Quotient inductive-inductive types and higher friends

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Motivation

Type theory in type theory:

- simple inductive types (ITs):
  - Abel–Öhman–Vezzosi, POPL 2018
- inductive-inductive types (IITs, Nordvall Forsberg PhD 2013):
  - Chapman: Type theory should eat itself, ENTCS 2009
- quotient inductive-inductive types (QIITs, this talk):
  - Altenkirch–Kaposi, POPL 2016

Other examples:

- real numbers (HoTT book)
- ordinal numbers (Lumsdaine–Shulman, 2019)
- partiality monad (Altenkirch–Danielsson–Kraus, FoSSaCS 2017)

Simple language of dependent types as a QIIT

$$
\begin{aligned}
\mathsf{Con} &: \mathsf{Set} \\
\mathsf{Ty} &: \mathsf{Con} \to \mathsf{Set} \\
\bullet &: \mathsf{Con} \\
- \triangleright - &: (\mathit{\Gamma} : \mathsf{Con}) \to \mathsf{Ty}\,\mathit{\Gamma} \to \mathsf{Con} \\
\mathsf{U} &: (\mathit{\Gamma} : \mathsf{Con}) \to \mathsf{Ty}\,\mathit{\Gamma} \\
\mathsf{El} &: (\mathit{\Gamma} : \mathsf{Con}) \to \mathsf{Ty}\,(\mathit{\Gamma} \triangleright \mathsf{U}\,\mathit{\Gamma}) \\
\Sigma &: (A : \mathsf{Ty}\,\mathit{\Gamma}) \to \mathsf{Ty}\,(\mathit{\Gamma} \triangleright A) \to \mathsf{Ty}\,\mathit{\Gamma} \\
\Sigma\triangleright &: \mathit{\Gamma} \triangleright A \triangleright B = \mathit{\Gamma} \triangleright \Sigma\,A\,B
\end{aligned}
$$

Simple language of dependent types as IITs

$$
\begin{aligned}
&\mathsf{Con} &&: \mathsf{Set}\\
&\mathsf{Ty} &&: \mathsf{Con} \to \mathsf{Set}\\
&\mathsf{Con}_\sim &&: \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set}\\
&\mathsf{Ty}_\sim &&: \mathsf{Con}_\sim\,\Gamma\,\Gamma' \to \mathsf{Ty}\,\Gamma \to \mathsf{Ty}\,\Gamma' \to \mathsf{Set}\\
&\bullet &&: \mathsf{Con}\\
&- \rhd - &&: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}\\
&\mathsf{U} &&: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma\\
&\mathsf{El} &&: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,(\Gamma \rhd \mathsf{U}\,\Gamma)\\
&\Sigma &&: (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Ty}\,(\Gamma \rhd A) \to \mathsf{Ty}\,\Gamma\\
&\Sigma\rhd &&: \mathsf{Con}_\sim\,(\Gamma \rhd A \rhd B)\,(\Gamma \rhd \Sigma\,A\,B)\\
&\bullet_\sim &&: \mathsf{Con}_\sim\,\bullet\,\bullet\\
&\rhd_\sim &&: (\overline{\Gamma} : \mathsf{Con}_\sim\,\Gamma\,\Gamma') \to \mathsf{Ty}_\sim\,\overline{\Gamma}\,A\,A' \to \mathsf{Con}_\sim\,(\Gamma \rhd A)\,(\Gamma' \rhd A')\\
&\ldots
\end{aligned}
$$

Simple language of dependent types as ITs

$$\Gamma ::= \bullet \mid \Gamma \triangleright A \qquad\qquad A, B ::= \mathsf{U}\,\Gamma \mid \mathsf{El}\,\Gamma \mid \Sigma\,A\,B$$

$$\boxed{\vdash \Gamma} \qquad \boxed{\Gamma \vdash A} \qquad \boxed{\Gamma \sim \Gamma'} \qquad \boxed{\Gamma \vdash A \sim A'}$$

$$\frac{}{\vdash \bullet} \qquad \frac{\vdash \Gamma \qquad \Gamma \vdash A}{\vdash \Gamma \triangleright A} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{U}\,\Gamma} \qquad \frac{\vdash \Gamma}{\Gamma \triangleright \mathsf{U}\,\Gamma \vdash \mathsf{El}\,\Gamma}$$

$$\frac{\vdash \Gamma \qquad \Gamma \vdash A \qquad \Gamma \triangleright A \vdash B}{\Gamma \vdash \Sigma\,A\,B}$$

$$\frac{\vdash \Gamma \qquad \Gamma \vdash A \qquad \Gamma \triangleright A \vdash B}{\Gamma \triangleright A \triangleright B \sim \Gamma \triangleright \Sigma\,A\,B} \qquad \frac{}{\bullet \sim \bullet} \qquad \frac{\Gamma \sim \Gamma' \qquad \Gamma \vdash A \sim A'}{\Gamma \triangleright A \sim \Gamma' \triangleright A'}$$

$$\frac{\Gamma \sim \Gamma'}{\Gamma \vdash \mathsf{U}\,\Gamma \sim \mathsf{U}\,\Gamma'} \qquad \ldots \qquad \frac{\Gamma \sim \Gamma' \qquad \Gamma \vdash A}{\Gamma' \vdash A} \qquad \ldots$$

Contents

- Formal specification of closed IITs
- Extension to QIITs
- Initial algebras
- HIITs
- Higher order abstract syntax (syntax with binding)

Contents

How do we specify a QIIT in Agda?

```
data Nat  : Set where
     zero : Nat
     suc  : Nat → Nat

data Int  : Set where
     zero : Int
     suc  : Int → Int
     pred : Int → Int
     β    : ∀{n} → pred (suc  n) ≡ n
     η    : ∀{n} → suc  (pred n) ≡ n

data Con : Set
data Ty  : Con → Set

_▷'_ : (Γ : Con) → Ty Γ → Con
Σ'   : {Γ : Con}(A : Ty Γ) → Ty (Γ ▷' A) → Ty Γ

data Con   where
     •   : Con
     _▷_ : (Γ : Con) → Ty Γ → Con
     Σ▷  : ∀{Γ A B} → Γ ▷' A ▷' B ≡ Γ ▷' Σ' A B
data Ty    where
     U   : {Γ : Con} → Ty Γ
     El  : {Γ : Con} → Ty (Γ ▷ U)
     Σ   : {Γ : Con}(A : Ty Γ) → Ty (Γ ▷ A) → Ty Γ

_▷'_ = _▷_
Σ'   = Σ
```

Theory of closed IIT signatures

A signature is a context in a type theory (Carette–O'Connor, 2012).
Theory of signatures (ToS): category with families (CwF)

$$\begin{aligned} \mathsf{Con} &: \mathsf{Set} & \mathsf{Ty} &: \mathsf{Con} \to \mathsf{Set} \\ \mathsf{Sub} &: \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set} & \mathsf{Tm} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Set} \\ -[-] &: \mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma & &\ldots \end{aligned}$$

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-[-] : \mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma & \dots
\end{array}
$$

with a universe:

$$\mathsf{U} : \mathsf{Ty}\,\Gamma \qquad \mathsf{El} : \mathsf{Tm}\,\Gamma\,\mathsf{U} \to \mathsf{Ty}\,\Gamma,$$

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\end{array}
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with a universe:

$$\mathsf{U} : \mathsf{Ty}\,\Gamma \qquad\qquad \mathsf{El} : \mathsf{Tm}\,\Gamma\,\mathsf{U} \to \mathsf{Ty}\,\Gamma,$$

Π types with small domain:

$$
\begin{array}{ll}
\Pi & : (a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a) \to \mathsf{Ty}\,\Gamma \\
- \mathbin{@} - & : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B) \to (u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) \to \mathsf{Tm}\,\Gamma\,(B[\mathsf{id}, u]),
\end{array}
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Π types with small domain:

$$\begin{array}{ll} \Pi & : (a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a) \to \mathsf{Ty}\,\Gamma \\ -\,@\,- & : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B) \to (u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) \to \mathsf{Tm}\,\Gamma\,(B[\mathsf{id}, u]), \end{array}$$

We will add more type formers for *open* and QIITs.

Closed IIT signatures: examples $((a \Rightarrow B) := \Pi\, a\, (B[\mathsf{p}]))$

- $\triangleright \quad \mathsf{U} \triangleright \quad \mathsf{El}\,\mathsf{q} \triangleright \quad \mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])$
- $\triangleright N : \mathsf{U} \triangleright \mathit{zero} : \mathsf{El}\, N \triangleright \mathit{suc} : N \Rightarrow \mathsf{El}\, N$

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- $\triangleright$

$\mathit{Con} : \mathsf{U} \triangleright$

$\mathit{Ty} : \mathit{Con} \Rightarrow \mathsf{U} \triangleright$

$\mathit{empty} : \mathsf{El}\, \mathit{Con} \triangleright$

$\mathit{ext} : \Pi(\Gamma : \mathit{Con}).\, \mathit{Ty} \,@\, \Gamma \Rightarrow \mathsf{El}\, \mathit{Con} \triangleright$

$U : \Pi(\Gamma : \mathit{Con}).\mathsf{El}\,(\mathit{Ty} \,@\, \Gamma) \triangleright$

$\mathit{El} : \Pi(\Gamma : \mathit{Con}).\mathsf{El}\,(\mathit{Ty} \,@\, (\mathit{ext} \,@\, \Gamma \,@\, (U \,@\, \Gamma))) \triangleright$

$\Sigma : \Pi(\Gamma : \mathit{Con}).\Pi(A : \mathit{Ty} \,@\, \Gamma).\, \mathit{Ty} \,@\, (\mathit{ext} \,@\, \Gamma \,@\, A) \Rightarrow \mathsf{El}\,(\mathit{Ty} \,@\, \Gamma)$

Closed IIT signatures: examples $((a \Rightarrow B) := \Pi\, a\, (B[\mathsf{p}]))$

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$\Sigma : \Pi(\Gamma : \mathit{Con}).\Pi(A : \mathit{Ty} \mathbin{@} \Gamma).\, \mathit{Ty} \mathbin{@} (\mathit{ext} \mathbin{@} \Gamma \mathbin{@} A) \Rightarrow \mathsf{El}\,(\mathit{Ty} \mathbin{@} \Gamma)$

Strict positivity is enforced.

Isn't this circular?

(Q)IIT signatures are defined using a type theory, but this type theory is itself a QIIT.

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(Q)IIT signatures are defined using a type theory, but this type theory is itself a QIIT.

We can bootstrap ToS using Church encoding (Awodey–Frey–Speight, LICS 2018).

Closed IIT signatures: semantics (i)

If $\mathcal{C}$ is a CwF, in $\hat{\mathcal{C}}$ we have (2-level type theory, Annenkov–Capriotti–Kraus–Sattler, 2019):

$$\mathsf{U}^\circ \;:\; \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma \qquad \text{interpreted } |\mathsf{U}^\circ|_I\,\gamma \;:=\; \mathsf{Ty}_{\mathcal{C}}\,I$$

$$\mathsf{El}^\circ \;:\; \mathsf{Tm}_{\hat{\mathcal{C}}}\,\Gamma\,\mathsf{U}^\circ \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma \qquad |\mathsf{El}^\circ\,a|_I\,\gamma \;:=\; \mathsf{Tm}_{\mathcal{C}}\,I\,(|a|_I\,\gamma)$$

$$\Pi^\circ : (a^\circ : \mathsf{Tm}_{\hat{\mathcal{C}}}\,\Gamma\,\mathsf{U}^\circ) \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,(\Gamma \rhd \mathsf{El}^\circ\,a^\circ) \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma$$

$$|\Pi^\circ\,a^\circ\,B|_I\,\gamma \;:=\; |B|_{I \rhd_{\mathcal{C}} |a|_I\,\gamma}(\gamma\mathsf{p}, \mathsf{q})$$

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$\mathsf{U}^\circ \; : \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma$ interpreted $|\mathsf{U}^\circ|_I\,\gamma \;:= \mathsf{Ty}_{\mathcal{C}}\,I$

$\mathsf{El}^\circ : \mathsf{Tm}_{\hat{\mathcal{C}}}\,\Gamma\,\mathsf{U}^\circ \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma$ $\quad |\mathsf{El}^\circ\,a|_I\,\gamma \;:= \mathsf{Tm}_{\mathcal{C}}\,I\,(|a|_I\,\gamma)$

$\Pi^\circ : (a^\circ : \mathsf{Tm}_{\hat{\mathcal{C}}}\,\Gamma\,\mathsf{U}^\circ) \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,(\Gamma \triangleright \mathsf{El}^\circ\,a^\circ) \to \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma$

$|\Pi^\circ\,a^\circ\,B|_I\,\gamma := |B|_{I \triangleright_{\mathcal{C}} |a|_I\,\gamma}(\gamma\mathsf{p},\mathsf{q})$

If $\mathcal{C}$ has Id types, U° is closed under Id.

Π $\qquad\qquad\qquad\qquad$ Π

Closed IIT signatures: semantics (ii)

We use Agda syntax to work in $\hat{\mathcal{C}}$.

$$\begin{aligned} \mathsf{U}^\circ &: \mathsf{Set} && (\mathsf{Ty}_{\mathcal{C}}) \\ \mathsf{El}^\circ &: \mathsf{U}^\circ \to \mathsf{Set} && (\mathsf{Tm}_{\mathcal{C}}) \\ \Pi^\circ &: (a^\circ : \mathsf{U}^\circ) \to (\mathsf{El}^\circ\ a^\circ \to \mathsf{Set}) \to \mathsf{Set} && (\triangleright_{\mathcal{C}}) \end{aligned}$$

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&\mathsf{U}^\circ\ : \mathsf{Set} && (\mathsf{Ty}_{\mathcal{C}})\\
&\mathsf{El}^\circ : \mathsf{U}^\circ \to \mathsf{Set} && (\mathsf{Tm}_{\mathcal{C}})\\
&\Pi^\circ\ : (a^\circ : \mathsf{U}^\circ) \to (\mathsf{El}^\circ\, a^\circ \to \mathsf{Set}) \to \mathsf{Set} && (\triangleright_{\mathcal{C}})
\end{aligned}$$

We define the standard model of ToS:

$$\begin{aligned}
&\mathsf{Con} && := \mathsf{Set}\\
&\mathsf{Ty}\, \Gamma && := \Gamma \to \mathsf{Set}\\
&\mathsf{Tm}\, \Gamma\, A && := (\gamma : \Gamma) \to A\, \gamma\\
&\mathsf{U}\, \gamma && := \mathsf{U}^\circ\\
&\mathsf{El}\, a\, \gamma && := \mathsf{El}^\circ\, (a\, \gamma)\\
&\Pi\, a\, B\, \gamma && := \Pi^\circ\, (a\, \gamma)\, (B\, (\gamma, -))
\end{aligned}$$

Example

Given the signature

$$\bullet \rhd \mathsf{U} \rhd \mathsf{El}\,\mathsf{q} \rhd \big(\mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])\big) : \mathsf{Con},$$

Example

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$$\bullet \rhd \mathsf{U} \rhd \mathsf{El}\,\mathsf{q} \rhd \big(\mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])\big) : \mathsf{Con},$$

in the standard model this is

$$(N : \mathsf{U}^\circ) \times (\mathsf{El}^\circ\, N) \times (N \Rightarrow^\circ \mathsf{El}^\circ\, N) : \mathsf{Set}$$

Example

Given the signature

$$\bullet \triangleright \mathsf{U} \triangleright \mathsf{El}\,\mathsf{q} \triangleright \big(\mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])\big) : \mathsf{Con},$$

in the standard model this is

$$(N : \mathsf{U}^\circ) \times (\mathsf{El}^\circ\, N) \times (N \Rightarrow^\circ \mathsf{El}^\circ\, N) : \mathsf{Set}$$

which is a presheaf over $\mathcal{C}$, and interpreting it at the empty context of $\mathcal{C}$, we get

$$(N : \mathsf{Ty}_{\mathcal{C}}\, \bullet) \times \mathsf{Tm}_{\mathcal{C}}\ \bullet\ N \times \mathsf{Tm}_{\mathcal{C}}\,(\bullet \triangleright N)\,(N[\mathsf{p}])$$

Closed IIT signatures: semantics (iii)

We use Agda syntax to work in $\hat{\mathcal{C}}$.

$$\begin{aligned}
&\mathsf{U}^\circ \;: \mathsf{Set} && (\mathsf{Ty}_{\mathcal{C}})\\
&\mathsf{El}^\circ : \mathsf{U}^\circ \to \mathsf{Set} && (\mathsf{Tm}_{\mathcal{C}})\\
&\Pi^\circ \;: (a^\circ : \mathsf{U}^\circ) \to (\mathsf{El}^\circ\, a^\circ \to \mathsf{Set}) \to \mathsf{Set} && (\triangleright_{\mathcal{C}})
\end{aligned}$$

We can extend the standard model to the graph model:

$$\begin{aligned}
&\mathsf{Con} &&:= (\Gamma^{\mathsf{A}} : \mathsf{Set}) && \times (\Gamma^{\mathsf{M}} : \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set})\\
&\mathsf{U} &&:= \big(\lambda\gamma.\mathsf{U}^\circ && ,\ \lambda_\, a^\circ\, a^{\circ\prime}.a^\circ \Rightarrow^\circ \mathsf{El}^\circ\, a^{\circ\prime}\big)\\
&\mathsf{El}\, a &&:= \big(\lambda\gamma.\mathsf{El}^\circ\,(a^{\mathsf{A}}\,\gamma) && ,\ \lambda_\,\alpha\,\alpha'.(a^{\mathsf{M}}\,_\,\alpha =_{\mathsf{El}^\circ\,(a\,\gamma')} \alpha')\big)\\
&\Pi\, a\, B &&:= \big(\lambda\gamma.\Pi^\circ\,(a^{\mathsf{A}}\,\gamma)\,(B^{\mathsf{A}}\,(\gamma, -)) && ,\ \lambda_\, f\, f'.\Pi^\circ(x : a^{\mathsf{A}}\,\gamma).\\
& && && \qquad B^{\mathsf{M}}\,_\,(f\,x)\,(f'\,(a^{\mathsf{M}}\,_\,x'))\big)
\end{aligned}$$

Example

Given the signature

$$\bullet \triangleright \mathsf{U} \triangleright \mathsf{El}\,\mathsf{q} \triangleright \big(\mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])\big) : \mathsf{Con},$$

in the graph model this is

$$(N : \mathsf{U}^\circ) \times (\mathsf{El}^\circ\, N) \times (N \Rightarrow^\circ \mathsf{El}^\circ\, N)$$

and for any two (N, z, s), (N', z', s') a set

$$(\overline{N} : N \Rightarrow^\circ \mathsf{El}^\circ\, N') \times (\overline{N}\, z = z') \times (\Pi^\circ (n : N).\overline{N}\,(s\, n) = s'\,(\overline{N}\, n)),$$

Example

Given the signature

$$\bullet \rhd \mathsf{U} \rhd \mathsf{El}\,\mathsf{q} \rhd \big(\mathsf{q}[\mathsf{p}] \Rightarrow \mathsf{El}\,(\mathsf{q}[\mathsf{p}])\big) : \mathsf{Con},$$

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and externally we obtain notions of $\mathbb{N}$-algebra

$$(N : \mathsf{Ty}_{\mathcal{C}}\, \bullet) \times \mathsf{Tm}_{\mathcal{C}}\, \bullet\, N \times \mathsf{Tm}_{\mathcal{C}}\,(\bullet \rhd N)\,(N[\mathsf{p}])$$

and homomorphism for any two algebras (N, z, s), (N', z', s'):

$$(\overline{N} : \mathsf{Tm}_{\mathcal{C}}\,(\bullet \rhd N)\, N') \times (\overline{N}[\epsilon, z] = z') \times (\overline{N}[\mathsf{p}, s] = s'[p, \overline{N}])$$

Closed IIT signatures: semantics (iv)

We use Agda syntax to work in $\hat{\mathcal{C}}$.

$$
\begin{aligned}
&\mathsf{U}^\circ : \mathsf{Set} && (\mathsf{Ty}_{\mathcal{C}})\\
&\mathsf{El}^\circ : \mathsf{U}^\circ \to \mathsf{Set} && (\mathsf{Tm}_{\mathcal{C}})\\
&\Pi^\circ : (a^\circ : \mathsf{U}^\circ) \to (\mathsf{El}^\circ\, a^\circ \to \mathsf{Set}) \to \mathsf{Set} && (\rhd_{\mathcal{C}})
\end{aligned}
$$

We can extend the graph model to the AMDS model

$$
\begin{aligned}
\mathsf{Con} :=\ & (\Gamma^{\mathsf{A}} : \mathsf{Set})\times\\
& (\Gamma^{\mathsf{M}} : \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set})\times\\
& (\Gamma^{\mathsf{D}} : \Gamma^{\mathsf{A}} \to \mathsf{Set})\times\\
& (\Gamma^{\mathsf{S}} : (\gamma : \Gamma^{\mathsf{A}}) \to \Gamma^{\mathsf{D}}\,\gamma \to \mathsf{Set})
\end{aligned}
$$

This is an inverse diagram model, see Shulman 2012, Lumsdaine 2018 HoTTEST talk, Lumsdaine–Kapulkin 2020.

Example

For natural numbers, the AMDS model gives notions of Algebras:

$$(N : \mathsf{Set}) \times N \times (N \to N),$$

Morphisms between algebras (N, z, s), (N', z', s'):

$$(\overline{N} : N \to N') \times (\overline{N}\, z = z') \times (\overline{N}\, (s\, n) = s'\, (\overline{N}\, n),$$

Displayed algebras over an algebra (N, z, s):

$$(\dot{N} : N \to \mathsf{Set}) \times (\dot{N}\, z) \times (\dot{N}\, n \to \dot{N}\, (s\, n)),$$

Sections of displayed algebras $(\dot{N}, \dot{z}, \dot{s})$:

$$(\overline{N} : (n : N) \to \dot{N}\, n) \times (\overline{N}\, z = z') \times (\overline{N}\, (s\, n) = s'\, (\overline{N}\, n).$$

A CwF $\mathcal{C}$ supports a closed IIT

Externally, for a QIIT signature Ω, from the AMDS model we get:

$$
\begin{aligned}
\Omega^{\mathsf{A}} &: \mathsf{Ty}_{\hat{\mathcal{C}}}\,\bullet \\
\Omega^{\mathsf{M}} &: \mathsf{Ty}_{\hat{\mathcal{C}}}\,(\bullet \rhd \Omega^{\mathsf{A}} \rhd \Omega^{\mathsf{A}}[\mathsf{p}]) \\
\Omega^{\mathsf{D}} &: \mathsf{Ty}_{\hat{\mathcal{C}}}\,(\bullet \rhd \Omega^{\mathsf{A}}) \\
\Omega^{\mathsf{S}} &: \mathsf{Ty}_{\hat{\mathcal{C}}}\,(\bullet \rhd \Omega^{\mathsf{A}} \rhd \Omega^{\mathsf{D}})
\end{aligned}
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\end{aligned}
$$

The CwF $\mathcal{C}$ supports a QIIT with signature Ω, if there is a

$$\mathsf{con} : \mathsf{Tm}_{\hat{\mathcal{C}}}\ \bullet\ \Omega^{\mathsf{A}}$$

A CwF $\mathcal{C}$ supports a closed IIT

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$$

The CwF $\mathcal{C}$ supports a QIIT with signature Ω, if there is a

$$\mathsf{con} : \mathsf{Tm}_{\hat{\mathcal{C}}}\ \bullet\ \Omega^{\mathsf{A}}$$

and an

$$\mathsf{elim} : \mathsf{Tm}_{\hat{\mathcal{C}}}\,(\bullet \triangleright \Omega^{\mathsf{D}}[\epsilon, \mathsf{con}])\,(\Omega^{\mathsf{S}}[\epsilon, \mathsf{con}[\mathsf{p}], \mathsf{q}]).$$

(This specifies definitional computation rules.)

Summary up to now

We showed what it means that a CwF $\mathcal{C}$ has closed IITs.

- A signature is a context in ToS.
- The AMDS model of ToS internal to $\hat{\mathcal{C}}$ uses U°, El°, Π°.
- Externally we get notions of constructors, eliminator.

Contents

External parameters

New type former in ToS (internal to $\hat{\mathcal{C}}$):

$$\begin{aligned}
\hat{\Pi} \quad &: (a^\circ : \mathsf{U}^\circ) \to (a^\circ \Rightarrow^\circ \mathsf{Ty}\,\mathit{\Gamma}) \to \mathsf{Ty}\,\mathit{\Gamma} \\
-\hat{@}- &: \mathsf{Tm}\,\mathit{\Gamma}\,(\hat{\Pi}\,a^\circ\,B) \to \Pi^\circ(x : a^\circ).\mathsf{Tm}\,\mathit{\Gamma}\,(B\,x)
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In the standard model,

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$$
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\bullet \rhd V \quad &: \mathbb{N}^\circ \hat{\Rightarrow} \mathsf{U} \rhd \\
\mathit{nil} \quad &: \mathsf{El}\,(V\,\hat{@}\,0) \rhd \\
\mathit{cons} &: a^\circ \hat{\Rightarrow} \hat{\Pi}(n : \mathbb{N}^\circ).V\,\hat{@}\,n \Rightarrow \mathsf{El}\,(V\,\hat{@}\,(1+n))
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and the Chapman-style syntax of type theory with an infinite hierarchy of universes.

Equations (identity type with reflection)

New type former in ToS:

$$\begin{aligned}
\mathsf{Eq} &: (a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a) \to \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a) \to \mathsf{Ty}\,\Gamma \\
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In the standard model:

$$\mathsf{Eq}_a\,u\,v\,\gamma :\equiv (u\,\gamma =_{\mathsf{El}^\circ\,a\,\gamma} v\,\gamma)$$

Now we can specify all strict QIITs (where the equations are definitional equalities). E.g. integers:

- $\triangleright Z : \mathsf{U} \triangleright \mathit{zero} : \mathsf{El}\,Z \triangleright \mathit{suc} : Z \Rightarrow \mathsf{El}\,Z \triangleright \mathit{pred} : Z \Rightarrow \mathsf{El}\,Z \triangleright$
$\beta : \Pi(i : Z).\mathsf{Eq}\,Z\,(\mathit{pred} \mathbin{@} (\mathit{suc} \mathbin{@} i))\,i \triangleright$
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or type theory as a QIIT.

Equations (U is closed under identity with J)

New type former in ToS:

$$\mathsf{Id} : (a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a) \to \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a) \to \mathsf{Tm}\,\Gamma\,\mathsf{U}$$

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Now we can specify all HIITs (Kaposi-Kovács 2020). E.g. the torus:

$$\bullet \rhd T : \mathsf{U} \rhd b : \mathsf{El}\,T \rhd p : \mathsf{El}\,(\mathsf{Id}_T\,b\,b) \rhd q : \mathsf{El}\,(\mathsf{Id}_T\,b\,b) \rhd$$
$$t : \mathsf{Id}_{\mathsf{Id}_T\,b\,b}\,(p \cdot q)\,(q \cdot p)$$

where $\cdot$ is defined using J.

Infinitary operators

New type former in ToS (internal to $\hat{\mathcal{C}}$):

$$\begin{aligned}
\tilde{\Pi} \quad &: (a^\circ : \mathsf{U}^\circ) \to (a^\circ \Rightarrow^\circ \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Tm}\,\Gamma\,\mathsf{U} \\
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$$\bullet \rhd T : \mathsf{U} \rhd \mathit{leaf} : \mathsf{El}\ T \rhd \mathit{node} : (\mathbb{N}^\circ \tilde{\Rightarrow} T) \Rightarrow \mathsf{El}\ T$$

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Now we can specify ToS itself, real numbers, the partiality monad.

Summary of operators

- U, El,
- Π with domain in U,
- $\hat{\Pi}$ with domain in U°,
- Eq: extensional identity,
- Id: intensional identity,
- $\tilde{\Pi}$ in U, with domain in U°.

Contents

flCwF model (i)

If $\mathcal{C}$ is a model of ETT, the AMDS model can be extended to a finite limit CwF model: CwF + Σ + Eq + K (Nordvall Forsberg PhD 2013, c.f. democracy, Dybjer–Clairambault 2014):

$$\mathsf{K} : \mathsf{Con} \to \mathsf{Ty}\,\Gamma \qquad\qquad \mathsf{mkK} : \mathsf{Sub}\,\Gamma\,\Delta \cong \mathsf{Tm}\,\Gamma\,(\mathsf{K}\,\Delta) : \mathsf{unK}$$

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The model (AMDS is the Con,Sub,Ty,Tm components):

- Contexts are flCwFs
- Substitutions strict flCwF morphisms
- Types are displayed flCwFs (c.f. Ahrens–Lumsdaine 2019)
- Terms are strict flCwF sections

this supports U, El, Π, $\hat{\Pi}$, Eq, but not $\tilde{\Pi}$, Id.
See Altenkirch–Kaposi-Kovács POPL 2019.

flCwF model (ii)

If $\mathcal{C}$ is a model of ETT, the AMDS model can be extended to a finite limit CwF model: CwF + Σ + Eq + K (Nordvall Forsberg PhD 2013, c.f. democracy, Dybjer–Clairambault 2014):

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The model (AMDS is the Con,Sub,Ty,Tm components):

- Contexts are flCwFs
- Substitutions weak flCwF morphisms (pseudomorphisms)
- Types are split flCwF isofibrations
- Terms are weak flCwF sections

this supports U, El, Π, $\hat{\Pi}$, Eq, $\tilde{\Pi}$, Id.
See Kovács–Kaposi LICS 2020.

Initiality $\leftrightarrow$ induction

For each signature, we obtain a CwF + Σ + Eq + K. We prove that initiality is equivalent to induction in the internal language. Assume a Θ : Con.

$$
\begin{aligned}
&\mathsf{rec} && : (\Gamma : \mathsf{Con}) \to \mathsf{Sub}\,\Theta\,\Gamma \\
&\mathsf{uni} && : (\sigma\,\delta : \mathsf{Sub}\,\Theta\,\Gamma) \to \sigma = \delta \\
&\mathsf{elim} && : (A : \mathsf{Ty}\,\Theta) \to \mathsf{Tm}\,\Theta\,A \\
&\mathsf{elim}\,A && := \mathsf{q}[\mathsf{rec}\,(\Theta \rhd A)] : \mathsf{Tm}\,\Theta\,(A[\underbrace{\mathsf{p} \circ \mathsf{rec}\,(\Theta \rhd A)}_{=\mathsf{id}\text{ by }\mathsf{uni}\,\mathsf{id}\,\_}]) \\
&\mathsf{rec}\,\Gamma && := \mathsf{unK}\,(\mathsf{elim}\,(\mathsf{K}\,\Gamma)) \\
&\mathsf{uni}\,\sigma\,\delta && := \mathsf{ap}\,\mathsf{unK}\,\underbrace{\Big(\mathsf{reflect}\,(\mathsf{elim}\,(\mathsf{Eq}\,(\mathsf{mkK}\,\sigma)\,(\mathsf{mkK}\,\delta)))\Big)}_{:\,\mathsf{mkK}\,\sigma = \mathsf{mkK}\,\delta}
\end{aligned}
$$

Initial algebras

If a model of ETT supports the ToS, then it supports all (Q)IITs specified by the ToS (for all combinations of ToS type formers).

Idea: natural numbers can be defined:

$$\begin{aligned}
\mathbb{N} &:= \mathsf{Tm}_{\mathsf{ToS}}\,(\bullet \rhd N : \mathsf{U} \rhd z : \mathsf{El}\,N \rhd s : N \Rightarrow \mathsf{El}\,N)\,(\mathsf{El}\,N) \\
\mathsf{zero} &:= z \\
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If we interpret the term in the standard model A, we get Church encoding (implementing the recursor):

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If interpret in the graph model AM, we get the Awodey-Frey-Speight encoding (LICS 2018).

Results on existence of initial algebras

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Negative result: certain infinitary QIITs cannot be defined in ETT + quotients (Lumsdaine–Shulman 2019).

A direct reduction (see Altenkirch–Kaposi–Kovács–Von Raumer, TYPES 2019) might work in intensional models and would give definitional computation rules.

Contents

- Formal specification of closed IITs
- Extension to QIITs
- Initial algebras
- <u>HIITs</u>
- Higher order abstract syntax (syntax with binding)

Categorical semantics of HIITs

Capriotti and Sattler (see abstract at TYPES 2020):

- construct a higher category of algebras from a signature
- support $\mathsf{U}, \mathsf{El}, \Pi, \hat{\Pi}, \tilde{\Pi}, \mathsf{Id}$
- define displayed algebras and sections
- show the equivalence of initiality and induction
- work in $\hat{\mathcal{C}}$ for a model of HoTT $\mathcal{C}$

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We know how to say that a CwF $\mathcal{C}$ supports a QIIT.

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- overhead: then our semantics says what it means that another CwF supports an (internal) CwF
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A possible solution, based on Capriotti's Rule Framework (TYPES 2017):

- the QIIT-ToS has Ty which we call Ty^0 from now on,
- new sort for Ty^1 types with, $\uparrow$: $\mathsf{Ty}^0\,\Gamma \to \mathsf{Ty}^1\,\Gamma$
- Ty^1 has a function space with domain in Ty^0 and Eq of Ty^0
- a signature is a context in this general ToS

Signatures for type theories (WIP) (ii)

Signature for Π with β:

- $$\begin{aligned}
\triangleright pi \quad &: \Pi^1(a : \mathsf{U}).(a \Rightarrow \mathsf{U}) \Rightarrow^1 \uparrow \mathsf{U} \triangleright \\
lam &: \Pi^1(a : \mathsf{U}).\Pi^1(b : a \Rightarrow \mathsf{U}). \\
&\qquad ((x : a) \Rightarrow \mathsf{El}\,(b \mathbin{@} x)) \Rightarrow^1 \uparrow (\mathsf{El}\,(pi \mathbin{@^1} a \mathbin{@^1} b)) \triangleright \\
app &: \Pi^1(a : \mathsf{U}).\Pi^1(b : a \Rightarrow \mathsf{U}). \\
&\qquad \mathsf{El}\,(pi \mathbin{@^1} a \mathbin{@^1} b) \Rightarrow^1 \uparrow ((x : a) \Rightarrow \mathsf{El}\,(b \mathbin{@} x)) \triangleright \\
\beta \quad &: \Pi^1(a : \mathsf{U}).\Pi^1(b : a \Rightarrow \mathsf{U}).\Pi^1(t : (x : a) \Rightarrow \mathsf{El}\,(b \mathbin{@} x)). \\
&\quad \mathsf{Eq}_{(x:a)\Rightarrow \mathsf{El}\,(b @ x)} \left(app \mathbin{@^1} a \mathbin{@^1} b \mathbin{@^1} (lam \mathbin{@^1} a \mathbin{@^1} b \mathbin{@^1} t)\right) t
\end{aligned}$$

Signatures for type theories (WIP) (iii)

Conversions:

- TT signature $\to$ QIIT signature:
  - adds substitution laws
  - obtain category of models, initiality
- QIIT signature $\to$ TT signature:
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We can generalise type theory signatures to arbitrary signatures with binding. In a CwF $\mathcal{C}$, $\mathsf{Ty}_{\mathcal{C}} : \mathsf{Ty}_{\hat{\mathcal{C}}}\,\bullet$, but $\mathsf{Tm}_{\mathcal{C}} : \overline{\mathsf{Ty}}_{\hat{\mathcal{C}}}\,(\bullet \rhd \mathsf{Ty}_{\mathcal{C}})$.

$$\overline{\mathsf{Ty}_{\hat{\mathcal{C}}}}\,\Gamma = (A : \mathsf{Ty}_{\hat{\mathcal{C}}}\,\Gamma) \times (- \rhd_A - : (I : |\mathcal{C}|) \to |\Gamma|_I \to |\mathcal{C}|) \times \\ \mathcal{C}(J, I \rhd_A \gamma) \cong (f : \mathcal{C}(J, I)) \times |A|_J\,(\gamma f)$$

See also: Bocquet–Kaposi–Sattler TYPES 2020, Awodey's natural models 2014, Uemura 2019, HoTTEST talks: Sterling, Bauer, Altenkirch.

Summary

- A QIIT/HIIT can be described as a context in a well chosen type theory of signatures.
- Models of the type theory of signatures provide semantics for QIITs/HIITs.
- In ETT, if we have the ToS, we get all QIITs.
- We can extend the theory of QIIT signatures to the theory of type theory signatures.