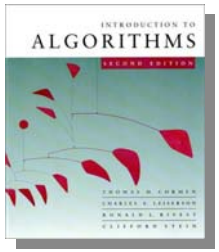
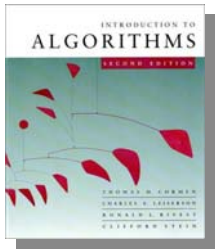


Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0 1 1 2 3 5 8 13 21 34 ⋯



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0 1 1 2 3 5 8 13 21 34 $\cdots$

Naive recursive algorithm: $\Omega(\phi^n)$ (exponential time), where $\phi = (1+\sqrt{5})/2$ is the ***golden ratio***.

Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$

for any two numbers a and b $\quad a > b > 0$

$$\phi = \frac{a+b}{a} = \frac{a}{b} \Rightarrow \text{quadratic in } \phi.$$

Relation to Fibonacci numbers:

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \phi$$

ϕ as continued fraction, $\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$ ∞

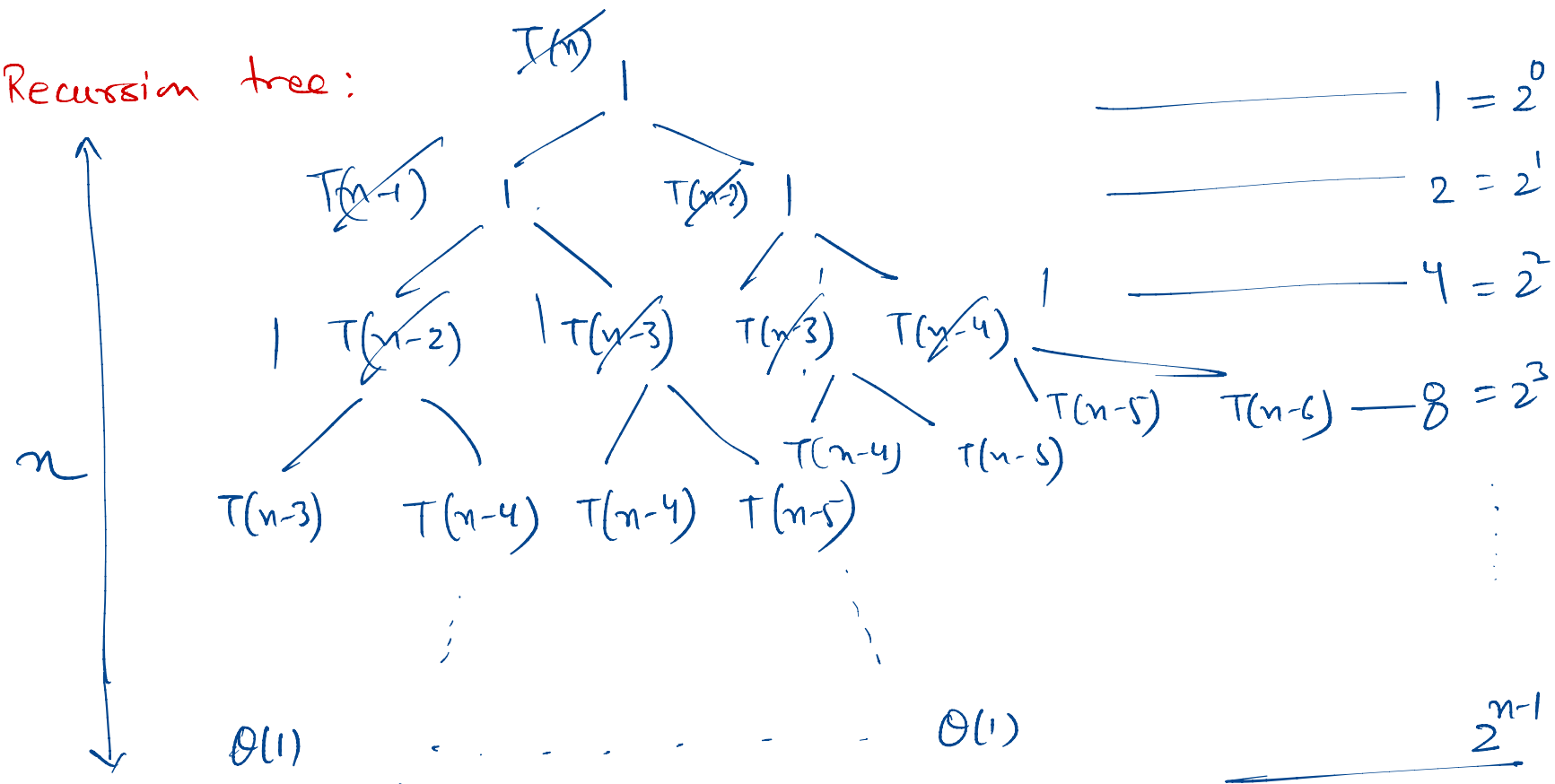
Naiive computation of Fibonacci numbers.

Open MP for multithreading

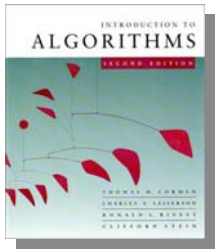
$$F_n = F_{n-1} + F_{n-2}$$

$$T(n) = T(n-1) + T(n-2) + \Theta(1) \quad \text{——— Rec. relation}$$

Recursion tree:



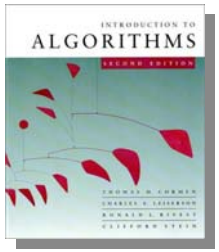
$$T(n) = \sum_{i=1}^{n} 2^{i-1} = 2^n = \boxed{O(2^n)}$$



Computing Fibonacci numbers

Bottom-up:

- Compute $F_0, F_1, F_2, \ldots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.



Computing Fibonacci numbers

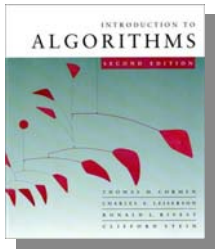
Bottom-up:

- Compute $F_0, F_1, F_2, \ldots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.

Naive recursive squaring: Binet's formula

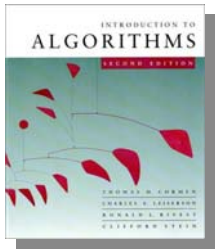
$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.



Recursive squaring

Theorem: $$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n .$$



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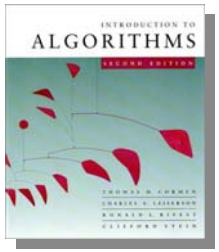
Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.



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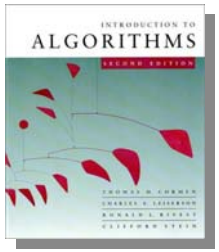
Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.

Proof of theorem. (Induction on n.)

Base ($n = 1$): $\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1$.



Recursive squaring

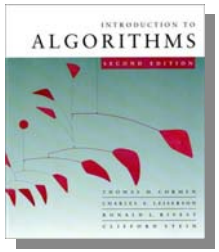
Inductive step ($n \geq 2$):

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n}$$

■



Matrix multiplication

Input: $A = [a_{ij}]$, $B = [b_{ij}]$.
Output: $C = [c_{ij}] = A \cdot B$.

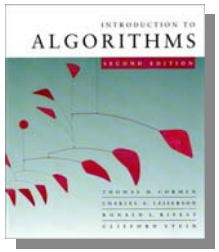
$\Big\}$ $i, j = 1, 2, \ldots, n$.

dot product = row^A · col^B

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

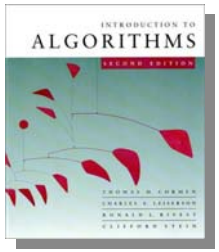
row^A · col^B

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



Standard algorithm

for $i \leftarrow 1$ **to** n
  do for $j \leftarrow 1$ **to** n
    do $c_{ij} \leftarrow 0$
      for $k \leftarrow 1$ **to** n
        do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$



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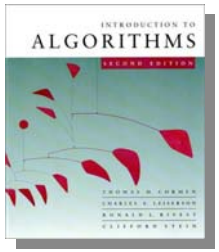
Running time = $\Theta(n^3)$



Divide-and-conquer algorithm

Cost of addition

$4\left(\left[\frac{n^2}{4}\right]_{n/2\times n/2} + \left[\frac{n^2}{4}\right]_{n/2\times n/2}\right)$

IDEA:

$n\times n$ matrix $= 2\times 2$ matrix of $(n/2)\times(n/2)$ submatrices:

block

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Block multiplication

$$C = A \cdot B$$

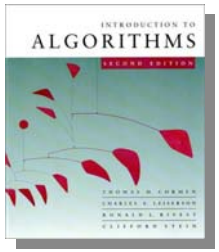
$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

cost of multiplication

$$\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}$$

8 mults of $(n/2)\times(n/2)$ submatrices

4 adds of $(n/2)\times(n/2)$ submatrices



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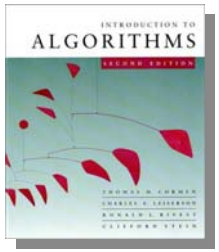
$$C = A \cdot B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dh \\ u &= cf + dg \end{aligned}\right\}$$

recursive

8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices



Analysis of D&C algorithm

$$T(n) = a\,T(n/b) + f(n)$$

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

$a = 8, \quad b = 2$

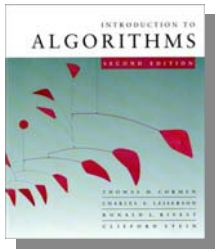
$f(n) = n^2$

$n^{\log_b a} = n^3$

 # submatrices

submatrix size

work adding submatrices



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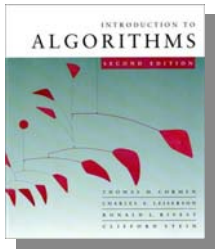
 # submatrices

submatrix size

work adding submatrices

$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$

$T(n) = 8\,T(n/2) + \Theta(n^2)$



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 # submatrices

submatrix size

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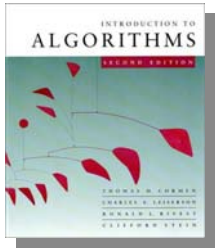
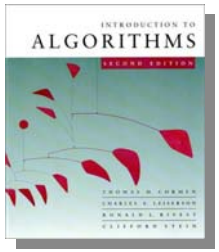
No better than the ordinary algorithm.



Strassen's idea

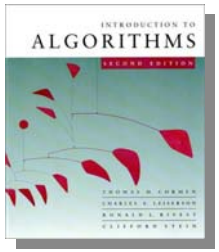
- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$



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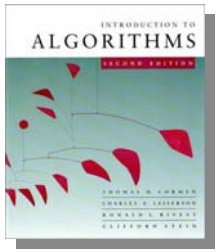
$$P_7 = (a - c) \cdot (e + f)$$

7 matrix mults

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$



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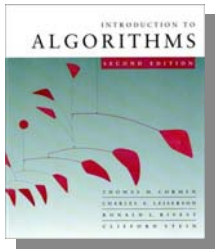
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!



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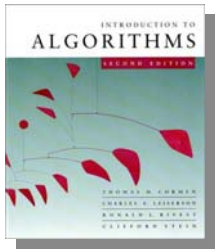
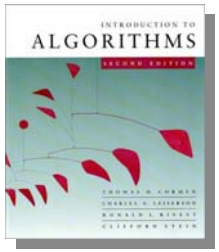
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$



Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.



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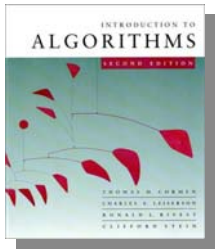
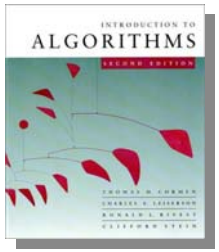
$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

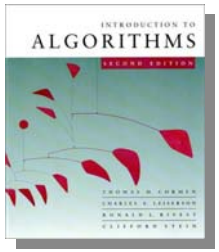
$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow$ $T(n) = \Theta(n^{\lg 7})$.



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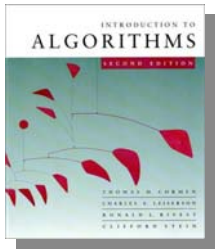
$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.



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$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.

(2024) $n^{2.3713}$

Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.

Tower of Hanoi

- A puzzle containg 3 rods and a number of discs of various sizes (diameters) which can slide onto any rod.
- The puzzle begins with all discs stacked on one rod in order of decreasing size; smallest on top.

Objective : Move the entire stack to one of the other empty rods.

Rules :
1. Only 1 disc can be moved at a time
2. discs can only be moved from/to top of a stack or on an empty rod.
3. Only a smaller disc may be placed on top of an existing disc.

# discs	# steps
$n=1$	1
$n=2$	3
$n=3$	7
$n=4$	15
⋮	
$n=k$	2^k-1

Complexity

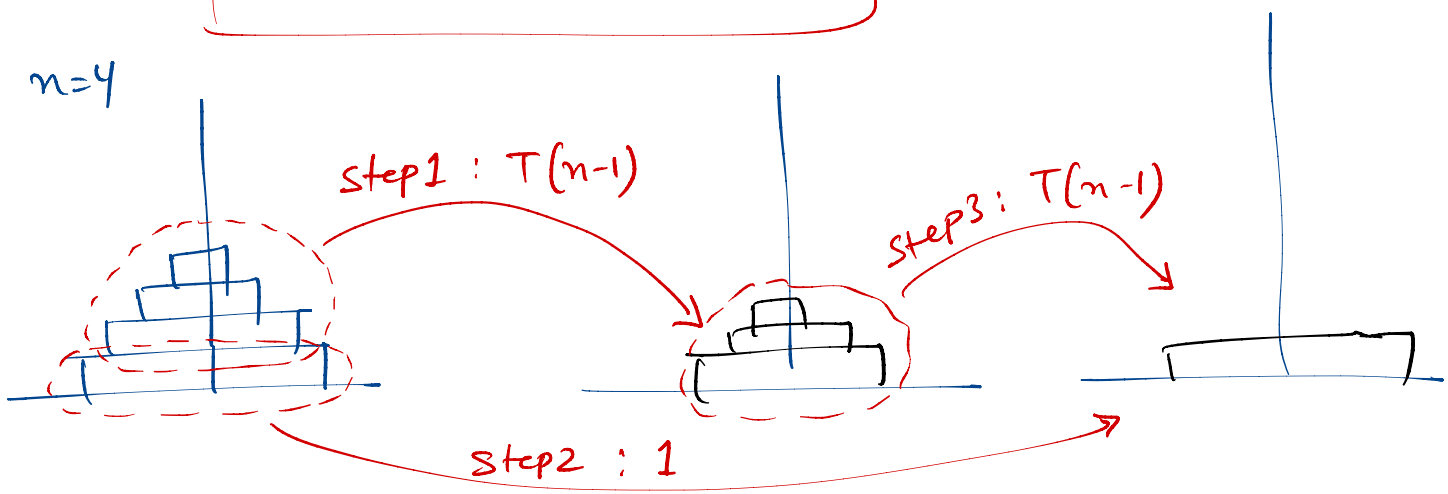
$$T(n) = 2^n - 1$$

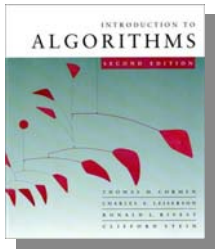
$$\Rightarrow T(n) = \theta(2^n)$$

Recursion:

$$T(n) = 2T(n-1) + 1$$

Lets take $n=4$





Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.

