Many processes, including the spread of an infectious disease through a small community, can be modeled as first-order exponential processes like

$$\frac{dV}{dt} = \frac{1}{\tau}(R-1)V, \qquad V(0) = 1 \tag{1}$$

where V is the tumor volume, measured in number of cells, and R is a constant.

This will either lead to exponential growth or exponential decay.

The constant R is different for every patient. Assume it has Gaussian distribution with mean 1 and standard deviation σ,

$$p_R(r) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(r-1)^2/2\sigma^2}.$$

i. Find the probability density function $p_V(v,t)$ of $V(t)$.

Intuitively, we expect half of the trajectories to grow exponentially, and half of the trajectories to decay exponentially.

ii. Sketch or plot the probability density you found for $p_V(v,t)$.

iii. What is the probability that a trajectory is above the initial condition at $V = 1$? In other words, what is $\mathbb{P}(V(t) > 1)$? Is it true that half the trajectories remain above the initial condition $V = 1$, and half remain below the initial condition $V = 1$?

iv. Suppose $\tau = 1$ months and $\sigma = 0.1$. What percent of patients have a tumor with more than 1000 cells after 10 months?

A slightly more complicated model that is a modified version of Equation 1, called the Gompertz model, is used to fit patient data.