BOOLEAN ALGEBRA

➢ Introduction:

- An algebra that deals with binary number system is called “Boolean Algebra”.
- It is very power in designing logic circuits used by the processor of computer system.
- The logic gates are the building blocks of all the circuit in a computer.
- Boolean algebra derives its name from the mathematician **George Boole** (1815-1864) who is considered the “**Father of symbolic logic**”.
- Boolean algebra deals with truth table TRUE and FALSE.
- It is also called as “**Switching Algebra**”.

➢ Binary Valued Quantities – Variable and Constants:

- A variable used in Boolean algebra or Boolean equation can have only one of two variables. The two values are FALSE (0) and TRUE (1)
- A Sentence which can be determined to be TRUE or FALSE are called **logical statements** or **truth functions** and the results TRUE or FALSE is called **Truth values**.
- The variables which can store the truth values are called **logical variables or binary valued variables**. These can store one of the two values 1 or 0.
- The decision which results into either YES (TRUE or 1) or NO (FALSE or 0) is called **Binary decision**.

➢ Truth Table:

- A truth table is a mathematical table used in logic to computer functional values of logical expressions.
- A truth table is a table whose columns are statements and whose rows are possible scenarios.
- Example: Consider the logical expression

  Logical Statement: Meals = “Ram prefer rice and roti for the meal”

  Y = A AND B (Logical Variables: Y, A, B, Logical Operator AND)

Ram Prefer Rice	Ram Prefer Roti	Meals
FALSE	FALSE	FALSE
FALSE	TRUE	FALSE
TRUE	FALSE	FALSE
TRUE	TRUE	TRUE

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	0

- If result of any logical statement or expression is always TRUE or 1, it is called **Tautology** and if the result is always FALSE or 0, it is called **Fallacy.**

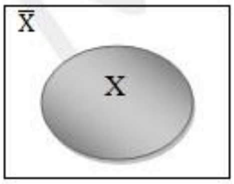
Logical Operators:

- There are three logical operator, NOT, OR and AND.
- These operators are now used in computer construction known as switching circuits.

NOT Operator:

- The Not operator is a unary operator. This operator operates on single variable.
- The operation performed by Not operator is called **complementation**.
- The symbol we use for it is bar.
- $\bar{X}$ means complementation of X
- If X=1, $\bar{X}$=0 If X=0, $\bar{X}$=1
- The Truth table and the Venn diagram for the NOT operator is:

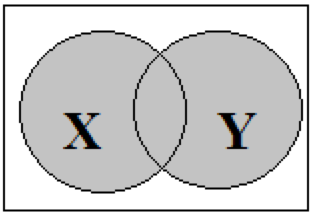
X	$\bar{X}$
1	0
0	1



OR Operator:

- The OR operator is a binary operator. This operator operates on two variables.
- The operation performed by OR operator is called **logical addition**.
- The symbol we use for it is **'+'**.
- Example: X + Y can be read as **X OR Y**
- The Truth table and the Venn diagram for the NOT operator is:

X	Y	X + Y
0	0	0
0	1	1
1	0	1
1	1	1

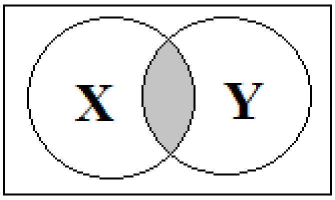


AND Operator:

- The AND operator is a binary operator. This operator operates on two variables.
- The operation performed by AND operator is called **logical multiplication**.
- The symbol we use for it is **'.'**.
- Example: X . Y can be read as **X AND Y**

- The Truth table and the Venn diagram for the NOT operator is:

X	Y	X . Y
0	0	0
0	1	0
1	0	0
1	1	1



➢ Evaluation of Boolean Expression using Truth Table:

- ***To create a truth table, follow the steps given below.***
- Step 1: Determine the number of variables, for n variables create a table with 2^n rows.
    - For two variables i.e. X, Y then truth table will need 2^2 or 4 rows.
    - For three variables i.e. X, Y, Z, then truth table will need 2^3 or 8 rows.
- Step 2: List the variables and every combination of 1 (TRUE) and 0 (FALSE) for the given variables
- Step 3: Create a new column for each term of the statement or argument.
- Step 4: If two statements have the same truth values, then they are equivalent.

➢ Example: Consider the following Boolean Expression $F=X+\overline{Y}$

- Step 1: This expression as two variables X and Y, then 2^2 or 4 rows.
- Step 2: List the variables and every combination of X and Y.
- Step 3: Create a new column $\overline{\mathbf{Y}}$ of the statement, and then fill the truth values of Y in that column.
- Step 4: The final column contain the values of $\mathbf{X+\overline{Y}}$.

X	Y	$\overline{Y}$	$\mathbf{X+\overline{Y}}$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

➢ Exercise Problems:

1. Prepare a table of combination for the following Boolean algebra expressions.

   a) $\overline{X}\,\overline{Y} + \overline{X}\,Y$ b) $XY\overline{Z}+\overline{X}\,\overline{Y}\,Z$

2. Verify using truth table for the following Boolean algebra.

   a) X + XY = X b) $\overline{X+Y} = \overline{X} + \overline{Y}$

➢ Boolean Postulates:

- The fundamental laws of Boolean algebra are called as the postulates of Boolean algebra.
- These postulates for Boolean algebra originate from the three basic logic functions AND, OR and NOT.
- **Properties of 0 and 1:**
  I. If $X \neq 0$ then $X = 1$, and If $X \neq 1$ then $X = 0$
  II. OR relation (Logical Addition)
     a. $0 + 0 = 0$
     b. $0 + 1 = 1$
     c. $1 + 0 = 1$
     d. $1 + 1 = 1$
  III. AND relation (Logical Multiplication)
     a. $0 . 0 = 0$
     b. $0 . 1 = 0$
     c. $1 . 0 = 0$
     d. $1 . 1 = 1$
  IV. Complement Rules
     a. $\overline{0} = 1$
     b. $\overline{1} = 0$

➢ Principle of Duality Theorem:

- This is very important principle used in Boolean algebra.
- Principle of Duality states that;
  - Changing each OR sign (+) to an AND sign (.)
  - Changing each AND sign (.) to an OR sign (+)
  - Replacing each 0 by 1 and each 1 by 0.
- The derived relation using duality principle is called dual of original expression.
- Example: Take postulate II, related to logical addition:

  1) $0 + 0 = 0$ 2) $0 + 1 = 1$ 3) $1 + 0 = 1$ 4) $1 + 1 = 1$

2. Now working according to above relations, + is changed to . and 0's replaced by 1's

  a) $1 . 1 = 1$ b) $1 . 0 = 0$ c) $0 . 1 = 0$ d) $0 . 0 = 0$

- which are nothing but same as that of postulate III related to logical multiplication.
- So 1, 2, 3, 4, are the duals of a, b, c, d.
- Example: Find the duals for the following Boolean Expression

Sl No	Boolean Expression	Duals
1	$X + 0 = X$	$X . 1 = X$
2	$X + 1 = 1$	$X . 0 = 0$
3	$X . \overline{X} = 0$	$X + \overline{X} = 1$
4	$X . (Y + Z)$	$X + (Y . Z)$
5	$X + X.Y = X + Y$	$X.(X + Y) = X.Y$

➢ Boolean Theorems:

- Boolean Theorem can be proved by substituting all possible values of the variable that are 0 and 1.
- This technique of proving theorem is called **<u>Proof by perfect induction.</u>**

Sl No	Theorem	Sl No	Theorem
Properties of 0 and 1		**Associative Law**	
1	$0 + X = X$	12	$X.(Y.Z) = (X.Y).Z$
2	$1 + X = 1$	13	$(X+Y).Z = X+(Y.Z)$
3	$0 . X = 0$	**Distributive Law**	
4	$1 . X = X$	14	$X.(Y+Z) = X.Y + X.Z$
Indempotence Law		15	$X+Y.Z = (X+Y).(X+Z)$
5	$X + X = X$	**Absorption Law**	
6	$X . X = X$	16	$X + XY = X$
Complementary Law		17	$X(X+Y) = X$
7	$X + \bar{X} = 1$	18	$XY + X\bar{Y} = X$
8	$X . \bar{X} = 0$	19	$(X+Y)(X+\bar{Y}) = X$
Involution Law		20	$X+\bar{X}Y = X+Y$
9	$\bar{\bar{X}} = X$	21	$X(\bar{X}+Y) = XY$
Commutative Law			
10	$X + Y = Y + X$		
11	$X . Y = Y . X$		

➢ Theorem 1: $0 + X = X$

Proof: If $X = 0$

then LHS $= 0 + X$

$= 0 + 0$

$= 0$

$=$ RHS

Proof: If $X = 1$

then LHS $= 0 + X$

$= 0 + 1$

$= 1$

$=$ RHS

Using Truth Table

0	X	0+X
0	0	0
0	1	1

➢ Theorem 2: $1 + X = 1$

Proof: If $X = 0$

then LHS $= 1 + X$

$= 1 + 0$

$= 1$

$=$ RHS

Proof: If $X = 1$

then LHS $= 1 + X$

$= 1 + 1$

$= 1$

$=$ RHS

Using Truth Table

1	X	1+X
1	0	1
1	1	1

- **Theorem 3: 0 . X = 0**

Proof: If X = 0
then LHS = 0 . X
= 0 . 0
= 0
= RHS

Proof: If X = 1
then LHS = 0 . X
= 0 . 1
= 0
= RHS

Using Truth Table

0	**X**	**0.X**
0	0	0
0	1	0

- **Theorem 4: 1 . X = X**

Proof: If X = 0
then LHS = 1 . X
= 1 . 0
= 0
= RHS

Proof: If X = 1
then LHS = 1 . X
= 1 . 1
= 1
= RHS

Using Truth Table

1	**X**	**0.X**
1	0	0
1	1	1

- **Indempotence Law:** "This law states that when a variable is combines with itself using OR or AND operator, the output is the same variable".

- **Theorem 5: X + X = X**

Proof: If X = 0
then LHS = X + X
= 0 + 0
= 0
= RHS

Proof: If X = 1
then LHS = X + X
= 1 + 1
= 1
= RHS

Using Truth Table

X	**X**	**X+X**
0	0	0
1	1	1

- **Theorem 6: X . X = X**

Proof: If X = 0
then LHS = X . X
= 0 + 0
= 0
= RHS

Proof: If X = 1
then LHS = X . X
= 1 . 1
= 1
= RHS

Using Truth Table

X	**X**	**X.X**
0	0	0
1	1	1

- **Complementary Law:** "This law states that when a variable is And ed with its complement is equal to 0 and a variable is OR ed with its complement is equal to 1".

- **Theorem 7:** $X + \bar{X} = 1$

Proof: If X = 0

then LHS $= X + \bar{X}$
$= 0 + 1$
$= 1$
= RHS

Proof: If X = 1

then LHS $= X + \bar{X}$
$= 1 + 0$
$= 1$
= RHS

Using Truth Table

X	$\bar{X}$	$X + \bar{X}$
0	1	1
1	0	1

- **Theorem 8:** $X . \bar{X} = 0$

Proof: If X = 0

then LHS $= X . \bar{X}$
$= 0 . 1$
$= 0$
= RHS

Proof: If X = 1

then LHS $= X . \bar{X}$
$= 1 . 0$
$= 0$
= RHS

Using Truth Table

X	$\bar{X}$	$X . \bar{X}$
0	1	0
1	0	0

- **Involution Law:** "This law states that when a variable is inverted twice is equal to the original variable".

- **Theorem 9:** $\bar{\bar{X}} = X$

Proof: If X = 0, then $\bar{X} = 1$

Take complement again, then $\bar{\bar{X}} = 0$ i.e. X

If X = 1, then $\bar{X} = 0$

Take complement again, then $\bar{\bar{X}} = 1$ i.e. X

Using Truth Table

X	$\bar{X}$	$\bar{\bar{X}}$
0	1	0
1	0	1

- **Commutative Law:** "This law states that the order in which two variable are Or ed or AND ed make no difference".

- **Theorem 10:** $X + Y = Y + X$

Proof: If Y = 0

then LHS $= X + Y$
$= X + 0$
$= X$

RHS $= Y + X$
$= 0 + X$
$= X$

Therefore LHS = RHS

Proof: If Y = 1

then LHS $= X + Y$
$= X + 1$
$= 1$

RHS $= Y + X$
$= 1 + X$
$= 1$

Therefore LHS = RHS

Using Truth Table

X	**Y**	**X+Y**	**Y+X**
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

- **Theorem 11:** **X . Y = Y . X**

Proof: If Y = 0

then LHS = X . Y
= X . 0
= 0

RHS = Y . X
= 0 . X
= 0

Therefore LHS = RHS

Proof: If Y = 1

then LHS = X . Y
= X . 1
= X

RHS = Y . X
= 1 . X
= X

Therefore LHS = RHS

Using Truth Table

X	**Y**	**X.Y**	**Y.X**
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

- **Associative Law:** "This law allows the removal of brackets from an expression and regrouping of the variables".

- **Theorem 12:** **X.(Y.Z) = (X.Y).Z**

Proof: If Y = 0

LHS = X.(Y.Z)
= X.(0.Z)
= X.0
= 0

RHS = (X.Y).Z
= (X.0).Z
= 0.Z
= 0

Therefore LHS = RHS

Proof: If Y = 1

LHS = X.(Y.Z)
= X.(1.Z)
= X.Z
= XZ

RHS = (X.Y).Z
= (X.1).Z
= X.Z
= XZ

Therefore LHS = RHS

Using Truth Table

X	**Y**	**Z**	**XY**	**YZ**	**X.(Y.Z)**	**(X.Y).Z**
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

- **Theorem 13:** **X+(Y+Z) = (X+Y)+Z**

Proof: If Y = 0

LHS = X+(Y+Z)
= X+(0+Z)
= X+Z

RHS = (X+Y)+Z
= (X+0)+Z
= X+Z

Therefore LHS = RHS

Proof: If Y = 1

LHS = X+(Y+Z)
= X+(1+Z)
= X+1
= 1

RHS = (X+Y)+Z
= (X+1).Z
= 1+Z
= 1

Therefore LHS = RHS

Using Truth Table

X	**Y**	**Z**	**X+Y**	**Y+Z**	**X+(Y+Z)**	**(X+Y)+Z**
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

- **Distributive Law:** "This law allows the multiplying or factoring out an expression".
- **Theorem 14:** **X.(Y+Z) = XY + XZ**

Proof: If X = 0	Proof: If X = 1
LHS = X.(Y+Z) = 0.(Y+Z) = 0 RHS = XY + XZ = 0.Y+0.Z = 0 Therefore LHS = RHS	LHS = X.(Y+Z) = 1.(Y+Z) = Y+Z RHS = XY + XZ = 1.Y+1.Z = Y+Z Therefore LHS = RHS

- **Theorem 15:** (X + Y) (X + Z) = X + YZ

LHS: $(X + Y)\ (X + Z) = XX + XZ + XY + YZ$
$= X + XZ + XY + YZ$
$= X(1 + Z) + XY + YZ$
$= X + XY + YZ$
$= X(1 + Y) + YZ$
$= X + YZ$
= RHS

- **Absorption Law:** "This law enables a reduction of complicated expression to a simpler one by absorbing common terms".

16) X+XY = X	**17) X (X+Y) = X**	**18)** $\mathbf{XY + X\overline{Y} = X}$
LHS = X + XY = X (1 + Y) = X = RHS	LHS = X (X+Y) = XX + XY = X + XY = X (1+Y) = X	LHS $= XY + X\overline{Y}$ $= X(Y+\overline{Y})$ = X.1 = X = RHS
19) $\mathbf{(X+Y)\ (X+\overline{Y}) = X}$ LHS $= (X+Y)\ (X+\overline{Y})$ $= XX + X\overline{Y} + XY + \overline{Y}Y$ $= X + X\overline{Y} + XY + 0$ $= X\ (1 + \overline{Y} + Y)$ = X. 1 = X	**20)** $\mathbf{X + \overline{X}Y = X+Y}$ LHS $= X + \overline{X}Y$ $= (X + \overline{X})\ (X+Y)$ = 1. (X+Y) = X+Y = RHS	**21)** $\mathbf{X\ (\overline{X}+Y) = XY}$ LHS $= X\ (\overline{X}+Y)$ $= X.\overline{X} + X.Y$ = 0 + XY = XY = RHS

➢ DeMorgan's Theorem:

- **DeMorgan's First Theorem:**
  - **Statement: "When the OR sum of two variables is inverted, this is same as inverting each variable individually and then AND ing these inverted variables**"
  - This can be written as $\overline{X+Y} = \bar{X} \,.\, \bar{Y}$
  - We can prove the DeMorgan's First theorem by using Truth Table is

X	Y	$\bar{X}$	$\bar{Y}$	X+Y	$\overline{X+Y}$	$\bar{X}\bar{Y}$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

  - Compare the column $\overline{X+Y}$ and $\bar{X}\,.\,\bar{Y}$. Both of these are identical. Hence the DeMorgan's first theorem is proved.

- **DeMorgan's Second Theorem:**
  - **Statement: "When the AND product of two variables is inverted, this is same as inverting each variable individually and then OR ing these inverted variables**"
  - This can be written as $\overline{XY} = \bar{X} + \bar{Y}$
  - We can prove the DeMorgan's Second theorem by using Truth Table is:

X	Y	$\bar{X}$	$\bar{Y}$	X.Y	$\overline{XY}$	$\overline{X+Y}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

  - Compare the column $\overline{XY}$ and $\bar{X}+\bar{Y}$. Both of these are identical. Hence the DeMorgan's Second theorem is proved.

- **Application of DeMorgan's Theorem:**
  - It is used in simplification of Boolean expression.
  - DeMorgan's law commonly apply to text searching using Boolean operators AND, OR and NOT.
  - It is useful in the implementation of the basic gates operations with alternative gates.