Introduction to Quantitative Finance

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Agenda

1. Basic Information
2. Course Structure
3. Possible Quants Career Paths
4. Basic Information About Financial Markets
5. Basic Financial Instruments
6. Derivatives
   - Definitions
   - Types
   - Usage
7. Options
   - Definition
   - Greeks
   - Replication & Stochastic Calculus
   - Options Usage
   - Strategies
   - Exotics
   - Structured Products

Basic Information

The goal of this course is to familiarize you with the basic concepts and theories of modern finance and derivatives.

What to expect?

- The course begins easy but gets tougher as we go along.
- Not very heavy on math, serious theoretical concepts will be explained in an accessible and (hopefully) intuitive way, but **requires regular, individual work.**
- In order to minimize the chance of being lost/disengaged, always try to **ask as soon as you realize you're not following.**
- Come prepared and regularly review your notes from the past weeks.
- There's no such thing as a stupid question!

Contact to coordinator:

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Basic Information

Grading criteria:

- Final exam - 60%
- Quiz – 30%
- Class activities – 10%

Grades scale:

[0%, 50%) = 2	[80%, 90%) = 4.5
[50%, 60%) = 3	[90%, 95%) = 5
[60%, 70%) = 3.5	[95%, 100%) = 5!
[70%, 80%) = 4	

To Pass the course you need to get:

- at least 50% of final exam;
- pass at least 5 out of 10 quizzes.

Class attendance is **not mandatory**: however there are going to be 10 quizes, absences on 6 result in FAILing the course.

All materials can be found on Moodle course webpage - https://elearning.wne.uw.edu.pl/course/view.php?id=3994

Course Plan

1. Introduction to Options and Other Financial Instruments
2. Future – Present Value Lemma Pricing of Basic Financial Derivatives
3. Martingale Theory & Risk Neutral Valuation
4. Binomial Option Pricing Model
5. Brownian Motion
6. Stochastic Calculus & Ito’s Lemma
7. Change of Numeraire & Girsanov’s Theorem
8. Derivations of Black-Scholes Formula
9. Black-Scholes Formula - Dive In
10. Other Stochastic Differential Equations
11. Exam

Possible Quants Career Paths

Quant types:

- **Quant Researchers**: focus on analysing data, recognize patterns and building new models for the business. Researches must be experienced in coding and understand the business in which they operate.

- **Quant Developers**: responsible for building production models, applications and managing the data used by the models. Developers must be experts in programming, but also have a solid knowledge of the underlying financial theory.

- **Quant Validators**: assess and challenge models independently from developers, making sure that the developed models perform as expected and within the firm risk appetite. Validators must have good coding skills and solid understanding of financial as well as math theory.

- **Quant Trader/Portfolio Managers**: a quant asset manager will have a large emphasis placed on portfolio construction and portfolio optimization techniques. Quant traders focus on improving automated market making and execution algorithms. These roles require a strong background in finance in order to understand the dynamics of the market.

Typical Institutions for Quants:

- **Investment Banks/Insurance Companies**: mainly sell-side firms which offer products and financial services to retail/institutional clients.

- **Investment Funds**: mainly buy-side firms which purchase investment products and profit from investing activities.

Time Value of Money

Money has time value because of the opportunity to invest it at some interest rate.

Time Value of Money investigates the following questions:

1) What is the future value, at time n, of any sum of money invested today at an interest rate r?
2) What is the present value, at time t_0, of any sum of money received in the future at time n?

Future Value

The future value of a sum of money P_0 invested today at rate r for n period is: $P_n = P_0 * (1 + r)^n$

Where:

n = number of periods,

P_n = future value of money

P_0 = principal or notional at time zero

r = interest rate

$(1 + r)^n$ = compounding factor

Present Value

Today's value of a sum of money P_n received in the future is: $P_0 = \frac{P_n}{(1+r)^n}$

Note that $1/(1 + r)^n$ is called discounting factor

Time Value of Money

Interest rate compounding (few remarks):

- Each interest rate has a frequency associated, i.e. annual/semi-annual/monthly/etc.
- The future/present value depends on how often the interest earned is added to the principal (**compounding** effect).
- If interest is compounded:

- Annually: $P_n = P_0 * (1+r)^n$
- Semi-annually: $P_n = P_0 * \left(1 + \frac{r_{freq}}{2}\right)^{2*n}$
- m times: $P_n = P_0 * \left(1 + \frac{r_{freq}}{m}\right)^{m*n}$
- Continuously: $P_n = P_0 \lim_{m \to \infty} \left(1 + \frac{r_{freq}}{m}\right)^{m*n} = P_0 e^{r*n}$

- Interest rate can be translated from one frequency to another by using the formula: $(1+r) = \left(1 + \frac{r_{freq}}{m}\right)^{m*n}$

1. Assume an annually compounded rate $r = 10\%$. Compute the semi-annual equivalent interest rate:

$(1+0.1) = \left(1 + \frac{r}{2}\right)^{2*1}$ -> $r_{semi-annual} = 2 * \left(\sqrt[2]{(1+0.1)} - 1\right) = 9.762\%$ - **Sanity check**: $\$100 * (1.1) = \$110 = \$100 * \left(1 + \frac{9.762\%}{2}\right)^2$

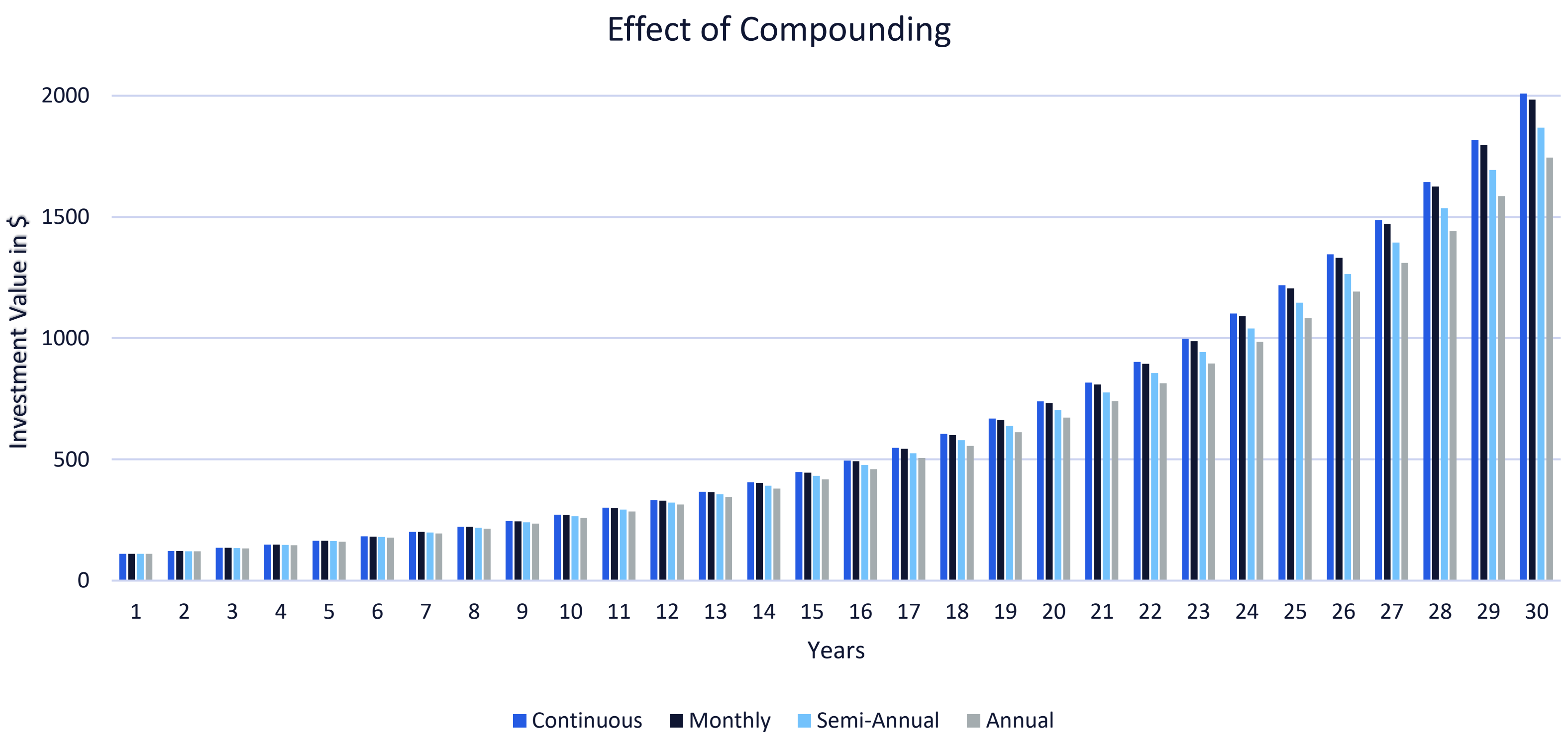
2. Assume a monthly compounded rate $r_{monthly} = 2\%$. Compute the annual equivalent interest rate:

$(1+r) = \left(1 + \frac{2\%}{12}\right)^{12*1}$ **->** $r = \left(1 + \frac{2\%}{12}\right)^{12*1} - 1 = 2.018\%$ - **Sanity check:** $\$100 * (1.02018) = \$102.018 = \$100 * \left(1 + \frac{2\%}{12}\right)^{12}$

Time Value of Money

- The higher the compounding frequency the higher the future value.
- Example: impact of different compounding frequencies on $100 investment. Interest rate = 10%

	Compounding Frequency			
Time	Continuous	Monthly	Semi-Annual	Annual
1	110.52	110.47	110.25	110.00
2	122.14	122.04	121.55	121.00
3	134.99	134.82	134.01	133.10
4	149.18	148.94	147.75	146.41
5	164.87	164.53	162.89	161.05
6	182.21	181.76	179.59	177.16
7	201.38	200.79	197.99	194.87
8	222.55	221.82	218.29	214.36
9	245.96	245.04	240.66	235.79
10	271.83	270.70	265.33	259.37
11	300.42	299.05	292.53	285.31
12	332.01	330.36	322.51	313.84
13	366.93	364.96	355.57	345.23
14	405.52	403.17	392.01	379.75
15	448.17	445.39	432.19	417.72
16	495.30	492.03	476.49	459.50
17	547.39	543.55	525.33	505.45
18	604.96	600.47	579.18	555.99
19	668.59	663.35	638.55	611.59
20	738.91	732.81	704.00	672.75
21	816.62	809.54	776.16	740.02
22	902.50	894.31	855.72	814.03
23	997.42	987.96	943.43	895.43
24	1102.32	1091.41	1040.13	984.97
25	1218.25	1205.69	1146.74	1083.47
26	1346.37	1331.95	1264.28	1191.82
27	1487.97	1471.42	1393.87	1311.00
28	1644.46	1625.50	1536.74	1442.10
29	1817.41	1795.71	1694.26	1586.31
30	2008.55	1983.74	1867.92	1744.94



Time Value of Money

Futures Value Examples

1. Assume you have \$10,000 to invest in *Investment Gurus of Giggles,* your pension fund. Assuming that *Investment Gurus of Giggles* achieves a stunning annual return of 10% (they are Gurus by the way...). What is the future value of the invested sum in 20 years?

$$P_0 = ?$$

2. Using data from the first exercise, assume that now the 10% interest rate is a) compounded semi-annually, b) continuously compounded. What is the future value of the invest sum?

   a. $P_0 = ?$

   b. $P_0 = ?$

Present Value Examples

1. *Please-Bail-Out Bank* will receive \$1,000 in 1 year from now and \$1,200 in 2 years from now. Assuming an annual interest rate r = 5 %, what is the present value of your future cashflow?

$$P_0 = ?$$

2. Assume now interest rates of 3% and 5% annually compounded for the first and second year respectively. What is the present value of future cashflow?

$$P_0 = ?$$

Time Value of Money

Futures Value Examples

1. Assume you have $10,000 to invest in *Investment Gurus of Giggles,* your pension fund. Assuming that *Investment Gurus of Giggles* achieves a stunning annual return of 10% (they are Gurus by the way...). What is the future value of the invested sum in 20 years?

$$\$10000 * (1.1)^{20} \cong \$67275$$

2. Using data from the first exercise, assume that now the 10% interest rate is a) compounded semi-annually, b) continuously compounded. What is the future value of the invest sum?

a. $\$10000 * \left(1 + \frac{10\%}{2}\right)^{40} \cong \$70400$

b. $\$10000 * e^{0.1*20} \cong \$73890$

Present Value Examples

1. *Please-Bail-Out Bank* will receive $1,000 in 1 year from now and $1,200 in 2 years from now. Assuming an annual interest rate r = 5 %, what is the present value of your future cashflow?

$$PV = \frac{\$1000}{(1.05)^1} + \frac{\$1200}{(1.05)^2} = \$952.381 + \$1088.435 \cong \$2040.816$$

2. Assume now interest rates of 3% and 5% annually compounded for the first and second year respectively. What is the present value of future cashflow?

$$PV = \frac{\$1000}{(1.03)^1} + \frac{\$1200}{(1.05)^2} = \$970.874 + \$1088.435 \cong \$2059.309$$

Time Value of Money – Alternative Way

Another way of deriving the continuously compounded result is via a differential equation. Suppose I have an amount M(t) in the bank at time t, how much does this increase in value from one day to the next? If I look at my bank account at time t and then again a short while later, time t + dt, the amount will have increased by:

$$M(t + dt) - M(t) \approx \frac{dM}{dt} dt + \cdots$$

where the right-hand side comes from a Taylor series expansion of *M(t + dt).* But I also know that the interest I receive must be proportional to the amount I have, *M*, the interest rate, *r*, and the time step, *dt*. Thus,

$$\frac{dM}{dt} dt = rM(t)\, dt$$

Dividing by *dt* gives the ordinary differential equation

$$\frac{dM}{dt} = rM(t)$$

the solution of which is

$$M(t) = M(0)e^{rt}$$

Time Value of Money – Alternative Way

This equation relates the value of the money I have now to the value in the future. Conversely, if I know I will get one dollar at time *T* in the future, its value at an earlier time *t* is simply

$$e^{-r(T-t)}$$

I can relate cashflows in the future to their **present value** by multiplying by this factor. As an example, suppose that *r* is 5% i.e. r = 0.05, then the present value of $1,000,000 to be received in two years *(T=2)* is

$$M(T) ** e^{-r*(T-t)} = \$1{,}000{,}000 * e^{-0.05*(2-0)} = \$904{,}837$$

The present value is clearly less than the future value. Interest rates are a very important factor determining the present value of future cashflows.

For the moment we assume constant interest rate, later we will generalize.

Financial Markets

A marketplace is intended to bring buyers and sellers together, facilitating them to trade. Supply and demand are not always balanced, markets have to rely on intermediaries to facilitate trading (brokers, dealers, liquidity providers).

Financial Markets Categories:

- **Primary market** - dedicated market for the issuance of new securities.
- **Secondary market** - already issued securities can be traded.

Secondary Markets:

Market Exchange:

- Centralized platform where buyers and sellers can trade securities according to pre-established rules and regulations.
- When two parties reach make a trade, the price at which the transaction is executed is communicated throughout the market.
- Market makers are obliged to provide liquidity.

Over-the-Counter (OTC):

- Decentralized market where securities are traded directly between two parties. Less regulated, transparent and liquid markets.
- The price at which securities a trade occurred may not be posted to the public.
- Market makers (dealers) are not obliged to provide liquidity.

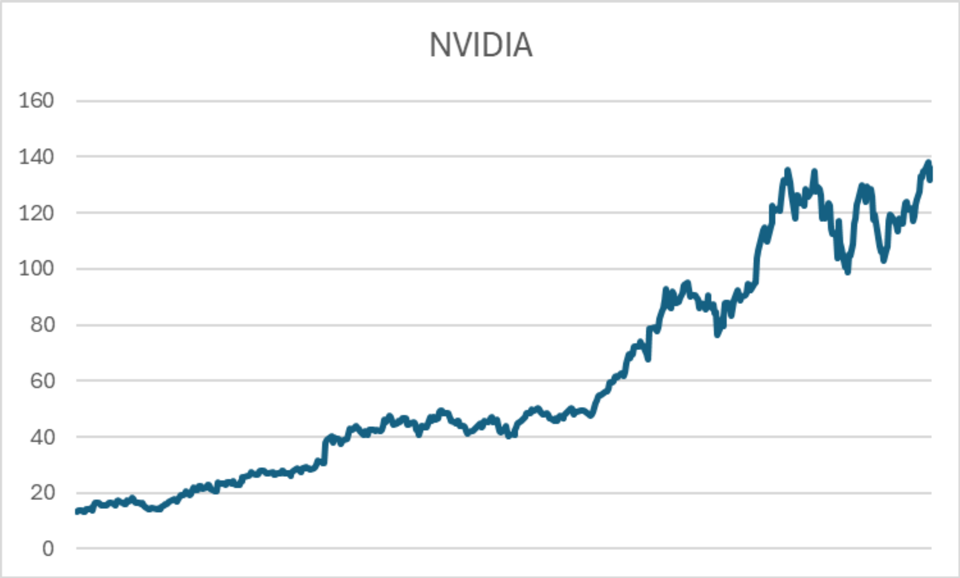
Financial Instruments - Equity

- **The equity** of a corporation represents the value of the remaining assets once all liabilities have been deducted. A share in a company (also known as **stock** or **share**) represents a portion of the total equity.
- Stocks allow companies to **finance themselves** by making their ownership public.
- A **finite number of shares** are issued by public corporations and traded in the markets.
- The owner of a stock is entitled of **voting** and **property rights**.
- Investors can profit from investing in a stock via **capital gain** and periodic payments known as **dividends**.
- The total value of outstanding shares is referred as the corporation's **market capitalization** (computed as price times total shares number).
- In U.S. stocks are grouped based on the following market cap:

  1) Large-cap if market cap. > \$10 billion
  2) Mid-cap if \$1 billion < market cap. < \$10 billion
  3) Small-cap if market cap.< \$1 billion
  4) Micro-caps if market cap. < \$100 million

Financial Instruments - Equity

The behaviour of the quoted prices of stocks is far from being predictable.

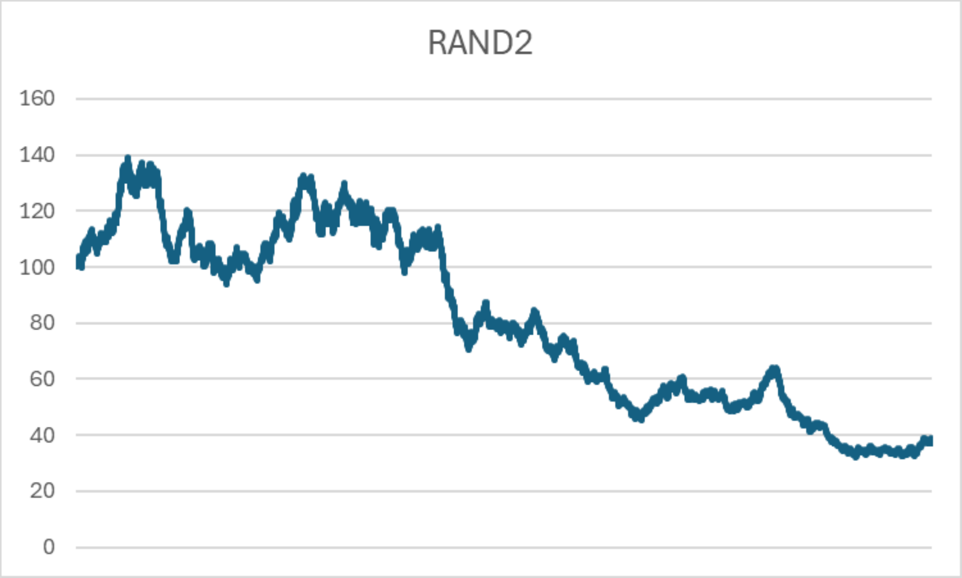


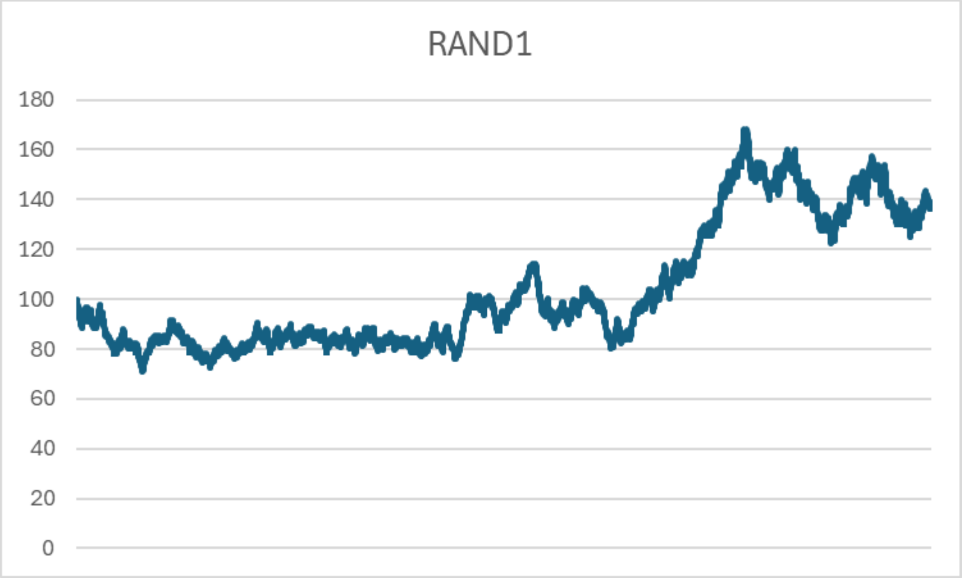
It does not mean we can not model the stock prices. It does mean we need to model equities` prices in **probabilistic sense**. No doubt the reality of the situation lies somewhere between complete predictability and perfect randomness, not least because there have been many cases of market manipulation where large trades have moved stock prices in a direction that was favorable to the person doing the moving.

Financial Instruments - Equity

We will go to proper mathematical modeling later. Nevertheless, the simplest way to simulate a random walk that looks something like a stock price is one of the simplest random processes **- tossing of a coin**.

Let`s assume initial price of stock (S_0) equal to \$100. If you throw a head multiply the number by 1.01, if you throw a tail multiply by 0.99. Continue this process and plot your value on a graph each time you throw the coin. Results of two particular experiments are shown below.





Financial Instruments – Fixed Income

- A fixed-income asset represents a debt for a specific amount (the **principal**), whereby the issuer is obliged to repay the holder at some future date (the **maturity date**).

- Debt can be issued by **governments**, **companies** and **agencies**.

- Fixed-Income securities (generally referred as **Bonds**) are traded in the markets (mostly **OTC**).

- Bondholders have **capital rights.**

- Investors can profit from buying a bond via **capital gain** and (possible) periodic payments known as **coupon.**

Define:

Zero-Coupon Bond: a bond that does not pay coupons but only the principal (at maturity).

Coupon Bond: a bond that does pay coupons and the principal (at maturity).

Financial Instruments – Fixed Income

The **price of a bond** is **the present value** of future **coupons and principal payment** (also called face value).

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{M}{(1+r)^T}$$

P_0 is today's price of the bond,

C_t is the coupon payment at time t,

M is the principal/face value paid at time T,

r is the interest rate.

The price and coupons of a bond are expressed as a percentage of its face value.

Coupon rates are expressed in annual terms.

Accrued Interest:

When an investor purchases a bond between coupon payments, the investor must compensate the seller of the bond for the coupon interest earned from the time of the last coupon payment to the settlement date of the bond.

Dirty Price:

The amount the buyer pays the seller is the agreed price plus accrued interest.

Clean Price:

The price of a bond without accrued. Bonds are generally quoted in terms of clean price.

Financial Instruments – Fixed Income – Pricing

Zero-Coupon Bond Pricing

The price of a ZCB (zero-coupon bond) is the present value of the principal repaid at maturity.

1. Consider a ZCB that pays $100 in 3 years. Compute the price of the bond assuming an annual interest rate of 5%.

$$P_0 = ?$$

Coupon Bond Pricing

1. Consider a 3-year bond that pays every six months a coupon of 2%. What is the price of the bond using a semi-annual interest rate of 5% and a face value of $ 100?

$$P_0 = ?$$

2. Consider a 3 year bond that pays 2% coupon annually. The interest rates are 3%, 5%, and 10% for the first, second and third year respectively. What is the price of the bond?

$$P_0 = ?$$

Financial Instruments – Fixed Income – Pricing

Zero-Coupon Bond Pricing

The price of a ZCB (zero-coupon bond) is the present value of the principal repaid at maturity.

1. Consider a ZCB that pays $100 in 3 years. Compute the price of the bond assuming an annual interest rate of 5%.

$$P_0 = \frac{\$100}{(1+0.05)^3} = \$86.38$$

Coupon Bond Pricing

1. Consider a 3 year bond that pays every six months a coupon of 2%. What is the price of the bond using a semi-annual interest rate of 5% and a face value of $ 100?

$$P_0 = \frac{100*0.01}{(1+0.025)^1} + \frac{100*0.01}{(1+0.025)^2} + \frac{100*0.01}{(1+0.025)^3} + \frac{100*0.01}{(1+0.025)^4} + \frac{100*0.01}{(1+0.025)^5} + \frac{100+100*0.01}{(1+0.025)^6} = \$91.74$$

2. Consider a 3 year bond that pays 2% coupon annually. The interest rates are 3%, 5%, and 10% for the first, second and third year respectively. What is the price of the bond?

$$P_0 = \frac{2}{(1+0.03)^1} + \frac{2}{(1+0.05)^2} + \frac{102}{(1+0.1)^3} = \$80.39$$

Financial Instruments – Fixed Income – Pricing

Dirty vs Clean Price

Consider the following bond:

Face Value = $100

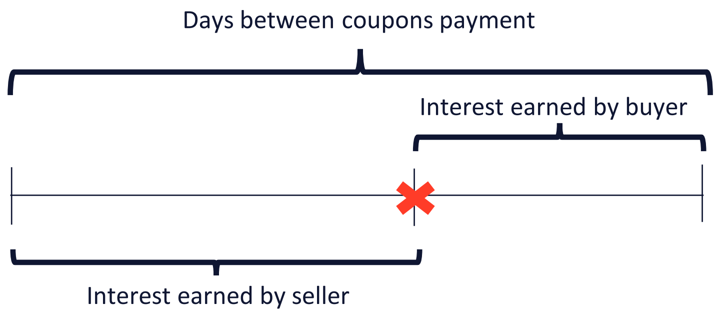
Interest Rate = 8% annually compounded

Coupon Rate = 10%

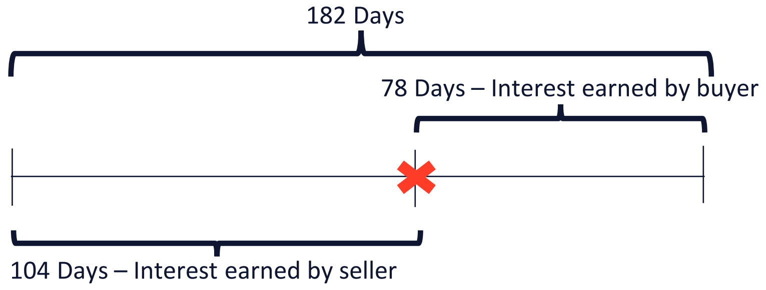
Coupon Frequency = semi-annual (182 days)

Residual Maturity = 2.5 years

Assuming that *Cash Cow Chuckles Llc.* wants to buy the bond between the two coupon payment, compute the accrued interest, dirty and clean price.



Financial Instruments – Fixed Income – Pricing



$$a)\ w\ periods = \frac{days\ left\ before\ next\ coupon\ payment}{days\ in\ a\ coupon\ period}$$

$$b)\ Present\ Value = \frac{Expected\ Cashflow}{(1+r)^{t-1+w}}$$

- Compute the Dirty Price:

$$P_0 = ?$$

- Compute the Accrued Interest:

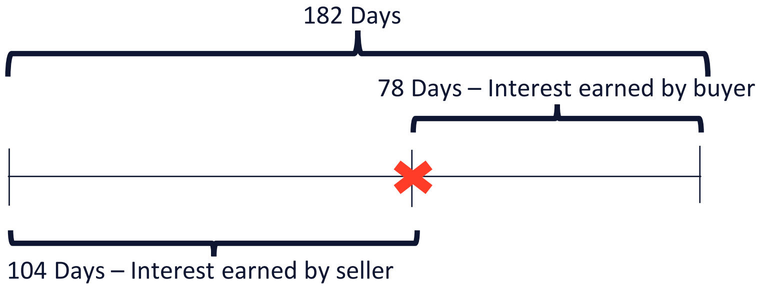
$$Days\ in\ accrued\ interest\ period = days\ in\ coupon\ period\ - days\ left\ before\ next\ coupon\ payment$$

$$Accrued\ Interest = Coupon\ * (1 - w)$$

- Compute the Clean Price:

$$Clean\ price = Dirty\ price\ - Accrued\ Interest$$

Financial Instruments – Fixed Income – Pricing



$$a)\ w\ periods = \frac{days\ left\ before\ next\ coupon\ payment}{days\ in\ a\ coupon\ period} = \frac{78}{182} = 0.4286$$

$$b)\ Present\ Value = \frac{Expected\ Cashflow}{(1+r)^{t-1+w}}$$

- Compute the Dirty Price:

$$P_0 = \frac{5}{(1+0.04)^{0.4286}} + \frac{5}{(1+0.04)^{1.4286}} + \frac{5}{(1+0.04)^{2.4286}} + \frac{5}{(1+0.04)^{3.4286}} + \frac{105}{(1+0.04)^{4.4286}} = \$106.819$$

- Compute the Accrued Interest:

$$Days\ in\ accrued\ interest\ period = days\ in\ coupon\ period\ - days\ left\ before\ next\ coupon\ payment = 182 - 78 = 104$$

$$Accrued\ Interest = Coupon\ * (1 - w) = \$5\ * (1 - 0.4286) = \$2.8570$$

- Compute the Clean Price:

$$Clean\ price = Dirty\ price\ - Accrued\ Interest = \$106.8192 - \$2.8570 = \$103.9622$$

Financial Instruments – Fixed Income – Pricing

Yield-to-Maturity (YTM)

The yield to maturity is the interest rate that will make the present value of a bond's cash flows equal to its market price plus accrued interest.

To find the yield to maturity, we first determine the expected cash flows and then search, by trial and error, for the interest rate that will make the present value of cash flows equal to the market price plus accrued interest.

Consider the following bond: $P_0 = \$98.50, Face\ Value = \$100, Coupon\ Rate = 10\%, Coupon\ Freq. = Semi - Annual, Maturity = 3\ years.$

Compute the YTM (y):

$$P_0 = \frac{5}{(1+y)^1} + \frac{5}{(1+y)^2} + \frac{5}{(1+y)^3} + \frac{5}{(1+y)^4} + \frac{5}{(1+y)^5} + \frac{105}{(1+y)^6} = \$98.50$$

The semi-annual YTM is equal to ?

Assumptions/Limitations of Yield-to-Maturity

1. Reinvestment risk: the coupon payments can be reinvested at the yield to maturity.
2. Interest rate risk: the bond is held to maturity.

Financial Instruments – Fixed Income – Pricing

Yield-to-Maturity (YTM)

The yield to maturity is the interest rate that will make the present value of a bond's cash flows equal to its market price plus accrued interest.

To find the yield to maturity, we first determine the expected cash flows and then search, by trial and error, for the interest rate that will make the present value of cash flows equal to the market price plus accrued interest.

Consider the following bond: $P_0 = \$98.50, Face\ Value = \$100, Coupon\ Rate = 10\%, Coupon\ Freq. = Semi - Annual, Maturity = 3\ years.$

Compute the YTM (y):

$$P_0 = \frac{5}{(1+y)^1} + \frac{5}{(1+y)^2} + \frac{5}{(1+y)^3} + \frac{5}{(1+y)^4} + \frac{5}{(1+y)^5} + \frac{105}{(1+y)^6} = \$98.50$$

The semi-annual YTM is equal to 5.298%. To express it in annual terms,

the convention used is to multiply it by the frequency,

in our example by 2. Thus, the YTM is 10.6%.

		Discounted Cashflows		
Time	**Cashflows**	**r=3%**	**r=5%**	**r=5.29835188%**
1	5	4.85	4.76	4.75
2	5	4.71	4.54	4.51
3	5	4.58	4.32	4.28
4	5	4.44	4.11	4.07
5	5	4.31	3.92	3.86
6	105	87.94	78.35	77.03
Bond Price		**110.83**	**100**	**98.5**

Remember! As the yield goes up, price goes down.

Assumptions/Limitations of Yield-to-Maturity

1. Reinvestment risk: the coupon payments can be reinvested at the yield to maturity.
2. Interest rate risk: the bond is held to maturity.

Financial Instruments – Fixed Income – Pricing

Par Yield (Par)

The par yield for a certain bond maturity is the coupon rate that causes the bond price to equal its par value. The par value is basicaly the principal value of the bond.

Consider the following bond: $Face\ Value = \$100,\ Coupon\ Rate = 15\%,\ Coupon\ Freq. = Semi - Annual,\ Maturity = 2\ years.$

To compute the Par Yield (y) we need to solve below equation:

$$Par\ Value = \sum_{i=1}^{n} \frac{c_i}{(1+y)^i}$$

Where, the bond have n cash flows (c_i) paid at time t_i, and yield of y.

Financial Instruments – Fixed Income – Pricing

Par Yield (Par)

The par yield for a certain bond maturity is the coupon rate that causes the bond price to equal its par value. The par value is basicaly the principal value of the bond.

Consider the following bond: $Face\ Value = \$100, Coupon\ Rate = 15\%, Coupon\ Freq. = Semi - Annual, Maturity = 2\ years.$

To compute the Par Yield (y) we need to solve below equation:

$$Par\ Value = \frac{7.5}{(1+y)^1} + \frac{7.5}{(1+y)^2} + \frac{7.5}{(1+y)^3} + \frac{107.5}{(1+y)^4} = \$100$$

The par yield is equal to 7.5%. So if YTM of this bond would be equal to 7.5%, it would be ***traded at Par***.

Duration

The duration (or Macaulay duration) of a bond, as its name implies, is a measure of how long on average the holder of the bond has to wait before receiving cash payments. A zero-coupon bond that lasts n years has a duration of n years. However, a coupon-bearing bond lasting *n* years has a duration of less than *n* years, because the holder receives some of the cash payments prior to year n.

Consider the bond with n cash flows (c_i) paid at time t_i, price P_0 and yield of y. *We know the price of the bond, assuming continuous compounding, is equal to:*

$$P_0 = \sum_{i=1}^{n} c_i \, e^{-y t_i}$$

Then the duration is defined as:

$$D = \frac{\sum_{i=1}^{n} t_i c_i \, e^{-y t_i}}{P_0} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y t_i}}{P_0} \right]$$

The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i. The sum of the weights is 1. Note that for the purposes of the definition of duration all discounting is done at the bond yield rate of interest, y.

Financial Instruments – Fixed Income – Duration

Duration

Consider the following bond: $Face\ Value = \$100$, $Coupon\ Rate = 10\%$, $Coupon\ Freq. = Semi - Annual$, $Maturity = 3\ years$, $yield = 12\%$ per annum with continuous compounding.

Compute the duration of this bond.

$$D = ?$$

Duration

Consider the following bond: $Face\ Value = \$100$, $Coupon\ Rate = 10\%$, $Coupon\ Freq. = Semi - Annual$, $Maturity = 3\ years$, $yield = 12\%$ per annum with continuous compounding.

Compute the duration of this bond.

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y t_i}}{P_0}\right] = 0.5\frac{5 * e^{-0.12*0.5}}{94.213} + 1\frac{5 * e^{-0.12*1}}{94.213} + 1.5\frac{5 * e^{-0.12*1.5}}{94.213} + 2\frac{5 * e^{-0.12*2}}{94.213} + 2.5\frac{5 * e^{-0.12*2.5}}{94.213} + 3\frac{105 * e^{-0.12*3}}{94.213} = 2.653$$

Time (years)	Cash Flow ($)	Present Value	Weight	Time x Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total:	130	94.213	1.000	**2.653**

Financial Instruments – Fixed Income – Duration

Duration Usage

When a small change Δy in the yield is considered, it is approximately true that:

$$\Delta P_0 = \frac{dP_0}{dy}\Delta y$$

From the previous slide we know:

$$P_0 = \sum_{i=1}^{n} c_i\, e^{-yt_i}$$

And so:

$$\Delta P_0 = \frac{d(\sum_{i=1}^{n} c_i\, e^{-yt_i})}{dy}\Delta y = \Delta y \sum_{i=1}^{n} -t_i c_i\, e^{-yt_i} = -\Delta y \sum_{i=1}^{n} t_i c_i\, e^{-yt_i} = -\Delta y \sum_{i=1}^{n} t_i c_i\, e^{-yt_i}\frac{P_0}{P_0} = -\Delta y \sum_{i=1}^{n} \frac{t_i c_i e^{-yt_i}}{P_0} P_0$$

Having that, we can conclude

$$\Delta P_0 = -D\Delta y P_0$$

or

$$\frac{\Delta P_0}{P_0} = -D\Delta y$$

Financial Instruments – Fixed Income – Duration

Duration Usage

Relationship we derived on the previous slide: $\frac{\Delta P_0}{P_0} = -D\Delta y$ can be modified to so cold Dolar Duration.

$$\frac{\Delta P_0}{P_0} = -D\Delta y \Leftrightarrow -\frac{\Delta P_0}{\Delta y} = \underbrace{DP_0}_{\textbf{Dollar Duration}}$$

Dollar duration can be used for a 1st order approximation (in the Taylor expansion) of bond price change w.r.t yield change. For a more precise approximation we have to take account of convexity, i.e., go to a 2nd order.

Bond price sensitivity w.r.t 1 bps (0.01%) yield change is called DV01.

Financial Instruments – Fixed Income – Duration

Duration Usage

Consider the bond we used before: $P_0 = \$94.213, Face\ Value = \$100,\ Coupon\ Rate = 10\%,\ Coupon\ Freq. = Semi - Annual,\ Maturity = 3\ years,\ yield = 12\%$ per annum with continuous compounding and duration $= 2.653$.

1. What is new approximated bond price if the yield on the bond increases by 10 basis points (0.1%)?

$$\Delta_1 P_0 = ?$$

$$P_{\Delta y_1} = ?$$

2. What is new approximated bond price if the yield on the bond increases by 220 basis points (2.2%)?

$$\Delta_2 P_0 = ?$$

$$P_{\Delta y_2} = ?$$

3. What is new approximated bond price if the yield on the bond increases by 1220 basis points (12.2%)?

$$\Delta_3 P_0 = ?$$

$$P_{\Delta y_3} = ?$$

Financial Instruments – Fixed Income – Duration

Duration Usage

Consider the bond we used before: $P_0 = \$94.213, Face\ Value = \$100,\ Coupon\ Rate = 10\%,\ Coupon\ Freq. = Semi-Annual,\ Maturity = 3\ years,\ yield = 12\%$ per annum with continuous compounding and duration $= 2.653$.

1. What is new approximated bond price if the yield on the bond increases by 10 basis points (0.1%)?

$$\Delta_1 P_0 = -P_0 D\Delta y = -94.213 * 2.653 * 0.001 = -0.250$$

$$P_{\Delta y_1} = P_0 + \Delta P_0 = 94.213 - 0.250 = 93.963$$

2. What is new approximated bond price if the yield on the bond increases by 220 basis points (2.2%)?

$$\Delta_2 P_0 = -P_0 D\Delta y = -94.213 * 2.653 * 0.022 = -5.499$$

$$P_{\Delta y_2} = P_0 + \Delta P_0 = 94.213 - 5.499 = 88.714$$

3. What is new approximated bond price if the yield on the bond increases by 1220 basis points (12.2%)?

$$\Delta_3 P_0 = -P_0 D\Delta y = -94.213 * 2.653 * 0.122 = -30.494$$

$$P_{\Delta y_3} = P_0 + \Delta P_0 = 94.213 - 30.494 = 63.719$$

Financial Instruments – Fixed Income – Duration

Duration Usage

Consider the bond we used before: $P_0 = \$94.213, Face\ Value = \$100,\ Coupon\ Rate = 10\%,\ Coupon\ Freq. = Semi - Annual,\ Maturity = 3\ years,\ yield = 12\%$ per annum with continuous compounding and duration $= 2.653$.

4. What are the errors of approximations?

$$abs.error_1 = P_1 - P_{\Delta y_1} = 93.963 - 93.963 = 0$$

$$rel.error_1 = \frac{P_1 - P_{\Delta y_1}}{P_1} = \frac{93.963 - 93.963}{93.963} = 0\%$$

$$abs.error_2 = P_2 - P_{\Delta y_2} = 88.883 - 88.714 = 0.169$$

$$rel.error_2 = \frac{P_2 - P_{\Delta y_2}}{P_2} = \frac{88.883 - 88.714}{88.883} = 0.19\%$$

$$abs.error_3 = P_3 - P_{\Delta y_3} = 68.449 - 63.719 = 4.730$$

$$rel.error_3 = \frac{P_3 - P_{\Delta y_3}}{P_3} = \frac{68.449 - 88.714}{68.449} = 6.91\%$$

Financial Instruments – Fixed Income – Modified Duration

Modified Duration

Macaulay duration is based on the assumption that y is expressed with continuous compounding. If y is expressed with m times per year compounding, it can be shown that change in bond price follows approximatelly below equation:

$$\Delta P_0 = \frac{-P_0 D \Delta y_c}{1 + \frac{y_m}{m}}$$

Thus, we can define so called $Modified\ Duration\ (MD)$, relaxing continous compounding assumption. MD is specified as follows:

$$MD = \frac{D}{1 + \frac{y_m}{m}}$$

It allows the duration relationship to be simplified to:

$$\Delta P_0 = -P_0 MD \Delta y$$

Financial Instruments – Fixed Income – Modified Duration

Modified Duration Usage

Consider our favorite bond: $P_0 = \$94.213, Face\ Value = \$100,\ Coupon\ Rate = 10\%,\ Coupon\ Freq. = Semi - Annual,\ Maturity = 3\ years,$ $yield = 12\%$ per annum with continuous compounding and duration $= 2.653$.

1. What is *Modified Duration* of such bond? The yield of the bond, expressed with semiannual compounding is 12.3673%

$$MD = ?$$

2. You are an analyst at *Diversify or Die* hedge fund. You were asked to preapre one of the risk sensitivity, Modified Duration, for a new Profit Approximation tool for fixed income trading desk. You want to first calculate it on an example bond with the following features: a 3-year bond that pays a coupon of 10% and a face value of $100. The interest rates are 3%, 5%, and 10% for the first, second and third year respectively.*

$$DV01 = ?$$

***hint:** The formula for converting a periodically compounded rate to a continuously compounded rate is $r_c = m * ln\left(1 + \frac{r_{freq}}{m}\right)$

The formula for converting a continuously compounded rate to a periodically compounded rate is $r_{freq} = m\left(e^{\frac{r_c}{m}} - 1\right)$

Financial Instruments – Fixed Income – Modified Duration

Modified Duration Usage

Consider our favorite bond: $P_0 = \$94.213, Face\ Value = \$100,\ Coupon\ Rate = 10\%,\ Coupon\ Freq. = Semi - Annual,\ Maturity = 3\ years,$ $yield = 12\%$ per annum with continuous compounding and duration $= 2.653.$

1. What is *Modified Duration* of such bond? The yield of the bond, expressed with semiannual compounding is 12.3673%

$$MD = \frac{D}{1 + \frac{y_m}{m}} = \frac{2.653}{1 + \frac{12.3673}{2}} = 2.498501417$$

Financial Instruments – Fixed Income – Modified Duration

Modified Duration Usage

2. You are an analyst at *Diversify or Die* hedge fund. You were asked to preapre one of the risk sensitivity for a new Profit Approximation tool for fixed income trading desk. The sensitivity should be based on annually coumponued yield. You want to first calculate your sensitivity based on an example bond with the following features: a 3-year bond that pays a coupon of 10% annually and a face value of $100. The spot interest rates are 3%, 5%, and 10% for the first, second and third year respectively.*

$$P_0 = \frac{10}{(1+0.03)^1} + \frac{10}{(1+0.05)^2} + \frac{110}{(1+0.1)^3} = \$101.4237$$

$$\frac{10}{(1+ytm_m)^1} + \frac{10}{(1+ytm_m)^2} + \frac{110}{(1+ytm_m)^3} = \$101.4237 \Rightarrow ytm_m \approx 9.4332\%$$

Our ytm is annually coumpanded. So we need to convert to continously coumpaunded.

$$ytm_c = m * ln\left(1 + \frac{ytm_{annual}}{m}\right) = 1 * ln\left(1 + \frac{0.094332}{1}\right) = 0.090144$$

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-ytm_c t_i}}{P_0}\right] = 1\frac{10 * e^{-0.090144*1}}{101.4237} + 2\frac{10 * e^{-0.090144*2}}{101.4237} + 3\frac{110 * e^{-0.090144*3}}{101.4237} = 2.737477$$

$$MD = \frac{D}{1 + \frac{y_m}{m}} = \frac{2.737477}{1 + \frac{0.094332}{1}} = 2.502 \Rightarrow Dollar\ Duration = MD * P_0 = 101.4237 * 2.502 = 253.7118 \Rightarrow DV01 = \frac{253.7118}{10{,}000} = 0.02537118$$

Financial Instruments – FRA

Forward Rates

Forward rate is an interest rate on which money can be lent from one future period to another, implied by the current zero-coupon (spot) rates.

Let`s consider following simple term structure for continously compounded interest rates, the 3% per annum rate for 1 year and the 4% per annum rate for 2 years. It means that a $100 invested today for a year is worth:

$$100 * e^{0.03*1} = 103.05$$

and a $100 invested today for two years is worth:

$$100 * e^{0.04*2} = 108.33$$

What then should be an one-year intrest rate in a year (forward rate)? In ideal world it should make no diffrence if we invest our $100 for two years or for a year and then reinvest for another year. Thus,

$$100 * e^{0.03*1} * e^{fwd_{1,1}*1} = 100 * e^{0.04*2}$$

or

$$103.05 * e^{fwd_{1,1}} = 108.33$$

Financial Instruments – FRA

Forward Rates

What then should be an one-year intrest rate in a year (forward rate)? In ideal world it should make no diffrence if we invest our $100 for two years or for a year and then reinvest for another year. Thus,

$$100 * e^{0.03*1} * e^{fwd_{1,1}*1} = 100 * e^{0.04*2}$$

or

$$103.05 * e^{fwd_{1,1}} = 108.33$$

thus

$$e^{fwd_{1,1}} = \frac{108.33}{103.05} = 1.051237$$

taking logharitm

$$ln\left(e^{fwd_{1,1}}\right) = ln(1.051237)$$

so

$$fwd_{1,1} \approx 0.05$$

Our one-year in a year intrest rate is approximetally equal to 5%.

Financial Instruments – FRA

Forward Rates

In general, if R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively, and R_{T_1,T_2} is the forward interest rate for the period of time between T_1 and T_2, then

$$R_{T_1,T_2} = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}$$

or

$$R_{T_1,T_2} = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1}$$

1. Consider a following term structure. The interest rates are 3%, 5%, and 10% for the first, second and third year respectively. What is a one-year forward rate in two years?

$$R_{T_1,T_2} = ?$$

Financial Instruments – FRA

Forward Rates

In general, if R_1 and R_2 are the continously zero rates for maturities T_1 and T_2, respectively, and R_{T_1,T_2} is the forward interest rate for the period of time

between T_1 and T_2, then

$$R_{T_1,T_2} = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}$$

or

$$R_{T_1,T_2} = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1}$$

1. Consider a following term structure. The interest rates are 3%, 5%, and 10% for the first, second and third year respectively. What is a one-year forward rate in two years?

$$R_{2,3} = 0.1 + (0.1 - 0.05)\frac{2}{3-2} = 0.2$$

Sanity check: $100e^{0.1*3} = 134.9859$ $100e^{0.05*2}e^{0.2*1} = 134.9859$

Financial Instruments – FRA

Forward Rate Agreement

A forward rate agreement (FRA) is an over-the-counter agreement designed to ensure that a certain interest rate will apply to either borrowing or lending a certain principal during a specified future period of time. The assumption underlying the contract is that the borrowing or lending would normally be done at given RFR (Risk Free Rate) e.g. SONIA (Sterling Overnight Index Average).

Note, RFRs replaced IBORs, e.g. SONIA (UK) and SOFR (US [Secured Overnight Financing Rate]) replaced LIBOR.

Consider a forward rate agreement where company X is agreeing to lend money to company Y for the period of time between T_1 and T_2. Define:

R_k- The rate of interest agreed to in the FRA

R_f- The forward RFR interest rate for the period between times T_1 and T_2, calculated today

R_m- The actual RFR interest rate observed in the market at time T_1 for the period between times T_1 and T_2

L- The principal underlying the contract.

We will depart from standard assumption of continuous compounding and assume that the rates R_k, R_f , and R_m are all measured with a compounding frequency reflecting the length of the period to which they apply. This means that if $T_2 - T_1 = 0.5$, they are expressed with semi-annual compounding; if $T_2 - T_1 = 0.25$, they are expressed with quarterly compounding; and so on. (This assumption corresponds to the usual market practice for FRAs.)

Financial Instruments – FRA

Forward Rate Agreement

Consider a forward rate agreement where company X is agreeing to lend money to company Y for the period of time between T_1 and T_2. Define:

R_k- The rate of interest agreed to in the FRA

R_f- The forward RFR interest rate for the period between times T_1 and T_2, calculated today

R_m- The actual RFR interest rate observed in the market at time T_1 for the period between times T_1 and T_2

L - The principal underlying the contract.

Normally company X would earn R_m from the RFR loan. The FRA means that it will earn R_k. The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $R_k - R_m$. The interest rate is set at time T_1 and paid at time T_2. The extra interest rate therefore leads to a cash flow to company X at time T_2 of

$$L(R_k - R_m)(T_2 - T_1)$$

and for Y

$$L(R_m - R_k)(T_2 - T_1)$$

From above, we see that there is another interpretation of the FRA. It is an agreement where company X will receive interest on the principal between T_2 and T_1 at the fixed rate of R_k and pay interest at the realized RFR rate of R_m. Company Y will pay interest on the principal between T_2 and T_1 at the fixed rate of R_k and receive interest at R_m.

Financial Instruments – FRA

Forward Rate Agreement

Usually, FRAs are settled at time T_1 rather than T_2 . So, the payoff must then be discounted from time T_2 to T_1.

$$\frac{L(R_k - R_m)(T_2 - T_1)}{1 + R_m(T_2 - T_1)}$$

and

$$\frac{L(R_m - R_k)(T_2 - T_1)}{1 + R_m(T_2 - T_1)}$$

Suppose that a *Cashflow Chaos Bank* enters into an FRA that is designed to ensure it will receive a fixed rate of 4% on a principal of $100 million for a 3-month period starting in 3 years. The FRA is an exchange where RFR is paid and 4% is received for the 3-month period. If 3-month RFR proves to be 4.5% for the 3-month period the cash flow to the lender will be?

$$CF_L = ?$$

Financial Instruments – FRA

Forward Rate Agreement

Usually, FRAs are settled at time T_1 rather than T_2 . So, the payoff must then be discounted from time T_2 to T_1.

$$\frac{L(R_k - R_m)(T_2 - T_1)}{1 + R_m(T_2 - T_1)}$$

and

$$\frac{L(R_m - R_k)(T_2 - T_1)}{1 + R_m(T_2 - T_1)}$$

Suppose that a *Cashflow Chaos Bank* enters into an FRA that is designed to ensure it will receive a fixed rate of 4% on a principal of $100 million for a 3-month period starting in 3 years. The FRA is an exchange where RFR is paid and 4% is received for the 3-month period. If 3-month RFR proves to be 4.5% for the 3-month period the cash flow to the lender will be?

$$CF_{L,3.25} = L(R_k - R_m)(T_2 - T_1) = 100{,}000{,}000(0.04 - 0.045)(3.25 - 3) = -125{,}000$$

which means

$$CF_{L,3} = \frac{CF_{L,3.25}}{1 + R_m(T_2 - T_1)} = \frac{-125{,}000}{1 + 0.045(3.25 - 3)} = -123{,}609$$

Note that cash flow received by the counterpart will be exactly opposite so +123,609.

Financial Instruments – FRA

Forward Rate Agreement Valuation

To value an FRA we first note that it is always worth zero when R_k= R_f, which is usually the case when the FRA is first initiated.

Compare two FRAs. The first promises that the RFR forward rate R_f will be received on a principal of L between times T_1 and T_2; the second promises that R_k will be received on the same principal between the same two dates. The two contracts are the same except for the interest payments received at time T_2. The excess of the value of the second contract over the first is, therefore, the present value of the difference between these interest payments, or for FRA, where R_k is received

$$P_0 = L\left(R_k - R_f\right)(T_2 - T_1)e^{-R_2 T_2}$$

where R_2 is the continuously compounded riskfree spot rate for a maturity T_2.

Similarly, the value of an FRA where R_k is paid is

$$P_0 = L\left(R_f - R_k\right)(T_2 - T_1)e^{-R_2 T_2}$$

Financial Instruments – FRA

Forward Rate Agreement Valuation

You are a trader at *Bull Market Busters.* You just received a proposition to enter a following Forward Rate Agreement where your company will receive a rate of 6%, measured with annual compounding, and pay RFR on a principal of $100 million between the end of year 1 and the end of year 2. In this case, the forward rate is 5% with continuous compounding or 5.127% with annual compounding. Your counterpart wants you to pay $835,000 to enter the contract. Should you accept the offer?

$$P_0 = ?$$

Financial Instruments – FRA

Forward Rate Agreement Valuation

You are a trader at *Bull Market Busters.* You just received a proposition to enter a following Forward Rate Agreement where your company will receive a rate of 6%, measured with annual compounding, and pay RFR on a principal of $100 million between the end of year 1 and the end of year 2. In this case, the forward rate is 5% with continuous compounding or 5.127% with annual compounding. Your counterpart wants you to pay $835,000 to enter the contract. Should you accept the offer?

$$P_0 = L\left(R_k - R_f\right)(T_2 - T_1)e^{-R_2 T_2}$$

$$P_0 = 100{,}000{,}000(0.06 - 0.05127)(2 - 1)e^{-0.04*2}$$

$$P_0 = 805{,}800$$

Since the proposed price is higher then present value

$$P_0 = 805{,}800 < 835{,}000$$

The proposed FRA is overpriced by the counterpart, thus you should not enter the trade.

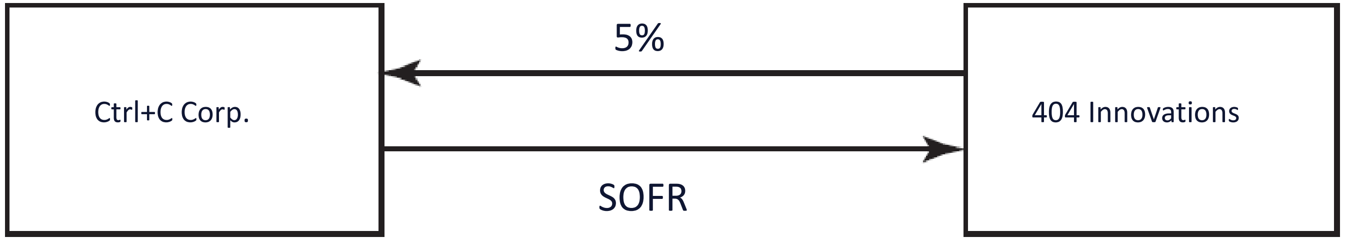
Financial Instruments – SWAP

SWAP

A swap is an over-the-counter agreement between two companies to exchange cash flows in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated. Usually, the calculation of the cash flows involves the future value of an interest rate, an exchange rate, or other market variable.

The most common type of swap is a ‘‘plain vanilla’’ interest rate swap (IRS). In this swap a company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a predetermined number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time.

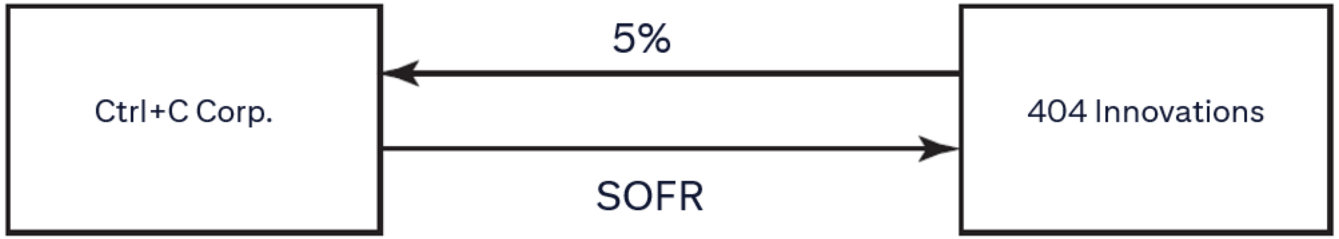
Consider a hypothetical 3-year swap initiated on March 5, 2024, between *404 Innovations* and *Ctrl+C Corp*. We suppose *404 Innovations* agrees to pay *Ctrl+C Corp.* an interest rate of 5% per annum on a principal of $100 million, and in return *Ctrl+C Corp.* agrees to pay *404 Innovations* the 6-month SOFR rate on the same principal. *404 Innovations* is the fixed-rate payer; *Ctrl+C Corp.* is the floating rate payer. We assume the agreement specifies that payments are to be exchanged every 6 months and that the 5% interest rate is quoted with semi-annual compounding.



Financial Instruments – SWAP

SWAP

Consider a hypothetical 3-year swap initiated on March 5, 2024, between two tech companies *404 Innovations* and *Ctrl+C Corp*. We suppose *404 Innovations* agrees to pay *Ctrl+C Corp.* an interest rate of 5% per annum on a principal of $100 million, and in return *Ctrl+C Corp.* agrees to pay *404 Innovations* the 6-month SOFR rate on the same principal. *404 Innovations* is the fixed-rate payer; *Ctrl+C Corp.* is the floating rate payer. We assume the agreement specifies that payments are to be exchanged every 6 months and that the 5% interest rate is quoted with semi-annual compounding.



So cash flows for *404 Innovations* are as follow:

$$\sum_{i=0}^{n} LR_{k,i}(T_{i+1} - T_i) - LR_{m,i}(T_{i+1} - T_i) = \sum_{i=0}^{n} L\left(R_{k,i} - R_{m,i}\right)(T_{i+1} - T_i)$$

and for *Ctrl+C Corp:*

$$\sum_{i=0}^{n} LR_{m,i}(T_{i+1} - T_i) - LR_{k,i}(T_{i+1} - T_i) = \sum_{i=0}^{n} L\left(R_{m,i} - R_{k,i}\right)(T_{i+1} - T_i)$$

Date	Six-month SOFR	Floating cash flow paid	Fixed cash flow paid	Net cash flow
05/03/2024	4.2			
05/09/2024	4.8	2,100,000	- 2,500,000	- 400,000
05/03/2025	5.3	2,400,000	- 2,500,000	- 100,000
05/09/2025	5.5	2,650,000	- 2,500,000	150,000
05/03/2026	5.6	2,750,000	- 2,500,000	250,000
05/09/2026	5.9	2,800,000	- 2,500,000	300,000
05/03/2027		2,950,000	- 2,500,000	450,000
Sum		15,650,000	-15,000,000	650,000

Financial Instruments – SWAP

SWAP valuation

Now, hom much swap should cost? An interest rate swap is worth close to zero when it is first initiated. After it has been in existence for some time, its value may be positive or negative. There are two valuation approaches. The first regards the swap as the difference between two bonds; the second regards it as a portfolio of FRAs.

For fix leg payer the value is equal to:

$$V_{swap,0} = B_{fix} - B_{flt}$$

or

$$V_{swap,0} = \sum_{i=0}^{n} L\left(R_{f,t_i} - R_{k,t_i}\right) T_{freq}\, e^{-r(T_{i+1} - T_0)}$$

While for float leg payer the value is equal to:

$$V_{swap,0} = B_{flt} - B_{fix}$$

or

$$V_{swap,0} = \sum_{i=0}^{n} L\left(R_{k,t_i} - R_{f,t_i}\right) T_{freq}\, e^{-r(T_{i+1} - T_0)}$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 12-month SONIA and receive 12% per annum (with annual compounding) on a notional principal of $100 million. The swap has a remaining life of 2 years. The SONIA rates with continuous compounding for a year and two-years maturities are 9.5% and 13%, respectively. What is the value of such SWAP using bond replication method?

$$V_{swap,0} = ?$$

2. Now what is the value of such SWAP using FRA replication method?

$$V_{swap,0} = ?$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 12-month SONIA and receive 12% per annum (with annual compounding) on a notional principal of $100 million. The swap has a remaining life of 2 years. The SONIA rates with continuous compounding for a year and two-years maturities are 9.5% and 13%, respectively. What is the value of such SWAP using bond replication method?

$$B_{fix} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rT} = 12{,}000{,}000 * e^{-0.095*1} + 112{,}000{,}000 * e^{-0.13*2} = 97{,}270{,}252.82$$

Because we know floating pays SONIA and we are at coupon payment date so:

$$B_{flt} = 100{,}000{,}000$$

Or we can calculate it manualy: $B_{flt} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rT}$ *But we do not know annual compounding rates and forward rate for a year in a year, so:*

First let`s calculate one-year rate with annual compounding: $R_m = m\left(e^{R_C/m} - 1\right) = 1\left(e^{0.095/1} - 1\right) = 0.0996588551$

Now we need to calculate one-year in a year rate: $R_{12M,24M} = \frac{R_{24M}T_{24M} - R_{12M}T_{12M}}{T_{24M} - T_{12M}} = \frac{0.13*2 - 0.095*1}{2-1} = 0.165$

And convert it into annual computing: $R_m = m\left(e^{R_C/m} - 1\right) = 1\left(e^{0.165/1} - 1\right) = 0.1793931187$

$$B_{flt} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rt} = 9{,}965{,}885.51e^{-0.095*1} + 117{,}939{,}311.87e^{-0.13*2} = 100{,}000{,}000$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 12-month SONIA and receive 12% per annum (with annual compounding) on a notional principal of $100 million. The swap has a remaining life of 2 years. The SONIA rates with continuous compounding for a year and two-years maturities are 9.5% and 13%, respectively. What is the value of such SWAP using bond replication method?

$$B_{fix} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rT} = 12{,}000{,}000 * e^{-0.095*1} + 112{,}000{,}000 * e^{-0.13*2} = 97{,}270{,}252.82$$

Because we know floating pays SONIA and we are at coupon payment date so:

$$B_{flt} = 100{,}000{,}000$$

$$V_{swap,0} = B_{flt} - B_{fix} = 100{,}000{,}000 - 97{,}270{,}252.82 = 2{,}729{,}747.18$$

2. Now what is the value of such SWAP using FRA replication method?

$$V_{swap,0} = ?$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 12-month SONIA and receive 12% per annum (with annual compounding) on a notional principal of $100 million. The swap has a remaining life of 2 years. The SONIA rates with continuous compounding for a year and two-years maturities are 9.5% and 13%, respectively. What is the value of such SWAP using bond replication method?

$$V_{swap,0} = B_{flt} - B_{fix} = 100{,}000{,}000 - 97{,}270{,}252.82 = 2{,}729{,}747.18$$

2. Now what is the value of such SWAP using FRA replication method?

$$R_m = m\left(e^{R_c/m} - 1\right) = 1\left(e^{0.095/1} - 1\right) = 0.0996588551$$

$$R_{12M,24M} = \frac{R_{24M}T_{24M} - R_{12M}T_{12M}}{T_{24M} - T_{12M}} = \frac{0.13 * 2 - 0.095 * 1}{2 - 1} = 0.165 \xRightarrow[need\ to\ convert\ from\ cont.\ to\ annual]{}$$

$$R_m = m\left(e^{R_c/m} - 1\right) = 1\left(e^{0.165/1} - 1\right) = 0.1793931187$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 12-month SONIA and receive 12% per annum (with annual compounding) on a notional principal of $100 million. The swap has a remaining life of 2 years. The SONIA rates with continuous compounding for a year and two-years maturities are 9.5% and 13%, respectively. What is the value of such SWAP using bond replication method?

$$V_{swap,0} = B_{flt} - B_{fix} = 100{,}000{,}000 - 97{,}270{,}252.82 = 2{,}729{,}747.18$$

2. Now what is the value of such SWAP using FRA replication method?

$$R_{12M} = 0.0996588551 \qquad R_{12M,24M} = 0.1793931187$$

$$V_{swap,0} = \sum_{i=0}^{n} L\left(R_{k,t_i} - R_{f,t_i}\right)T_{freq}\, e^{-r(T_{i+1}-T_0)} =$$

$$= 100{,}000{,}000(0.0996588551 - 0.12) * 1 * e^{-0.095*1} + 100{,}000{,}000(0.1793931187 - 0.12) * 1 * e^{-0.13*1} =$$

$$= -1{,}849{,}768.66 + 4{,}579{,}515.84 = 2{,}729{,}747.17$$

We can observe small differences because of roundings.

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA+0.2% and receive 11% per annum (with semi-annual compounding) on a notional principal of $50 million. The swap has a remaining life of 3 years. The SONIA rates with continuous compounding for a 6-months, 12-months, 18-months, 24-months, 30-months and 36-months maturities are 2%, 3%, 4%, 5%, 6%, and 7%, respectively. What is the value of such SWAP using bond replication method?

$$B_{fix} = \sum_{t=1}^{T} C_t e^{-rt} + M e^{-rT} = 50{,}000{,}000 * \frac{0.11}{2} * e^{-0.02*0.5} + 2{,}750{,}000 * e^{-0.03*1} + 2{,}750{,}000 * e^{-0.04*1.5} +$$

$$+2{,}750{,}000 * e^{-0.05*2} + 2{,}750{,}000 * e^{-0.06*2.5} + 52{,}750{,}000 * e^{-0.07*3} = 55{,}594{,}783.54$$

For floating bond we need 6-month spot rate with semi-annual compounding and forward rates. We can not use the „payment date" trick in this case as floating pays SONIA+spread (0.2%).

$$R_{6M,semi-annual} = m\left(e^{R_c/m} - 1\right) = 2\left(e^{0.02/2} - 1\right) = 0.0201003341683359$$

$$R_{6M,12M} = \frac{R_{12M}T_{12M} - R_{6M}T_{6M}}{T_{12M} - T_{6M}} = \frac{0.03 * 1 - 0.02 * 0.5}{1 - 0.5} = 0.04 \quad R_{6M,12M\ semi-annual} = 2\left(e^{0.04/2} - 1\right) = 0.0404026800535116$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA+0.2% and receive 11% per annum (with semi-annual compounding) on a notional principal of $50 million. The swap has a remaining life of 3 years. The SONIA rates with continuous compounding for a 6-months, 12-months, 18-months, 24-months, 30-months and 36-months maturities are 2%, 3%, 4%, 5%, 6%, and 7%, respectively. What is the value of such SWAP using bond replication method?

$$R_{6M,12M} = \frac{R_{12M}T_{12M} - R_{6M}T_{6M}}{T_{12M} - T_{6M}} = \frac{0.03 * 1 - 0.02 * 0.5}{1 - 0.5} = 0.04 \quad R_{6M,12M\ semi-annual} = 2\left(e^{0.04/2} - 1\right) = 0.0404026800535116$$

$$R_{12M,18M} = \frac{R_{18M}T_{18M} - R_{12M}T_{12M}}{T_{18M} - T_{12M}} = \frac{0.04 * 1.5 - 0.03 * 1}{1.5 - 1} = 0.06 \quad R_{12M,18Msemi-annual} = 2\left(e^{0.06/2} - 1\right) = 0.0609090679070339$$

$$R_{18M,24M} = \frac{R_{24M}T_{24M} - R_{18M}T_{18M}}{T_{24M} - T_{18M}} = \frac{0.05 * 2 - 0.04 * 1.5}{2 - 1.5} = 0.08 \quad R_{18M,24Msemi-annual} = 2\left(e^{0.08/2} - 1\right) = 0.0816215483847764$$

$$R_{24M,30M} = \frac{R_{30M}T_{30M} - R_{24M}T_{24M}}{T_{30M} - T_{24M}} = \frac{0.06 * 2.5 - 0.05 * 2}{2.5 - 2} = 0.1 \quad R_{24M,30Msemi-annual} = 2\left(e^{0.1/2} - 1\right) = 0.102542192752048$$

$$R_{30M,36M} = \frac{R_{36M}T_{36M} - R_{30M}T_{30M}}{T_{36M} - T_{30M}} = \frac{0.07 * 3 - 0.06 * 2.5}{3 - 2.5} = 0.12 \quad R_{30M,36Msemi-annual} = 2\left(e^{0.12/2} - 1\right) = 0.123673093090719$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA+0.2% and receive 11% per annum (with semi-annual compounding) on a notional principal of $50 million. The swap has a remaining life of 3 years. The SONIA rates with continuous compounding for a 6-months, 12-months, 18-months, 24-months, 30-months and 36-months maturities are 2%, 3%, 4%, 5%, 6%, and 7%, respectively. What is the value of such SWAP using bond replication method?

$$B_{fix} = 55{,}594{,}783.54$$

$$R_{6M,semi-annual} = 0.0201003341683359 \quad R_{6M,12M\ semi-annual} = 0.0404026800535116$$

$$R_{12M,18Msemi-annual} = 0.0609090679070339 \quad R_{18M,24Msemi-annual} = 0.0816215483847764$$

$$R_{24M,30Msemi-annual} = 0.102542192752048 \quad R_{30M,36Msemi-annual} = 0.123673093090719$$

$$B_{flt} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rt} = 50{,}000{,}000 * \frac{0.0201003341683359 + 0.002}{2} * e^{-0.02*0.5} +$$

$$+50{,}000{,}000 * \frac{0.0404026800535116 + 0.002}{2} * e^{-0.03*1} + 50{,}000{,}000 * \frac{0.0609090679070339 + 0.002}{2} * e^{-0.04*1.5} +$$

$$+50{,}000{,}000 * \frac{0.0816215483847764 + 0.002}{2} * e^{-0.05*2} + 50{,}000{,}000 * \frac{0.102542192752048 + 0.002}{2} * e^{-0.06*2.5} +$$

$$+(50{,}000{,}000 + 50{,}000{,}000 * \frac{0.123673093090719 + 0.002}{2}) * e^{-0.07*3} = 50{,}273{,}919.48$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA+0.2% and receive 11% per annum (with semi-annual compounding) on a notional principal of $50 million. The swap has a remaining life of 3 years. The SONIA rates with continuous compounding for a 6-months, 12-months, 18-months, 24-months, 30-months and 36-months maturities are 2%, 3%, 4%, 5%, 6%, and 7%, respectively. What is the value of such SWAP using bond replication method?

$$B_{fix} = 55{,}594{,}783.54$$

$$R_{6M,semi-annual} = 0.0201003341683359 \quad R_{6M,12M\ semi-annual} = 0.0404026800535116$$

$$R_{12M,18Msemi-annual} = 0.0609090679070339 \quad R_{18M,24Msemi-annual} = 0.0816215483847764$$

$$R_{24M,30Msemi-annual} = 0.102542192752048 \quad R_{30M,36Msemi-annual} = 0.123673093090719$$

$$B_{flt} = 50{,}273{,}919.48$$

$$V_{swap,0} = B_{flt} - B_{fix} = 50{,}273{,}919.48 - 55{,}594{,}783.54 = -5{,}320{,}864.06$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA+0.2% and receive 11% per annum (with semi-annual compounding) on a notional principal of $50 million. The swap has a remaining life of 3 years. The SONIA rates with continuous compounding for a 6-months, 12-months, 18-months, 24-months, 30-months and 36-months maturities are 2%, 3%, 4%, 5%, 6%, and 7%, respectively. What is the value of such SWAP using bond replication method?

$$V_{swap,0} = B_{flt} - B_{fix} = 50{,}273{,}919.48 - 55{,}594{,}783.54 = -5{,}320{,}864.06$$

2. Now what is the value of such SWAP using FRA replication method?

$$R_{6M,semi-annual} = 0.0201003341683359 \quad R_{6M,12M\ semi-annual} = 0.0404026800535116$$
$$R_{12M,18Msemi-annual} = 0.0609090679070339 \quad R_{18M,24Msemi-annual} = 0.0816215483847764$$
$$R_{24M,30Msemi-annual} = 0.102542192752048 \quad R_{30M,36Msemi-annual} = 0.123673093090719$$

$$V_{swap,0} = \sum_{i=0}^{n} L\left(R_{k,t_i} - R_{f,t_i}\right)T_{freq}\, e^{-r(T_{i+1}-T_0)} = 50{,}000{,}000((0.0201003341683359 + 0.002) - 0.11) * 0.5 * e^{-0.02*0.5} +$$
$$+50{,}000{,}000((0.0404026800535116 + 0.002) - 0.11) * 0.5 * e^{-0.03*1} + 50{,}000{,}000((0.0609090679070339 + 0.002) - 0.11) * 0.5 * e^{-0.04*1.5} +$$
$$+50{,}000{,}000((0.0816215483847764 + 0.002) - 0.11) * 0.5 * e^{-0.05*2} + 50{,}000{,}000((0.102542192752048 + 0.002) - 0.11) * 0.5 * e^{-0.06*2.5} +$$
$$+50{,}000{,}000((0.123673093090719 + 0.002) - 0.11) * 0.5 * e^{-0.07*3} = -2{,}175626.24 + (-1{,}639{,}987.93) + (-1{,}108{,}714.24) + (-596{,}705.25) +$$
$$+(-117{,}439.46) + 317{,}609.06 = -5{,}320{,}864.06$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA and receive 8% per annum (with semi-annual compounding) on a notional principal of $100 million. The swap has a remaining life of 1.25 years. The SONIA rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month SONIA rate at the last payment date was 10.2% (with semi-annual compounding). What is the value of such SWAP using bond replication method?

$$V_{swap,0} = ?$$

2. Now what is the value of such SWAP using FRA replication method?

$$V_{swap,0} = ?$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA and receive 8% per annum (with semi-annual compounding) on a notional principal of $100 million. The swap has a remaining life of 1.25 years. The SONIA rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month SONIA rate at the last payment date was 10.2% (with semi-annual compounding). What is the value of such SWAP using bond replication method?

$$B_{fix} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rT} = 4{,}000{,}000 * e^{-0.1*0.25} + 4{,}000{,}000 * e^{-0.105*0.75} + 104{,}000{,}000 * e^{-0.11*1.25} = 98{,}237{,}895.90$$

$B_{flt} = \sum_{t=1}^{T} C_t e^{-rt} + Me^{-rT}$ *or in that case, we know the next coupon and that all the rest is equal to principal at the next payment date so:*

$$B_{flt} = C_t e^{-rt} + Me^{-rt} = 5{,}100{,}000e^{-0.1*0.25} + 100{,}000{,}000e^{-0.1*0.25} = 102{,}505{,}071.75$$

$$V_{swap,0} = B_{flt} - B_{fix} = 102{,}505{,}071.75 - 98{,}237{,}895.90 = 4{,}267{,}175.85$$

2. Now what is the value of such SWAP using FRA replication method?

$$V_{swap,0} = ?$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA and receive 8% per annum (with semi-annual compounding) on a notional principal of $100 million. The swap has a remaining life of 1.25 years. The SONIA rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month SONIA rate at the last payment date was 10.2% (with semi-annual compounding). What is the value of such SWAP using bond replication method?

$$V_{swap,0} = 4{,}267{,}175.85$$

2. Now what is the value of such SWAP using FRA replication method?

$$R_{3M,9M} = \frac{R_{9M}T_{9M} - R_{3M}T_{3M}}{T_{9M} - T_{3M}} = \frac{0.105 * 0.75 - 0.1 * 0.25}{0.75 - 0.25} = 0.1075 \xRightarrow[need\ to\ convert\ from\ cont.\ to\ semi-annual]{}$$

$$R_m = m\left(e^{R_c/m} - 1\right) = 2\left(e^{0.1075/2} - 1\right) = 0.11044$$

$$R_{9M,15M} = \frac{R_{15M}T_{15M} - R_{9M}T_{9M}}{T_{15M} - T_{9M}} = \frac{0.11 * 1.25 - 0.105 * 0.75}{1.25 - 0.75} = 0.1175 \xRightarrow[need\ to\ convert\ from\ cont.\ to\ semi-annual]{}$$

$$R_m = m\left(e^{R_c/m} - 1\right) = 2\left(e^{0.1175/2} - 1\right) = 0.12102$$

Financial Instruments – SWAP

SWAP valuation

1. Suppose that a financial institution has agreed to pay 6-month SONIA and receive 8% per annum (with semi-annual compounding) on a notional principal of $100 million. The swap has a remaining life of 1.25 years. The SONIA rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month SONIA rate at the last payment date was 10.2% (with semi-annual compounding). What is the value of such SWAP using bond replication method?

$$V_{swap,0} = 4{,}267{,}175.85$$

2. Now what is the value of such SWAP using FRA replication method?

$$R_{3M,9M} = 0.11044 \qquad R_{9M,15M} = 0.12102$$

$$V_{swap,0} = \sum_{i=0}^{n} L\left(R_{k,t_i} - R_{f,t_i}\right)T_{freq}\, e^{-r(T_{i+1}-T_0)} =$$

$$= 100{,}000{,}000(0.102 - 0.08) * 0.5 * e^{-0.1*0.25} + 100{,}000{,}000(0.11044 - 0.08) * 0.5 * e^{-0.105*0.75} +$$

$$+100{,}000{,}000(0.12102 - 0.08) * 0.5 * e^{-0.11*1.25} = 1{,}072{,}840.90 + 1{,}406{,}811.02 + 1{,}787{,}523.93 =$$

$$= 4{,}267{,}175.85$$

Financial Instruments – Commodity and Foreign Exchange (FX)

Commodity:

A good sold for **production** or **consumption** (Metals, Energy, Agricultural products, etc.).

A commodity transaction may be **physical** (delivery of the commodity) or **financial** (a cashflow from one party to the other).

Nowadays, most of the commodity transaction are done through derivative contracts which are traded in exchanges and OTC markets.

Foreign Exchange:

Foreign exchange (FX) represents the trading of **currencies**, essentially by the transfer of ownership of deposits.

FX transaction **lock-in the exchange** rate for cash.

A U.S Dollar/Japanese Yen trade consists of selling U.S. dollars (base currency) to another counterparty in exchange for a specific amount of Yen (quoting/counter currency).

How many Euro can I buy with PLN 10000 today? The €1 = PLN 4.63, thus $\frac{PLN}{PLN/€}\frac{10000}{4.63/1} = €2159.83$

Derivatives

Definition:

Financial derivatives are a type of financial contract whose value is dependent on an underlying asset, group of assets, or benchmark. A derivative is set between two or more parties that can trade on an exchange or over-the-counter (OTC).

Derivatives Underlying:

There are numerous underlyings for derivatives available right now, and new ones are being developed every year.

– **Equities** of companies listed on public exchanges, such as Microsoft or Citigroup.

– **Fixed income instruments**, such as government bonds, corporate bonds, credit spreads, or baskets of mortgages.

– **Commodities**, such as gold, oil, silver, cotton, electricity or weather.

– **Indices**, such as the FTSE 100, Hang Seng of Hong Kong, or Nikkei of Japan.

– **Foreign exchange**.

– **Events**, such as football games or shipping catastrophes.

Derivatives – Definition of Common Terms

The most common terms in finance world:

- **Premium**: The amount paid for the contract initially. How to find this value is the subject of much of this book.
- **Underlying (asset)**: The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.
- **Strike (price) or exercise price**: The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. We will see more complicated contracts in later chapters and the definition of strike or exercise price will be extended.
- **Expiration (date) or expiry (date)**: Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights. This will be denoted by T .
- **Intrinsic value**: The payoff that would be received if the underlying is at its current level when the option expires.
- **Time value**: Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.
- **In the money**: An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.
- **Out of the money**: An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.
- **At the money**: A call or put with a strike that is close to the current asset level.
- **Long position**: A positive amount of a quantity, or a positive exposure to a quantity.
- **Short position**: A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

Derivatives

The most common types of derivative contracts:

Forwards: a customized OTC derivative contract obligating counterparties to buy (receive) or sell (deliver) an asset at a specified price on a future date.

Futures: exchange-listed financial derivatives contracts that oblige the buyer to purchase some underlying asset (or the seller to sell that asset) at a predetermined future price and date.

Options: OTC or exchange-listed financial derivatives that give buyers the right, but not the obligation, to buy or sell an underlying asset at an agreed-upon price and date. **Call Options** allow the holder to buy the asset at a stated price within a specific timeframe. **Put Options**, on the other hand, allow the holder to sell the asset at a stated price within a specific timeframe.

Swaps: OTC agreement between two parties to exchange cash flows in future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be computed.

Asset Backed Securities (ABS): bonds or notes backed by financial assets. The asset pool is usually a group of small and illiquid assets that would be difficult to sell individually. Pooling these assets into financial instruments allows them to be sold to investors.

The use of derivatives:

Hedging or mitigating risk in an underlying: by entering into a derivative contract whose value moves in the opposite direction to their underlying position, hedgers aim to reduce their risk.

Speculate and make a profit: if the value of the underlying asset moves the way a trader expects.

Obtain **exposure to an underlying** which cannot be traded directly (e.g. indexes, weather derivatives).

Define the risk: traders can use options to give them quite defined risk exposures, such as setting a maximum loss for a position.

Tailored exposures: derivatives traders can take positions that profit if an underlying moves in a given direction, stays in or out of a specified range, or reaches a certain level.

Arbitrage: derivatives can be used to make a risk-free profit.

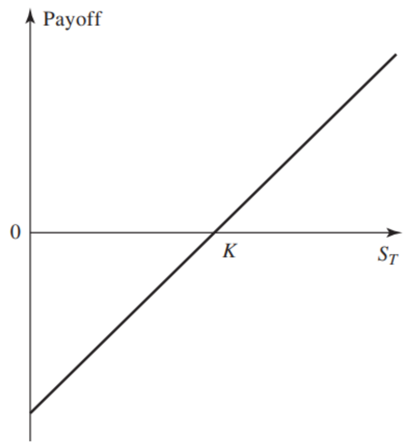
Futures and Forwards

Definition:

Forward contracts and **futures contracts** are derivatives arrangements that involve two parties who agree to buy or sell a specific asset at a set price at a certain date in the future.

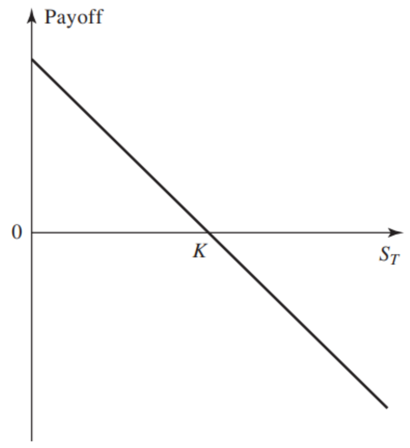
Forwards contracts are OTC arrangements that are settled only once, at the expiration (collateral).

Futures are standardized exchange-listed contracts that are settled on a daily basis (magin).



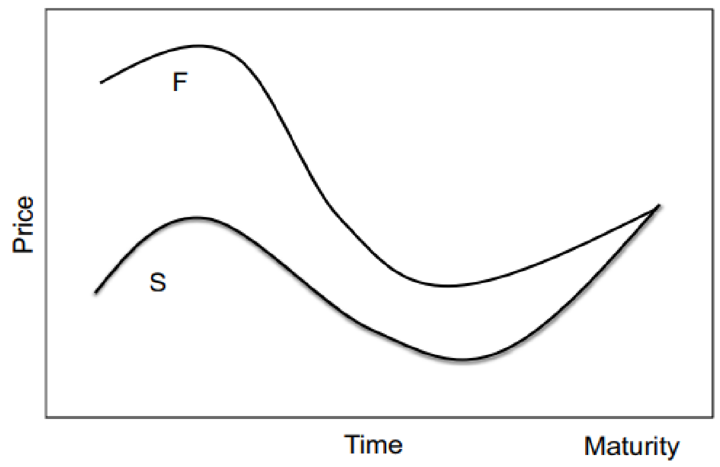
Long Position

Payoff: $S_T - \mathrm{K}$



Short Position

Payoff: $\mathrm{K} - S_T$



Future convergence to Spot Price (Basis risk)

A First Example of No Arbitrage

Consider a forward contract that obliges us to hand over an amount *\$F* at time *T* to receive the underlying asset. Today's date is *t* and the price of the asset is currently *\$S(t)*, this is the spot price, the amount for which we could get immediate delivery of the asset. When we get to maturity we will hand over the amount *\$F* and receive the asset, then worth *\$S(T).*

How much profit we make cannot be known until we know the value *\$S(T),* and we can't know this until time *T?*

A First Example of No Arbitrage

Consider a forward contract that obliges us to hand over an amount *\$F* at time *T* to receive the underlying asset. Today's date is *t* and the price of the asset is currently *\$S(t)*, this is the spot price, the amount for which we could get immediate delivery of the asset. When we get to maturity we will hand over the amount *\$F* and receive the asset, then worth *\$S(T).*

How much profit we make cannot be known until we know the value *\$S(T),* and we can't know this until time *T?*

Consider following portfolio, enter into the forward contract. This costs us nothing up front but exposes us to the uncertainty in the value of the asset at maturity. Simultaneously sell the asset. It is called going short when you sell something you don't own. We now have an amount *S(t)* in cash due to the sale of the asset, a forward contract, and a short asset position. But **our net position is zero.** Put the cash in the bank, to receive interest.

A First Example of No Arbitrage

Consider a forward contract that obliges us to hand over an amount *$F* at time *T* to receive the underlying asset. Today's date is *t* and the price of the asset is currently *$S(t)*, this is the spot price, the amount for which we could get immediate delivery of the asset. When we get to maturity we will hand over the amount *$F* and receive the asset, then worth *$S(T).*

How much profit we make cannot be known until we know the value *$S(T),* and we can't know this until time *T?*

Consider following portfolio, enter into the forward contract. This costs us nothing up front but exposes us to the uncertainty in the value of the asset at maturity. Simultaneously sell the asset. It is called going short when you sell something you don't own. We now have an amount *S(t)* in cash due to the sale of the asset, a forward contract, and a short asset position. But **our net position is zero.** Put the cash in the bank, to receive interest.

When we get to maturity we hand over the amount *F* and receive the asset, this cancels our short asset position regardless of the value of S(T). At maturity we are left with a guaranteed −*F* in cash as well as the bank account. The word 'guaranteed' is important because it emphasizes that it is independent of the value of the asset. The bank account contains the initial investment of an amount *S(t)* with added interest, which has a value at maturity of

$$S(t)e^{r(T-t)}$$

Our net position at maturity is therefore

$$S(t)e^{r(T-t)} - F$$

Thus,

$$F = S(t)e^{r(T-t)}$$

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
−Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$

Financial Instruments – Forward/Futures

Forward/Futures valuation

1. Suppose that a financial institution has agreed to buy a Tesla stock for $50 per stock in a year. Today TSLA is traded at $73 dollars. SOFR rate with continuous compounding for a year is 4.5%. What is the value of such Forward contract?

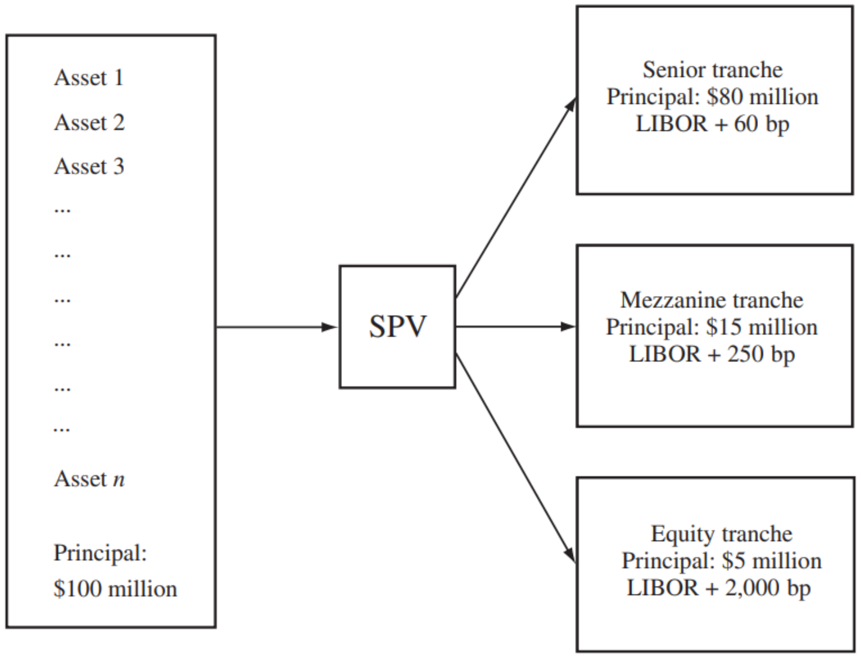
$$F = S(t)e^{r(T-t)}$$

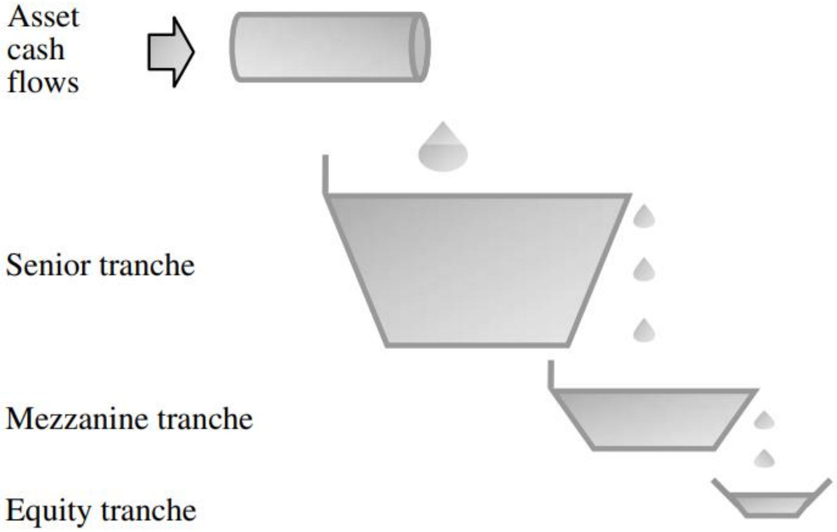
$$F = 73 * e^{0.045(1-0)} = \$76.36$$

Asset Backed Securities (ABS)

Definition:

Asset-Backed Security (ABS) is a type of financial investment that is **collateralized** by an underlying **pool of assets**—usually ones that generate a cash flow from debt, such as loans, leases, credit card balances or receivables. It takes the **form of a bond** or note, paying income at a fixed rate for a set amount of time, until maturity.





Options

Definition:

The **buyer** of the **option** gains the **right**, but **not the obligation**, **to engage in that transaction**, while the **seller** incurs the corresponding **obligation** to **fulfil the transaction**.

Call Options:

- Derivatives contract that gives the **owner the right**, but **not the obligation**, **to buy** a specified amount of an **underlying security** at a **specified price** within a **specified time**.

Put Options:

- Derivatives contract that gives the **owner the right**, but **not the obligation**, **to sell** a specified amount of an **underlying security** at a **specified price** within a **specified time**.

Options Types:

- **European Options:** can be exercised **only on the expiration date** itself.
- **American Options:** can be exercised at **any time up to the expiration date**.

Options

Options – Payoff at Maturity

At maturity, the payoff of an option can be expressed as:

$$Call_T = max(S_T - K, 0)$$

$$Put_T = max(K - S_T, 0)$$

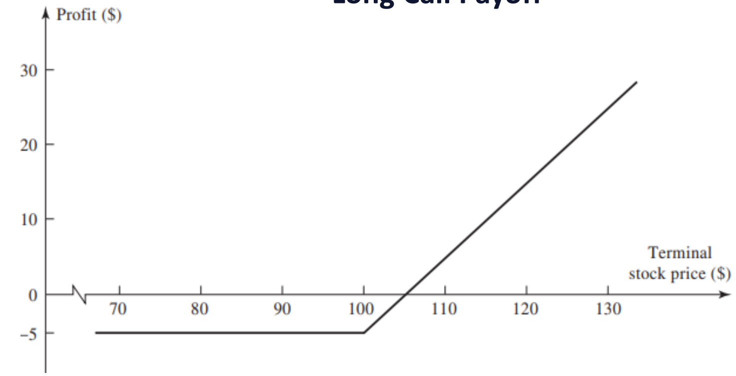
Where S_T is the underlying price at maturity T and K is the agreed price when the option was bought (the strike/exercise price).

As all the other financial instruments, long and short positions can be taken on options.

Option price = $5

Strike Price= $100

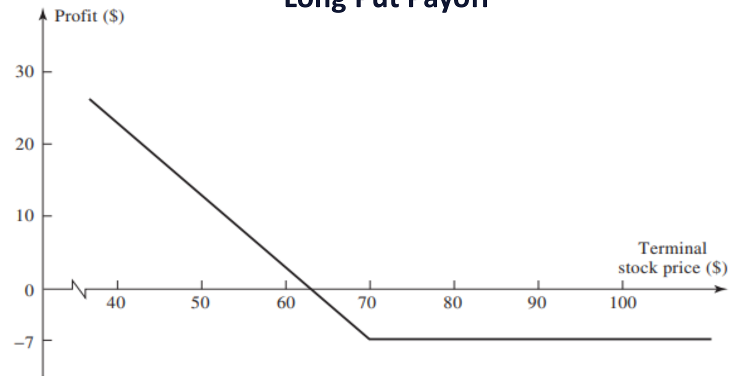
Long Call Payoff



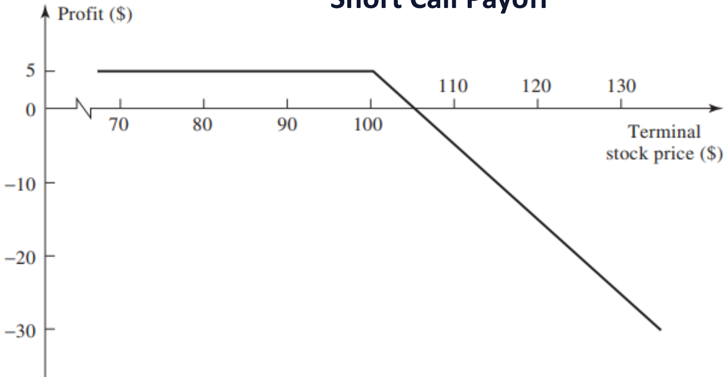
Long Put Payoff

Option price = $7

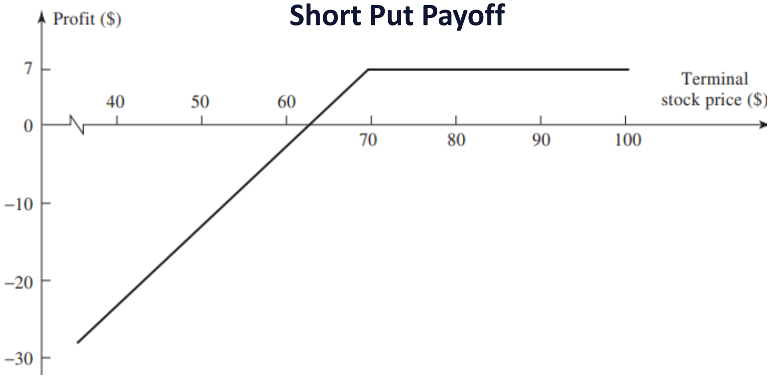
Strike Price= $70



Short Call Payoff



Short Put Payoff



Derivatives Usage - Speculate

Speculate: wish to take a position in the market, making profit if the value of the underlying asset moves in the expected way.

- **Example 1**

Today's $/£ exchange rate is equal to 1.5470 and Penny Pincher Investment thinks that £ will strengthen respect to $ over the next 2 months. Assume Penny Pincher Investment want to trade £250,000.

1) Penny Pincher Investment can buy £250,000 in the spot market and invest them at the 0.194% monthly interest-bearing account that your bank offers for GBP accounts.

2) Penny Pincher Investment can buy 4 Futures on sterling with 2-month expiration. The 2-month futures is quoted at 1.5410 $/£ and each futures costs $5000.

Payoff: the two strategies give the same payoff. However, the futures strategy requires less capital since derivatives are leveraged instruments.

	Strategy 1	Strategy 2
Investment	$ 386750	$ 20000
After 2 months - Scenario 1: $/£ 1.6	$ 13250 + $ 1500 = $ 14750	$ 14750
After 2 months - Scenario 2: $/£ 1.5	$ -11750 + $ 1500 = $ -10250	$ -10250

Derivatives Usage - Hedging Risks

Hedging: by entering into a derivative contract whose value moves in the opposite direction to their underlying position, hedgers aim to reduce the risk.

- **Example 1**

Suppose *Innovative Procrastination Technologies*, a US company, knows that today, October 2023, will have to pay £10 million in 3 months, for goods it bought from UK. What risks is *Innovative Procrastination Technologies* exposed to and how could it protect itself from these risks?

Foreign Exchange Risk: the company is exposed to the depreciation of the USD against the GBP, if USD decreases its value respect to GBP *Innovative Procrastination Technologies* will need more USD to buy 1 GBP.

Hedging strategy: knowing that today the 3-month Fwd (Forward) on \$/£, Bid/Ask price are 1.5533/1.5538, *Innovative Procrastination Technologies* could **buy today** 1 **GBP for** 1.5538 **USD**. Thus, the company decides to **buy the forward contract**.

Payoff: in such way *Innovative Procrastination Technologies* **covers its potential loss (and gain)** from the \$/£ fluctuations and locks its payment at \$ 15,538,000 $= £10\text{ M} * \frac{\$1.5538}{£1}$.

If on the **day** of the **transaction** (in 3 months), the **GBP strengthen** (depreciates) respect to \$1.5538, *Innovative Procrastination Technologies* would have incurred in a **loss** (gain) **without hedging**.

Derivatives Usage - Hedging Risks

- **Example 2**

Suppose *It-is What-it-is Investment Partners* owns 1000 stocks of ABC company that will report earnings in the next month. Today's price of ABC stock is $28 and the investment committee of *It-is What-it-is* forecasted that the ABC earnings will be disappointing.

What risks is *It-is What-it-is Investment Partners* exposed to and how could it protect itself from these risks?

Market Risk: *It-is What-it-is Investment Partners* is exposed to the decrease in market value of ABC stocks.

Hedging strategy: Knowing that each option is written on 100 stocks, *It-is What-it-is Investment Partners* **could buy** 10 **put options** expiring in 1 month with $27.50 strike price. Assuming market price of the option is $1, the hedging strategy will cost $1000.

Payoff: **if at expiration the stock price falls below $27.50** ($26.50 considering the cost) the put option will be exercised. **The loss on the stock will be offset by the gain on the put option**, covering *It-is What-it-is Investment Partners* losses on ABC stocks.

Derivatives Usage - Arbitrage

Arbitrage: derivatives can be used to make a risk-free profit.

Consider a long forward contract to purchase a non-dividend-paying stock in 3 months. Assume the current stock price, S_0, is $40 and the 3-month risk-free interest rate, r, is 5% per annum.

Under no arbitrage the following statement should hold:

$$F_0 = S_0 e^{rT} \Rightarrow \$40e^{0.05\times 3/12} = \$40.5$$

$F_0 > S_0 e^{rT}$	$F_0 < S_0 e^{rT}$
Forward Price F_0= $43	**Forward Price F_0 = $39**
Action Now: • Borrow $40 at 5% for 3 months • Buy one unit of asset • Enter into forward contract to sell asset in 3 months for $43	***Action Now:*** • Short 1 unit of asset to realize $40 • Invest $40 at 5% for 3 months • Enter into a forward contract to buy asset in 3 months for $39
Action In 3 Months: • Use $40.50 to repay loan with interest • Sell asset for $43	***Action In 3 Months:*** • Close short position • Receive $40.50 from investment • Buy asset for $39
Profit Realized = $ 2.50	***Profit Realized*** = $ 1.50

Derivatives Usage - Arbitrage

Consider a long forward contract to purchase a coupon-bearing bond whose current price is $900. Suppose that the forward contract matures in 9 months and that a coupon payment of $40 is expected after 4 months. We assume that the 4-month and 9-month risk-free interest rates (continuously compounded) are, respectively, 3% and 4% per annum.

When an investment asset will provide income with a present value of I during the life of a forward contract, under no arbitrage the following statement should hold:

$$F_0 = (S_0 - I)e^{rT} \Rightarrow (900 - 39.60)e^{0.04\times\frac{9}{12}} = \$886.60\,, \quad \text{where } I = 40e^{-0.03\times\frac{4}{12}} = \$39.60$$

$F_0 > (S_0 - I)e^{rT}$ **Forward Price = $910**	$F_0 < (S_0 - I)e^{rT}$ **Forward Price = $870**
Action Now: • Borrow $900: $39.60 for 4 months and $860.40 for 9 months • Buy one unit of asset • Enter into forward contract to sell asset in 9 months for $910	***Action Now:*** • Short 1 unit of asset to realize $900 • Invest $39.6 for 4 months and 860.40 for 9 months • Enter into a forward contract to buy asset in 9 months for $870
Action In 4 Months: • Receive $40 of income on asset • Use $40 to repay first loan with interest	***Action In 4 Months:*** • Receive $40 from 4-month investment • Pay income $40 on asset
Action In 9 Months: • Sell asset for $910 • Use $886.60 to repay second loan with interest	***Action In 9 Months:*** • Receive $886.60 from 9-month investment • Buy asset for $870 • Close out short position
Profit Realized = $ 23.40	***Profit Realized*** = $ 16.60

Options: Intrinsic Value and Time Value (Extrinsic Value)

The option premium/price can be divided into two main components:

Intrinsic value: the option value if there were no time to maturity, computed as the difference between underlying and strike prices.

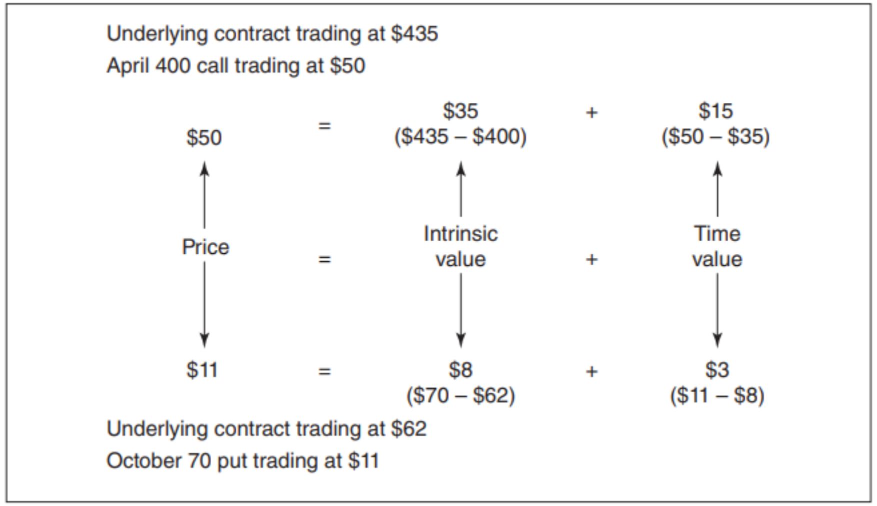
Time Value: the option value arising from the time left to maturity, computed as the difference between an option's price minus its intrinsic value.

Out-of-the-money (OTM): options are considered "out of the money" when they have no intrinsic value. A call option is out of the money when the strike price is above the spot price of the underlying security. A put option is out of the money when the strike price is below the spot price.

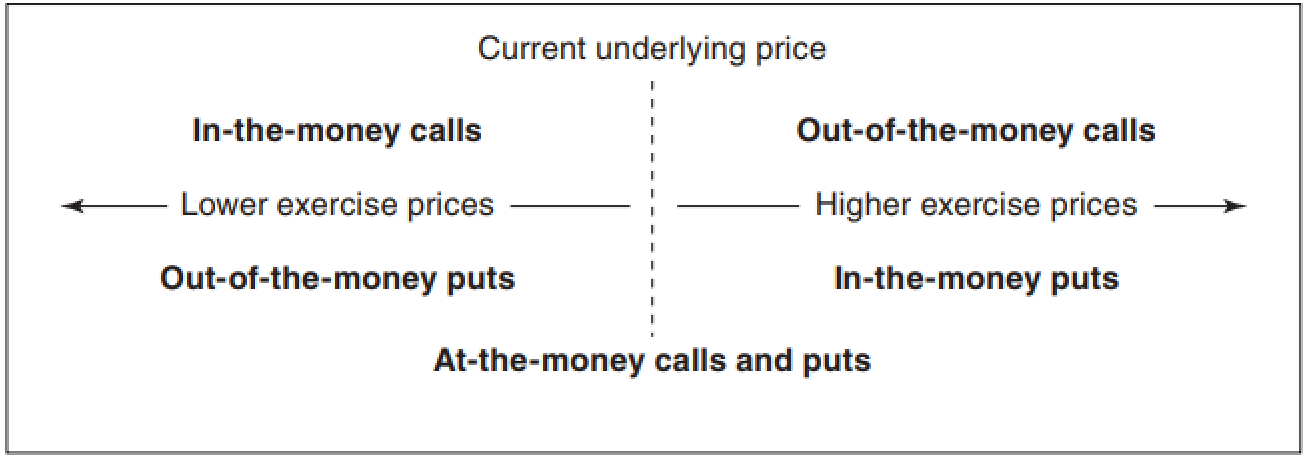
At-the-money (ATM): options are considered "at the money" if the strike price is the same as the current spot price of the underlying security. An at the money option has no intrinsic value, only time value.

In-the-money (ITM): options are considered "in the money" if the option has positive intrinsic value. A call option is in the money when the strike price is below the spot price. A put option is in the money when the strike price is above the spot price.

Intrinsic/Extrinsic Option Value



ITM/ATM/OTM Option



Option Greeks

Options Inputs:

1. The current stock price, S_0
2. The strike price, K
3. The time to expiration, T
4. The volatility of the stock price, σ
5. The risk-free interest rate, r
6. The dividends that are expected to be paid, D (if discrete) or q (if dividend yield)

Effects of inputs increase on option value:

Variable	Greeks	Eurpoean Call	Eurpean Put	American Call	American Put
Stock Price	Delta, Gamma	+	-	+	-
Strike Price		-	+	-	+
Time to Expiration	Theta	+	+	+	+
Volatility	Vega	+	+	+	+
Risk-Free Rate	Rho	+	-	+	-
Dividends		-	+	-	+
+ Indicates that an increase in the variable causes the option price to increase					
- Indicates that an increase in the variable causes the option price to decrease					

Replication and Stochastic Calculus

Replication

Derivatives pricing is based on the concept of replication.

Replication means that you can recreate an asset's payoff by combining other financial instruments.

Examples:

- Static Replication - Put-Call Parity: $p + S_0 = c + Ke^{-rT} + D$
- Dynamic Replication – Black and Scholes

Stochastic Calculus

To price a derivative, we need to come up with a range of values and associated probabilities that the underlying can assume at maturity.

Therefore, we need a model to govern the distribution of underlying values.

Why do we need Stochastic Calculus?

Stochastic calculus is the area of mathematics that deals with processes containing a stochastic component and thus allows the modelling of random systems.

A derivative price depends on the random motion of the underlying price, interest rate, dividends and volatility.

Option Replication – Put-Call Parity

The main idea of replication is based on the **Law of One Price.**

- Intuitively, the Law of One Price tells us that, if **no arbitrage** exists, **portfolios** or assets with the **same** expected **payoff** at maturity **must have** the **same value** on each point in time.
- Based on this idea you can **combine** different financial **instruments** to **replicate** the **payoff of other instruments** (**synthetic derivatives**).

Example: Put-Call Parity relationship

Consider the following two portfolios:

- Portfolio A: one European call option (Strike K) + zero-coupon bond that provides a payoff of K at time T, $\text{Ptf A} = c + Ke^{-rT}$
- Portfolio C: one European put option (Strike K) + one share of the stock, $\text{Ptf C} = p + S_0$

Portfolios Value at Maturity T

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

- The two portfolios return the same payoff at maturity, hence they must have the same value today.
- Equating the two portfolio we obtain the Put-Call Parity relationship: $p + S_0 = c + Ke^{-rT}$

Option Replication – Put-Call Parity

Put-Call Parity numerical example:

Inputs:

Stock price at time zero, $S_0 = \$100$

Strike price, $K = \$100$

Time to expiration, $\mathrm{T} = 1$ year

Volatility of the stock price, $\sigma = 20\%$

Risk-free interest rate, $\mathrm{r} = 1\%$

Dividends, $\mathrm{D} = 0$

European Call, $\mathrm{c} = \$8.4333$

European Put, $\mathrm{p} = \$7.4383$

- Portfolio A: European call option (Strike \$100) + zero-coupon bond that provides a payoff of K at time T =

$$= \mathrm{c} + Ke^{-rT} = \$8.4333 + 100e^{-0.01*1} = \$107.44$$

- Portfolio C: one European put option (Strike \$100) + one share of the stock =

$$= \mathrm{p} + S_0 = \$7.4383 + 100 = \$107.44$$

Question: the relation depends on the put and call prices, how to compute such fair values?

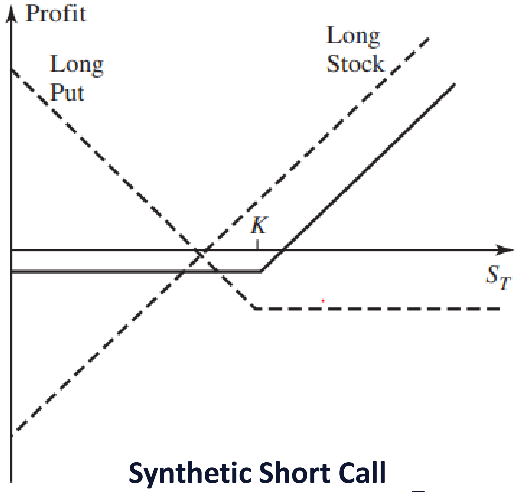
Option Replication – Put-Call Parity

Put-Call Parity: $p + S_0 = c + Ke^{-rT}$

Based on Put-Call Parity, you can replicate put, call options and futures (futures replication: $c - p + Ke^{-rT} = Fe^{-rT}$)

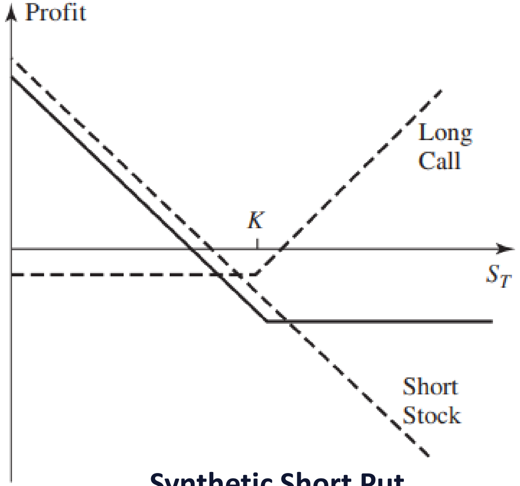
Synthetic Long Call

$p + S_0 = c + Ke^{-rT}$



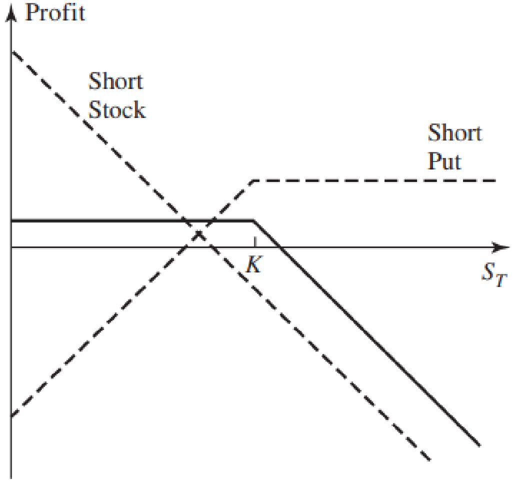
Synthetic Long Put

$-S_0 + c = -Ke^{-rT} + p$



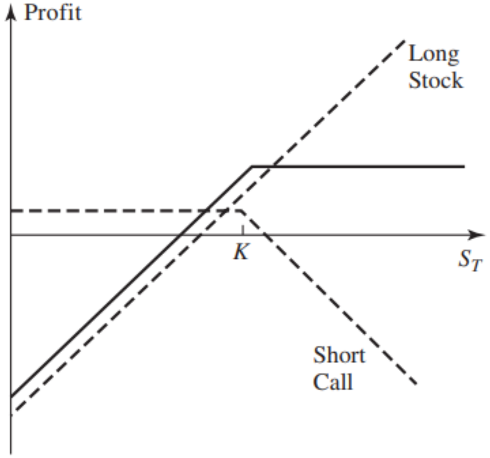
Synthetic Short Call

$-p - S_0 = -c - Ke^{-rT}$



Synthetic Short Put

$S_0 - c = +Ke^{-rT} - p$



Motivation for Stochastic Calculus

Simple Option Pricing Model Example

The underlying price at expiration **T** can take one of five equally likely prices: $\mathbf{S_T} = \{\$80,\ \$90,\ \$100,\ \$110,\ \$120\}$

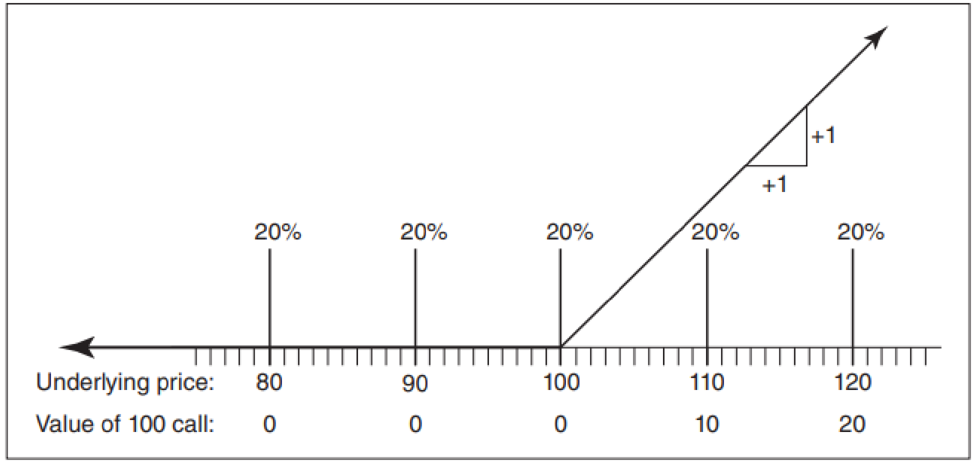
Expected Value of the **underlying** at expiration: $\mathbf{E}[\boldsymbol{S_T}]$ = $(20\% * \$80) + (20\% * \$90) + (20\% * \$100) + (20\% * \$110) + (20\% * \$120) = \$100$

Consider a Call option to buy the underlying at strike price $\boldsymbol{K} = \$100$ at maturity $\mathbf{T}$. What is the expected value of the Call option?

Expected Value of the **Call option** at expiration: $\mathbf{E}[\boldsymbol{c_T}] = \sum_{\mathrm{i}=1}^{n} p_i \cdot \max(S_i - K, 0)$

$\mathbf{E}[\mathbf{c_T}] = 20\% * [\max(\$80 - \$100, 0) + \max(\$90 - \$100, 0) + \max(\$100 - \$100, 0) + \max(\$110 - \$100, 0) + \max(\$110 - \$100, 0) + \max(\$120 - \$100, 0)]$

$\mathbf{E}[\mathbf{c_T}] = (20\% * 0) + (20\% * 0) + (20\% * 0) + (20\% * \$10) + (20\% * \$20) = \$6$



Question: How to build a model that takes in consideration every possible price outcome with their associated probability?

Option Strategies

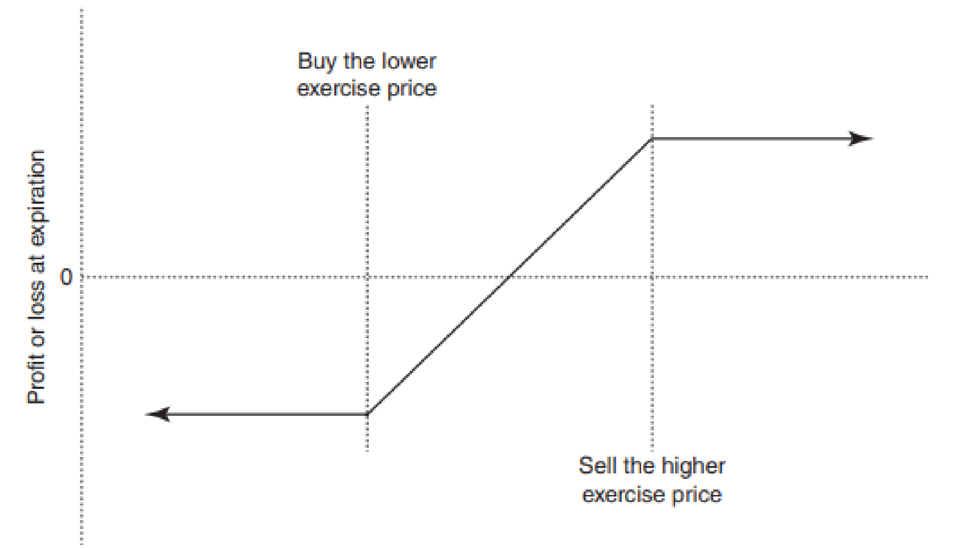
Bull and Bear Spread

One option is purchased and one option is sold, where both options are the same type (either both calls or both puts) and expire at the same time.

The options are distinguished only by their different exercise prices. Regardless of whether a spread consists of calls or puts, whenever you buy the lower exercise price and sell the higher exercise price, the position is bullish, and whenever you buy the higher exercise price and sell the lower exercise price, the position is bearish.

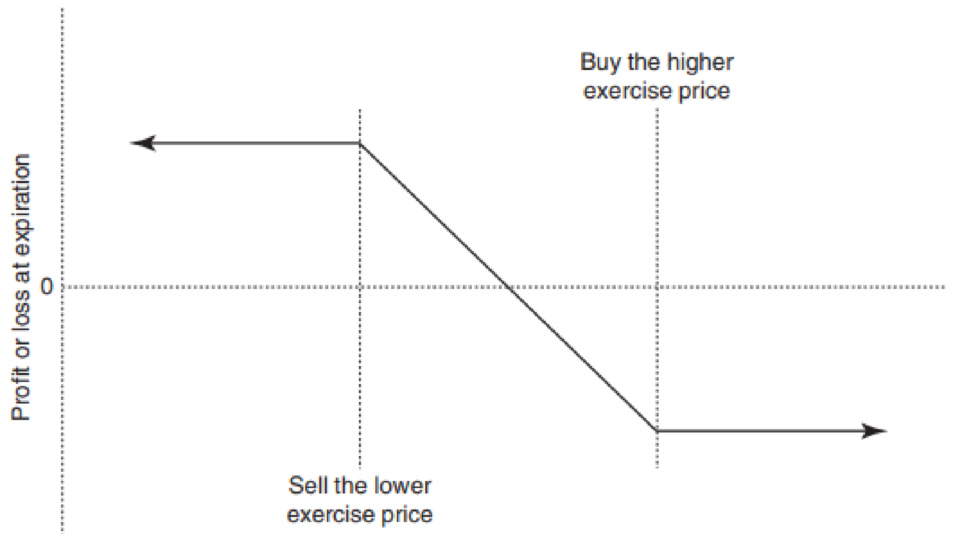
Usage of Bull and Bear Spread: you focus primarily on the direction of the underlying market.

Bull Spread Payoff



Buy an option at a lower exercise price and sell an option at a higher exercise price, where both options are the same type (both calls or both puts) and expire at the same time

Bear Spread Payoff



Buy an option at a higher exercise price and sell an option at a lower exercise price, where both options are the same type (both calls or both puts) and expire at the same time

Option Strategies

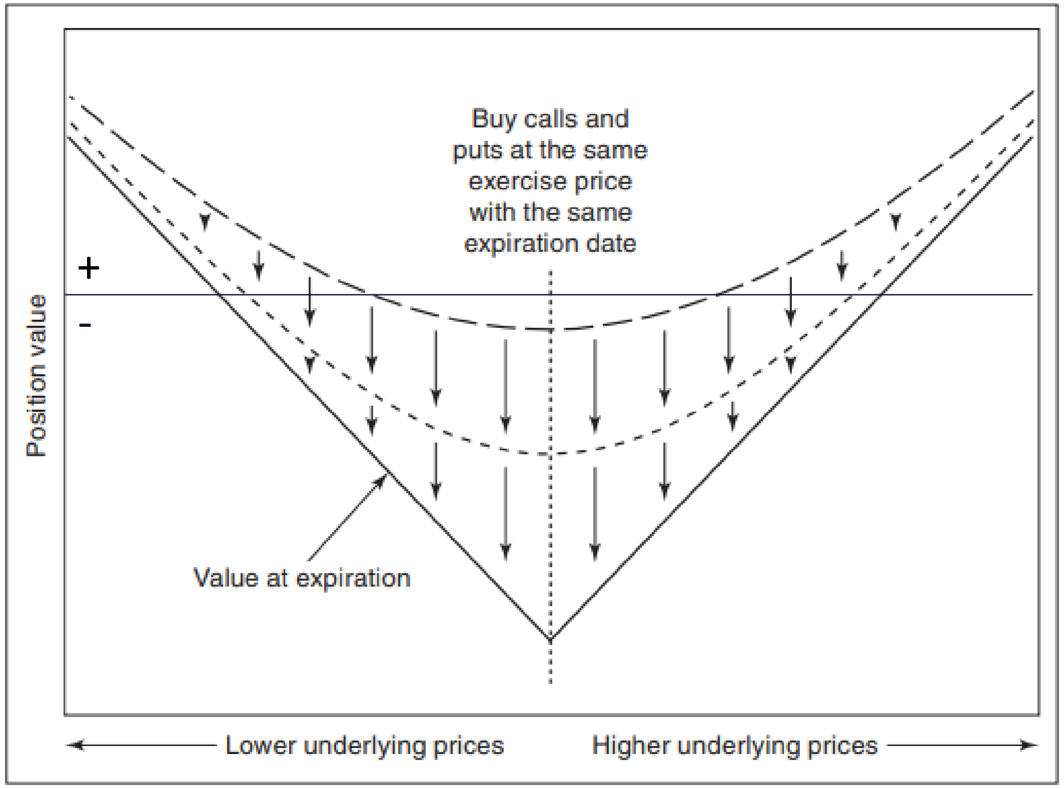
Straddle

A straddle consists of a call and a put where both options have the same exercise price and expiration date.

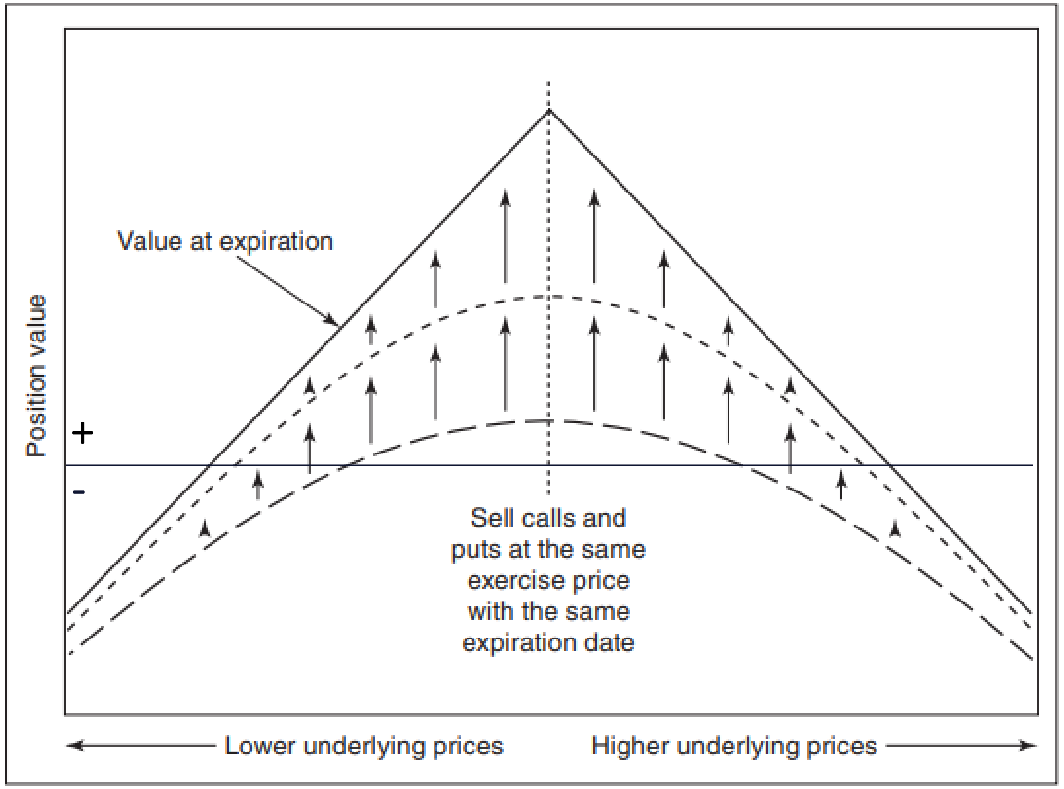
In a straddle, both options are either purchased (a long straddle) or sold (a short straddle).

Usage of Long (Short) Straddle: you think the underlying will (will not) move a lot but you don't care about the direction.

Long Straddle Payoff



Short Straddle Payoff



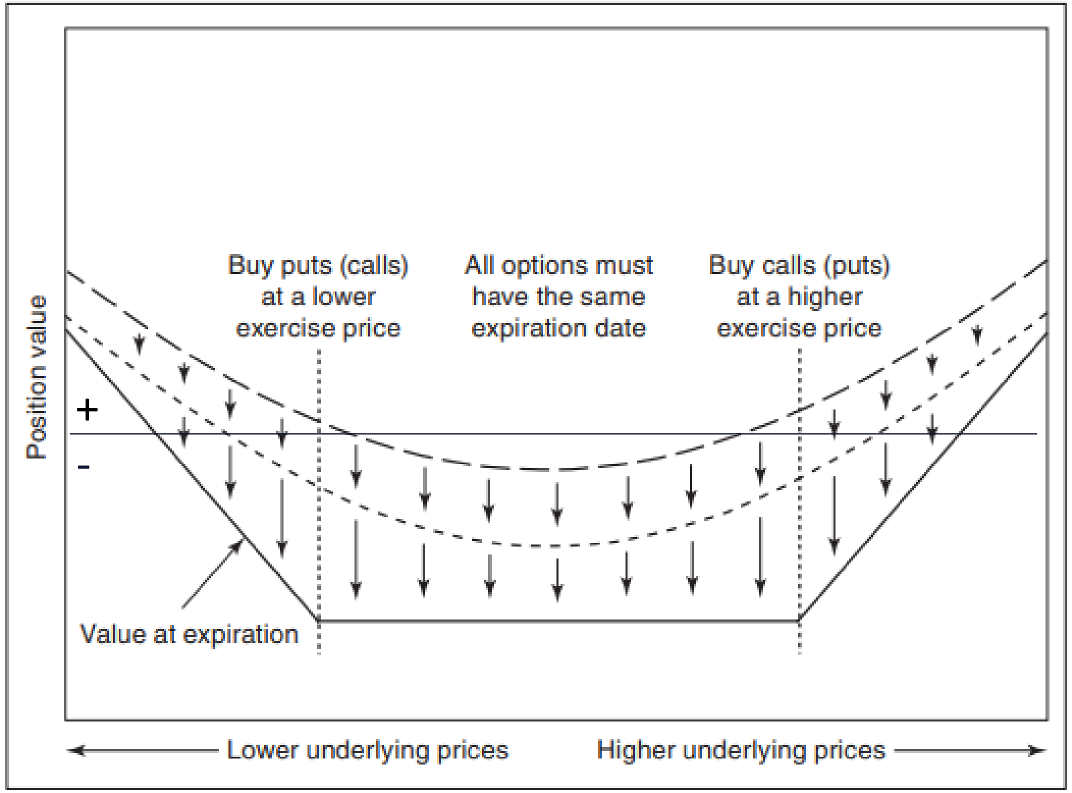
Option Strategies

Strangle

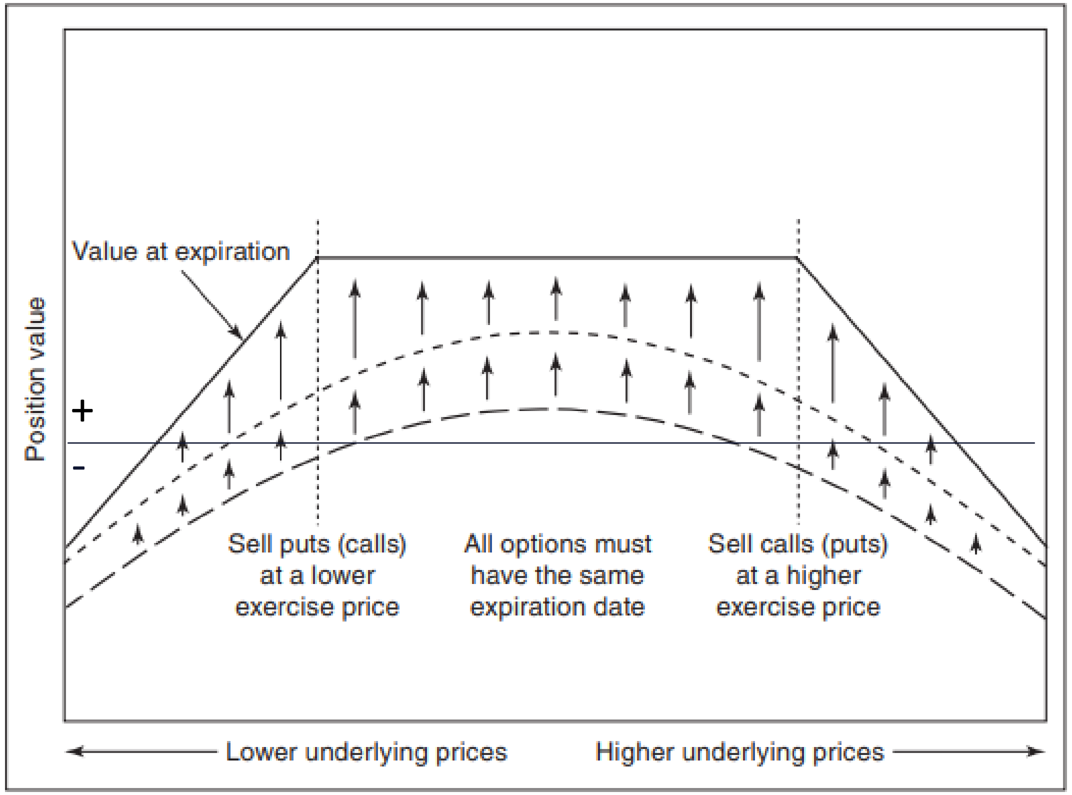
A strangle consists of a long call and a long put (a long strangle) or a short call and a short put (a short strangle), where both options expire at the same time. In a strangle the options have different exercise prices.

Usage of Long (Short) Strangle: same as Straddle but less expensive. You are willing to give up a bit of profit to pay less.

Long Strangle Payoff



Short Strangle Payoff



Option Strategies

Butterfly Spread

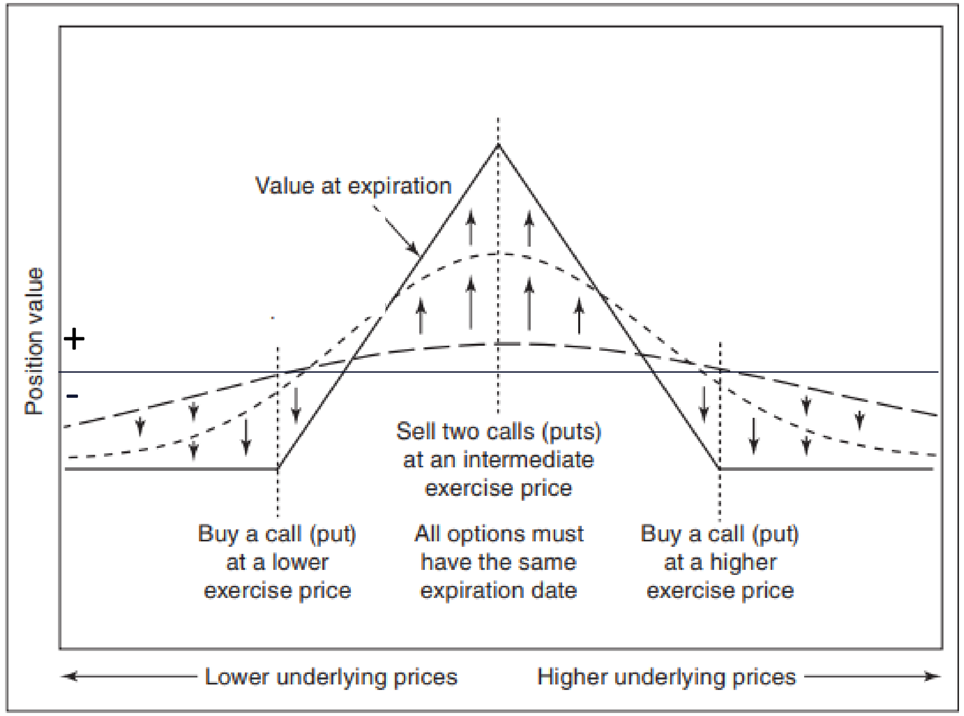
A butterfly is a three-sided spread consisting of options with equally spaced exercise prices, where all options are of the same type (either all calls or all puts) and expire at the same time.

In a long butterfly, the outside exercise prices are purchased and the inside exercise price is sold, and vice versa for a short butterfly.

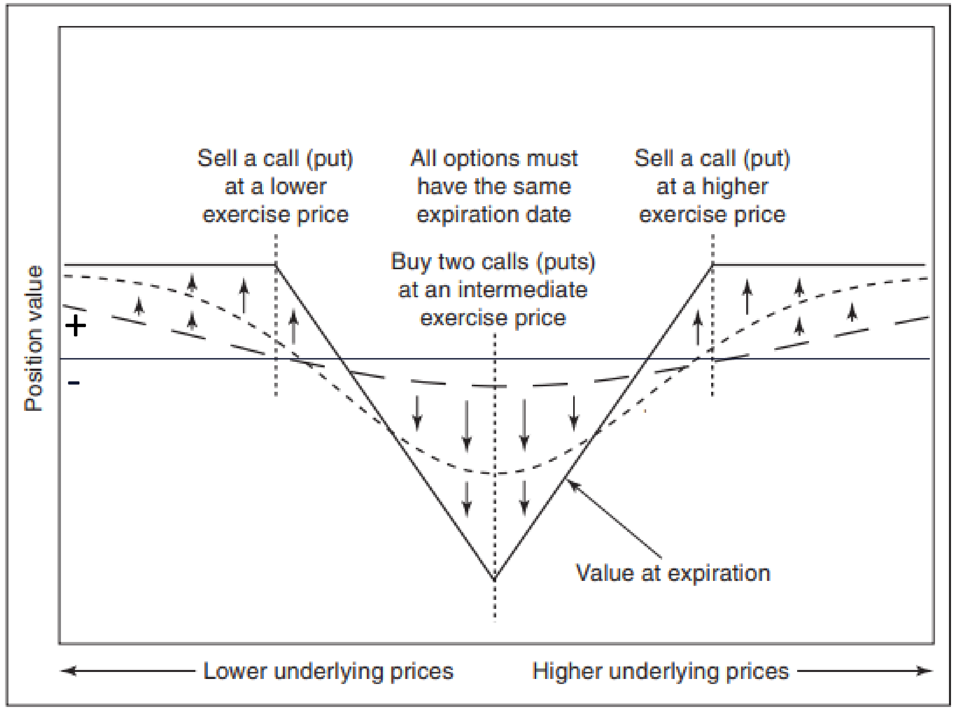
A butterfly position is always 1 × 2 × 1, with two of each inside exercise price traded for each one of the outside exercise prices.

Usage of Long (Short) Butterfly: same as Short (Long) Straddle but less expensive. You are willing to give up a bit of profit to bear less risk.

Long Butterfly Payoff



Short Butterfly Payoff



Exotic Options

Exotic options are non-standard products that have been created to meet some client needs (hedging need in the market, reflect a view on potential future movements in particular market variables, etc.).

Most Common Exotic Options include:

1. **Digital Options:** options with discontinuous payoffs which pay only if $S_T > K$ (in case of call) at expiration. Two common types of Digitals:
   a) **Cash-or-Nothing**: pay a specific amount of money if the condition is met.
   b) **Asset-or-Nothing**: pay the underlying price if the condition is met.

2. **Barrier Options:** are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time. Two common types of Barrier Options:
   a) **Knock-Out**: the option ceases to exist when the underlying asset price reaches a certain barrier.
   b) **Knock-In**: the option comes into existence only when the underlying asset price reaches a barrier.

3. **Asian Options:** options where the payoff depends on the arithmetic average of the price of the underlying asset during the life of the option.

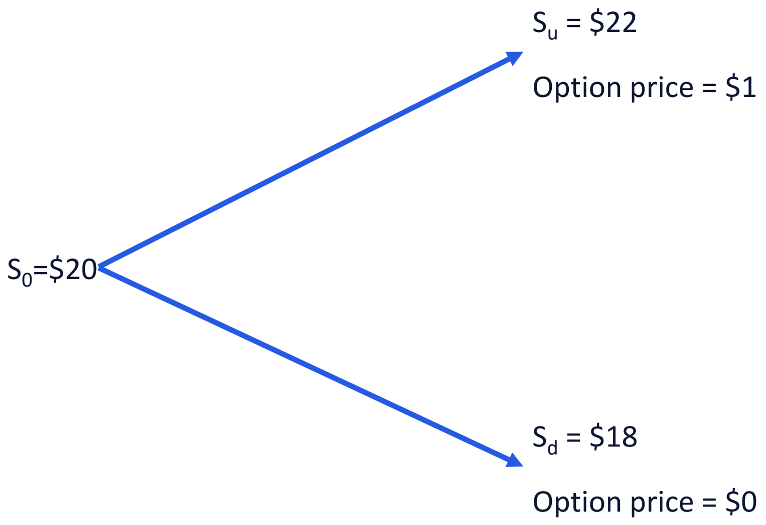
4. **Basket Options:** options where the payoff is dependent on the value of a portfolio (or basket) of assets.

Options – Binomial Trees

Options valuation

A useful and very popular technique for pricing an option involves constructing a binomial tree. This is a diagram representing different possible paths that might be followed by the stock price over the life of an option. The underlying assumption is that the stock price follows **a random walk** (Brownian Motion). In each time step, it has a certain probability of moving up by a certain percentage amount and a certain probability of moving down by a certain percentage amount. In the limit, as the time step becomes smaller, this model is the same as the Black—Scholes—Merton model.

Let`s consider following situation. A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months. This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be zero.



Options – Binomial Trees

Options valuation

How to price such option? This simple exampe may help us to understand how to do so. The only assumption needed is that arbitrage opportunities do not exist. If we set up a portfolio of the stock and the option in such a way that there is no uncertainty about the payoff at the end of the 3 months. We then can argue that, because the portfolio has no risk, the return it earns must equal the risk-free interest rate. This enables us to work out the cost of setting up the portfolio and therefore the option's price. Because there are two securities (the stock and the stock option) and only two possible outcomes, it is always possible to set up the riskless portfolio.

Consider a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option (Δ is the capital Greek letter "delta"). We calculate the value of Δ that makes the portfolio riskless. If the stock price moves up from \$20 to \$22, the value of the shares is 22Δ and the value of the option is 1, so that the total value of the portfolio is $22\Delta - 1$. If the stock price moves down from \$20 to \$18, the value of the shares is 18Δ and the value of the option is zero, so that the total value of the portfolio is 18Δ. The portfolio is riskless if the value of Δ is chosen so that the final value of the portfolio is the same for both alternatives. This means that

$$22\Delta - 1 = 18\Delta$$

Thus:

$$\Delta = 0.25$$

A riskless portfolio is therefore: long 0.25 shares and short 1 option.

If the stock price moves up to \$22, the value of the portfolio is $22 * 0.25 - 1 = 4.5$

If the stock price moves down to \$18, the value of the portfolio is $18 * 0.25 - 0 = 4.5$

Options – Binomial Trees

Options valuation

Regardless of whether the stock price moves up or down, the value of the portfolio is always \$4.5 at the end of the life of the option. This shows that Δ is the number of shares necessary to hedge a short position in one option.

Riskless portfolios must, in the absence of arbitrage opportunities, earn the risk-free rate of interest. Suppose that, in this case, the risk-free rate is 12% per annum. It follows that the value of the portfolio today must be the present value of \$4.5, or:

$$4.5e^{-0.12*0.25} = 4.367$$

The value of the stock price today is known to be \$20. Suppose the option price is denoted by f. The value of the portfolio today is

$$S_0 * \Delta - f = 20*0.25 - f = 4.367$$

So

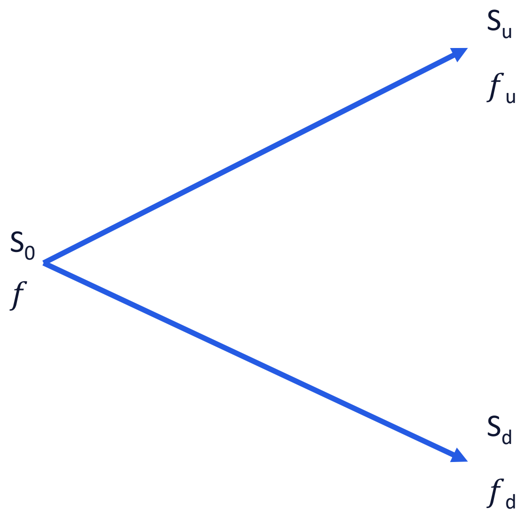
$$f = 0.633$$

This shows that, in the absence of arbitrage opportunities, the current value of the option must be 0.633. If the value of the option were more than 0.633, the portfolio would cost less than \$4.367 to set up and would earn more than the risk-free rate. If the value of the option were less than 0.633, shorting the portfolio would provide a way of borrowing money at less than the risk-free rate.

Options – Binomial Trees

Options valuation

We can generalize the no-arbitrage argument just presented by considering a stock whose price is S_0 and an option on the stock (or any derivative dependent on the stock) whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, where $u > 1$, or down from S_0 to a new level, S_0d, where $d < 1$. The percentage increase in the stock price when there is an up movement is $u-1$; the percentage decrease when there is a down movement is $1-d$. If the stock price moves up to S_0u, we suppose that the payoff from the option is f_u; if the stock price moves down to S_0d, we suppose the payoff from the option is f_d.



Options – Binomial Trees

Options valuation

As before, we imagine a portfolio consisting of a long position in Δ shares and a short position in one option. We calculate the value of A that makes the portfolio riskless. If there is an up movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S_0 u\Delta - f_u$$

If there is a down movement in the stock price, the value becomes

$$S_0 \mathrm{d}\Delta - f_d$$

The two are equal when

$$S_0 u\Delta - f_u = S_0 \mathrm{d}\Delta - f_d$$

Or

$$\Delta = \frac{f_u - f_d}{S_0 \mathrm{u} - S_0 \mathrm{d}}$$

In this case, the portfolio is riskless and, for there to be no arbitrage opportunities, it must earn the risk,-free interest rate. Equation above shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between the nodes at time T.

Options – Binomial Trees

Options valuation

If we denote the risk-free interest rate by r, the present value of the portfolio is

$$(S_0 u\Delta - f_u)e^{-rT}$$

The cost of setting up the portfolio is

$$S_0\Delta - f$$

So

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

Or

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

Substituting Δ , we obtain

$$f = S_0 \frac{f_u - f_d}{S_0\mathrm{u} - S_0\mathrm{d}}(1 - ue^{-rT}) + f_u e^{-rT}$$

Or

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{\mathrm{u} - \mathrm{d}}$$

or

$$f = e^{-rT}[pf_u + (1-p)f_d] \qquad where \qquad \mathrm{p} = \frac{e^{rT} - d}{\mathrm{u} - \mathrm{d}}$$

Options – Binomial Trees

Options valuation

$$f = e^{-rT}[pf_u + (1-p)f_d] \qquad where \qquad \mathrm{p} = \frac{e^{rT} - d}{\mathrm{u} - \mathrm{d}}$$

The above equation enables an option to be priced when stock price movements are given by a one-step binomial tree. The only assumption needed for the equation is that there are no arbitrage opportunities in the market.

In the numerical example considered previously u=1.1, d=0.9, r=0.12, T=0.25, f_u=1, and $f_d = 0$.

$$\mathrm{p} = \frac{e^{rT} - d}{\mathrm{u} - \mathrm{d}} = \mathrm{p} = \frac{e^{0.12*0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

And

$$f = e^{-rT}[pf_u + (1-p)f_d] = e^{-0.12*0.25}[0.6523*1 + 0.3477 * 0] = 0.633$$

The result agrees with the answer obtained earlier.

Options – Binomial Trees

Options valuation – probabilities do not matter

The option pricing formula we come up with does not involve the probabilities of the stock price moving up or down. For example, we get the same option price when the probability of an upward movement is 0.5 as we do when it is 0.9. This is surprising and seems counterintuitive. It is natural to assume that, as the probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option on the stock decreases. This is not the case. The key reason is that we are not valuing the option in absolute terms. We are calculating its value in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the stock price: we do not need to take them into account again when valuing the option in terms of the stock price.

Options – Binomial Trees

Options valuation – risk neutral valuation

A risk-neutral valuation states that, when valuing a derivative, we can make the assumption that investors are risk-neutral. This assumption means investors do not increase the expected return they require from an investment to compensate for increased risk. A world where investors are risk-neutral is referred to as a risk-neutral world. The world we live in is, of course, not a risk-neutral world. The higher the risks investors take, the higher the expected returns they require. However, it turns out that assuming a risk-neutral world gives us the right option price for the world we live in, as well as for a risk-neutral world. Almost miraculously, it finesses the problem that we know hardly anything about the risk aversion of the buyers and sellers of options.

Risk-neutral valuation seems a surprising result when it is first encountered. Options are risky investments. Should not a person's risk preferences affect how they are priced? The answer is that, when we are pricing an option in terms of the price of the underlying stock, risk preferences are unimportant. As investors become more risk averse, stock prices decline, but the formulas relating option prices to stock prices remain the same.

A risk-neutral world has two features that simplify the pricing of derivatives:

1. The expected return on a stock (or any other investment) is the risk-free rate.
2. The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.

Options – Binomial Trees

Options valuation – risk neutral valuation

The parameter p should be interpreted as the probability of an up movement in a risk-neutral world, so that $1 - p$ is the probability of a down movement in this world. The expression

$$pf_u + (1 - p)f_d$$

is the expected future payoff from the option in a risk-neutral world and equation

$$f = e^{-rT}[pf_u + (1 - p)f_d]$$

states that the value of the option today is its expected future payoff in a risk-neutral world discounted at the risk-free rate. This is an application of risk-neutral valuation.

Options – Binomial Trees

Options valuation – risk neutral valuation

To prove the validity of our interpretation of p, we note that, when p is the probability of an up movement, the expected stock price $E(S_T)$ at time T is given by

$$\mathrm{E}(S_T) = pS_0u + (1-p)S_0d \qquad or \qquad \mathrm{E}(S_T) = pS_0(u-d) + S_0d$$

Using $\mathrm{p} = \frac{e^{rT}-d}{\mathrm{u-d}}$

$$\mathrm{E}(S_T) = S_0e^{rT}$$

This shows that the stock price grows, on average, at the risk-free rate when p is the probability of an up movement. In other words, the stock price behaves exactly as we would expect it to behave in a risk-neutral world when p is the probability of an up movement.

Risk-neutral valuation is a very important general result in the pricing of derivatives. It states that, when we assume the world is risk-neutral, we get the right price for a derivative in all worlds, not just in a risk-neutral one. We have shown that risk-neutral valuation is correct when a simple binomial model is assumed for how the price of the the stock evolves. It can be shown that the result is true regardless of the assumptions we make about the evolution of the stock price.

To apply risk-neutral valuation to the pricing of a derivative, we first calculate what the probabilities of different outcomes would be if the world were risk-neutral. We then calculate the expected payoff from the derivative and discount that expected payoff at the risk-free rate of interest.

Options – Binomial Trees

Options valuation – risk neutral valuation

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Options – Binomial Trees

Options valuation – risk neutral valuation

Let`s check if risk-neutral valuation gives the same answer as no-arbitrage arguments. Following our previous example. The stock price is currently $20 and will move either up to $22 or down to $18 at the end of 3 months. The option considered is a European call option with a strike price of $21 and an expiration date in 3 months. The risk-free interest rate is 12% per annum.

We define p as the probability of an upward movement in the stock price in a risk neutral world. We can calculate p from equation we delivered before. Alternatively, we can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 12%. This means that p must satisfy

$$22p + 18(1-p) = 20e^{0.12*0.25}$$

Or

$$4p = 20e^{0.12*0.25} - 18$$

That is, p must be 0.6523.

At the end of the 3 months, the call option has a 0.6523 probability of being worth 1 and a 0.3477 probability of being worth zero. Its expected value is therefore

$$0.6523 * 1 + 0.3477 * 0 = 0.6523$$

In a risk-neutral world this should be discounted at the risk-free rate. The value of the option today is therefore

$$0.6523e^{-0.12*0.25} = \$0.633$$

This is the same as the value obtained earlier, demonstrating that no arbitrage arguments and risk-neutral valuation give the same answer.

Options – Binomial Trees

Risk neutral vs. Real world

It should be emphasized that p is the probability of an up movement in a risk-neutral world. In general, this is not the same as the probability of an up movement in the real world. In our example $p = 0.6523$. When the probability of an up movement is 0.6523, the expected return on both the stock and the option is the risk-free rate of 12%. Suppose that, in the real world, the expected return on the stock is 16% and p^* is the probability of an up movement in this world. It follows that

$$22p^* + 18(1 - p^*) = 20e^{0.16*0.25}$$

So that, $p^* = 0.7041$

The expected payoff from the option in the real world is then given by

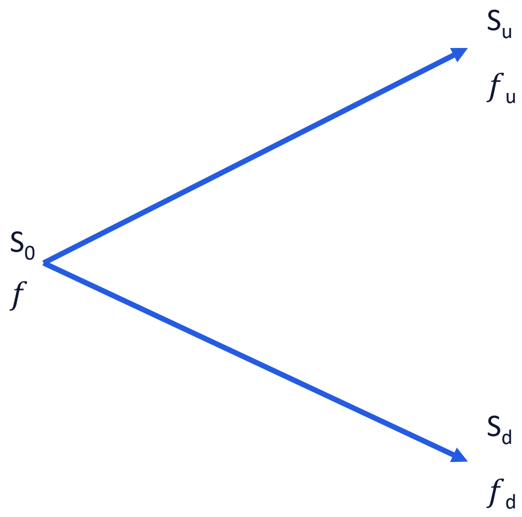
$$p^* * 1 + (1 - p^*) * 0$$

or 0.7041. Unfortunately, it is not easy to know the correct discount rate to apply to the expected payoff in the real world. The return the market requires on the stock is 16% and this is the discount rate that would be used for the expected cash flows from an investment in the stock. A position in a call option is riskier than a position in the stock. As a result, the discount rate to be applied to the payoff from a call option is greater than 16%, but we do not know how much greater than 16% it should be. Using risk-neutral valuation solves this problem because we know that in a risk-neutral world the expected return on all assets (and therefore the discount rate to use for all expected payoffs) is the risk-free rate.

Options – Binomial Trees

Options valuation

We can generalize the no-arbitrage argument just presented by considering a stock whose price is S_0 and an option on the stock (or any derivative dependent on the stock) whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, where $u > 1$, or down from S_0 to a new level, S_0d, where $d < 1$. The percentage increase in the stock price when there is an up movement is $u-1$; the percentage decrease when there is a down movement is $1-d$. If the stock price moves up to S_0u, we suppose that the payoff from the option is f_u; if the stock price moves down to S_0d, we suppose the payoff from the option is f_d.



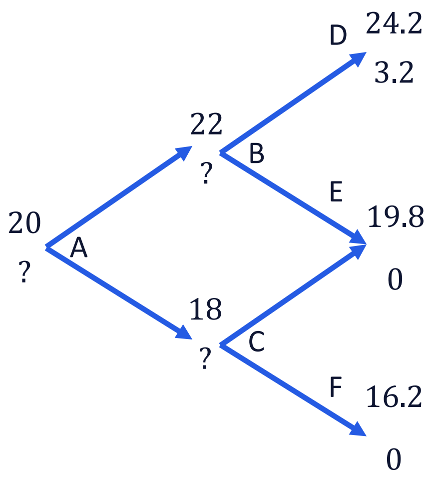
Options – Binomial Trees

Options valuation

Of course, we can extend one step analysis to a two-step binomial tree. Here the stock price starts at $20 and in each of two time steps may go up by 10% or down by 10%. Each time step is 3 months long and the risk-free interest rate is 12% per annum. We consider a 6-month option with a strike price of $21.

The objective of the analysis is to calculate the option price at the initial node of the tree. This can be done by repeatedly applying the principles established earlier.

The option prices at the final nodes of the tree are easily calculated. They are the payoffs from the option. At node D the stock price is 24.2 and the option price is $24.2 - 21 = 3.2$; at nodes E and F the option is out of the money and its value is zero.

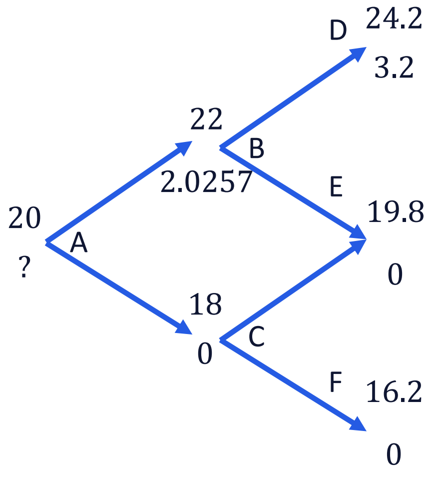


Options – Binomial Trees

Options valuation

At node C the option price is zero, because node C leads to either node E or node F and at both of those nodes the option price is zero. We calculate the option price at node B by focusing our attention on the BDE part of the tree. Using the notation introduced earlier, $u = 1.1$, $\mathrm{d} = 0.9$, $\mathrm{r} = 0.12$, $\mathrm{T} = 0.25$, so that $\mathrm{p} = 0.6523$, and the value of the option at node B is

$$f = e^{-rt}(p * f_u + (1-p) * f_d) = e^{-0.12*0.25}(0.6523 * 3.2 + (1-0.6523) * 0) = 2.0257$$



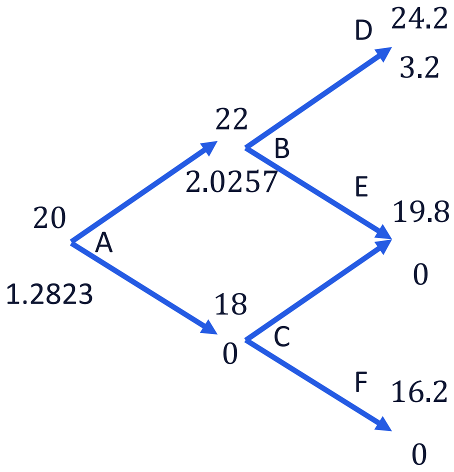
Options – Binomial Trees

Options valuation

It remains for us to calculate the option price at the initial node A. We do so by focusing on the first step of the tree. We know that the value of the option at node B is 2.0257 and that at node C it is zero. Therefore, the value at node A is

$$f = e^{-rt}(p * f_u + (1-p) * f_d) = e^{-0.12*0.25}(0.6523 * 2.0257 + (1-0.6523) * 0) = 1.2823$$

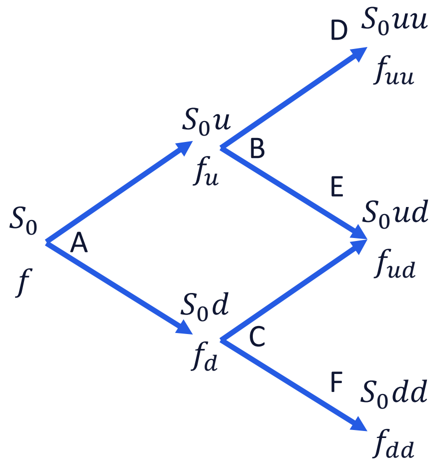
The value of the option is $1.2823. Note that this example was constructed so that u and d (the proportional up and down movements) were the same at each node of the tree and so that the time steps were of the same length. As a result, the risk-neutral probability, p, is the same at each node.



Options – Binomial Trees

Options valuation - Generalization

We can generalize the case of two time steps by considering the above situation. The stock price is initially S_0. During each time step, it either moves up to u times its initial value or moves down to d times its initial value. The notation for the value of the option is shown on the tree. (For example, after two up movements the value of the option is f_{uu}.) We suppose that the risk-free interest rate is r and the length of the time step is Δt years.



Options – Binomial Trees

Options valuation - Generalization

Because the length of a time step is now Δt rather than T, earlier delivered equations become

$$f = e^{-r\Delta t}[p * f_u + (1-p) * f_d]$$

$$p = \frac{e^{-r\Delta t} - d}{u - d}$$

Repeated application of above equation gives

$$f_u = e^{-r\Delta t}[p * f_{uu} + (1-p) * f_{ud}]$$
$$f_d = e^{-r\Delta t}[p * f_{ud} + (1-p) * f_{dd}]$$

Thus,

$$f = e^{-2r\Delta t}[p^2 * f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 * f_{dd}]$$

This is consistent with the principle of risk-neutral valuation mentioned earlier. The variables p^2, $2p(1-p)$, $(1-p)^2$ are the probabilities that the upper, middle, and lower final nodes will be reached. The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate. As we add more steps to the binomial tree, the risk-neutral valuation principle continues to hold. The option price is always equal to its expected payoff in a risk neutral world discounted at the risk-free interest rate.

Options – Binomial Trees

Options valuation – Put

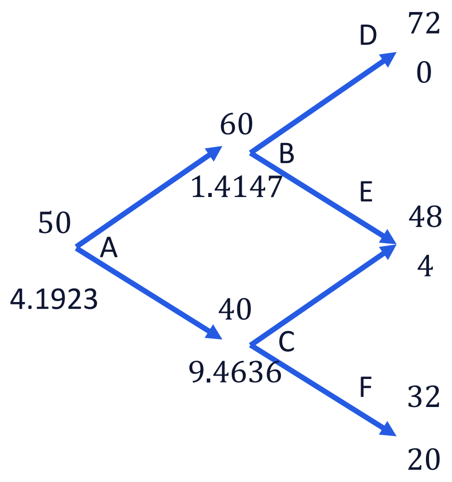
The procedures described can be used to price puts as well as calls. Consider a 2-year European put with a strike price of $52 on a stock whose current price is $50. We suppose that there are two time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. We also suppose that the risk-free interest rate is 5%. In this case $u = 1.2$, $\mathrm{d} = 0.8$, $\mathrm{r} = 0.05$, $\Delta\mathrm{t} = 1$.

The value of the risk-neutral probability, p, is given by

$$p = \frac{e^{-r\Delta t} - d}{u - d} = \frac{e^{-0.05*1} - 0.8}{1.2 - 0.8} = 0.6282$$

The value of the option is

$$f = e^{-2r\Delta t}[p^2 * f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 * f_{dd}] = e^{-2*0.05*1}[0.6282^2 * 0 + 2 * 0.6282(1 - 0.6282) * 4 + (1 - 0.6282)^2 * 20] = 4.1923$$



Options – Binomial Trees

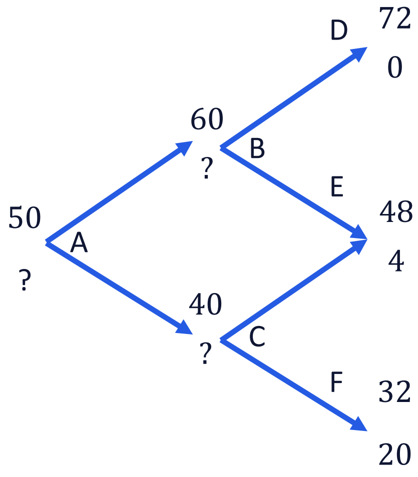
Options valuation – American Options

Up to now all the options we have considered have been European. We now move on to consider how American options can be valued using a binomial tree. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of

1. The value given by $f = e^{-r\Delta t}[p * f_u + (1-p) * f_d]$

2. The payoff from early exercise.

In case of American options the stock prices and their probabilities are unchanged. The values for the option at the final nodes are also unchanged. Thus, for put option we considered earlier two step tree looks as follow:



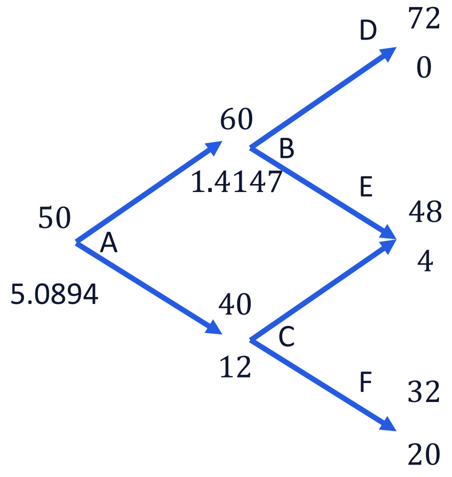
Options – Binomial Trees

Options valuation – American Options

At node B, equation $f = e^{-r\Delta t}[p * f_u + (1-p) * f_d]$ gives the value of the option as 1.4147, whereas the payoff from early exercise is negative (-8). Clearly early exercise is not optimal at node B, and the value of the option at this node is 1.4147. At node C, the value of the option is 9.4636, whereas the payoff from early exercise is 12. In this case, early exercise is optimal and the value of the option at the node is 12. At the initial node A, the value is:

$$f = e^{-r\Delta t}[p * f_u + (1-p) * f_d] = e^{-0.05*1}[0.6282 * 1.4147 + 0.3718 * 12] = 5.0894$$

and the payoff from early exercise is 2. In this case early exercise is not optimal. The value of the option is therefore $5.0894.



Options – Binomial Trees

Delta

At this stage, it is appropriate to introduce delta, an important parameter (sometimes referred to as a ‘‘Greek letter’’ or simply a ‘‘Greek’’) in the pricing and hedging of options. The delta (Δ) of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. It is the number of units of the stock we should hold for each option shorted in order to create a riskless portfolio. The construction of a riskless portfolio is sometimes referred to as delta hedging. The delta of a call option is positive, whereas the delta of a put option is negative.

We can calculate the value of the delta of the call option being considered in the first binomial tree example:

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} = \frac{1 - 0}{22 - 18} = 0.25$$

This is because when the stock price changes from \$18 to \$22, the option price changes from \$0 to \$1.

Options – Binomial Trees

Delta

For the two step tree on the call option example the delta corresponding to stock price movements over the first time step is:

$$\Delta_1 = \frac{f_u - f_d}{S_0 u - S_0 d} = \frac{2.0257 - 0}{22 - 18} = 0.5064$$

The delta for stock price movements over the second time step, if there is an upward movement over the first time step, is

$$\Delta_{2u} = \frac{f_{uu} - f_{ud}}{S_0 uu - S_0 ud} = \frac{3.2 - 0}{24.2 - 19.8} = 0.7273$$

And if there is a downward movement over the first time step:

$$\Delta_{2d} = \frac{f_{du} - f_{dd}}{S_0 du - S_0 dd} = \frac{0 - 0}{19.8 - 16.2} = 0$$

Options – Binomial Trees

Delta

For the two step tree on the put option example the delta corresponding to stock price movements over the first time step is:

$$\Delta_1 = \frac{f_u - f_d}{S_0 u - S_0 d} = \frac{1.4147 - 9.4636}{60 - 40} = -0.4024$$

The delta for stock price movements over the second time step, if there is an upward movement over the first time step, is

$$\Delta_{2u} = \frac{f_{uu} - f_{ud}}{S_0 uu - S_0 ud} = \frac{0 - 4}{72 - 48} = -0.1667$$

And if there is a downward movement over the first time step:

$$\Delta_{2d} = \frac{f_{du} - f_{dd}}{S_0 du - S_0 dd} = \frac{4 - 20}{48 - 32} = -1$$

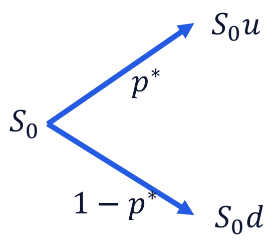
The two-step examples show that delta changes over time. In call example, delta changes from 0.5064 to either 0.7273 or 0; and, in put example, it changes from -0.4024 to either -0.1667 or -1.) Thus, in order to maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically.

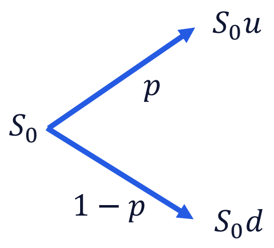
Options – Binomial Trees

Options valuation – Matching Volatility with u and d

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters u and d to match the volatility of the stock price. A question that arises is whether we should match volatility in the real world or the risk-neutral world. As we will now show, this does not matter. For small Δt and particular values of u and d, the volatility being assumed is the same in both the real world and the risk-neutral world.

Left figure shows stock price movements over one step of a binomial tree in the real world and right one shows these movements in a risk-neutral world. The step is of length Δt. The stock price starts at S_0 and moves either up to S_0u or down to S_0d. These are the only two possible outcomes in both the real world and the risk-neutral world. The probability of an up movement in the real world is denoted by p^* and, consistent with our earlier notation, in the risk-neutral world this probability is p.





Options – Binomial Trees

Options valuation – Matching Volatility with u and d

The expected stock price at the end of the first time step in the real world is $S_0 e^{\mu \Delta t}$, where μ is the expected return. On the tree the expected stock price at this time is

$$p^* S_0 u + (1 - p^*) S_0 d$$

In order to match the expected return on the stock with the tree's parameters, we must therefore have

$$p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\mu \Delta t} \quad or \quad p^* = \frac{e^{\mu \Delta t} - d}{u - d}$$

As we will explain later, the volatility σ of a stock price is defined so that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time of length Δt. Equivalently, the variance of the return is $\sigma^2 \Delta t$. On the real world tree, the variance of the stock price return is

$$E(X^2) - [E(X)]^2 = p^* u^2 + (1 - p^*) d^2 - [p^* u + (1 - p^*) d\,]^2$$

In order to match the stock price volatility with the tree's parameters, we must therefore have

$$p^* u^2 + (1 - p^*) d^2 - [p^* u + (1 - p^*) d\,]^2 = \sigma^2 \Delta t$$

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In order to match the stock price volatility with the tree's parameters, we must therefore have

$$p^* u^2 + (1 - p^*) d^2 - [p^* u + (1 - p^*) d\,]^2 = \sigma^2 \Delta t$$

Knowing that $p^* = \frac{e^{\mu \Delta t} - d}{u - d}$

$$e^{\mu \Delta t}(u + d) - ud - e^{2\mu \Delta t} = \sigma^2 \Delta t$$

Using Taylor expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

When terms in Δt^2 and higher powers of Δt are ignored, one solution to this equation is

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = e^{-\sigma \sqrt{\Delta t}}$$

Options – Binomial Trees

Options valuation – Matching Volatility with u and d

In risk neutral world, the expected stock price at the end of the time step is $S_0e^{r\Delta t}$. The variance of the stock price return in the risk-neutral world is

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}$$

Subsituting $\mathrm{u} = e^{\sigma\sqrt{\Delta t}}$ and $\mathrm{d} = e^{-\sigma\sqrt{\Delta t}}$, we find this equals $\sigma^2\Delta t$ when terms in Δt^2 and higher powers of Δt are ignored.

This analysis shows that when we move from the real world to the risk-neutral world the expected return on the stock changes, but its volatility remains the same (at least in the limit as Δt tends to zero). This is an illustration of an important general result known as **Girsanov's theorem**. When we move from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rates in variables change, but their volatilities remain the same. Moving from one set of risk preferences to another is sometimes referred to as changing the measure or change of numeraire. The real-world measure is sometimes referred to as the P-measure, while the risk-neutral world measure is referred to as the Q-measure. With the notation we have been using, p is the probability under the Q-measure, while p^* is the probability under the P-measure.

Options – Binomial Trees

Options valuation – Increasing the number of steps

The binomial model presented above is unrealistically simple. Clearly, an analyst can expect to obtain only a very rough approximation to an option price by assuming that stock price movements during the life of the option consist of one or two binomial steps.

When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps. In each time step there is a binomial stock price movement. With 30 time steps there are 31 terminal stock prices and 2^{30}, or about 1 billion, possible stock price paths are implicitly considered.

The equations, we delivered, defining the tree holds regardless of the number of time steps. As the number of time steps is increased (so that Δt becomes smaller), the binomial tree model makes the same assumptions about stock price behavior as the Black–Scholes–Merton model. When the binomial tree is used to price a European option, the price converges to the Black–Scholes–Merton price, as expected, as the number of time steps is increased.

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

One way of deriving the famous Black–Scholes–Merton result for valuing a European option on a non-dividend-paying stock is by allowing the number of time steps in a binomial tree to approach infinity.

Suppose that a tree with n time steps is used to value a European call option with strike price K and life T. Each step is of length T/n. If there have been n upward movements and $n-j$ downward movements on the tree, the final stock price is $S_0 u^j d^{n-j}$, where u is the proportional up movement, d is the proportional down movement, and S_0 is the initial stock price. The payoff from a European call option is then

$$\max\left(S_0 u^j d^{n-j} - K, 0\right)$$

From the properties of the binomial distribution, the probability of exactly j upward and $n-j$ downward movements is given by

$$\frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \ \ \textit{where p is probability of up jump}$$

It follows that the expected payoff from the call option is

$$\sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

As the tree represents movements in a risk-neutral world, we can discount this at the risk-free rate r to obtain the option price:

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$$

The terms in above equation re nonzero when the final stock price is greater than the strike price, that is, when

$$S_0 u^j d^{n-j} > K$$

Or

$$\ln\left({S_0}/{K}\right) > j * ln(u) - (n-j)\ln(d)$$

Since $u = e^{\sigma\sqrt{T/n}}$ and $\mathrm{d} = e^{-\sigma\sqrt{T/n}}$, this condition becomes

$$\ln\left({S_0}/{K}\right) > n\sigma\sqrt{T/n} - 2j\sigma\sqrt{T/n} \quad or \quad j > \frac{n}{2} - \frac{\ln({S_0}/{K})}{2\sigma\sqrt{T/n}}$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

Therefore formula to the option price can be written as:

$$c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K)$$

Where $\alpha = \frac{n}{2} - \frac{\ln({}^{S_0}/_{K})}{2\sigma\sqrt{{}^{T}/_{n}}}$

For convenience, we define

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

And

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j}$$

So that

$$c = e^{-rT} (S_0 U_1 - K U_2)$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

Consider first U_2. As is well known, the binomial distribution approaches a normal distribution as the number of trials approaches infinity. Specifically, when there are n trials and p is the probability of success, the probability distribution of the number of successes is approximately normal with mean np and standard deviation $\sqrt{np(1-p)}$. The variable U_2 is the probability of the number of successes being more than α. From the properties of the normal distribution, it follows that, for large n,

$$U_2 = N\left(\frac{np-\alpha}{\sqrt{np(1-p)}}\right)$$

where N is the cumulative normal distribution function. Substituting for α, we obtain

$$U_2 = N\left(\frac{\ln({}^{S_0}/_K)}{2\sigma\sqrt{T}\sqrt{p(1-p)}} + \frac{\sqrt{n}(p-\frac{1}{2})}{\sqrt{p(1-p)}}\right)$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

From equations for p (in Q-measure), u and d, we have

$$p = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

By expanding the exponential functions in a series, we see that, as n tends to infinity, $p(1-p)$ tends to $\frac{1}{4}$ and $\sqrt{n}\left(p - \frac{1}{2}\right)$ tends to $\frac{(r - \sigma^2/2)\sqrt{T}}{2\sigma}$. So that in the limit, as n tends to infinity, U_2 becomes

$$U_2 = N\left(\frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

Now let`s move on to evaluate U_1

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} pu^j [(1-p)d]^{n-j}$$

Define

$$p^{**} = \frac{pu}{pu + (1-p)d}$$

It then follows that

$$1 - p^{**} = \frac{(1-p)d}{pu + (1-p)d}$$

So we can rewrite U_1 into

$$U_1 = pu + (1-p)d^n \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^{**} u^j [(1-p^{**})d]^{n-j}$$

Since the expected return in the risk-neutral world is the risk-free rate r, it follows that $pu + (1-p)d = e^{rT/n}$ and

$$U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^* u^j [(1-p^*)d]^{n-j}$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

This shows that U_1 involves a binomial distribution where the probability of an up movement is p^{**} rather than p. Approximating the binomial distribution with a normal distribution, we obtain

$$U_1 = e^{rT} N\left(\frac{np^{**} - \alpha}{\sqrt{np^{**}(1-p^{**})}}\right)$$

and substituting for α gives,

$$U_1 = e^{rT} N\left(\frac{\ln\left({S_0}/{K}\right)}{2\sigma\sqrt{T}\sqrt{p^{**}(1-p^{**})}} + \frac{\sqrt{n}(p^{**} - \frac{1}{2})}{\sqrt{p^{**}(1-p^{**})}}\right)$$

Substituting for u and d in equation for p^{*}

$$p^{**} = \left(\frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}\right)\left(\frac{e^{\sigma\sqrt{T/n}}}{e^{rT/n}}\right)$$

Black-Scholes-Merton Model Derivation from Binomial Trees

Options valuation – B-S model derivation

By expanding the exponential functions in a series we see that, as n tends to infinity, $p^{**}(1-p^{**})$ tends to $\frac{1}{4}$ and $\sqrt{n}\left(p^{**}-\frac{1}{2}\right)$ tends to $\frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma}$. So that in the limit, as n tends to infinity, U_1 becomes

$$U_1 = e^{rT} N\left(\frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

So using our formula for call option, U_1 and U_2, we have

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

Where

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

And

$$d_2 = \frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

This is the Black–Scholes–Merton formula for the valuation of a European call option.

Wiener Processes and Ito`s Lemma

Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. Stochastic processes can be classified as discrete time or continuous time.

A discrete-time stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a **continuous-time stochastic process** is one where changes can take place at any time. Stochastic processes can also be classified as continuous variable or discrete variable. In a continuous-variable process, the underlying variable can take any value within a certain range, whereas in a discrete variable process, only certain discrete values are possible.

Now we will consider a continuous-variable, continuous-time stochastic process for stock prices. Learning about this process is the first step to understanding the pricing of options and other more complicated derivatives. It should be noted that, in practice, we do not observe stock prices following continuous-variable, continuous time processes. Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open for trading. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes.

Wiener Processes and Ito`s Lemma

The Markov Propery

A Markov process is a particular type of stochastic process where only the current value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant.

Stock prices are usually assumed to follow a Markov process. Suppose that the price of IBM stock is $100 now. If the stock price follows a Markov process, our predictions for the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is that the price is now $100. Predictions for the future are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

The Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past prices. If the weak form of market efficiency were not true, technical analysts could make above-average returns by interpreting charts of the past history of stock prices. There is very little evidence that they are in fact able to do this.

It is competition in the marketplace that tends to ensure that weak-form market efficiency holds. There are many investors watching the stock market closely. Trying to make a profit from it leads to a situation where a stock price, at any given time, reflects the information in past prices. Suppose that it was discovered that a particular pattern in stock prices always gave a 65% chance of subsequent steep price rises. Investors would attempt to buy a stock as soon as the pattern was observed, and demand for the stock would immediately rise. This would lead to an immediate rise in its price and the observed effect would be eliminated, as would any profitable trading opportunities.

Wiener Processes and Ito`s Lemma

Square root of time rule

Consider a variable that follows a Markov stochastic process. Suppose that its current value is 10 and that the change in its value during a year is $\phi(0,1)$, where $\phi(m, v)$, denotes a probability distribution that is normally distributed with mean m and variance v. What is the probability distribution of the change in the value of the variable during 2 years?

The change in 2 years is the sum of two normal distributions, each of which has a mean of zero and variance of 1. Because the variable is Markov, the two probability distributions are independent. When we add two independent normal distributions, the result is a normal distribution where the mean is the sum of the means and the variance is the sum of the variances. The mean of the change during 2 years in the variable we are considering is, therefore, zero and the variance of this change is 2. Hence, the change in the variable over 2 years has the distribution $\phi(0,2)$. The standard deviation of the distribution is $\sqrt{2}$.

Consider next the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months. We assume these are the same. It follows that the variance of the change during a 6-month period must be 0.5. Equivalently, the standard deviation of the change is $\sqrt{0.5}$. The probability distribution for the change in the value of the variable during 6 months is $\phi(0,0.5)$.

A similar argument shows that the probability distribution for the change in the value of the variable during 3 months is $\phi(0,0.25)$. More generally, the change during any time period of length T is $\phi(0, T)$. In particular, the change during a very short time period of length Δt is $\phi(0, \Delta t)$.

Note that, when Markov processes are considered, the variances of the changes in successive time periods are additive. The standard deviations of the changes in successive time periods are not additive. The variance of the change in the variable in our example is 1 per year, so that the variance of the change in 2 years is 2 and the variance of the change in 3 years is 3. The standard deviations of the changes in 2 and 3 years are $\sqrt{2}$ and $\sqrt{3}$, respectively. Strictly speaking, we should not refer to the standard deviation of the variable as 1 per year. The results explain why uncertainty is sometimes referred to as being proportional to the square root of time.

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