A. Many processes, including the spread of an infectious disease through a small community, can be modeled as first-order exponential processes like

$$\frac{dY}{dt} = (R-1)\,Y \quad Y(0) = 1$$

where R is a constant. This will either lead to exponential growth or exponential decay.

Assume instead that R is a random variable that is different for each community. Assume it has Gaussian distribution with mean 1 and standard deviation σ,

$$p_R(r) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(r-1)^2/2\sigma^2}.$$

(a) Find the probability density function $p_Y(y,t)$ of $Y(t)$.

Intuitively, we expect half of the trajectories to grow exponentially, and half of the trajectories to decay exponentially.

(b) Sketch or plot the probability density you found for $p_Y(y,t)$.

(c) What is the probability that a trajectory is above the initial condition at $Y = 1$? In other words, what is $\mathbb{P}(Y(t) > 1)$? Is it true that half the trajectories remain above the initial condition $Y = 1$, and half remain below the initial condition $Y = 1$?