Numerical Mathematics II: Boundary Value Problem

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Definition

In the following we consider a
non-linear two point boundary value problem (BCP):

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ R(y(a), y(b)) &= 0 \end{aligned}$$

with $R : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

Different kinds of BVP

- **higher order BVP**

$$
\begin{aligned}
y^{(m)}(x) &= f(t, y(x), ..., y^{(m-1)}(x)) \\
R(y(a), y(b)) &= 0
\end{aligned}
$$

- **linear BVP**

$$
\begin{aligned}
y'(x) &= A(x)y(x) + q(x); \quad a \leq x \leq b \\
R(y(a), y(b)) &= B_a y(a) + B_b y(b) - c = 0, \qquad B_a, B_b \in \mathbb{R}^{n \times n}
\end{aligned}
$$

- **BVP with separate boundary values**

$$
\begin{aligned}
R_1(y(a)) &= 0, \quad R_1 : \mathbb{R}^n \to \mathbb{R}^p \\
R_2(y(b)) &= 0, \quad R_2 : \mathbb{R}^n \to \mathbb{R}^q, \qquad p + q = n
\end{aligned}
$$

Different kinds of BVP (cont'd)

- **free BVPs**

$$\begin{aligned} y'(x) &= f(x, y(x)), \quad a \le x \le b, \ b \text{ free} \\ R(y(a), y(b)) &= 0 \end{aligned}$$

- **multiple BCs**

$$y'(x) = f(x, y(x)), y \in \mathbb{R}^n \quad a \le x \le b$$
$$0 = R(y(x_0), ..., y(x_s))$$
$$\text{where } a = x_0 < x_1 < ... < x_s = b$$

Picard Lindelöf for BVPs ?

$$y'(x) = f(x, y(x)), \quad R(y(a), y(b)) = 0 \qquad \textbf{(III.1)}$$

Definition (III.1)

A sloution $y(x)$ of the BVP (III.1) is **local unique**, if there ex itst a neighbourhood around $y(x)$, such that $y(x)$ is the only solution, i.e.

$$\exists \delta > 0 : \forall u(x) \neq y(x) : \{\|u - y\| \geq \delta \vee u \text{ is no soution of } \textbf{(III.1)}\}$$

Isolated Solution & Variational Problem

$$y'(x) = f(x, y(x)), \quad R(y(a), y(b)) = 0 \qquad \textbf{(III.1)}$$

Definition (III.2)

A sloution $y(x)$ of the BVP (III.1) is **isolated**, if the *variational problem*

$$z' = \frac{\partial f(x, y(x))}{\partial y} z \qquad \textbf{(III.2)}$$

$$\frac{\partial R(y(a), y(b))}{\partial y(a)} z(a) + \frac{\partial R(y(a), y(b))}{\partial y(b)} z(b) = 0 \ \textbf{(III.3)}$$

has the unique solution $z(x) \equiv 0$.

Theorem (III.3)

Let $f \in C^2$, such that (III.2) is well defined. If then y is an isolated solution of (III.1), then it is also unique.

reminder:

Definition (III.2)

A sloution $y(t)$ of the BVP (III.1) is **isolated**, if the *variational problem* (**III.2+3**)

$$z' = \frac{\partial f(t, y(x))}{\partial y} z$$

$$\frac{\partial R(y(a), y(b))}{\partial y(a)} z(a) + \frac{\partial R(y(a), y(b))}{\partial y(b)} z(b) = 0$$

has the unique solution $z(x) \equiv 0$.

need to show:

$$z' = \frac{\partial f(x, y(x))}{\partial y} z$$

$$\underbrace{\frac{\partial R(y(a), y(b))}{\partial y(a)}}_{:=B_a} z(a) + \underbrace{\frac{\partial R(y(a), y(b))}{\partial y(b)}}_{:=B_b} z(b) = 0$$

has a unique solution.

little more general:

Theorem (III.4)

Let A and q be continuous on $[a,b]$. The linear BVP

$$\begin{aligned} y'(x) &= A(x)y(x) + q(x); \quad a \le x \le b \\ R(y(a), y(b)) &= B_a y(a) + B_b y(b) - c = 0, \end{aligned}$$

has a unique solution, iff

$$Q := B_a Y(a) + B_b Y(b)$$

is non singular, where $Y(x)$ is a fundamental matrix of the homogeneous ODE

$$\mathbf{y}'(x) = \mathbf{A}(x)\mathbf{y}(x)$$