Towards Higher Observational Type Theory

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Eötvös Loránd University, Budapest

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TYPES 2022
Nantes
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- Ordinary type theory: inductively by

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2. definitional injectivity and disjointness of constructors of inductive types
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$$\begin{aligned}
&\mathsf{Id}_{A \times B}\, (a_0, b_0)\, (a_1, b_1) = \mathsf{Id}_A\, a_0\, a_1 \times \mathsf{Id}_B\, b_0\, b_1 \\
&\mathsf{Id}_{A \to B}\, f\;\; g \;= (x : A) \to \mathsf{Id}_B\, (f\, x)\, (g\, x) \\
&\mathsf{Id}_{\mathsf{Bool}}\;\; a\;\; b \;= \text{if } a \text{ then } (\text{if } b \text{ then } \top \text{ else } \bot) \text{ else } (\text{if } b \text{ then } \bot \text{ else } \top) \\
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- We just need to change from an external viewpoint to an internal.

1. Mike fixed our old syntax.

Internal standard model

In the presheaf model over the syntax of type theory, we have

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\begin{aligned}
\mathsf{Ty}^\circ &: \mathsf{Set} \\
\mathsf{Tm}^\circ &: \mathsf{Ty}^\circ \to \mathsf{Set} \\
\Sigma^\circ &: (A : \mathsf{Ty}^\circ) \to (\mathsf{Tm}^\circ\, A \to \mathsf{Ty}^\circ) \to \mathsf{Ty}^\circ
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1. syntax of type theory forms a category
2. two-level type theory ($^\circ$ notation), HOAS
3. translate everything to external in words
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We define the standard model of type theory internally to presheaves over the syntax.

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\mathrm{Ty}\,\Gamma &:= \mathrm{Tm}^\circ\,\Gamma \to \mathrm{Ty}^\circ \\
\mathrm{Tm}\,\Gamma\,A &:= (\gamma : \mathrm{Tm}^\circ\,\Gamma) \to \mathrm{Tm}^\circ\,(A\,\gamma) \\
(\Gamma, A) &:= \Sigma^\circ\,\Gamma\,A
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Internal parametricity

$$\frac{\Gamma : \mathsf{Con}}{\Gamma^{\mathsf{R}} : \mathsf{Ty}\,(\Gamma, \Gamma)}$$

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1. We replace Con, Ty, ... by the standard model

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1. We rename the operations.
2. This is the core of the syntax of H.O.T.T.

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Summary

- The syntax for internal parametricity is the internal Bernardy logical relation interpretation.
  - Internal to presheaves over the syntax a.k.a. two level type theory, HOAS, logical framework.
  - Logical relation over the internal standard model.
- Work in progress!
- To get H.O.T.T., we need: transport, symmetries.
  - See Mike's talks at the CMU HoTT seminar (click!)
- Compared to cubical type theory, cubical internal parametricity:
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1. More precisely, section of the logical relation displayed model over the standard model.