Solutions — Quiz 6 (Discrete Structures)

Full marks: 25 Date: November 17, 2025

Problem 1

Question. Prove or disprove that: if 9 integers are selected from the first 14 positive integers, then there are at least two numbers whose sum is 15.
Solution. Partition the set $\{1, 2, \dots, 14\}$ into the following 7 disjoint pairs whose elements sum to 15:

$$\{1, 14\},\ \{2, 13\},\ \{3, 12\},\ \{4, 11\},\ \{5, 10\},\ \{6, 9\},\ \{7, 8\}.$$

Think of these 7 pairs as 7 boxes. Every chosen integer belongs to exactly one box. If we choose 9 integers (pigeons) and place each into the box corresponding to its complementary partner, then by the pigeonhole principle at least one box must contain at least

$$\left\lceil \frac{9}{7} \right\rceil = 2$$

elements. Any box that contains two elements contains both members of a complementary pair, hence their sum is 15. Thus the statement is true.
(Indeed, 8 chosen numbers already force a complementary pair because 8 pigeons into 7 boxes implies some box has at least 2 pigeons.)

Problem 2

Question. Prove or disprove that: if 6 positive integers are selected from the first 10 positive integers, then there are two pairs of integers with the sum 11.
Solution. The pairs of numbers from $\{1, 2, \dots, 10\}$ that sum to 11 are

$$\{1, 10\},\ \{2, 9\},\ \{3, 8\},\ \{4, 7\},\ \{5, 6\}.$$

We show the statement is false by giving a counterexample. Take the six-element set

$$S = \{1, 2, 3, 4, 5, 7\}.$$

Among these elements only the pair $\{4, 7\}$ sums to 11. The other complementary partners $10, 9, 8, 6$ are not all present, so there is exactly one pair summing to 11 in S. Therefore it is *not* guaranteed that six chosen numbers contain two (disjoint) pairs summing to 11. This single counterexample disproves the claim.

Problem 3

Question. Prove by combinatorial arguments that for integers n and k with $1 \le k \le n$,

$$k\binom{n+1}{k} = (n+1)\binom{n}{k-1}.$$

Solution. We give a counting interpretation of both sides as counting the same object: a *k-subset of* $[n+1] = \{1, 2, \ldots, n+1\}$ *together with a distinguished (marked) element of that k-subset.*

- Count in the left-to-right way: first choose the k-subset from $[n+1]$ in $\binom{n+1}{k}$ ways, then choose which one of the k chosen elements will be the distinguished one. This gives
$$\binom{n+1}{k} \cdot k.$$
- Count in the right-to-left way: first choose which element of $[n+1]$ will be the distinguished element (there are $n+1$ choices), then choose the remaining $k-1$ elements of the k-subset from the remaining n elements. This gives
$$(n+1) \cdot \binom{n}{k-1}.$$

Since both procedures count the same set of marked k-subsets, the two expressions are equal, proving
$$k\binom{n+1}{k} = (n+1)\binom{n}{k-1}.$$
This is a purely combinatorial proof, as required.

Problem 4

Question. How many different three-letter initials can people have? Formally prove your answer by using counting argument. (Consider both cases: repetition allowed and repetition not allowed.)

Solution. Let the alphabet consist of 26 letters (English alphabet). An initial is an ordered triple of letters corresponding to (first, middle, last) names, so order matters.

(A) Repetition allowed. For each of the three positions (first, middle, last) we may independently choose any of the 26 letters. By the rule of product (multiplication principle), the total number of possible initials is
$$26 \times 26 \times 26 = 26^3 = 17576.$$
A short justification: there are 26 choices for the first initial, for each such choice 26 choices for the middle initial, and for each of those 26 choices for the last initial.

(B) Repetition not allowed. If letters may not repeat in an individual's three initials, then choose the first initial in 26 ways, the middle initial in 25 remaining ways, and the last initial in 24 remaining ways. Thus
$$26 \cdot 25 \cdot 24 = 15600.$$
This can also be written as the permutation $P(26, 3) = \dfrac{26!}{(26-3)!} = 26 \cdot 25 \cdot 24.$
