Notes on the FedAVG algorithm

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1 Setting and Algorithm

1.1 Problem definition

TODOs

- Setting description
- Cost function

1.2 The Federated Averaging Algorithm

We consider the Federate Algorithm (*FedAVG*) which we formally describe below.

```
Algorithm 1: Federated Averaging
  Input :  initial model $\boldsymbol{x}^(0)$, learning rate $\eta$
  for $t \in \{0, 1, ..., T-1\}$ do
      for $i \in \{1, ..., M\}$ clients in parallel do
          Initialize local model $\boldsymbol{x}_i^{(t,0)} \leftarrow \boldsymbol{x}^{(t)}$
          for $k \in \{0, ..., \tau\}$ do
              Compute local stochastic gradient $g_i(\boldsymbol{x}^{(t,k)})$
              Compute local step $\boldsymbol{x}_i^{(t,k+1)} \leftarrow \boldsymbol{x}_i^{(t,k)} - \eta g_i(\boldsymbol{x}^{(t,k)})$
          Compute local change over round $\Delta_i^{(t)} \leftarrow \boldsymbol{x}_i^{(t,\tau)} - \boldsymbol{x}_i^{(t,0)}$
      Average local updates $\Delta^{(t)} = \frac{1}{M} \cdot \sum_{i=1}^{M} \Delta_i^{(t)}$
      Update global model $\boldsymbol{x}^{(t+1)} \leftarrow \boldsymbol{x}^{(t)} + \Delta^{(t)}$
  Return : $\boldsymbol{x}^{(t)}$
```

2 Analysis of the Algorithm

2.1 Setting and assumptions

As a preliminary step to the analysis of algorithm 1, we make the following 7 assumptions:

1. At any round t each *client* takes $\tau \in \mathbb{N}$ local SGD steps with constant learning rate η (which we denote as $\boldsymbol{x}_i^{(t,k+1)} \leftarrow \boldsymbol{x}_i^{(t,k)} - \eta g_i(\boldsymbol{x}_i^{(t,k)})$ with g_i is one draw of the stochastic gradient of F_i and $k \in [0, k)$.
2. The *server step* is computed as $\boldsymbol{x}^{(t+1)} \leftarrow \boldsymbol{x}^{(t)} + \Delta^{(t)}$.
3. There are (M) clients labelled $i \in \{0, 1, ..., M\}$ and each client contributes a uniform share of the global objective $F(\boldsymbol{x}) = \frac{1}{M}\sum_{i=1}^{M} F_i(\boldsymbol{x})$.
4. Each clients takes part in every round.
5. Each local objective F_i is convex and L-smooth.
6. Each client queries an unbiased stochastic gradient with σ^2-uniformly bounded variance in l_2 norm, i.e.

$$\mathbb{E}[g_i(\boldsymbol{x}_i^{(t,k)}) | \boldsymbol{x}_i^{(t,k)}] = \nabla F_i(\boldsymbol{x}_i^{(t,j)}), \tag{1}$$

$$\mathbb{E}[\|g_i(\boldsymbol{x}_i^{(t,k)}) - F_i(\boldsymbol{x}_i^{(t,j)})\|^2 | \boldsymbol{x}_i^{(t,k)}] \leq \sigma^2. \tag{2}$$

7. The difference of local gradient $\nabla F_i(\boldsymbol{x})$ and the global gradient $\nabla F(\boldsymbol{x})$ is ζ-uniformly bounded in l_2 norm, i.e.

$$\max_i \sum_{\boldsymbol{x}} \|\nabla F_i(\boldsymbol{x}) - \nabla F(\boldsymbol{x})\| \leq \zeta. \tag{3}$$

First, let us define the shadow sequence which we will use to make the notation a bit more readable as we go through the proof:

Notation 2.1. *(Shadow sequence) We call the sequence described by* $\bar{x}^{t,k} = \frac{1}{M}\sum_{i=1}^{M} \boldsymbol{x}_i^{(t,k)}$ *the shadow sequence.*

As we often do in the optimization literature we will try to show a result of the form:

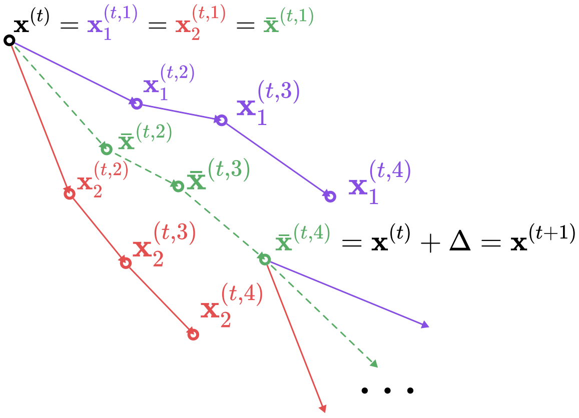
$$\mathbb{E}\left[\frac{1}{\tau T}\sum_{t=0}^{T-1}\sum_{k=1}^{\tau} F(\bar{\boldsymbol{x}}^{(t,k)}) - F(\boldsymbol{x}^*)\right] = O\left(\frac{1}{\tau T}\right).$$

Which one can read as *as we progress, in expectation we are guaranteed to have an error that goes to some small constant.* In order to get to a bound we will have to prove two lemmas to show that. Showing that this results is true amounts to finding a relevant upper bound decreasing in $\frac{1}{\tau T}$. We split our proving effort in two steps:

1. We are making progress in each round $\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau} F(\bar{\boldsymbol{x}}^{(t,k)}) - F(\boldsymbol{x}^*)\right]$ is bounded by some term decreasing when t increases.

2. All client iterates remain close to the global average (the shadow sequence), i.e. $\|\boldsymbol{x}_i^{(t,k)} - \hat{\boldsymbol{x}}^{(t,k)}\|_{l_2}$ is bounded in expectation.

Formally we will write our proof using one theorem that relies on two lemmas (showing both properties discussed above). The formal proofs are detailed in the next section.



$\mathbf{x}^{(t)} = \mathbf{x}_1^{(t,1)} = \mathbf{x}_2^{(t,1)} = \bar{\mathbf{x}}^{(t,1)}$
$\mathbf{x}_1^{(t,2)}$
$\mathbf{x}_1^{(t,3)}$
$\bar{\mathbf{x}}^{(t,2)}$
$\mathbf{x}_2^{(t,2)}$
$\bar{\mathbf{x}}^{(t,3)}$
$\mathbf{x}_1^{(t,4)}$
$\mathbf{x}_2^{(t,3)}$
$\bar{\mathbf{x}}^{(t,4)} = \mathbf{x}^{(t)} + \Delta = \mathbf{x}^{(t+1)}$
$\mathbf{x}_2^{(t,4)}$
$\dots$

Figure 1: Illustration of the progression of one 4 step round of algorithm 1 with the shadow sequence represented on green.

2.2 Convergence of FedAVG

We will prove the following results.

Theorem 2.1. *(Convergence for Convex Functions) under the assumptions and assuming* $\eta \leq \frac{1}{4L}$ *one has:*

$$\mathbb{E}\left[\frac{1}{\tau T}\sum_{t=0}^{T-1}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right]\leq\frac{D^2}{2\eta\tau T}+\frac{\eta\sigma^2}{M}+4\tau\eta^2L\sigma^2+18\tau^2\eta^2L\zeta^2 \\ =O\left(\frac{1}{\tau T}\right) \tag{4}$$

To which we will get using two lemmas:

Lemma 2.2. *(Per round progress) Assuming $\eta\leq\frac{1}{4L}$, for one round t of the algorithm, one has:*

$$\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right] \\ \leq\frac{1}{2\eta\tau}\left(\|\bar{\boldsymbol{x}}^{(t,0)}-\boldsymbol{x}^*\|^2-\mathbb{E}\left[\|\bar{\boldsymbol{x}}^{(t,\tau)}-\boldsymbol{x}^*\|^2|\mathcal{F}^{(t,0)}\right]\right) \\ +\frac{\eta\sigma^2}{M}+\frac{1}{M\tau}\sum_{i=1}^{M}\sum_{k=1}^{\tau-1}\mathbb{E}\left[\|\boldsymbol{x}_i^{(t,k)}-\bar{\boldsymbol{x}}^{(t,k)}\|^2|\mathcal{F}^{(0,t)}\right] \tag{5}$$

Lemma 2.3. *(Bounded client drift) Assuming $\eta\leq\frac{1}{4L}$, for one round t of the algorithm, one has:*

$$\mathbb{E}\left[\|\boldsymbol{x}_i^{(t,k)}-\bar{\boldsymbol{x}}^{(t,k)}\|^2|\mathcal{F}^{(0,t)}\right]\leq18\tau^2\eta^2\zeta^2+4\tau\eta^2\sigma^2 \tag{6}$$

Proving Theorem 2.1 (convergence of FedAVG)

Most of the technical work will lie in proving the two lemmas, but first we will focus on proving theorem 2.1, while assuming that lemmas 2.2 and 2.3 are true.

Proof. (Of Theorem 2.1.) We want to find a bound for the quantity

$$\mathbb{E}\left[\frac{1}{\tau T}\sum_{t=0}^{T-1}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right],$$

to do so we will use the bound on $\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right]$ which is given by lemma 2.2. First, let's write out the sum on which we will take the expectation and express it as a function of the per round progress which we bounded in lemma 2.2:

$$\mathbb{E}\left[\frac{1}{\tau T}\sum_{t=0}^{T-1}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right], \\ =\mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1}\overbrace{\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau}F(\bar{\boldsymbol{x}}^{(t,k)})-F(\boldsymbol{x}^*)\right]}^{(\triangledown)}\right].$$

Observing that the term ($\triangledown$) is the left side of the inequality (5) of lemma 2.2, we use the lemma to bound our expectation. Using linearity of expectation we split this expression in three different terms which we will then discuss separately.

$$\leq \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{2\eta\tau}\left(\|\bar{\boldsymbol{x}}^{(t,0)}-\boldsymbol{x}^*\|^2-\mathbb{E}\left[\|\bar{\boldsymbol{x}}^{(t,\tau)}-\boldsymbol{x}^*\|^2|\mathcal{F}^{(t,0)}\right]\right)\right.$$
$$\left.+\frac{\eta\sigma^2}{M}+\frac{1}{M\tau}\sum_{i=1}^{M}\sum_{k=1}^{\tau-1}\mathbb{E}\left[\|\boldsymbol{x}_i^{(t,k)}-\bar{\boldsymbol{x}}^{(t,k)}\|^2|\mathcal{F}^{(0,t)}\right]\right]$$

$$=\overbrace{\mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{2\eta\tau}\left(\|\bar{\boldsymbol{x}}^{(t,0)}-\boldsymbol{x}^*\|^2-\mathbb{E}\left[\|\bar{\boldsymbol{x}}^{(t,\tau)}-\boldsymbol{x}^*\|^2|\mathcal{F}^{(t,0)}\right]\right)\right]}^{(\bigstar)}$$
$$+\overbrace{\frac{\eta\sigma^2}{M}}^{(\diamond)}+\overbrace{\frac{1}{M\tau}\sum_{i=1}^{M}\sum_{k=1}^{\tau-1}\mathbb{E}\left[\|\boldsymbol{x}_i^{(t,k)}-\bar{\boldsymbol{x}}^{(t,k)}\|^2|\mathcal{F}^{(0,t)}\right]}^{(\dagger)}$$

Let us now consider the three terms. Terms ($\diamond$) and ($\dagger$) gives a bound on individual client drift (i.e. how far do the clients get from the shadow sequence), term ($\bigstar$) gives a bound on the global progression. Here our goal is to show that ($\diamond$) and ($\dagger$) can be arbitrarily bounded as a function of the algorithm's parameters and that ($\bigstar$) goes to 0 with $T\cdot\tau$. We now discuss bounds for every single term.

1. Term ($\diamond$) is already a function of our algorithm's parameters, there is nothing to show here.

2. Now we consider term ($\dagger$):

$$\frac{1}{M\tau}\sum_{i=1}^{M}\sum_{k=1}^{\tau-1}\overbrace{\mathbb{E}\left[\|\boldsymbol{x}_i^{(t,k)}-\bar{\boldsymbol{x}}^{(t,k)}\|^2|\mathcal{F}^{(0,t)}\right]}^{(\spadesuit)}$$

it is a sum over term ($\spadesuit$), which is the left side the inequality (6) of lemma 2.3. We plug the right side of (6) and as it is not a function of the sum variables we can drop the sums as well.

$$(\dagger)\leq 18\tau^2\eta^2\zeta^2+4\tau\eta^2\sigma^2$$

3. Finally we consider term ($\bigstar$),

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{2\eta\tau}\left(\|\bar{\boldsymbol{x}}^{(t,0)}-\boldsymbol{x}^*\|^2-\mathbb{E}\left[\|\bar{\boldsymbol{x}}^{(t,\tau)}-\boldsymbol{x}^*\|^2|\mathcal{F}^{(t,0)}\right]\right)\right],$$

here we have to use a few tricks. First using that expectation is linear we will separate our terms and using that $\mathbb{E}\left[\mathbb{E}[x]\right]=\mathbb{E}[x]$ we will drop the double expectation in the sum:

$$\frac{1}{2\eta\tau T}\sum_{t=0}^{T-1}\left(\mathbb{E}\left[\|\overbrace{\bar{\boldsymbol{x}}^{(t,0)}}^{\bar{\boldsymbol{x}}^{(t)}}-\boldsymbol{x}^*\|^2\right]-\mathbb{E}\left[\|\overbrace{\bar{\boldsymbol{x}}^{(t,\tau)}}^{\bar{\boldsymbol{x}}^{(t+1)}}-\boldsymbol{x}^*\|^2\right]\right).$$

At this point we first observe (as denoted above) that by definition of the algorithm we have $\bar{\boldsymbol{x}}^{(t+1)} = \bar{\boldsymbol{x}}^{(t+1,0)} = \bar{\boldsymbol{x}}^{(t,\tau)}$ and $\bar{\boldsymbol{x}}^{(t)} = \bar{\boldsymbol{x}}^{(t,0)}$, then for readability we make use of the notation $d(\boldsymbol{x}) = \mathbb{E}\left[\|\boldsymbol{x} - \boldsymbol{x}^*\|^2\right]$ and write out the sum over t:

$$\begin{aligned}&\sum_{t=0}^{T-1}\Big(d(\boldsymbol{x}^{(t)}) - d(\boldsymbol{x}^{(t+1)})\Big)\\ &= d(\boldsymbol{x}^{(0)}) - d(\boldsymbol{x}^{(1)}) + d(\boldsymbol{x}^{(1)}) - d(\boldsymbol{x}^{(2)})\\ &+ \cdots + d(\boldsymbol{x}^{(T-2)}) - d(\boldsymbol{x}^{(T-1)}) + d(\boldsymbol{x}^{(T-1)}) - d(\boldsymbol{x}^{(T)}).\end{aligned}$$

Observing that the terms cancel out we get the following expression:

$$\begin{aligned}(\bigstar) &= \frac{1}{2\eta\tau T}\Big(d(\boldsymbol{x}^{(0)}) - \cancel{d(\boldsymbol{x}^{(1)})} + \cancel{d(\boldsymbol{x}^{(1)})} - \cancel{d(\boldsymbol{x}^{(2)})}\\ &+ \cdots + \cancel{d(\boldsymbol{x}^{(T-2)})} - \cancel{d(\boldsymbol{x}^{(T-1)})} + \cancel{d(\boldsymbol{x}^{(T-1)})} - d(\boldsymbol{x}^{(T)})\Big)\\ &= \frac{1}{2\eta\tau T}\Big(d(\boldsymbol{x}^{(0)}) - d(\boldsymbol{x}^{(T)})\Big) \leq \frac{d(\boldsymbol{x}^{(0)})}{2\eta\tau T} = \frac{D^2}{2\eta\tau T}.\end{aligned}$$

Where $d(\boldsymbol{x}^{(0)}) = \mathbb{E}[\|\boldsymbol{x}^{(0)} - \boldsymbol{x}^*\|^2] = \|\boldsymbol{x}^{(0)} - \boldsymbol{x}^*\|^2 = D^2$ with D the diameter to the global opt at the beginning of the gradient descent (this is a constant).

Putting all bounds back together we get the following convergence bound (which concludes the proof):

$$\mathbb{E}\left[\frac{1}{\tau T}\sum_{t=0}^{T-1}\sum_{k=1}^{\tau} F(\bar{\boldsymbol{x}}^{(t,k)}) - F(\boldsymbol{x}^*)\right] \leq \overbrace{\frac{D^2}{2\eta\tau T}}^{(\bigstar)} + \overbrace{\frac{\eta\sigma^2}{M}}^{(\diamond)} + \overbrace{4\tau\eta^2 L\sigma^2 + 18\tau^2\eta^2 L\zeta^2}^{(\dagger)}$$

□

We now get in the (somewhat) more technical parts of the proof, proving the lemmas.

Proving Lemma 2.2 (per round progression)

Let's start with the per round progress lemma, we want to bound the quantity :

$$\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau} F(\bar{\boldsymbol{x}}^{(t,k)}) - F(\boldsymbol{x}^*)\right],$$

in $O(\frac{1}{\tau})$. Similarly to what we just did for the theorem (and to how most of these convergence proofs are computed), we will to try to bound a single term of the sum (in expectation) by the previous term and then we will telescope the sum to get a serviceable bound. In other words we are trying to bound the expectation:

$$\mathbb{E}\Big[\overbrace{F(\bar{\boldsymbol{x}}^{(t,k+1)}) - F(\boldsymbol{x}^*)}^{(\heartsuit)}\,|\mathcal{F}^{(t,k)}\Big].$$

Since it is an expectation, which we will do by bounding the (♡) term. Writing it out we can first split it between the separate client objective functions:

$$(\heartsuit) = F(\bar{\boldsymbol{x}}^{(t,k+1)}) - F(\boldsymbol{x}^*) = \frac{1}{M}\sum_{i=1}^{M}\Big(F_i(\bar{\boldsymbol{x}}^{(t,k+1)}) - F(\boldsymbol{x}^*)\Big).$$

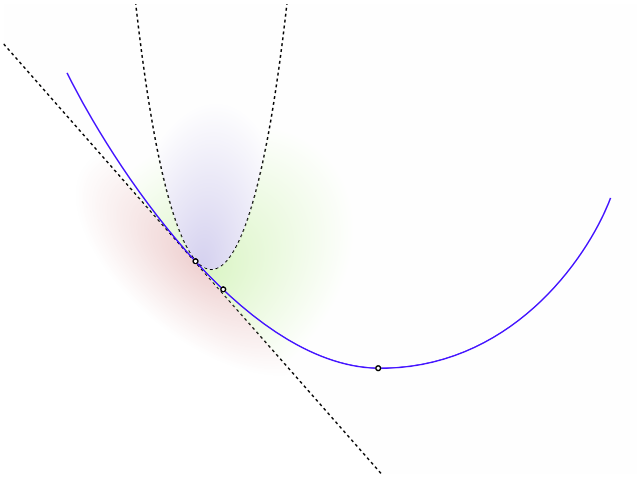


Figure 2: Illustration of the L-smooth/convex properties of F_i.

Looking at the expression above, we clearly see that bounding (♡) will amount to bound $F_i(\bar{\boldsymbol{x}}^{(t,k+1)})$, which we will do using the smoothness (property 2.3) and convexity (property 2.1) properties from the assumptions made in section 2.1. First, we get an upper bound from the L-smoothness of F_i:

$$F_i(\bar{\boldsymbol{x}}^{(t,k+1)}) \leq F_i(\boldsymbol{x}) + \langle \nabla F_i(\boldsymbol{x}), \bar{\boldsymbol{x}}^{(t,k+1)} - \boldsymbol{x}\rangle + \frac{L}{2}\|\bar{\boldsymbol{x}}^{(t,k+1)} - \boldsymbol{x}\|^2,\ \forall \boldsymbol{x}.$$

This inequality is true for all $\boldsymbol{x} \in \boldsymbol{dom}(f)$ but we care about the algorithm's step and for each $F_i(\bar{\boldsymbol{x}}^{(t,k+1)})$ the step of the algorithm depends of $\boldsymbol{x}_i^{(t,k)}$ (the parameters of client i at the previous time step).

$$\begin{aligned} F_i(\bar{\boldsymbol{x}}^{(t,k+1)}) &\leq F_i(\boldsymbol{x}_i^{(t,k)}) + \langle \nabla F_i(\boldsymbol{x}_i^{(t,k)}), \bar{\boldsymbol{x}}^{(t,k+1)} - \boldsymbol{x}_i^{(t,k)}\rangle + \frac{L}{2}\|\bar{\boldsymbol{x}}^{(t,k+1)} - \boldsymbol{x}_i^{(t,k)}\|^2 \\ &\leq F_i(\boldsymbol{x}^*) + \langle \nabla F_i(\boldsymbol{x}_i^{(t,k)}), \boldsymbol{x}^* - \boldsymbol{x}_i^{(t,k)}\rangle + \frac{L}{2}\|\bar{\boldsymbol{x}}^{(t,k+1)} - \boldsymbol{x}_i^{(t,k)}\|^2 \end{aligned}$$

2.3 Properties used for the proof

In the following subsection, we list the different properties and lemmas that are useful to the proofs of the convergence of algorithm 1. First, we consider convexity and smoothness lemmas:

Property 2.1. *(First order characterization of convexity) suppose* $\boldsymbol{dom}(f)$ *is open and that* $f : \boldsymbol{dom}(f) \to \mathbb{R}$ *is differentiable (it's gradient* $\nabla f(x)$ *exists* $\forall \boldsymbol{x} \in \boldsymbol{dom}(f)$*), then* f *is convex* $\iff$

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle, \ \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{dom}(f).$$

This first property is equivalent to the *monotonicity of the gradient property below* which will be more useful to our proofs.

Property 2.2. *(Monotonicity of the gradient for differentiable convex function) suppose* $\boldsymbol{dom}(f)$ *is open and that* $f : \boldsymbol{dom}(f) \to \mathbb{R}$ *is differentiable (it's gradient* $\nabla f(x)$ *exists* $\forall \boldsymbol{x} \in \boldsymbol{dom}(f)$*), then* f *is convex* $\iff$

$$\langle \nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{dom}(f).$$

We will also use the definition of L-smoothness:

Property 2.3. *(L-Smoothness) let* $f : \boldsymbol{dom}(f) \to \mathbb{R}$ *be a differentiable function and* $X \subseteq \boldsymbol{dom}(f)$ *convex and* $L \in \mathbb{R}_+$*. Function* f *is called smooth over* X *if:*

$$f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle + \frac{L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2, \ \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{dom}(f).$$