# Constructing inductive-inductive types using a domain-specific type theory[1]

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BEFEKTETÉS A JÖVŐBE

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Constructing them.

# How to specify inductive types?

By listing their constructors.

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Inductive-inductive types allow multiple sorts indexed over each other, e.g.

$$\begin{aligned} Con &: \text{Type} \\ Ty &: Con \to \text{Type} \\ \bullet &: Con \\ - \triangleright - &: (\Gamma : Con) \to Ty\,\Gamma \to Con \\ U &: (\Gamma : Con) \to Ty\,\Gamma \\ Pi &: (\Gamma : Con)(A : Ty\,\Gamma) \to Ty\,(\Gamma \triangleright A) \to Ty\,\Gamma \end{aligned}$$

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- Variables
- Empty universe U with underline for El:

$$\frac{}{\Gamma \vdash \mathrm{U}} \qquad \frac{\Gamma \vdash a : \mathrm{U}}{\Gamma \vdash \underline{a}}$$

- Restricted function space:

$$\frac{\Gamma \vdash a : \mathrm{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t\, u : B[x \mapsto u]}$$

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$\Theta := (\cdot,\ \mathit{Nat} : \mathsf{U},\ \mathit{zero} : \underline{\mathit{Nat}},\ \mathit{suc} : \mathit{Nat} \Rightarrow \underline{\mathit{Nat}})$

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Not possible: $(\cdot, T : \mathrm{U}, \mathit{suc} : (T \Rightarrow \underline{T}) \Rightarrow \underline{T})$

# Standard interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{A}} \in \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{A}} \in \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash t : A}{t^{\mathsf{A}} \in (\gamma \in \Gamma^{\mathsf{A}}) \to A^{\mathsf{A}}(\gamma)}$$

$$\begin{aligned}
\cdot^{\mathsf{A}} &:= \top \\
(\Gamma, x : A)^{\mathsf{A}} &:= (\gamma \in \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}(\gamma) \\
\mathsf{U}^{\mathsf{A}}(\gamma) &:= \mathsf{Set} \\
(\underline{a})^{\mathsf{A}}(\gamma) &:= a^{\mathsf{A}}(\gamma) \\
((x : a) \Rightarrow B)^{\mathsf{A}}(\gamma) &:= (\alpha \in a^{\mathsf{A}}(\gamma)) \to B^{\mathsf{A}}(\gamma, \alpha) \\
(t\, u)^{\mathsf{A}}(\gamma) &:= t^{\mathsf{A}}(\gamma)(u^{\mathsf{A}}(\gamma))
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\end{aligned}$$

$-^{\mathsf{A}}$ on a context gives algebras for that signature.
E.g. $\Theta^{\mathsf{A}} = (N \in \mathsf{Set}) \times N \times (N \to N)$

# Logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\,\gamma\,\gamma' \to A^{\mathsf{A}}\,\gamma \to A^{\mathsf{A}}\,\gamma' \to \mathsf{Set}}$$

$$\frac{\Gamma \vdash t : A}{t^{\mathsf{M}} \in (\gamma_M \in \Gamma^{\mathsf{M}}(\gamma, \gamma')) \to A^{\mathsf{M}}(\gamma_M, t^{\mathsf{A}}(\gamma), t^{\mathsf{A}}(\gamma'))}$$

$$\begin{aligned}
(\Gamma, x : A)^{\mathsf{M}}((\gamma, \alpha), (\gamma', \alpha')) &:= (\gamma_M : \Gamma^{\mathsf{M}}(\gamma, \gamma')) \times A^{\mathsf{M}}(\gamma_M, \alpha, \alpha') \\
\mathsf{U}^{\mathsf{M}}(\gamma_M, a, a') &:= a \to a' \to \mathsf{Set} \\
(\underline{a})^{\mathsf{M}}(\gamma_M, \alpha, \alpha') &:= a^{\mathsf{M}}(\gamma_M, \alpha, \alpha') \\
((x : a) \Rightarrow B)^{\mathsf{M}}(\gamma_M, f, f') &:= (\alpha_M \in a^{\mathsf{M}}(\gamma, \alpha, \alpha')) \to \\
&\qquad B^{\mathsf{M}}\Big((\gamma_M, \alpha_M), f(\alpha), f'(\alpha')\Big)
\end{aligned}$$

# Tweaked logical relation interpretation

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{M}} \in \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}} \qquad \frac{\Gamma \vdash A}{A^{\mathsf{M}} \in \Gamma^{\mathsf{M}}\,\gamma\,\gamma' \to A^{\mathsf{A}}\,\gamma \to A^{\mathsf{A}}\,\gamma' \to \mathsf{Set}}$$

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\end{aligned}$$

$-^{\mathsf{M}}$ on a context gives homomorphisms of algebras. E.g.

$$\begin{aligned}
\Theta^{\mathsf{M}}((N, z, s), (N', z', s')) = {} & (N_M : N \to N') \times (N_M(z) = z') \times \\
& ((\alpha \in N) \to N_M(s(\alpha)) = s'(N_M(\alpha)))
\end{aligned}$$

# Constructing inductive types à la Awodey-Frey-Speight

System F impredicative encoding:

$$\mathbb{N} := ((x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x))$$

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Problem: given an $f \in \Theta^{\mathsf{M}}(x, x')$ and an $n \in \mathbb{N}$, $f(n(x)) \stackrel{?}{=} n(x')$

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Solution: let's build this in the definition!

$$\mathbb{N} := (n \in (x : \Theta^{\mathsf{A}}) \to \mathsf{proj}_1(x)) \times (\forall x, x', f. f(n(x)) = n(x'))$$

# Another way

Natural numbers using the domain-specific type theory:

$$\mathbb{N} := \{t \mid \Theta \vdash t : \underline{Nat}\}$$

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Constructors

$$\mathsf{zero} := zero \in \mathbb{N}$$

$$\mathsf{suc}(n \in \mathbb{N}) := suc\ n \in \mathbb{N}$$

# Recursion

We need:

$$\mathsf{rec}_{\mathbb{N}} : (x : \Theta^{\mathsf{A}}) \to \mathbb{N} \to \mathsf{proj}_1(x)$$

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The standard interpretation of $t \in \mathbb{N}$, i.e. $\Theta \vdash t : \underline{Nat}$:

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$$\mathsf{rec}_{\mathbb{N}}(x, t) := t^{\mathsf{A}}(x)$$

# Model for the initial algebra

Specification (fix a $\Theta$):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{C}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{A}}}$$

$$\frac{\Gamma \vdash A}{A^{\mathsf{C}} \in (\Theta \vdash \nu : \Gamma)(\Theta \vdash t : A[\nu]) \to A^{\mathsf{A}}(\Gamma^{\mathsf{C}}(\nu))}$$

On the universe:

$$\mathsf{U}^{\mathsf{C}}(\nu, a) := \{t \mid \Theta \vdash t : \underline{a}\}$$

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Initial algebra for $\Theta$ is $\Theta^{\mathsf{C}}(\mathsf{id}_\Theta) \in \Theta^{\mathsf{A}}$.

# Model for the recursor

Specification (fix a $\Theta$ and an $\theta \in \Theta^{\mathsf{A}}$):

$$\frac{\vdash \Gamma}{\Gamma^{\mathsf{R}} \in (\Theta \vdash \nu : \Gamma) \to \Gamma^{\mathsf{M}}(\Gamma^{\mathsf{C}}(\nu), \nu^{\mathsf{A}}(\theta))}$$

$$\frac{\Gamma \vdash A}{A^{\mathsf{R}} \in (\Theta \vdash \nu : \Gamma)(\Theta \vdash t : A[\nu]) \to A^{\mathsf{M}}(\Gamma^{\mathsf{R}}(\nu), A^{\mathsf{C}}(\nu, t), t^{\mathsf{A}}(\theta))}$$

On the universe:

$$\mathsf{U}^{\mathsf{R}}(\nu, a)(t) := t^{\mathsf{A}}(\theta)$$

# Model for the recursor

Specification (fix a $\Theta$ and an $\theta \in \Theta^{\mathsf{A}}$):

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Recursor is given by $\Theta^{\mathsf{R}}(\mathsf{id}_\Theta) \in \Theta^{\mathsf{M}}(\text{initial algebra for }\Theta, \theta)$.

# Dependent eliminator

- Families over an algebra are given by the logical predicate interpretation.
- Sections of a family by a tweaked dependent logical relation interpretation.
- The dependent eliminator uses the logical predicate interpretation on terms.

# Summary

Domain-specific type theory.

Contexts in this type theory are signatures for IITs.

We can do universal algebra by defining models of this type theory.

| | |
|---|---|
| Standard model: | algebras |
| Logical predicates: | families |
| Tweaked logical relations: | algebra homomorphisms |
| Tweaked dependent logical relations: | sections |
| Model where U is terms: | initial algebra |
| Model where U is the standard model: | recursor |
| Model where U is the logical predicates: | eliminator |

All of this extends to quotient inductive-inductive types.
Challenge: what about higher inductive-inductive types?

If the models are syntactic, we can iterate: e.g. a signature for categories, log.rel. gives us functors, log.rel. again natural transformations etc.