Proof: (Last Class)

Theorem: Let n be an integer. If n^3+5 is odd, then n is even.

$A(x)$: x is odd

Domain $\mathbb{Z}$
set of all integers

$\neg A(x)$: x is even

Original statement

$$\forall x\,(\underbrace{A(x^3+5)}_{p} \rightarrow \underbrace{\neg A(x)}_{q}).$$

What did we prove

Equivalent contrapositive $\quad (p \rightarrow q)$ $\quad (\neg q \rightarrow \neg p)$

$$\boxed{\forall x\,(A(x) \rightarrow \neg A(x^3+5))}$$

Methods of proofs:

Direct proof: Assume the premise (whatever the assumption means) Then use mathematical reasonings to conclude that the claimed conclusion is true.

Give a direct proof of its equivalent contrapositive statement.

We gave a direct proof of this contrapositive statement.

Proof by contraposition: Assume the negation of the claimed conclusion. Then, use a sequence of logical reasoning to conclude the negation of premise

Theorem 1: Let n be an integer. Then $\boxed{n^2 \text{ is even}}$ if and only if $\boxed{n \text{ is even}}$.

Proof: $\forall x\,(A(x^2) \rightarrow A(x))$ and

$\forall x\,(A(x) \rightarrow A(x^2))$ → needs to prove.

$\forall x\,(A(x^2) \rightarrow A(x))$

$(\Rightarrow)$ Forward direction

Domain is $\mathbb{Z}$ $\qquad$ biconditional

$A(x)$: x is even

$$\boxed{\forall x\,(A(x^2) \leftrightarrow A(x))} \quad (p \leftrightarrow q)$$

How to prove? $(p \leftrightarrow q)$ is equivalent to $((p \rightarrow q) \wedge (q \rightarrow p))$

$\boxed{\text{Exercise}}$

Prove that $(p \rightarrow q)$, and $(q \rightarrow p)$

$\forall x\,(A(x) \rightarrow A(x^2))$

$(\Leftarrow)$ Backward direction.

Formal proof:

$(\Rightarrow)$ Forward direction.

(for all integers x, [if x^2 is

$(\Leftarrow)$ Backward direction

(for all integers x, [if x is even, then x^2 is even])

even, then x is even)

By contraposition, we assume that x is odd.
Then there exists an integer k such that $\boxed{x = 2k+1}$

Then $x^2 = 4k^2+4k+1 = 2(2k^2+2)+1$

Then, x^2 is odd which is the negation of the premise.
This completes the proof of forward direction ($\Rightarrow$)

Direct proof:
Assume that x is an even integer. Then there exists an integer K such that $x = 2k$.

Then $x^2 = 4k^2 = 2.2k^2$.

As k^2 is an integer, therefore x^2 is an even integer.
This completes the proof of backward direction ($\Leftarrow$)

What did we do in proof by contraposition?

We assume the negation of conclusion ($\neg q$) and used mathematical reasonings to conclude negation of premise ($\neg p$)

Assume p (the premise) and the negation of the conclusion ($\neg q$).

Then using the assumption $\neg q$ either you conclude $\neg p$. This will imply $\boxed{p \wedge \neg p} \to$ contradiction

OR, you conclude some thing that violates some basic/fundamental

Coprime: Two distinct natural numbers x and y are coprime if they do not have any common divisor apart from 1.

$\boxed{4 \text{ and } 6} \to$ both divisible by 2
Hence, not coprime

$\boxed{4 \text{ and } 5} \to$ coprime

$\boxed{9 \text{ and } 10} \to$ 9 is divisible by 1, 3, and 9.
10 is divisible 1, 2, 5, 10.
9 and 10 are coprime.

conventions in mathematics

PROOF BY CONTRADICTION

Rational Numbers

$\frac{p}{q}$ $p, q \in \mathbb{Z}$
$q \neq 0$

Theorem 2: $\sqrt{2}$ is an irrational number.

Proof: Assume that $\sqrt{2}$ is a rational number.

Then there exists $x, y \in \mathbb{Z}$ such that $y \neq 0$, x and y are coprime, and $\sqrt{2} = \frac{x}{y}$.

Then, $\frac{x^2}{y^2} = 2$, it means that

$x^2 = 2y^2$.

Then, x^2 is even.

Then, by Theorem 1, $\boxed{x \text{ is even.}}$

Then there exists integer k such that $x = 2k$.

Hence, $x^2 = 4k^2 = 2y^2$

It means that $y^2 = 2k^2$.

Then, y^2 is even.

By Theorem 1, $\boxed{y \text{ is even}}$

Then, x and y both are divisible by 2.

This contradicts our assumption that x and y are coprime.

$(x, y \in \mathbb{Z}, y \neq 0) \wedge (x \text{ and } y \text{ are coprime}) \wedge (\sqrt{2} = x/y)$

Tut question: $\boxed{\sqrt{7} \text{ is irrational}}$

Think about. $\sqrt{20}$ is irrational

Exercise

Theorem 3: If n is an integer, then $n^2 \geq n$.

Proof: Let $n \in \mathbb{Z}$.

Three cases arise.

Case 1: $n = 0$

Case 2: $n > 0$ $(n \geq 1)$

Case 3: $n < 0$ $(n \leq -1)$

Case 1: If $n = 0$, then

$n^2 = 0 = 0 = n$. Hence proved. $\boxed{n^2 = n}$

Case 2: $n \geq 1$.

Then $n^2 \geq n$. Hence statement is true.

Case 3: $\boxed{n \leq -1.}$

Then $n^2 \geq 1 > -1 \geq n$.

Then $n^2 > n$. Hence the statement true.

PROOF BY CASES

Divide the premise into

Case 2: $n \geq 1$.

Then $n^2 = n \cdot n \geq 1 \cdot n$
$= n$

Statement: If n^2 is a natural number then n is a natural number.

$\forall x \left(A(x^2) \to A(x) \right)$

Then find an integer x such that $A(x^2)$ is true but $A(x)$ is false.

x is called $\boxed{\text{COUNTER-EXAMPLE}}$

multiple cases.

Then give proof for each of the cases.

$A(x)$: $x \geq 0$

Domain set of all integers

~~$B(x)$: $x \geq 0$~~

How to DISPROVE a statement?

$\exists x \left(A(x^2) \wedge \neg A(x) \right)$ is true.

$\boxed{x = -2}$ $A(x^2)$ is true $4 \geq 0$

$A(x)$ is false $-2 < 0$

How to disprove that two compound propositions are logically equivalent?

Equivalent if and only if in the truth table, the values are the same

Both compound propositions are obtained from the same set of atomic propositions.

Find a combination of truth values for which they give different outcome.

Theorem 4: Let x and y be two integers. If both xy and $x+y$ are even, then both x and y are even.

Proof: Let $x, y \in \mathbb{Z}$

Proof by contrapositive.

Case 1: $\boxed{x \text{ is odd}}$ and y is even.

$x = 2k+1$

Premise: xy is even and $x+y$ is even.

Conclusion: x is even and y is even.

Negation of conclusion: x is odd or y is odd.

Case 2: x is even and y is odd

(This case can be proved

Then. for some integer k and $y = 2l$ for some integer l.

Then, $x+y = 2(k+l)+1$.

Then $x+y$ is odd.

Negation of premise is true.
(Hence Statement is true)

Without loss of generality assume that x is odd

Then two cases
- y is even
- y is odd

by switching the role of x and y in Case 1.)

Arguments are similar.
(Hence Statement is true)

Case 3: $\boxed{x \text{ is odd}}$ and y is odd.

Then there are integers a, b such that $x = 2a+1$
$y = 2b+1$.

$$xy = 4ab + 2a + 2b + 1$$
$$= 2(ab + a + b) + 1$$

Then xy is odd

Implies the negation of premise. Hence Statement is true

In each of the three cases,