

Levels of abstraction when defining type theory in type theory

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Workshop on type theory in type theory,
Gödel 90 conference
Nürtingen near Tübingen, 6 July 2021

What is the syntax of type theory?

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(1) A term is a string:

"($\lambda x. x+x$) 3"

"($\lambda x. x+x$) 3a"

"(($\lambda x. x+x$) 3) "

"($\lambda x. x + x$) 3"

"($\lambda x. x+x$ 3"

"6"

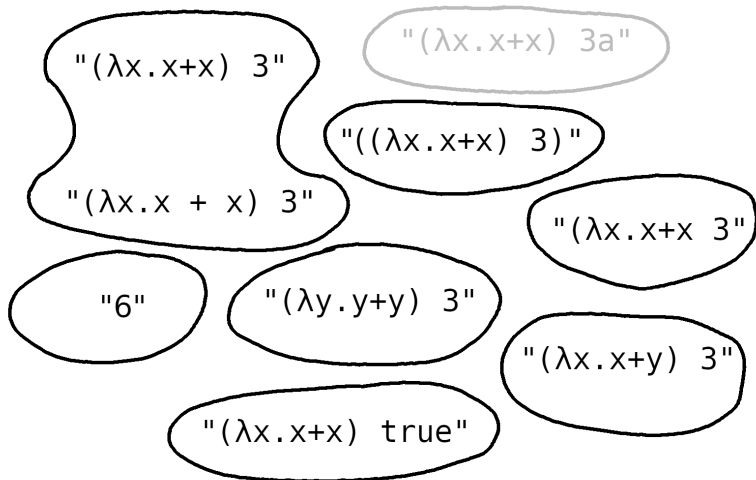
"($\lambda y. y+y$) 3"

"($\lambda x. x+y$) 3"

"($\lambda x. x+x$) true"

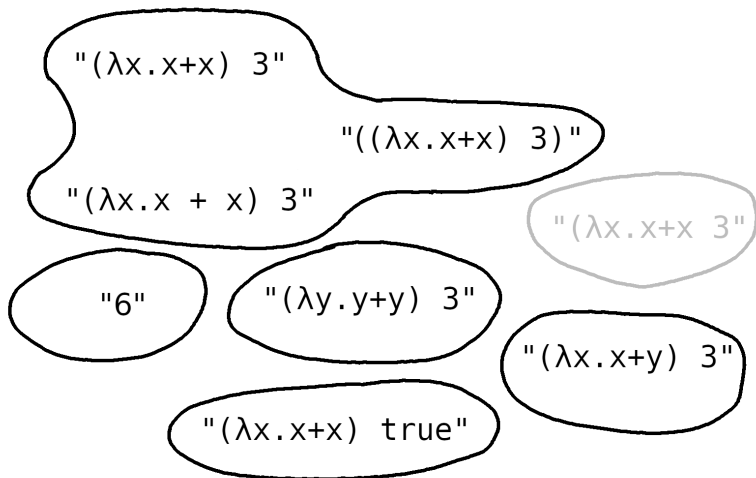
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(2) A term is a list of lexical elements:



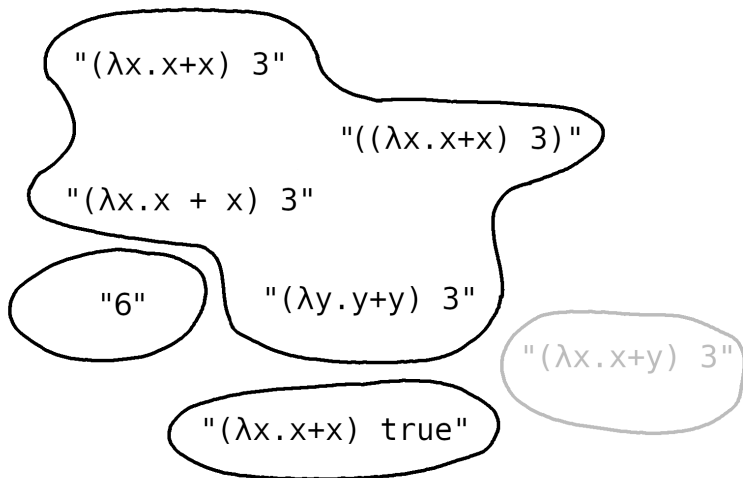
What is the syntax of type theory?

(3) A term is a tree (AST):



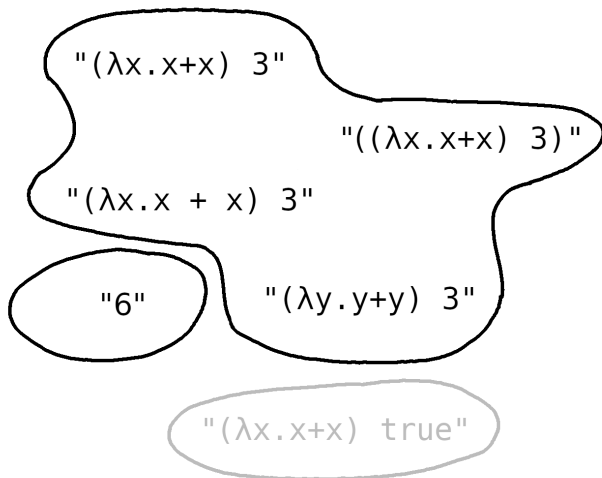
What is the syntax of type theory?

(4) A term is a well-scoped syntax tree (ABT):



What is the syntax of type theory?

(5) A term is a well-typed syntax tree (intrinsic):



What is the syntax of type theory?

(6) A term is a well-typed syntax tree quotiented by conversion (algebraic, equational theory, model-theoretic):

"($\lambda x. x+x$) 3"

"(($\lambda x. x+x$) 3)"

"($\lambda x. x + x$) 3"

"6"

"($\lambda y. y+y$) 3"

Going abstract

(1) string

↳ lexical analysis

(2) list of lexical elements

↳ parsing

(3) syntax tree

↳ scope checking

(4) well scoped syntax tree

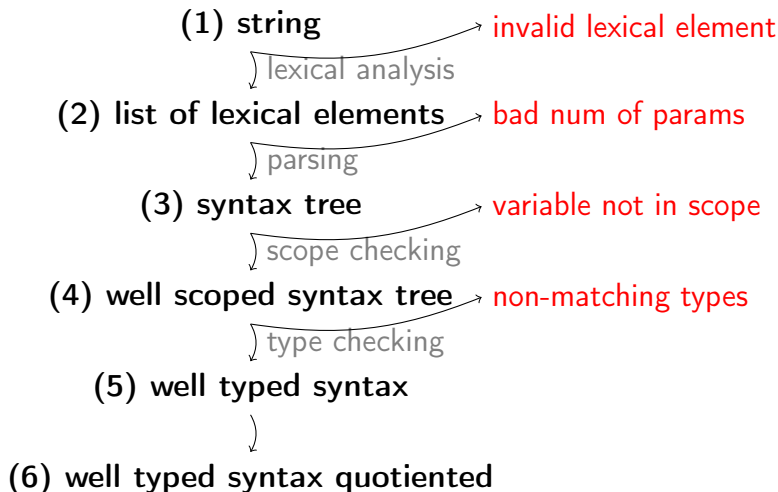
↳ type checking

(5) well typed syntax

↳

(6) well typed syntax quotiented

Going abstract: errors



Going abstract: quotienting

(1) string

↓ lexical analysis

(2) list of lexical elements

↓ parsing

(3) syntax tree

↓ scope checking

(4) well scoped syntax tree

↓ type checking

(5) well typed syntax

↓

(6) well typed syntax quotiented

"1+1" = "1 + 1"

[1, +, 1] = [(, 1, +, 1,)]

$\lambda x.x = \lambda y.y$

$(\lambda x.x + x) 3 = 3 + 3$

Going concrete (i)

(1) string

add spaces $\uparrow \downarrow$ lexical analysis

(2) list of lexical elements

add brackets $\uparrow \downarrow$ parsing

(3) syntax tree

pick var names $\uparrow \downarrow$ scope checking

(4) well scoped syntax tree

$\uparrow \downarrow$ type checking

(5) well typed syntax

normalise $\uparrow \downarrow$

(6) well typed syntax quotiented

Non-theorems (and another level)

(1) string



(2) list of lexical elements



(3) syntax tree



(4) well scoped syntax tree



(5) well typed syntax



(6) well typed syntax quotiented

α -renaming preserves
matching brackets

α -renaming preserves
typing

conversion preserves
typing

normalisation is sound

Non-theorems (and another level)

(1) string



(2) list of lexical elements



(3) syntax tree



(4) well scoped syntax tree



(5) well typed syntax



(6) well typed syntax quotiented



(7) higher order abstract syntax

α -renaming preserves
matching brackets

α -renaming preserves
typing

conversion preserves
typing

normalisation is sound

everything is stable un-
der substitution

What do we need to define the syntax?

(1) string

strings

(2) list of lexical elements

lists

(3) syntax tree

inductive types (ITs)

(4) well scoped syntax tree

indexed ITs

(5) well typed syntax

inductive-inductive types (IITs)

(6) well typed syntax quotiented

quotient IITs (QIITs)

(7) higher order abstract syntax

QIITs with bindings

T.T. in T.T. at levels (1)–(4)

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- ▶ tree: inductive type given by the BNF grammar
(Abel–Öhman–Vezzosi POPL 2018)

$$v ::= x \mid y \mid z \mid \dots$$

$$t ::= v \mid \lambda v. t \mid t \$ t$$

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- ▶ well-scoped tree: indexed inductive type

$$\mathsf{Tm} \quad : \mathbb{N} \rightarrow \mathsf{Set}$$

$$\mathsf{var} \quad : (i : \mathbb{N}) \rightarrow i < n \rightarrow \mathsf{Tm} \, n$$

$$\mathsf{lam} \quad : \mathsf{Tm} \, (1 + n) \rightarrow \mathsf{Tm} \, n$$

$$- \$ - : \mathsf{Tm} \, n \rightarrow \mathsf{Tm} \, n \rightarrow \mathsf{Tm} \, n$$

T.T. in T.T. at level (5)

- ▶ well-typed tree: inductive-inductive type¹
(Chapman: Type theory should eat itself 2009)

Con : Set

Ty : Con \rightarrow Set

· : Con

-, - : (Γ : Con) \rightarrow Ty $\Gamma \rightarrow$ Con

...

Tm : (Γ : Con) \rightarrow Ty $\Gamma \rightarrow$ Set

$- \Rightarrow -$: Ty $\Gamma \rightarrow$ Ty $\Gamma \rightarrow$ Ty Γ

lam : Tm (Γ, A) $B \rightarrow$ Tm Γ ($A \Rightarrow B$)

$- \$ -$: Tm Γ ($A \Rightarrow B$) \rightarrow Tm Γ $A \rightarrow$ Tm Γ B

¹the first B in the type of lam needs to be weakened, also in the next slide

T.T. in T.T. at level (6)

- well-typed tree quotiented: QIIT

(Dybjer CwF 1996, Altenkirch–Kaposi POPL 2016)

$\text{Con} : \text{Set}$

$\text{Ty} : \text{Con} \rightarrow \text{Set}$

$\cdot : \text{Con}$

$-, - : (\Gamma : \text{Con}) \rightarrow \text{Ty } \Gamma \rightarrow \text{Con}$

$\text{Sub} : \text{Con} \rightarrow \text{Con} \rightarrow \text{Set}$

\dots

$\text{Tm} : (\Gamma : \text{Con}) \rightarrow \text{Ty } \Gamma \rightarrow \text{Set}$

$-[-] : \text{Ty } \Gamma \rightarrow \text{Sub } \Delta \Gamma \rightarrow \text{Ty } \Delta$

$-[-] : \text{Tm } \Gamma A \rightarrow (\sigma : \text{Sub } \Delta \Gamma) \rightarrow \text{Tm } \Delta (A[\sigma])$

$\text{lam} : \text{Tm } (\Gamma, A) B \rightarrow \text{Tm } \Gamma (A \Rightarrow B)$

$- \$ - : \text{Tm } \Gamma (A \Rightarrow B) \rightarrow \text{Tm } \Gamma A \rightarrow \text{Tm } \Gamma B$

$\beta : \text{lam } t \$ u = t[\text{id}, u]$

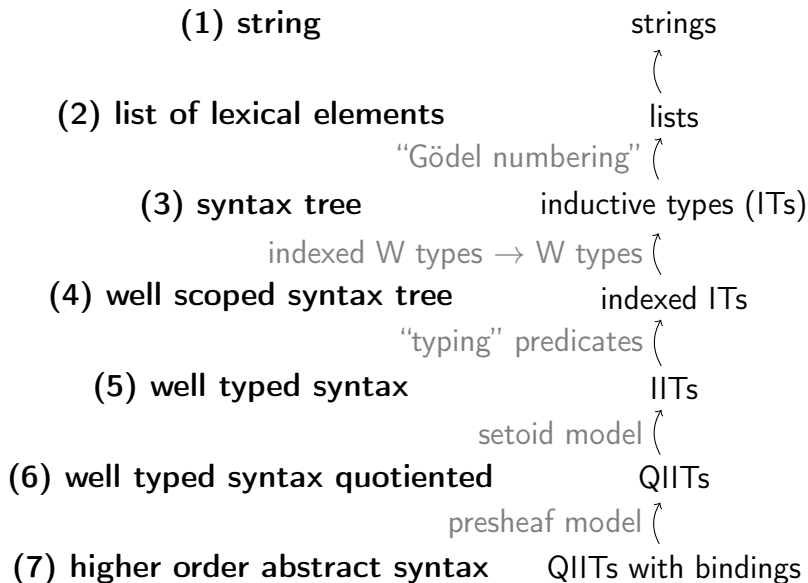
T.T. in T.T. at level (7)

► higher order abstract syntax

(Hofmann 1999, Awodey's natural models 2014,
Bocquet–Kaposi–Sattler 2021)

$$\text{Ty} \quad : \text{Set}$$
$$\text{Tm} \quad : \text{Ty} \rightarrow \overline{\text{Set}}$$
$$- \Rightarrow - : \text{Ty} \rightarrow \text{Ty} \rightarrow \text{Ty}$$
$$\text{lam} \quad : (\text{Tm } A \Rightarrow \text{Tm } B) \rightarrow \text{Tm } (A \Rightarrow B)$$
$$- \$ - : \text{Tm } (A \Rightarrow B) \rightarrow \text{Tm } A \rightarrow \text{Tm } B$$
$$\beta \quad : \text{lam } t \$ u = \overline{t} \, u$$

Going concrete (ii)



What has been done at levels (6)/(7)?

- ▶ Normalisation (canonicity, decidability of equality).
 - ▶ statement: $Tm \Gamma A \cong Nf \Gamma A$
 - ▶ normalisation by evaluation, logical predicates
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Some of the above at level (7): (Bocquet–Kaposi–Sattler 2021)

What is hard at levels (6)/(7)?

- ▶ Compilation to lower level language: the low level language needs a matching equational theory.
- ▶ Level (7) cannot formalise calculi where some operations are not stable under substitution (e.g. Martin-Löf's first presentation of t.t.)
- ▶ Level (6) formalisation is still hard because QIITs are not supported (except Cubical Agda).
- ▶ Level (7) needs modalities when moving between models, e.g. multi-modal type theory (Gratzer–Kavvos–Nuyts–Birkedal 2021).

Why not normal forms instead of quotienting?

(1) string

$\uparrow \downarrow$

(2) list of lexical elements

$\uparrow \downarrow$

(3) syntax tree

$\uparrow \downarrow$

(4) well scoped syntax tree

$\uparrow \downarrow$

(5) well typed syntax

$\uparrow \downarrow$

(6) well typed syntax quotiented

$\uparrow \downarrow$

(7) higher order abstract syntax

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1. not easier to formalise

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(7) higher order abstract syntax

1. not easier to formalise

2. We want to write non-normal proofs and programs

Questions

- ▶ Is there a presentation of normal forms of t.t. that does not refer to the equational theory?
- ▶ What features of programming languages cannot be described at the algebraic level? E.g. small step semantics.
- ▶ Can we reproduce (Abel–Öhman–Vezzosi POPL 2018) at level (6) without UIP?
- ▶ What is the best calculus for level (7)? Binding and names built-in, maybe multi-modal t.t.?