First-Order Linear Equations $\dot{x} + a(t)x = b(t)$ integrating factor $e^{A(t)}$ $\boldsymbol{x = c \cdot e^{-at} + \frac{b}{a}}$

Bernoulli's equation $\dot{x} + a(t)x = b(t)x^r$ $z = x^{1-r}$

characteristic equation

$$r^2 + ar + b = 0.$$

If the characteristic equation has two distinct real roots r_1, r_2 $x = Ae^{r_1 t} + Be^{r_2 t}$

(II) If the characteristic equation has one real double root r $x = (A + Bt)e^{rt}$

(III) If the characteristic equation has no real roots $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t), \quad \alpha = -\frac{1}{2}a, \quad \beta = \sqrt{b - \frac{1}{4}a^2}$

λ_1 and λ_2 are real and different $\{e^{\lambda_1 t} \cdot \boldsymbol{v}^1, \quad e^{\lambda_2 t} \cdot \boldsymbol{v}^2\}$

$\lambda_1 = \lambda_2$ are real double root $\{e^{\lambda_1 t} \cdot \boldsymbol{v}^1, \quad te^{\lambda_1 t} \cdot \boldsymbol{v}^1\}$

$\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are complex numbers $\{Re(e^{\lambda_1 t} \cdot \boldsymbol{v}^1), \quad Im(e^{\lambda_1 t} \cdot \boldsymbol{v}^1)\}$

The equilibrium point for the linear system is globally asymptotically stable if

$tr\,A = a_{11} + a_{22} < 0$ and $\det A > 0$ / both eigenvalues of A have negative real parts.

Linearization of nonlinear systems $\begin{aligned}\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y)\end{aligned}$ $\begin{aligned}\dot{x} &= f_1'(a,b)x + f_2'(a,b)y + b_1 \\ \dot{y} &= g_1'(a,b)x + g_2'(a,b)y + b_2\end{aligned}$

Basic control

Hamiltonian

$$H(t,x,u,p) = f(t,x,u) + p \cdot g(t,x,u)$$

$u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ $\dot{p}(t) = -H_x'(t, x^*(t), u^*(t), p(t)),\ p(t_1) = 0$

First-order linear equation $x_{t+1} = ax_t + b$

$x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$, when $a \neq 1$

$x_t = x_0 + bt$, when $a = 1$

Fundamental equations of dynamic programming

Let $J_s(x)$ be the value function for the problem

$$\max \sum_{t=0}^{T} f(t, x_t, u_t) \quad \text{subject to } x_{t+1} = g(t, x_t, u_t), \quad x_0 \text{ given}, u \in U.$$

$$J_s(x) = \max_{u \in U}[f(s,x,u) + J_{s+1}(g(s,x,u))], \quad s = 0, 1, \dots T-1$$

$$J_T(x) = \max_{u \in U} f(T,x,u)$$

The Euler Equation

$x_{t+1} = g(t, x_t, u_t)$ has a unique solution $u_t = \varphi(t, x_t, x_{t+1})$

$F(t, x_t, x_{t+1}) = f(t, x_t, \varphi(t, x_t, x_{t+1}))$, for $t < T$

$F(T, x_T, x_{T+1}) = \max_u f(T, x_T, u)$ for $t = T$

Let $\{x_0^*, \dots, x_{T+1}^*\}$ be and optimal solution for the dynamic programming problem

If we define $F(T+1, x_{T+1}, x_{T+2}) = 0$, then $\{x_0^*, \dots, x_{T+1}^*\}$ satisfies the Euler equation

$$F_2'(t+1, x_{t+1}, x_{t+2}) + F_3'(t, x_t, x_{t+1}) = 0$$