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Quotient inductive-inductive types

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Overview

Inductive types by examples

- Universal inductive type

Indexed inductive types by examples

- Universal indexed inductive type

Quotient inductive types (QITs) by examples

- UNIVERSAL QIT

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Inductive types

are specified by their constructors.

E.g.

$$\begin{aligned} \mathsf{Bool} &: \mathsf{Type} \\ \mathsf{true} &: \mathsf{Bool} \\ \mathsf{false} &: \mathsf{Bool} \end{aligned}$$

means

$$\mathsf{Bool} = \{\mathsf{true}, \mathsf{false}\}.$$

Another example

$$\begin{aligned} \mathbb{N} &: \mathsf{Type} \\ \mathsf{zero} &: \mathbb{N} \\ \mathsf{suc} &: \mathbb{N} \to \mathbb{N} \end{aligned}$$

means

$$\mathbb{N} = \{\mathsf{zero},\ \mathsf{suc}\,\mathsf{zero},\ \mathsf{suc}\,(\mathsf{suc}\,\mathsf{zero}),\ \mathsf{suc}\,(\mathsf{suc}\,(\mathsf{suc}\,\mathsf{zero})),\ \ldots\},$$

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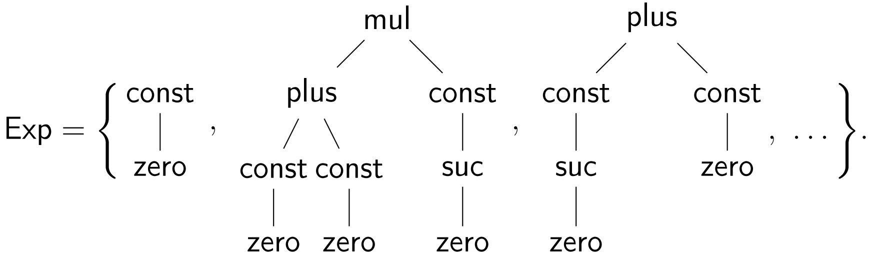
usually written

$$\mathbb{N} = \{0,\ 1,\ 2,\ \dots\}.$$

Another example

$$
\begin{aligned}
\mathsf{Exp} &: \mathsf{Type} \\
\mathsf{const} &: \mathbb{N} \to \mathsf{Exp} \\
\mathsf{plus} &: \mathsf{Exp} \to \mathsf{Exp} \to \mathsf{Exp} \\
\mathsf{mul} &: \mathsf{Exp} \to \mathsf{Exp} \to \mathsf{Exp}
\end{aligned}
$$

means



Another example

$$\begin{aligned}\mathsf{Exp} &: \mathsf{Type}\\ \mathsf{const} &: \mathbb{N} \to \mathsf{Exp}\\ \mathsf{plus} &: \mathsf{Exp} \to \mathsf{Exp} \to \mathsf{Exp}\\ \mathsf{mul} &: \mathsf{Exp} \to \mathsf{Exp} \to \mathsf{Exp}\end{aligned}$$

written in a linear notation as

$$\begin{aligned}&\mathsf{Exp} =\\ &\ \Big\{\mathsf{const\,zero},\\ &\quad \mathsf{mul}\,\big(\mathsf{plus}\,(\mathsf{const}\,(\mathsf{suc\,zero}))\,(\mathsf{const}\,(\mathsf{suc\,zero}))\big)\,\big(\mathsf{const}\,(\mathsf{suc\,zero})\big),\\ &\quad \mathsf{plus}\,\big(\mathsf{const}\,(\mathsf{suc\,zero})\big)\,(\mathsf{const\,zero}),\ \ldots\Big\}.\end{aligned}$$

Another example

$$
\begin{aligned}
\mathbb{N}' &: \text{Type} \\
\text{suc} &: \mathbb{N}' \to \mathbb{N}'
\end{aligned}
$$

means

Another example

$$\begin{aligned} \mathbb{N}' &: \mathsf{Type} \\ \mathsf{suc} &: \mathbb{N}' \to \mathbb{N}' \end{aligned}$$

means

$$\mathbb{N}' = \{\}.$$

Why *inductive*?

Why *inductive*? We can do induction!

On Bool: $(P : \mathsf{Bool} \to \mathsf{Type}) \to P\,\mathsf{true} \to P\,\mathsf{false} \to$
$(b : \mathsf{Bool}) \to P\,b$

On $\mathbb{N}$: $(P : \mathbb{N} \to \mathsf{Type}) \to P\,\mathsf{zero} \to$
$((n : \mathbb{N}) \to P\,n \to P\,(\mathsf{suc}\,n)) \to (n : \mathbb{N}) \to P\,n$

On Exp: $(P : \mathsf{Exp} \to \mathsf{Type}) \to ((n : \mathbb{N}) \to P\,(\mathsf{const}\,n)) \to$
$((e\,e' : \mathsf{Exp}) \to P\,e \to P\,e' \to P\,(\mathsf{plus}\,e\,e')) \to$
$((e\,e' : \mathsf{Exp}) \to P\,e \to P\,e' \to P\,(\mathsf{mul}\,e\,e')) \to$
$(e : \mathsf{Exp}) \to P\,e$

Not an inductive type

$$\begin{aligned}\mathsf{Neg} &: \mathsf{Type}\\ \mathsf{con} &: (\mathsf{Neg} \to \bot) \to \mathsf{Neg}\end{aligned}$$

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The induction principle:

$$\begin{aligned} \mathsf{elimNeg} : {} & (P : \mathsf{Neg} \to \mathsf{Type}) \to ((f : \mathsf{Neg} \to \bot) \to P\,(\mathsf{con}\, f)) \to \\ & (n : \mathsf{Neg}) \to P\; n \end{aligned}$$

Not an inductive type

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The induction principle:

$$\begin{aligned} \mathrm{elimNeg} : {} & (P : \mathrm{Neg} \to \mathrm{Type}) \to ((f : \mathrm{Neg} \to \bot) \to P\,(\mathrm{con}\, f)) \to \\ & (n : \mathrm{Neg}) \to P\, n \end{aligned}$$

Now we can do something bad:

$$\begin{aligned} \mathrm{probl} &: \mathrm{Neg} \to \bot := \lambda n.\mathrm{elimNeg}\,(\lambda\_.\mathrm{Neg} \to \bot)\,(\lambda f.f)\, n\, n \\ \mathrm{PROBL} &: \bot \qquad\qquad := \mathrm{probl}\,(\mathrm{con}\ \mathrm{probl}) \end{aligned}$$

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What is a generic definition?

We have $\bot$, $\top$, $+$ and $\times$ types.

Universal inductive type (Martin-Löf, 1984): for every

$$S : \mathrm{Type} \qquad \text{and} \qquad P : S \to \mathrm{Type}$$

there is an inductive type

$$\begin{aligned} \mathrm{W} &: \mathrm{Type} \\ \mathrm{sup} &: (s : S) \to (P\, s \to \mathrm{W}) \to \mathrm{W} \end{aligned}$$

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E.g. $\mathbb{N}$ is given by

$$S := \top + \top \qquad P\,(\mathsf{inl}\,\mathsf{tt}) := \bot \qquad P\,(\mathsf{inr}\,\mathsf{tt}) := \top.$$

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An indexed inductive type

$$\begin{aligned}
\mathsf{Vec} &: \mathbb{N} \to \mathsf{Type} \\
\mathsf{nil} &: \mathsf{Vec}\,\mathsf{zero} \\
\mathsf{cons} &: (n : \mathbb{N}) \to \mathsf{Bool} \to \mathsf{Vec}\,n \to \mathsf{Vec}\,(\mathsf{suc}\,n)
\end{aligned}$$

means

$$\begin{aligned}
\mathsf{Vec}\,\mathsf{zero} &= \{\mathsf{nil}\} \\
\mathsf{Vec}\,(\mathsf{suc}\,\mathsf{zero}) &= \{\mathsf{cons}\,\mathsf{zero}\,\mathsf{true}\,\mathsf{nil},\ \mathsf{cons}\,\mathsf{zero}\,\mathsf{false}\,\mathsf{nil}\} \\
\mathsf{Vec}\,(\mathsf{suc}\,(\mathsf{suc}\,\mathsf{zero})) &= \{\mathsf{cons}\,(\mathsf{suc}\,\mathsf{zero})\,\mathsf{true}\,(\mathsf{cons}\,\mathsf{zero}\,\mathsf{true}\,\mathsf{nil}),\ \ldots\} \\
\ldots
\end{aligned}$$

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\end{aligned}$$

usually written as

$$\begin{aligned}
\mathsf{Vec}\,\mathsf{zero} &= \{[]\} \\
\mathsf{Vec}\,(\mathsf{suc}\,\mathsf{zero}) &= \{[\mathsf{true}],\ [\mathsf{false}]\} \\
\mathsf{Vec}\,(\mathsf{suc}\,(\mathsf{suc}\,\mathsf{zero})) &= \{[\mathsf{true}, \mathsf{true}],\ [\mathsf{true}, \mathsf{false}],\ [\mathsf{false}, \mathsf{true}],\ \ldots\} \\
&\ldots
\end{aligned}$$

A mutual inductive type

$$
\begin{aligned}
\mathsf{Cmd} &: \mathsf{Type}\\
\mathsf{Block} &: \mathsf{Type}\\
\mathsf{skip} &: \mathsf{Cmd}\\
\mathsf{ifelse} &: \mathsf{Exp} \to \mathsf{Block} \to \mathsf{Block} \to \mathsf{Cmd}\\
\mathsf{assign} &: \mathbb{N} \to \mathsf{Exp} \to \mathsf{Cmd}\\
\mathsf{single} &: \mathsf{Cmd} \to \mathsf{Block}\\
\mathsf{semicol} &: \mathsf{Cmd} \to \mathsf{Block} \to \mathsf{Block}
\end{aligned}
$$

BNF definitions are usually mutual inductive types.

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Universal indexed/mutual inductive type

Mutual inductive types can be reduced to indexed ones.

$$\text{Cmd}, \text{Block} \qquad \text{becomes} \qquad \text{CmdOrBlock} : \text{Bool} \to \text{Type}$$

Universal indexed/mutual inductive type

Mutual inductive types can be reduced to indexed ones.

$$\mathsf{Cmd}, \mathsf{Block} \qquad \text{becomes} \qquad \mathsf{CmdOrBlock} : \mathsf{Bool} \to \mathsf{Type}$$

Altenkirch–Ghani–Hancock–McBride, 2015: for every

$$S : \mathsf{Type} \qquad \text{and} \qquad P : S \to \mathsf{Type} \qquad \text{and}$$

$$out : S \to I \qquad \text{and} \qquad in : (s : S) \to P\, s \to I$$

there is the indexed inductive type

$$\begin{aligned} \mathsf{W} \quad &: I \to \mathsf{Type} \\ \mathsf{sup} &: (s : S)((p : P\, s) \to \mathsf{W}\,(in\, s\, p)) \to \mathsf{W}\,(out\, s) \end{aligned}$$

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Integers

$$\begin{aligned}
\mathbb{Z} &: \text{Type} \\
\text{pair} &: \mathbb{N} \to \mathbb{N} \to \mathbb{Z} \\
\text{quot} &: (a\, b\, a'\, b' : \mathbb{N}) \to a + b' = a' + b \to \text{pair}\, a\, b = \text{pair}\, a'\, b'
\end{aligned}$$

means

$$\begin{aligned}
\mathbb{Z} = \{ &\{\text{pair}\,0\,0,\ \text{pair}\,1\,1,\ \text{pair}\,2\,2,\ \ldots\}, \\
&\{\text{pair}\,0\,1,\ \text{pair}\,1\,2,\ \text{pair}\,2\,3,\ \ldots\}, \\
&\{\text{pair}\,1\,0,\ \text{pair}\,2\,1,\ \text{pair}\,3\,2,\ \ldots\}, \\
&\{\text{pair}\,0\,2,\ \text{pair}\,1\,3,\ \text{pair}\,2\,4,\ \ldots\}, \\
&\ldots\}
\end{aligned}$$

Quotients

Given $A : \mathsf{Type}$, $R : A \to A \to \mathsf{Type}$, the quotient type is

$$\begin{aligned} A/R &: \mathsf{Type} \\ [-] &: A \to A/R \\ \mathsf{quot} &: (a\, a' : A) \to R\, a\, a' \to [a] = [a'] \end{aligned}$$

Cauchy Real numbers

$$
\begin{array}{ll}
\mathbb{R} & : \mathsf{Type} \\
\mathsf{P} & : \mathbb{Q}_+ \to \mathbb{R} \to \mathbb{R} \to \mathsf{Type} \\
\mathsf{rat} & : \mathbb{Q} \to \mathbb{R} \\
\mathsf{lim} & : (f : \mathbb{Q}_+ \to \mathbb{R}) \to \big((\delta\, \epsilon : \mathbb{Q}_+) \to \mathsf{P}\,(\delta + \epsilon)\,(f\,\delta)\,(f\,\epsilon)\big) \to \mathbb{R} \\
\mathsf{eq} & : (u\,v : \mathbb{R}) \to \big((\epsilon : \mathbb{Q}_+) \to \mathsf{P}\,\epsilon\,u\,v\big) \to u = v \\
\mathsf{ratrat} & : (q\,r : \mathbb{Q})(\epsilon : \mathbb{Q}_+)(-\epsilon < q - r < \epsilon) \to \mathsf{P}\,\epsilon\,(\mathsf{rat}\,q)\,(\mathsf{rat}\,r) \\
\mathsf{ratlim} & : \mathsf{P}\,(\epsilon - \delta)\,(\mathsf{rat}\,q)\,(g\,\delta) \to \mathsf{P}\,\epsilon\,(\mathsf{rat}\,q)\,(\mathsf{lim}\,g) \\
\mathsf{limrat} & : \mathsf{P}\,(\epsilon - \delta)\,(f\,\delta)\,(\mathsf{rat}\,r) \to \mathsf{P}\,\epsilon\,(\mathsf{lim}\,f)\,(\mathsf{rat}\,r) \\
\mathsf{limlim} & : \mathsf{P}\,(\epsilon - \delta - \eta)\,(f\,\delta)\,(g\,\eta) \to \mathsf{P}\,\epsilon\,(\mathsf{lim}\,f)\,(\mathsf{lim}\,g) \\
\mathsf{trunc} & : (\xi\,\zeta : \mathsf{P}\,\epsilon\,u\,v) \to \xi = \zeta
\end{array}
$$

(Homotopy Type Theory book, 2013)

Partiality monad for non-terminating programs

(Altenkirch–Danielsson–Kraus, 2017)

$$
\begin{array}{ll}
A_\perp & : \mathsf{Type} \\
- \sqsubseteq - & : A_\perp \to A_\perp \to \mathsf{Type} \\
\eta & : A \to A_\perp \\
\perp & : A_\perp \\
\bigsqcup & : (f : \mathbb{N} \to A_\perp)((n : \mathbb{N}) \to f\ n \sqsubseteq f\ (n+1)) \to A_\perp \\
\mathsf{refl} & : d \sqsubseteq d \\
\mathsf{inf} & : \perp \sqsubseteq d \\
\mathsf{in} & : ((n : \mathbb{N}) \to f\ n \sqsubseteq d) \to \bigsqcup f\ p \sqsubseteq d \\
\mathsf{out} & : \bigsqcup f\ p \sqsubseteq d \to (n : \mathbb{N}) \to f\ n \sqsubseteq d \\
\mathsf{antisym} & : (d\ d' : A_\perp) \to d \sqsubseteq d' \to d' \sqsubseteq d \to d = d' \\
\mathsf{trunc} & : (\xi\ \zeta : d \sqsubseteq d') \to \xi = \zeta
\end{array}
$$

Algebraic syntax for a programming language

$$
\begin{aligned}
&\mathsf{Ty} &&: \mathsf{Type} \\
&\mathsf{Tm} &&: \mathsf{Ty} \to \mathsf{Type} \\
&\mathsf{Bool},\ \mathsf{Nat} &&: \mathsf{Ty} \\
&\mathsf{true},\ \mathsf{false} &&: \mathsf{Tm}\,\mathsf{Bool} \\
&\mathsf{if}\!-\!\mathsf{then}\!-\!\mathsf{else}\!- &&: \mathsf{Tm}\,\mathsf{Bool} \to \mathsf{Tm}\,A \to \mathsf{Tm}\,A \to \mathsf{Tm}\,A \\
&\mathsf{num} &&: \mathbb{N} \to \mathsf{Tm}\,\mathsf{Nat} \\
&\mathsf{isZero} &&: \mathsf{Tm}\,\mathsf{Nat} \to \mathsf{Tm}\,\mathsf{Bool} \\
&\mathsf{if}\beta_1 &&: \mathsf{if}\,\mathsf{true}\,\mathsf{then}\,t\,\mathsf{else}\,t' = t \\
&\mathsf{if}\beta_2 &&: \mathsf{if}\,\mathsf{false}\,\mathsf{then}\,t\,\mathsf{else}\,t' = t' \\
&\mathsf{isZero}\beta_1 &&: \mathsf{isZero}\,(\mathsf{num}\,0) = \mathsf{true} \\
&\mathsf{isZero}\beta_2 &&: \mathsf{isZero}\,(\mathsf{num}\,(1+n)) = \mathsf{false}
\end{aligned}
$$

(Altenkirch–Kaposi, 2016)

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A domain-specific language for QIT signatures

$$\frac{}{\vdash \cdot} \qquad \frac{\Gamma \vdash A}{\vdash \Gamma, x : A} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U}}{\Gamma \vdash \underline{a}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t \mathbin{@} u : B[x \mapsto u]}$$

$$\frac{\Gamma \vdash u : \underline{a} \qquad \Gamma \vdash v : \underline{a}}{\Gamma \vdash u = v} \qquad \ldots$$

A domain-specific language for QIT signatures

$$\frac{}{\vdash \cdot} \qquad \frac{\Gamma \vdash A}{\vdash \Gamma, x : A} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U}}{\Gamma \vdash \underline{a}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \underline{a} \vdash B}{\Gamma \vdash (x : a) \Rightarrow B} \qquad \frac{\Gamma \vdash t : (x : a) \Rightarrow B \qquad \Gamma \vdash u : \underline{a}}{\Gamma \vdash t \,@\, u : B[x \mapsto u]}$$

$$\frac{\Gamma \vdash u : \underline{a} \qquad \Gamma \vdash v : \underline{a}}{\Gamma \vdash u = v} \qquad \dots$$

A signature is a context Γ, e.g.

$$(\cdot,\ N : \mathsf{U},\ zero : \underline{N},\ suc : N \Rightarrow \underline{N})$$

$$(\cdot,\ Ty : \mathsf{U},\ Tm : Ty \Rightarrow \mathsf{U},\ Bool : \underline{Ty},\ true : \underline{Tm \,@\, Bool},\ \dots)$$

This is a QIT itself

$$
\begin{aligned}
&\mathsf{Con} && : \mathsf{Type} \\
&\mathsf{Ty} && : \mathsf{Con} \to \mathsf{Type} \\
&\mathsf{Var} && : \mathsf{Con} \to \mathsf{Type} \\
&\mathsf{Tm} && : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type} \\
&\cdot && : \mathsf{Con} \\
&(-, - : -) && : (\Gamma : \mathsf{Con}) \to \mathsf{Var}\,\Gamma \to \mathsf{Ty}\,\Gamma \to \mathsf{Con} \\
&\mathsf{U} && : \mathsf{Ty}\,\Gamma \\
&\underline{-} && : \mathsf{Tm}\,\Gamma\,\mathsf{U} \to \mathsf{Ty}\,\Gamma \\
&(- : -) \Rightarrow - && : \mathsf{Var}\,\Gamma \to (a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) \to \mathsf{Ty}\,(\Gamma, x : \underline{a}) \to \mathsf{Ty}\,\Gamma \\
&- \mathbin{@} - && : \mathsf{Tm}\,\Gamma\,((x : a) \Rightarrow B) \to (u : \mathsf{Tm}\,\Gamma\,\underline{a}) \to \\
& && \quad \mathsf{Tm}\,\Gamma\,(B[x \mapsto u])
\end{aligned}
$$

$\ldots$

Results

- A generic definition of signatures for QITs which includes all the known examples
- Description of the induction principle
  - Kaposi–Kovács, FSCD 2018
- If the universal QIT exists, then all of them exist
  - Kaposi–Kovács–Altenkirch, POPL 2019
- Existence of the universal QIT
  - People proved this in different settings, e.g. Brunerie
  - Part without quotients done (by Ambroise Lafont), full version further work

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