb{N}}\,(\mathsf{suc}\,m)\,& (\mathsf{suc}\,n) && := \mathsf{Id}_N\,m\,n \\
\mathsf{Id}_{\mathbb{N}}\,0 \quad & (\mathsf{suc}\,n) && := \bot \\
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$$\mathsf{Id}_{A\times B}\,(a_0, b_0)\,(a_1, b_1) :\equiv \mathsf{Id}_A\,a_0\,a_1 \times \mathsf{Id}_B\,b_0\,b_1$$

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$$\mathsf{Id}_{A\to B}\,f_0\,f_1 :\equiv \forall a_0\,a_1\,.\,\mathsf{Id}_A\,a_0\,a_1 \to \mathsf{Id}_B\,(f_0\,a_0)\,(f_1\,a_0)$$

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  Promises:

  - explainable: no interval, only low dimensional operations
  - computational univalence (unlike C.T.T.)
  - efficient (?): a simple extension of Martin-Löf's type theory

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- For now we see things in a glass, darkly; but then face to face. (1 Cor 13:12)
  - explainability, computation, simple extension

Semantics

The semantics is Bezem-Coquand-Huber cubes

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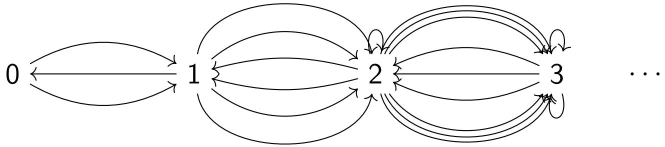

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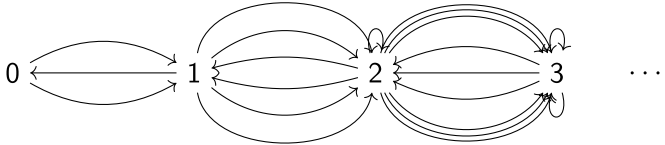

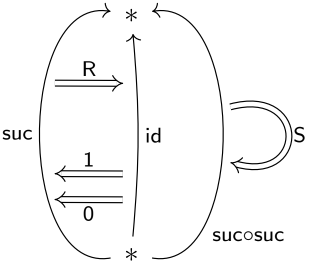

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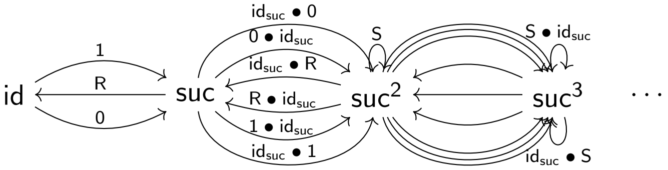
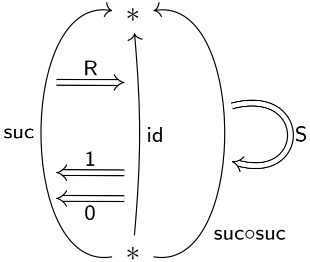
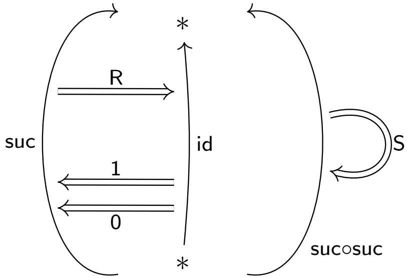




Syntax from semantics

The cube category $\square$:



Syntax from semantics

Structure on presheaves
over $\square$:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{PSh}(\square) \arrow[d, bend right=80, "\mathrm{suc}^*"'] \arrow[d, "\mathrm{id}"] \arrow[d, bend left=80, "\mathrm{suc}^* \circ \mathrm{suc}^*"] \arrow[d, phantom, shift right=6ex, "\overset{R}{\Leftarrow} \; \overset{1}{\Rightarrow} \; \underset{0}{\Rightarrow}"] \\
\mathrm{PSh}(\square)
\end{tikzcd}
$$

S

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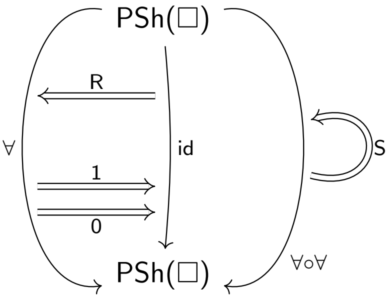


A new kind of modal theory (substructural):

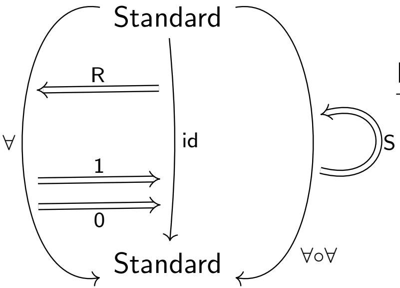
$$\frac{\vdash \Gamma}{\vdash \forall\Gamma} \qquad \frac{\sigma : \Delta \Rightarrow \Gamma}{\forall\sigma : \forall\Delta \Rightarrow \forall\Gamma}$$

$$\frac{\Gamma \vdash A}{\forall\Gamma \vdash \forall A} \qquad \frac{\Gamma \vdash t : A}{\forall\Gamma \vdash \forall t : \forall A}$$

$$\frac{\vdash \Gamma}{\begin{array}{c} \mathsf{R}_\Gamma : \Gamma \Rightarrow \forall\Gamma \\ \mathsf{k}_\Gamma : \forall\Gamma \Rightarrow \Gamma \\ \mathsf{S}_\Gamma : \forall\forall\Gamma \Rightarrow \forall\forall\Gamma \end{array}}$$

Syntax from semantics

Structure on the standard model internal to PSh(□):



Our final theory (structural):

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall A} \qquad \frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \mathrm{ap}\, f : \forall A \to \forall B}$$

$$\frac{\Gamma, x : A \vdash B \quad a_2 : \forall A}{\Gamma \vdash \forall\!\mathrm{d}(x.B)\, a_2} \qquad \frac{\Gamma \vdash t : \Pi(x : A).B}{\Gamma \vdash \mathrm{apd}\, t : \Pi(a_2 : \forall A).\forall\!\mathrm{d}(x.B)\, a_2}$$

$$\frac{\Gamma \vdash A}{\begin{array}{c}\Gamma \vdash \mathrm{R_A} : A \to \forall A \\ \Gamma \vdash \mathrm{k_A} : \forall A \to A \\ \Gamma \vdash \mathrm{S_A} : \forall\!\!\forall A \to \forall\!\!\forall A\end{array}}$$

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- We proved canonicity: every closed boolean is convertible to true or false.
- Ongoing and future work:
  - Prove normalisation
  - Replace spans by relations (Reedy fibrancy)
  - Add Kan operations = transport rule = symmetry, transitivity of Id
  - Implementation