Discrete Structures-2025: Tutorial-8

Relations and Partial Orderings

(1) Let $\mathcal{A}$ be the set of all nonempty finite subsets of $\mathbb{Z}$. For every finite subset $S \subseteq \mathbb{Z}$, $\min(S)$ is defined as the smallest element among all elements of S. Define a relation

$$T = \{(X, Y) \mid X, Y \in \mathcal{A} \text{ such that } \min(X) = \min(Y)\}$$

(a) Prove that T is an equivalence relation.

(b) For every $X \in \mathcal{A}$, provide a precise definition of the equivalence class with respect to the equivalence relation T.

(2) Let $f : A \to B$ be a function from A to B and S be an equivalence relation on B. Define a relation $T = \{(x, y) \mid x, y \in A \text{ such that } (f(x), f(y)) \in S\}$.

(a) Prove that T be an equivalence relation

(b) How do you define the equivalence class containing an element x?

(3) Let $\mathcal{A}$ be the set of all nonempty finite subsets of $\mathbb{Z}$. Consider the following two relations

(a) $T_1 = \{(X, Y) \mid X, Y \in \mathcal{A} \text{ such that } \min(X) \leq \min(Y)\}$.

(b) $T_2 = \{(X, Y) \mid X, Y \in \mathcal{A} \text{ such that } X = Y \text{ or } \min(X) < \min(Y)\}$.

Is T_1 a partial order? If yes, then prove it. Otherwise, disprove it. Is T_2 a partial order? If yes, then prove it. Otherwise disprove it.

(4) Let $\mathcal{B}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the following relations on $\mathcal{B}$ as follows.

(a) $S_1 = \{(f, g) \mid \text{for all } a \in \mathbb{R},\ f(a) \leq g(a)\}$.

(b) $S_2 = \{(f, g) \mid f(0) \leq g(0)\}$.

(c) $S_3 = \{(f, g) \mid f(0) = g(0)\}$.

Which of the relations S_1, S_2, S_3 is/are equivalence relation(s)? Which of these relations S_1, S_2, S_3 is/are partial order(s)? For each of them, give formal proofs.

(5) Prove that if a relation S on a set A is antisymmetric, then any subset of $S' \subseteq S$ is also an antisymmetric relation on A.

(6) A relation S on A is irreflexive if for every $x \in A$, $(x, x) \notin S$. Suppose that R_1 and R_2 are reflexive relations on a set A. Prove that $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$ is irreflexive relation.