27th Aug: **Proof:**

Theorem 1: If n is an odd integer then n^2 is an odd integer

$A(n)$: n is an odd integer

$B(n)$: n^2 is an odd integer.

Domain = set of all integers

$\forall n \left(A(n) \rightarrow B(n)\right)$

Proof: Let n be an odd integer.

Then there exists $k \in \mathbb{Z}$ such that $n = 2k+1$.

Then $n^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$.

As $k \in \mathbb{Z}$, hence $2k^2 + 2k \in \mathbb{Z}$

Then n^2 is an odd integer.

(Proved)

Odd integer: An integer n is odd if there exist an integer k such that $n = 2k+1$

Even Integer: An integer n is even if there exists an integer k such that $n = 2k$.

"n is an odd integer" = $A(n)$ premise of the statement

"n^2 is an odd integer" = $B(n)$ conclusion of the statement

DIRECT PROOF

$(p \rightarrow q)$ statement to be proved. Assume the premise p, use a sequence of mathematical arguments to justify that q is true.

Theorem 2: If m and n are perfect squares then mn is a perfect square.

Proof: Let m and n be two perfect squares. Then, there exists $x, y \in \mathbb{Z}$ such that $m = x^2$ and $n = y^2$.

Then $mn = x^2y^2 = (xy)^2$

Since $x, y \in \mathbb{Z}$, $xy \in \mathbb{Z}$

Hence mn is a perfect square

(Proved)

An integer m is a perfect square if there exists $y \in \mathbb{Z}$ such that $m = y^2$.

$A(m)$: m is a perfect square.

$$\boxed{\forall m \, \forall n \left((A(m) \wedge A(n)) \rightarrow A(mn) \right)}$$

Domain: set of all integers.

$\boxed{\text{DIRECT PROOF}}$

Theorem 3: Let n be an integer. If $3n+2$ is odd, then n is odd.

Proof: Let $3n+2$ be an odd integer (by premise). Then there exists $k \in \mathbb{Z}$ such that $3n+2 = 2k+1$

Then $3n+1 = 2k$.

Can we really conclude that n is odd?

DIRECT PROOF ATTEMPT FAILS.

Proof: Let n be an integer and assume that n is even.

Then there exists $k \in \mathbb{Z}$ such that $n = 2k$.

Then $3n+2 = 3.2k+2 = 6k+2$
$= 2(3k+1)$.

As $3k+1 \in \mathbb{Z}$, $3n+2$ is even

This is the negation of the premise

(Proved)

Domain: Set of all positive integers $\mathbb{N} \setminus \{0\}$.

$\boxed{n = ab.}$ $P(n,a,b)$: $n = ab$

$R(a,n)$: $a \leq \sqrt{n}$

$R(b,n)$: $b \leq \sqrt{n}$

$$\forall n \forall a \forall b \left(P(n,a,b) \rightarrow (R(a,n) \vee R(b,n)) \right)$$

$A(n)$: n is odd

Domain is set of all integers

$$\forall n \left(A(3n+2) \rightarrow A(n) \right)$$

$(p \rightarrow q)$ can be proved by assuming $\neg q$, then use mathematical arguments to conclude $\neg p$ which is the negation of premise.

INDIRECT PROOF

PROOF BY CONTRAPOSITION

$(\neg q \rightarrow \neg p)$ contrapositive of $p \rightarrow q$.

Theorem 4: If $n = ab$ where a and b are positive integers then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof: We will prove that

$$\forall n \forall a \forall b \left((\neg R(a,n) \wedge \neg R(b,n)) \rightarrow \neg P(n,a,b) \right)$$

Assume that $a > \sqrt{n}$ and $b > \sqrt{n}$.

Then, $ab > n$ when both $a, b > 0$

It implies that $ab \neq n$.

which is the negation of the premise.

VACOUS PROOF: $(p \to q)$ is a statement if p is false, then no matter of whether q is true or false. $(p \to q)$ is true. $\boxed{\text{VACOUSLY TRUE}}$

Example: If n is an integer with $10 \leq n \leq 15$ that is a perfect square, then n is a perfect cube.

10, 11, 12, 13, 14, 15

There does not exist any perfect square between 10 and 15.

Hence, the premise is false.

Therefore, the statement is vacously true.

Theorem 5: Let n be an integer. If n^3+5 is odd, then n is even.

Proof: Assume that the conclusion is false. Then n is odd.

Then, there exists $k \in \mathbb{Z}$ such that $n = 2k+1$.

Then, $n^3 = 8k^3 + 12k^2 + 6k + 1$

Then $n^3 + 5 = 8k^3 + 12k^2 + 6k + 6$

$$= 2(4k^3 + 6k^2 + 3k + 3)$$

Then, n^3+5 is even.

which is the negation of the premise.

Example: Every positive integer is the sum of squares of two integers. (Exercise)

$A(n)$: n is an odd integer

$\forall n \left(A(n^3+5) \to \neg A(n) \right)$.

Domain integers

Theorem 6: The sum of two rational numbers is a rational number.

$\mathbb{Q}$: a number x is rational if there exist $p, q \in \mathbb{Z}$ with $q \neq 0$ such that

$$x = \frac{p}{q}$$

Proof: Domain: set of all rational numbers.

$$\forall x \forall y \left((x \in \mathbb{Q}) \land (y \in \mathbb{Q}) \to x+y \in \mathbb{Q} \right)$$

Proof: Let $x = \frac{a}{b}$ such that $a, b \in \mathbb{Z}$ and $b \neq 0$,

Similarly $y = \frac{p}{q}$ such that $p, q \in \mathbb{Z}$ and $q \neq 0$.

Then $x + y = \frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}$

Since $b, q \neq 0$, therefore $bq \neq 0$. $aq + bp \in \mathbb{Z}$. Hence $x + y \in \mathbb{Q}$

DIRECT PROOF

A positive integer n is prime if $n \neq 1$ and n is only divisible by 1 and n itself.

Definition implies if and only if. If a positive integer n is divisible by n and itself, then n is prime.

If a natural number n is divided by k, then the set of possible remainders is $\{0, 1, 2, \ldots, k-1\}$.

$k = 3$ remainders are 0, 1, 2

$$\boxed{\begin{matrix} 3x+1 \\ 3x+2 \\ 3x \end{matrix}}$$

division by k

$$\boxed{\begin{matrix} kx+1 \\ kx+2 \\ \vdots \\ kx+(k-1) \\ kx \end{matrix}}$$

Basic Set Theory:

A and B are two sets.

Set is a collection of some well-defined objects. $x \in A$

x is an element of A.

$A = \{0, 4, 7, 10, 15\}$ $\boxed{4 \in A}$

$\boxed{25 \notin A}$

Property:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

$\boxed{\mathbb{N} \setminus \{0\}}$ elements that are in A but are not in B.

Properties: associative

$A \cup (B \cup C) = (A \cup B) \cup C$

Distributive property

needs a proof.

$(p \to q)$ $(\neg q \to \neg p)$

contrapositive