Runge 4th Order Method

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Algorithm

- Define f(x,y)
- Enter the value of xo,y0, xn, h
- Repeat
  - k1= h.f(x0, y0)
  - k2 = h.f(x0+h/2,y0+k1/2)
  - k3 = h.f(x0 + h/2, y0 + k2/2)
  - k4 = h.f(x0 + h, y0 + k3)
  - k = (k1 + 2.k2 + 2.k3 + k4)/6
  - y1 = y0 + k
  - print x0, y0
  - y0 = y1
  - x0 =x0 +h
- Until (x0>=xn)
- stop

Problem

- Solve dy/dx = x+ y,

when y= 1, at x = 0.

Find solution for x = 0.1, 0.2, by Runge-Kutta fourth order method. Take step size h= 0.1

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method.

Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f(t, \theta) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f(t_0, \theta_o) = f(0,1200) = -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$
$$= f(120,653.05) = -2.2067\times10^{-12}\left(653.05^4 - 81\times10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$
$$= f(120,1154.0) = 2.2067\times10^{-12}\left(1154.0^4 - 81\times10^8\right) = -3.8954$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h) = f(0 + (240), 1200 + (-3.984)240)$$
$$= f(240,265.10) = 2.2067\times10^{-12}\left(265.10^4 - 81\times10^8\right) = 0.0069750$$

Solution Cont

$$\theta_1 = \theta_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 1200 + \frac{1}{6}\left(-4.5579 + 2\left(-0.38347\right) + 2\left(-3.8954\right) + \left(0.069750\right)\right)240$$

$$= 1200 + \frac{1}{6}\left(-2.1848\right)240$$

$$= 675.65K$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta\left(240\right) \approx \theta_1 = 675.65K$$

Solution Cont

Step 2: $i=1, t_1=240, \theta_1=675.65K$

$$k_1=f(t_1,\theta_1)=f(240,675.65)=-2.2067\times10^{-12}\left(675.65^4-81\times10^8\right)=-0.44199$$

$$k_2=f\left(t_1+\frac{1}{2}h,\theta_1+\frac{1}{2}k_1h\right)=f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.44199)240\right)$$
$$=f(360,622.61)=-2.2067\times10^{-12}\left(622.61^4-81\times10^8\right)=-0.31372$$

$$k_3=f\left(t_1+\frac{1}{2}h,\theta_1+\frac{1}{2}k_2h\right)=f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.31372)240\right)$$
$$=f(360,638.00)=2.2067\times10^{-12}\left(638.00^4-81\times10^8\right)=-0.34775$$

$$k_4=f(t_1+h,\theta_1+k_3h)=f(240+(240),675.65+(-0.34775)240)$$
$$=f(480,592.19)=2.2067\times10^{-12}\left(592.19^4-81\times10^8\right)=-0.25351$$

Solution Cont

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 675.65 + \frac{1}{6}\left(-0.44199 + 2\left(-0.31372\right) + 2\left(-0.34775\right) + \left(-0.25351\right)\right)240$$

$$= 675.65 + \frac{1}{6}\left(-2.0184\right)240$$

$$= 594.91K$$

θ_2 is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta\left(480\right) \approx \theta_2 = 594.91K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

Comparison with exact results

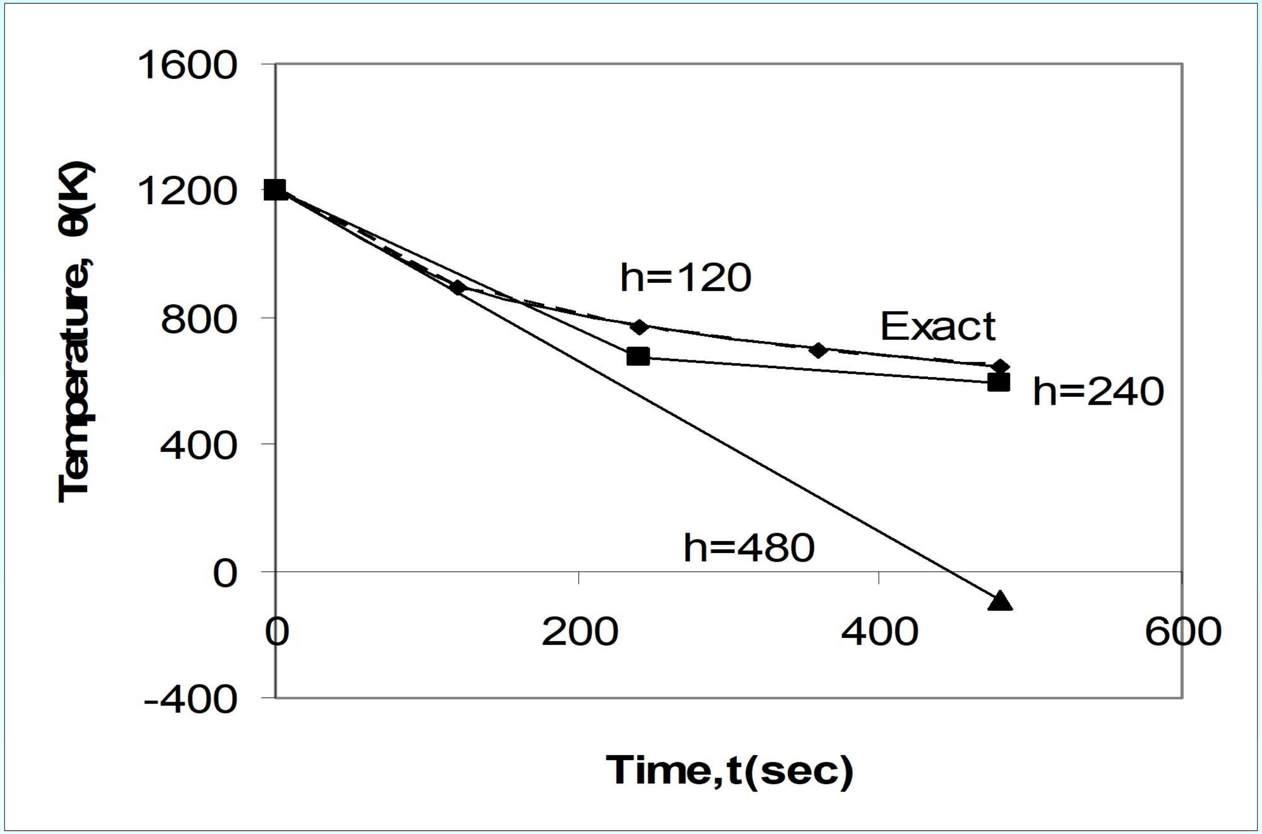

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, *h*	θ (480)	E_t	$\lvert\epsilon_t\rvert$%
480	−90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

$\theta(480) = 647.57K$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

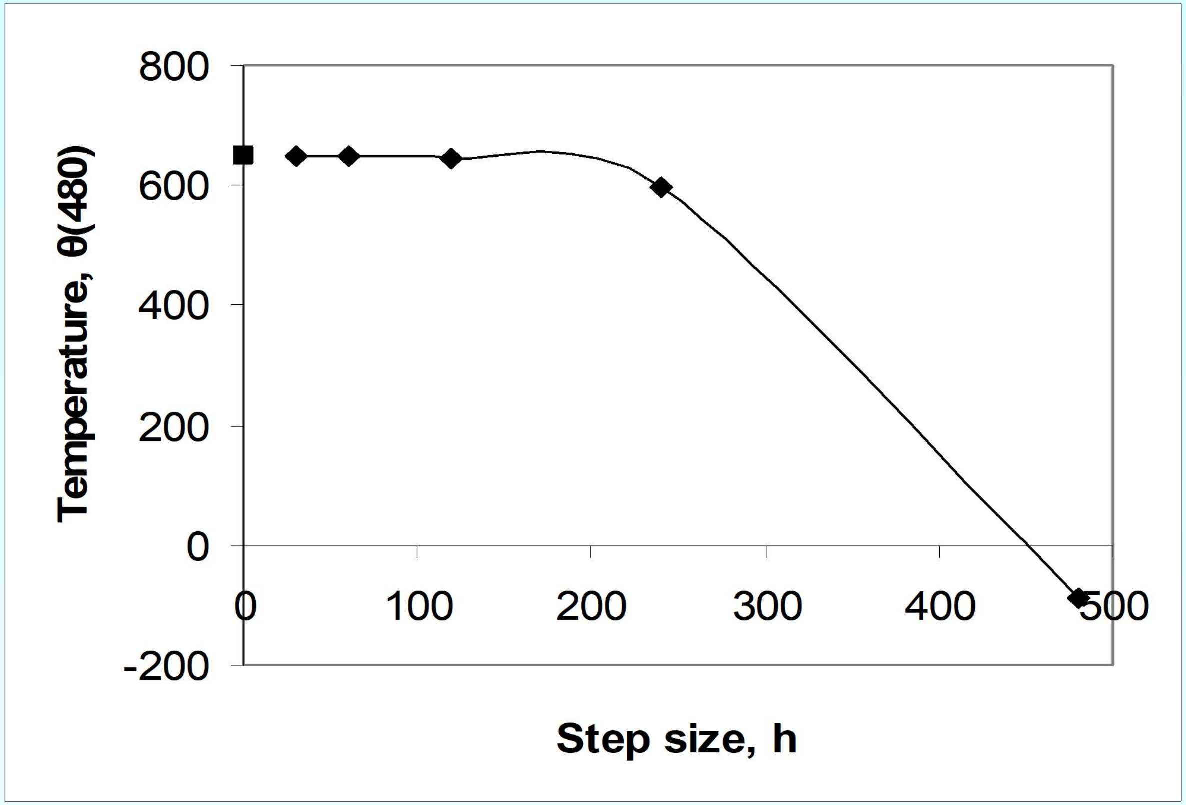

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

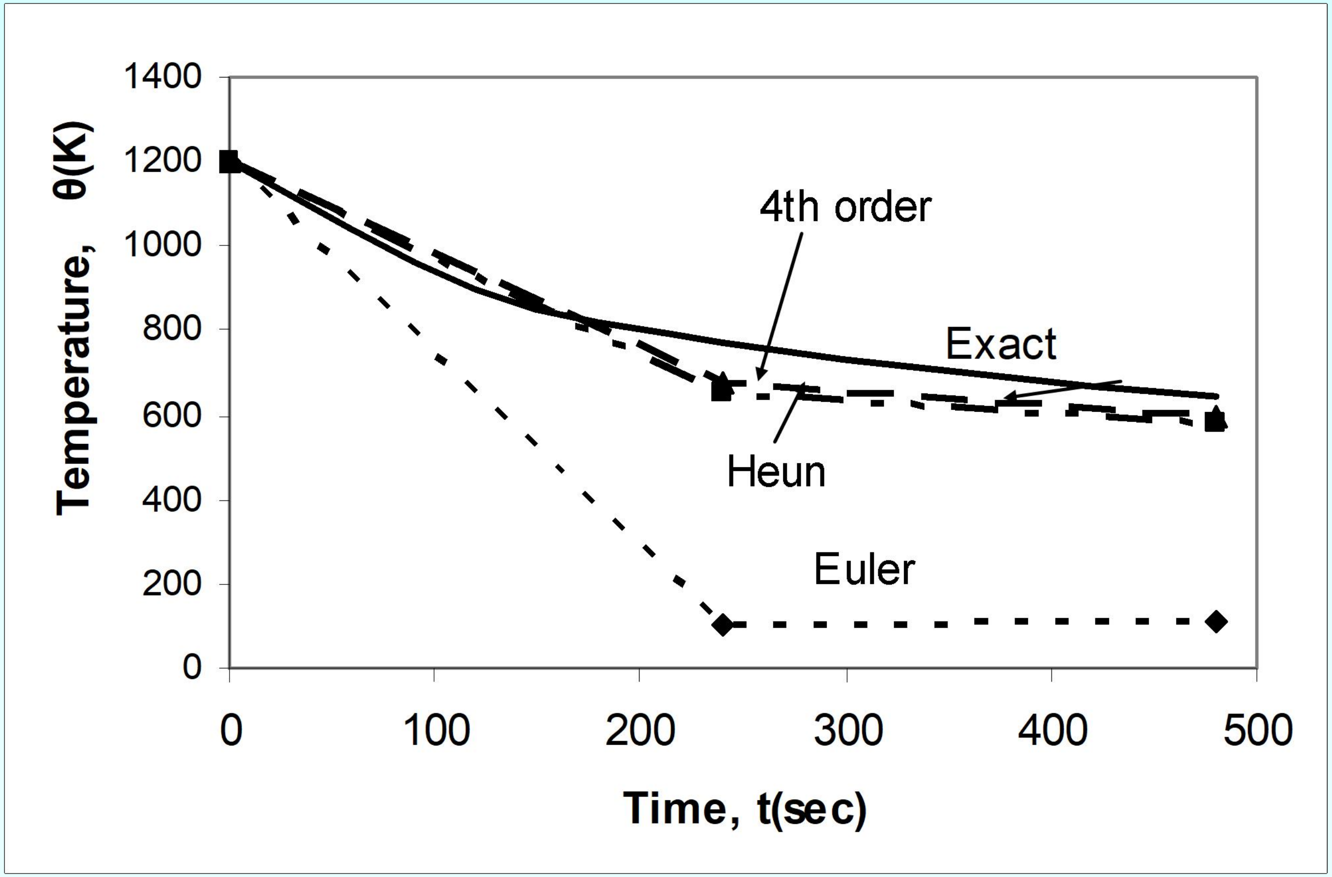

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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