

Many processes, including the spread of an infectious disease through a small community, can be modeled as first-order exponential processes like

$$\frac{dV}{dt} = \frac{1}{\tau} (R - 1) V, \quad V(0) = 1 \quad (1)$$

where  $V$  is the tumor volume, measured in number of cells, and  $R$  is a constant.

This will either lead to exponential growth or exponential decay.

The constant  $R$  is different for every patient. Assume it has Gaussian distribution with mean 1 and standard deviation  $\sigma$ ,

$$p_R(r) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(r-1)^2/2\sigma^2}.$$

- i. Find the probability density function  $p_V(v, t)$  of  $V(t)$ .

Intuitively, we expect half of the trajectories to grow exponentially, and half of the trajectories to decay exponentially.

- ii. Sketch or plot the probability density you found for  $p_V(v, t)$ .
- iii. What is the probability that a trajectory is above the initial condition at  $V = 1$ ? In other words, what is  $\mathbb{P}(V(t) > 1)$ ? Is it true that half the trajectories remain above the initial condition  $V = 1$ , and half remain below the initial condition  $V = 1$ ?
- iv. Suppose  $\tau = 1$  months and  $\sigma = 0.1$ . What percent of patients have a tumor with more than 1000 cells after 10 months?

A slightly more complicated model that is a modified version of Equation 1, called the Gompertz model, is used to fit patient data.