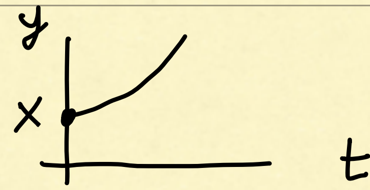
# HETEROGENEITY / PARAMETRIC NOISE

EX

$$\frac{dy}{dt} = Ay$$

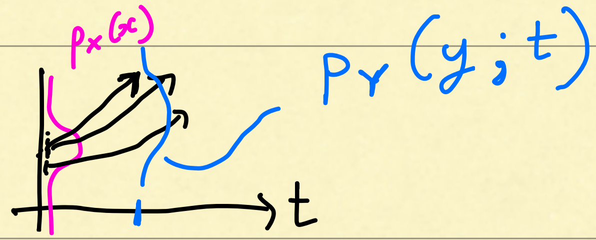
$$y(0) = X$$




$$y(t) = X e^{At}$$

SUPPOSE

$$X \sim p_X(x)$$




$$Y \sim p_Y(y;t)$$

WHAT IS $p_Y(y;t)$ ?

---

ASIDE SUPPOSE $X \sim p_X(x)$

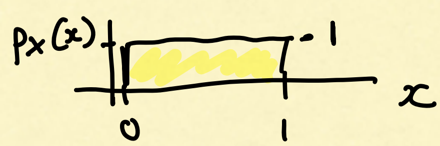
$$Y = g(X)$$

KNOWN FUNCTION

THEN WHAT IS $p_Y(y)$ ?

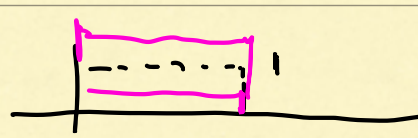
EX $Y = cX$

$X \sim$ UNIF $(0,1)$




IT IS **NOT** TRUE THAT

$$p_Y(y) = c\, p_X(x)$$



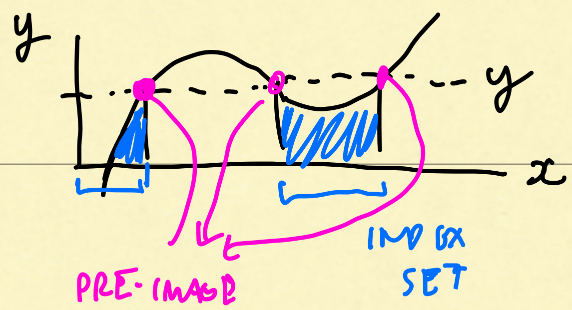
FOR A GENERAL FUNCTION $g$,

DEFINE $I_y = \{x : g(x) \leq y\}$

INDEX SET OF $y$

$\{x_k\} = \{x : g(x) = y\}$

PRE-IMAGE OF $y$




CUMULATIVE $F_Y(y) = \mathbb{P}(Y \leq y)$

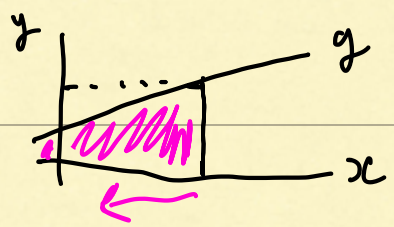
$F_Y(y) = \mathbb{P}(x \in I_y)$

$= \int_{I_y} p_X(x)\, dx$

EX $y = ax + b$ $\quad Y = aX + b$

$a > 0$

$g(x) = ax + b$




$I_y = \{x : g(x) \leq y\}$

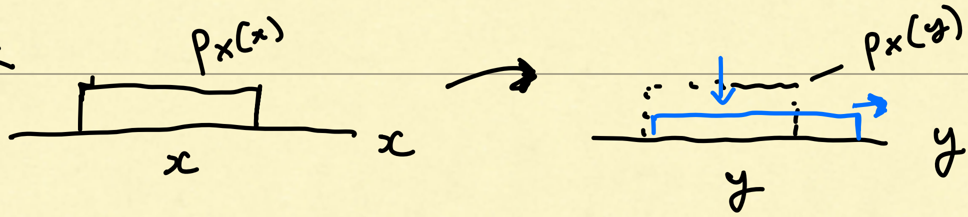
$= \{x \leq \frac{y-b}{a}\}$

IF GIVEN $F_X(x)$, THEN

$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$

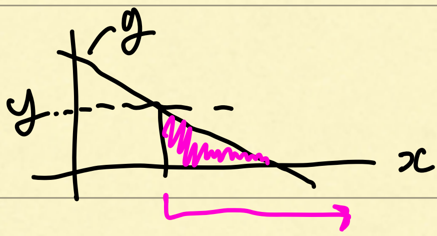
$$p_Y(y) = \frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

CHECK




EX $\quad Y = aX + b$

$a < 0$




$$I_y = \{ x : g(x) \leq y \}$$

$$= \left\{ x \geq \frac{y-b}{a} \right\}$$

CUMULATIVE $\quad F_Y(y) = \mathbb{P}(Y \leq y)$

$$= \mathbb{P}\left(x \geq \frac{y-b}{a}\right)$$

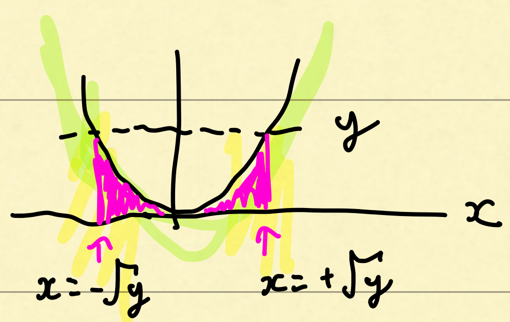
$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

DENSITY $\quad p_Y(y) = -\frac{1}{a} p_X\left(\frac{y-b}{a}\right)$

↑ RECALL $a < 0$

IN GENERAL $a \quad p_Y(y) = \left|\frac{1}{a}\right| p_X\left(\frac{y-b}{a}\right)$

EX $\quad Y = X^2 \qquad g(x) = x^2$




$$I_y = \begin{cases} [-\sqrt{y}, \sqrt{y}] & \text{IF } y \geq 0 \\ \text{EMPTY} & y < 0 \end{cases}$$

$$F_Y(y) = \begin{cases} F_X(+\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} p_X(+\sqrt{y}) + \frac{1}{2\sqrt{y}} p_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

GENERAL FORMULA

LET $X \sim p_X(x)$ AND $Y = g(X)$

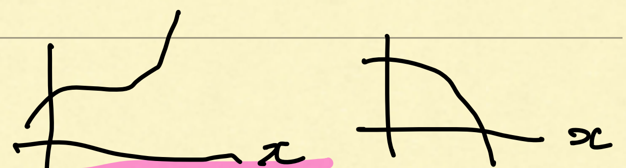
THEN
$$\boxed{p_Y(y) = \sum_k p_X(x_k(y)) \cdot \left|\frac{dx_k}{dy}\right|}$$

WHERE $\{x_k\}$ IS THE PRE-IMAGE

IF $\frac{dg}{dx} \neq 0$ THEN

THEN
$$p_Y(y) = \sum_k p_X(g_k^{-1}(y)) \cdot \left|\left(\frac{dg}{dx}\right)^{-1}\right|$$

IF $g$ IS MONOTONIC




THEN
$$\boxed{p_Y(y) = p_X(g^{-1}(y)) \left|\left(\frac{dg}{dx}\right)^{-1}\right|}$$

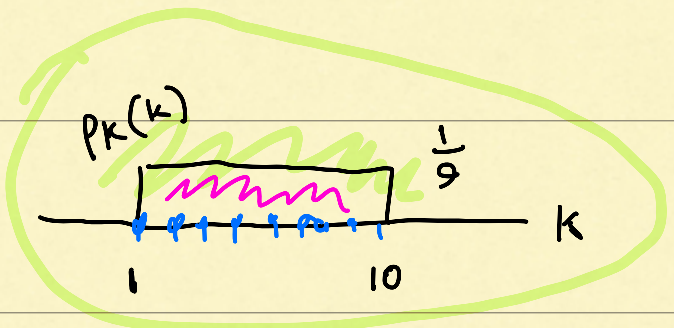
EX $K \sim \text{UNIF}(1, 10)$

$K$ - RATE CONSTANT $\left(\frac{1}{\text{TIME}}\right)$

$T$ - MEAN TIME

$T = \frac{1}{K}$ $\qquad g(K) = \frac{1}{K}$

MONOTONIC

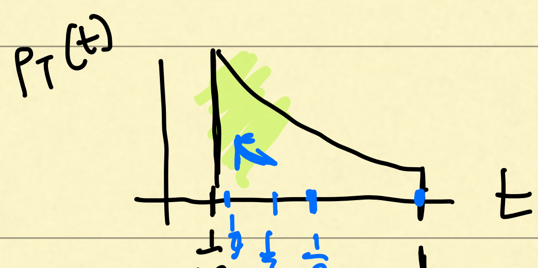



$$p_T(t) = p_K\left(g^{-1}(t)\right) \cdot \left| \left(\frac{dg}{dk}\right)^{-1} \right|$$

$g^{-1}(t) = \frac{1}{t}$ $\qquad \frac{dg}{dk} = \frac{-1}{K^2}$

$$p_T(t) = \frac{1}{9} \cdot \left| \frac{-1}{K^2} \right|^{-1}$$

$$= \frac{1}{9} K^2 = \frac{1}{9t^2}$$



10 3 2