Indian Institute of Information Technology Allahabad
Probability and Statistics
Problem Set 09

1. Check weather the following functions are distribution functions of 2-dim random vector or not.

   (a) $F(x, y) = \begin{cases} 1 & \text{if } x + 2y \geq 1 \\ 0 & \text{if } x + 2y < 1. \end{cases}$

   (b) $F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1 & \text{otherwise.} \end{cases}$

2. Let $F(\cdot, \cdot)$ be the distribution functions of a two-dimensional random vector (X, Y), and let $F_1(\cdot)$ and $F_2(\cdot)$, respectively, be the marginal d.f.s of X and Y. Define $U(x, y) = \min\{F_1(x), F_2(y)\}$ and $L(x, y) = \max\{F_1(x) + F_2(y) - 1, 0\}$. Prove the followings.

   (a) $L(x, y) \leq F(x, y) \leq U(x, y)$.

   (b) $L(x, y)$ and $U(x, y)$ are distribution functions of 2-dimensional random vector.

   (c) The marginal distributions of $L(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are same as that of $F(\cdot, \cdot)$.

3. Let the random variable X have distribution function $F_1(\cdot)$ and let $Y = g(X)$ have distribution function $F_2(\cdot)$, where $g(\cdot)$ is some function. Prove that

   (a) If $g(\cdot)$ is increasing, $F_{X,Y}(x, y) = \min\{F_1(x), F_2(y)\}$.

   (b) If $g(\cdot)$ is decreasing, $F_{X,Y}(x, y) = \max\{F_1(x) + F_2(y) - 1, 0\}$.

4. Consider the following joint p.m.f. of the random vector (X, Y).

y \ x	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

   (a) Find the conditional p.m.f. of X given $Y = 5$.

   (b) Find the probabilities $P(X + Y < 8)$, $P(X + Y > 7)$, $P(XY \leq 14)$.

   (c) Find the $Corr\,(X, Y)$

5. For the bivariate negative binomial distribution, the p.m.f. is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x \theta_2^y (1 - \theta_1 - \theta_2)^k & \text{if } x \in \{0, 1, 2, \ldots\},\ y \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise,} \end{cases}$$

   k is a positive integer, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the marginal distributions and both the conditional distributions.

6. Suppose that the number, X, of eggs raid by a bird has the Poisson(λ) distribution with $\lambda > 0$, the probability that an egg would finally develop is $p \in (0,\, 1)$. Under the assumption of independence of development of eggs, show here that the number, Y, of eggs surviving has the Poisson(λp) distribution. Find the conditional distribution of X given $Y = y$.

7. Three balls are randomly placed in three empty boxes B_1, B_2, and B_3. Let N denote the total number of boxes which are occupied and let X_i denote the number of balls in the box B_i, $i = 1,\, 2,\, 3$.

   (a) Find the joint p.m.f. of $(N,\, X_1)$.

   (b) Find the joint p.m.f. of $(X_1,\, X_2)$.

   (c) Find the marginal distributions of N and X_2.

   (d) Find the marginal p.m.f. of X_1 from the joint p.m.f. of $(X_1,\, X_2)$.

8. Five cards are drawn at random without replacement from a deck of 52 cards. Let the random variables X_1, X_2, and X_3, respectively, denote the number of spades, the number of hearts, and the number of diamonds among the five drawn cards.

   (a) Find the joint p.m.f. of $(X_1,\, X_2,\, X_3)$.

   (b) Are random variables X_1, X_2, and X_3 are independent?

9. Suppose that the lifetime of electric bulbs manufactured by a manufacturer follows exponential distribution with mean of 50 hours. Eight such bulbs are chosen at random.

   (a) Find the probability that, among eight chosen bulbs, 2 will last less than 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours.

   (b) Find the expected number of bulbs in the lot of chosen B bulbs with lifetime between 60 and 80 hours.

   (c) Find the expected number of bulbs in the lot of 8 chosen bulbs with lifetime between 60 and 80 hours, given that the number of bulbs in the lot with lifetime anywhere between 40 and 60 hours is 2.

10. Suppose that $\underline{X} = (X_1,\, X_2,\, X_3,\, X_4) \sim \text{Mult}(30,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4)$. Find the conditional probability mass function of $(X_1,\, X_2,\, X_3,\, X_4)$ given that $X_5 = 2$, where $X_5 = 30 - \sum_{i=1}^{4} X_i$.

11. Suppose that $X_1,\, \ldots,\, X_n$ are independent and identicaliy distributed random variables such that $P(X_i = 0) = 1 - p = 1 - P(X_i = 1)$, $i = 1,\, \ldots,\, n$, for some $p \in (0,\, 1)$. Let X be the number of $X_1,\, \ldots,\, X_n$ that are as large as X_1. Find the p.m.f. of X.

12. For the bivariate beta random vector $(X,\, Y)$ having p.d.f.

$$f_{X,Y}(x,\, y) = \begin{cases} \frac{\Gamma(\theta_1+\theta_2+\theta_3}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} y^{\theta_2-1} (1-x-y)^{\theta_3-1} & \text{if } x > 0,\, y > 0,\, x + y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_i > 0$, $i = 1,\, 2,\, 3$. Find both the marginal p.d.f.s and conditional p.d.f.s.

13. The joint p.d.f. of $(X,\, Y)$ is given by

$$f_{X,Y}(x,\, y) = \begin{cases} 4xy & \text{if } 0 < x < 1,\, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal p.d.f.s of X and Y.

(b) Verify whether X and Y are independent.

(c) Find $P(\{0 < X < 0.5,\, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$.

14. Let $(X,\, Y)$ be a random vector such that the p.d.f. of X is given by

$$f_X(x) = \begin{cases} 4x(1-x^2) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For fixed $x \in (0,\, 1)$, the conditional p.d.f. of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{if } x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional p.d.f. of X given $Y = y$ for appropriate values of y.

(b) Find $E(X|Y = 0.5)$ and $E(Y|X = 0.5)$.

(c) Find $P(\{0 < Y < 1/3\})$ and $P(\{1/3 < Y < 2/3\} \,|\, \{X = 0.5\})$.

15. Let $X = (X_1,\, X_2,\, X_3)$ be a random vector with joint p.d.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}\right) \quad \text{if } (x_1, x_2, x_3) \in \mathbb{R}^3$$

(a) Are X_1, X_2, and X_3 independent?

(b) Are X_1, X_2, and X_3 pairwise independent?

(c) Find the marginal p.d.f.s of $(X_1,\, X_2)$, $(X_1,\, X_3)$ and $(X_2,\, X_3)$.

16. With the help of a counter example, show that if the random variables X_1 and X_2 are uncorrelated, then this does not, in general, imply that X_1 and X_2 are independent.

17. Let $(X_1,\, X_2)$ be a 2-dimensional absolutely continuous random vector with joint p.d.f. $f_{X_1,\, X_2}(\cdot,\, \cdot)$. Show that X_1 and X_2 are independent if and only if, for some functions $g_1(\cdot)$ and $g_2(\cdot)$,

$$f_{X_1,\, X_2}(x_1,\, x_2) = g_1(x_1) g_2(x_2).$$

18. Let X and Y be independent random variables. Show that $h_1(X)$ and $h_2(Y)$ are also independent random variables, for Borel functions $h_1(\cdot)$ and $h_2(\cdot)$.

19. Let $(X,\, Y)$ be a random vector with p.d.f.

$$f_{X,Y}(x,\, y) = \begin{cases} 15e^{-(2x+3y)} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find $P\left(\{X \leq x\}|\{4.999 < Y \leq 5.001\}\right)$.

20. Let us choose at random a point from the interval (0,1) and let the random variable X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0,\ X_1)$ and let X_2 be equal to the number which corresponds to this point. Compute $P\left(X_1 + X_2 \geq 1\right)$ and find the conclitional mean $E\left(X_1|X_2 = x_2\right)$ for $x_2 \in (0, 1)$.