Generalizations of Hedberg's Theorem

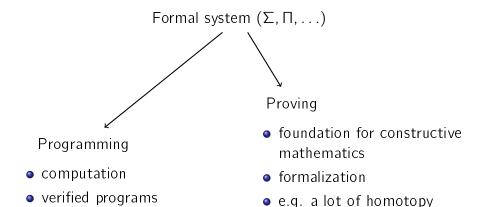
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Overview

- Views on Type Theory
- Reminder: Equality in Type Theory
- Hedberg's Theorem
- Generalizations

Views on Type Theory



theory has been formalized

in HoTT

Reminder: Equality

Definitional Equality

Decidable equality for typechecking & computation; e.g. $(\lambda a.b)x \equiv_{\beta} b[x/a]$

Reminder: Equality

Definitional Equality

Decidable equality for typechecking & computation; e.g. $(\lambda a.b)x \equiv_{\beta} b[x/a]$

Propositional Equality

Equality needing a proof, e.g.

$$\forall m n . (m+n) = (n+m)$$

Propositional equality

...is just an inductive type

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Formation

a, b : A

 $a =_A b$: type

Propositional equality

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$$a =_A b$$
: type

Introduction

$$refl_a$$
: $a =_A a$

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Introduction

a : A

$$refl_a$$
: $a =_A a$

Elimination (J - Paulin-Mohring) for any a : A

$$P: (b:A) \rightarrow a =_A b \rightarrow \mathcal{U}$$

 $m: P(a, refl_a)$

$$J_{(m,b,q)}: P(b,q)$$

Propositional equality

...is just an inductive type

Formation

$$a =_A b$$
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Introduction

$$refl_a: a =_A a$$

Elimination (J - Paulin-Mohring)for any a : A

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 $m: P(a, refl_a)$

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Computation (β)

$$J_{(m,a,refl_a)} \equiv_{\beta} m$$

Given a:A.

• Can we show $(b, c: A) \rightarrow (p: a = b) \rightarrow (q: a = c) \rightarrow (b, p) = (c, q)$?

• Can we show $(p, q : a = a) \rightarrow p = q$?

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Yes! Induction/
$$J$$
/"pattern matching" on (b, p) and (c, q)
 $\Rightarrow (a, refl_a) = (a, refl_a).$

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 \Rightarrow $(a, refl_a) = (a, refl_a)$.

• Can we show $(p, q : a = a) \rightarrow p = q$?

No!

Axiom UIP (or K)

$$\begin{array}{c} p, \ q \ : \ a = b \\ \hline \text{UIP} \ : \ p = q \end{array}$$

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$$\frac{p, q : a = b}{\text{UIP} : p = q}$$

Advantages

- simple
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Disadvantages

- impossible to use the rich equality structure (as Homotopy Type Theory does to formalize axiomatic homotopy theory)
- incompatible with univalence (which allows us to identify isomorphic types)

Which types satisfy UIP naturally?

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First, a definition:

Decidable Equality

Decidable Equality $a := \forall ab \cdot (a = b + \neg a = b)$

- Examples: \mathbb{N} , List_A if A has decidable equality
- Counterexamples: Colists (over a nonempty type), universes

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constant function

$$const(f) :\equiv \forall ab. f(a) = f(b)$$

$DecidableEquality_A$



there is a family $g_{ab}: a = b \rightarrow a = b$ of **constant** endofunctions



 UIP_A

DecidableEquality is a very strong property. How about something weaker? For example:

Separated

$$\forall ab. \neg \neg (a = b) \rightarrow a = b$$

With function extensionality,

$$separated_A \rightarrow UIP_A$$

We can still do better if we have truncation, aka squash types or bracket types (Awodey / Bauer).

Think of ||A|| as the "squashed" version of A where we cannot distinguish the different inhabitants any more (similar to $\neg \neg A$).

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H-Separated

$$\forall ab . \|a = b\| \rightarrow a = b$$

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$$\forall ab . \|a = b\| \rightarrow a = b$$

h-separated_A \Leftrightarrow UIP_A

Generalizations

h-separated_A, i. e.

$$||a=b|| \rightarrow a=b$$



there is a family

$$g_{ab}$$
: $a = b \rightarrow a = b$ of **constant** endofunctions

 UIP_A , i. e.

$$(p, q : a = b) \rightarrow p = q$$

Generalizations

h-separated_A, i. e.

$$||a = b|| \rightarrow a = b$$

 \updownarrow

there is a family $g_{ab}: a = b \rightarrow a = b$ of **constant** endofunctions



$$UIP_A, i. e.$$

$$(p, q : a = b) \rightarrow p = q$$



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there is a **constant**

$$g:X\to X$$



all inhabitants of X are equal,

i. e.
$$(a, b : X) \to a = b$$

Generalizations

h-separated_A, i. e.

$$||a = b|| \rightarrow a = b$$



there is a family $g_{ab}: a = b \rightarrow a = b$ of **constant** endofunctions



$$UIP_A, i. e.$$

$$(p, q : a = b) \rightarrow p = q$$





there is a **constant**

$$g: X \to X$$



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$$(a, b : X) \to a = b$$

Applications

- We can define a new notion of anonymous existence that behaves similar to truncation $\|\cdot\|$, but is definable in Type Theory.
- We can show related theorems, such as:

If we have $||X|| \to X$ for all types, then all equalities are decidable.

Questions?

Thank you!