MTH 201: Probability and Statistics

Quiz 1

10/02/2025

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No books, notes, or devices are allowed. Just a pen and eraser. Any exchange of information related to the quiz with a human or machine will be deemed as cheating. Institute rules will apply. Explain your answers. Show your steps. Approximate calculations are fine as long as the approximations are reasonable. Don't bother simplifying products and sums that are too time consuming. Leave your answer as products and sums, if you don't have the time. You have 50 minutes.

Question 1. 30 **marks** You are given a bundle of coins that contains four 2 rupee coins, four 3 rupee coins, eight 5 rupee coins, and sixteen 10 rupee coins. Answer the following (show your work).

(a) Suppose you choose a coin randomly from the bundle. Calculate the probability that you choose 10 rupees.

(b) Suppose you choose two coins randomly and without replacement from the bundle. Calculate the probability that you collect a total of 10 rupees from the two chosen coins.

(c) Derive the PMF of the total rupees you take away at the end of choosing two coins randomly and without replacement.

(d) Suppose choosing two coins randomly and without replacement results in a total less than or equal to 5 rupees. Calculate the probability that the total is 4 rupees?

(e) Consider a game that requires you to choose two coins randomly and without replacement. At the end of a game you take away the total of coins you chose in the game. You play the game repeatedly till your current game attempt results in at least 5 rupees. Outcomes of games are independent of each other. Derive the distribution (Probability Mass Function) of the number of games you end up playing.

Question 2. 35 **marks** A randomly chosen person in Delhi voted in the recent elections with probability 0.6. A person who votes must choose one from options A, B, C, and N. A voter chooses A with probability 0.4, B with probability 0.3, C with probability 0.2, and N with probability 0.1. Note that a non-voter, chooses from neither of the above options with probability 1.

Voters have their left hand's index finger stained with an electoral ink and often use images of their fingers to declare that they voted. Non-voters, with probability 0.2, stain their fingers to resemble that of actual voters.

Given that a randomly chosen person's finger is stained with ink, derive the probability that the person voted for B.

Question 3. 35 **marks** A coin gives the first heads in N trials where N is a random variable with PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, 9, \\ (1-p)^9 & n = 10, \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < p < 1$. Suppose the coin gives tails in the first three trials. Derive the probability that the first heads is obtained in the sixth trial? [Hint: What's the given event in terms of random variable N?]

Question 1. 30 **marks** You are given a bundle of coins that contains four 2 rupee coins, four 3 rupee coins, eight 5 rupee coins, and sixteen 10 rupee coins. Answer the following (show your work).

(a) Suppose you choose a coin randomly from the bundle. Calculate the probability that you choose 10 rupees.

(b) Suppose you choose two coins randomly and without replacement from the bundle. Calculate the probability that you collect a total of 10 rupees from the two chosen coins.

(c) Derive the PMF of the total rupees you take away at the end of choosing two coins randomly and without replacement.

(d) Suppose choosing two coins randomly and without replacement results in a total less than or equal to 5 rupees. Calculate the probability that the total is 4 rupees?

(e) Consider a game that requires you to choose two coins randomly and without replacement. At the end of a game you take away the total of coins you chose in the game. You play the game repeatedly till your current game attempt results in at least 5 rupees. Outcomes of games are independent of each other. Derive the distribution (Probability Mass Function) of the number of games you end up playing.

(a), (b), (c), (d): 5 marks each.

(e): 10 marks.

(a) We have a total of 32 coins of which 16 are ten rupee coins. — Some valid reasoning 2.5/5

$P[\text{Chosen coin is a 10 rupee coin}]$

$= \frac{16}{32} = 1/2.$ — Final answer 2.5/5

(b) Given the bundle, one obtains a total of 10 rupees from two coins if both chosen coins are 5 rupee coins.

$P[\text{We obtain a total of 10 rupees from two coins}]$

$= P[\text{First coin is a 5 rupee coin, Second coin is a 5 rupee coin}]$ — Valid reasoning 2.5/5

$= \frac{8}{32} \times \frac{7}{31}.$ — Answer demonstrating choosing without replacement. 2.5/5

(c) The sample space of outcomes is

$\{(2,2), (2,3), (2,5), (2,10),$
$(3,2), (3,3), (3,5), (3,10),$
$(5,2), (5,3), (5,5), (5,10),$
$(10,2), (10,3), (10,5), (10,10)\}.$

Let R be the total rupees.

$S_R = \{4, 5, 6, 7, 8, 10, 12, 13, 15, 20\}$ ← Capturing all valid sums in the PMF — Up to 1/5.

The PMF

$$P_R(x) = \begin{cases} \left(\frac{4}{32}\right)\left(\frac{3}{31}\right) & x=4 \\ \left(\frac{4}{32}\right)\left(\frac{4}{31}\right)+\left(\frac{4}{32}\right)\left(\frac{4}{31}\right) & x=5 \\ \left(\frac{4}{32}\right)\left(\frac{3}{31}\right) & x=6 \\ \left(\frac{4}{32}\right)\left(\frac{8}{31}\right)+\left(\frac{8}{32}\right)\left(\frac{4}{31}\right) & x=7 \\ \left(\frac{4}{32}\right)\left(\frac{8}{31}\right)+\left(\frac{8}{32}\right)\left(\frac{4}{31}\right) & x=8 \\ \left(\frac{8}{32}\right)\left(\frac{7}{31}\right) & x=10 \\ \left(\frac{4}{32}\right)\left(\frac{16}{31}\right)+\left(\frac{16}{32}\right)\left(\frac{4}{31}\right) & x=12 \\ \left(\frac{4}{32}\right)\left(\frac{16}{31}\right)+\left(\frac{16}{32}\right)\left(\frac{4}{31}\right) & x=13 \\ \left(\frac{8}{32}\right)\left(\frac{16}{31}\right)+\left(\frac{16}{32}\right)\left(\frac{8}{31}\right) & x=15 \\ \left(\frac{16}{32}\right)\left(\frac{15}{31}\right) & x=20 \\ 0 & \text{otherwise} \end{cases}$$

— Up to 0.3/5 for each x in S_R. — otherwise 1/5

Sample Outcomes.

(2,2) occurs
(2,3) or (3,2)
(3,3)
(2,5) or (5,2)
(3,5) or (5,3)
(5,5)
(2,10) or (10,2)
(3,10) or (10,3)
(5,10) or (10,5)
(10,10)

(d) We are given that $\{R \le 5\}$.

We require $P[R=4 \mid R \le 5]$

$= \frac{P[R=4, R \le 5]}{P[R \le 5]}$

$= \frac{P[R=4]}{P[R=4]+P[R=5]}$ (Given S_R, $\{R \le 5\} = \{R=4\} \cup \{R=5\}$) — Up to 4/5

$= \frac{\left(\frac{4}{32}\right)\left(3/31\right)}{\left(\frac{4}{32}\right)\left(3/31\right) + 2\left(\frac{4}{32}\right)\left(\frac{4}{31}\right)}$ (Substituting values from the PMF) — 1/5 for the correct values.

(e) Probability that a game returns at least 5 rupees is $P[R \ge 5] = 1 - P[R=4].$ — Correctly capturing the success event & its probability. 2/10

Let N be the no. of games you play.

$P[N=1] = P[R \ge 5] = 1 - P[R=4]$
$P[N=2] = P[R=4]\left(1 - P[R=4]\right)$
$\vdots$

Note that the games result in independent outcomes.

We require

$$P_N(n) = \begin{cases} \left(P[R=4]\right)^{n-1}\left(1-P[R=4]\right) & n=1,2,\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

— Up to 8/10. The correct PMF. One may also say that N is geometric $(P[R \ge 5])$ & go from there.

Question 2. 35 **marks** A randomly chosen person in Delhi voted in the recent elections with probability 0.6. A person who votes must choose one from options A, B, C, and N. A voter chooses A with probability 0.4, B with probability 0.3, C with probability 0.2, and N with probability 0.1. Note that a non-voter, chooses from neither of the above options with probability 1.

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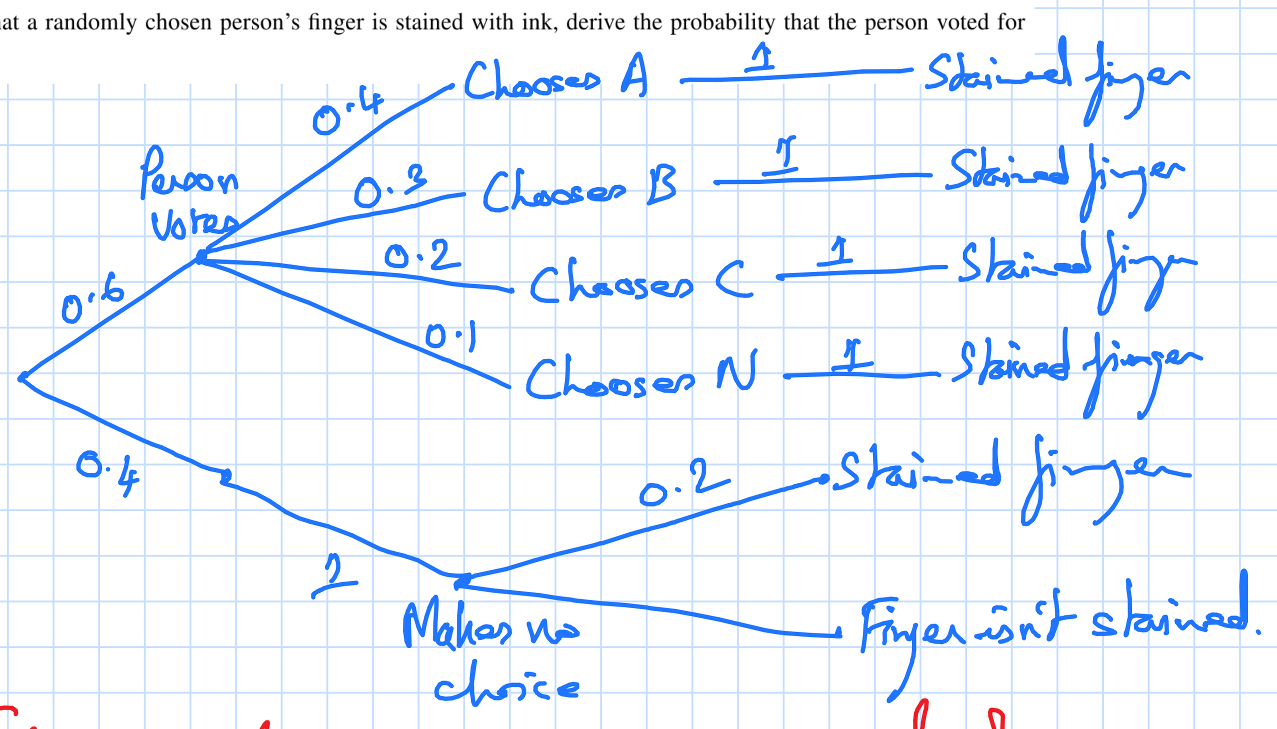
Given that a randomly chosen person's finger is stained with ink, derive the probability that the person voted for B.



[A tree diagram or any equivalent way of capturing all required probabilities Up to 10/35; This is to reward an attempt, even if incorrect.]

$$P[\text{Chooses } B \mid \text{Finger Stained}]$$

$$= \frac{P[\text{Chooses } B, \text{Finger Stained}]}{P[\text{Finger Stained}]}$$

15/35 Correctly mentioning the conditional prob. of interest & rewriting it.

$$= \frac{0.18}{P[\text{Finger Stained}]}$$ (See second path from top in the tree diagram)

$$= \frac{0.18}{P[\text{Votes, Finger Stained}] + P[\text{Doesn't vote, Finger Stained}]}$$

15/35 Correctly calculating the numerator (using tree diagram or otherwise) & correctly expanding the denominator

$$= \frac{0.18}{(0.6)(1) + (0.4)(0.2)}$$

$$= \frac{0.18}{0.6 + 0.08} = \frac{0.18}{0.68}$$

$$= \frac{9}{34}$$

Correct answer 5/35

Alternately:

$$P[\text{Chooses } B \mid \text{Finger Stained}]$$

$$= \frac{P[\text{Chooses } B, \text{Finger Stained}]}{P[\text{Finger Stained}]}$$

$$= \frac{P[\text{Finger Stained} \mid \text{Chooses } B]\, P[\text{Chooses } B]}{P[\text{Finger Stained}]}$$

$$= \frac{(1)\, P[\text{Chooses } B]}{P[\text{Finger Stained}]}$$

15/35

$$= \frac{(0.6)(0.3)}{P[\text{Finger Stained}]}$$

The rest is as before.

Rest as before.

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$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, 9, \\ (1-p)^9 & n = 10, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < p < 1$. Suppose the coin gives tails in the first three trials. Derive the probability that the first heads is obtained in the sixth trial? [Hint: What's the given event in terms of random variable N?]

The coin gives tails in the first three trials. Since N counts trials up to and including the first heads, we are given that $N > 3$.

Identifying & correctly expressing in terms of N the given event.

15/35

We require $P[N=6 \mid N>3]$

$$= \frac{P[N=6, N>3]}{P[N>3]}$$

$$= \frac{P[N=6]}{P[N>3]}$$

$$= \frac{(1-p)^5 p}{1-\left(P[N=1]+P[N=2]+P[N=3]\right)}$$

$$= \frac{(1-p)^5 p}{1-\left(p+(1-p)p+(1-p)^2 p\right)}$$

15/35

$$= \frac{(1-p)^5 p}{1-(2p-p^2)-p(1-p)^2}$$

$$= \frac{p(1-p)^5}{(p^2-2p+1)-p(1-p)^2} = \frac{p(1-p)^5}{(1-p)^3}$$

$$= p(1-p)^2.$$

5/35

final answer