## 19th Nov:

### Job Assignment:

A set of $n$ jobs $J$

A set of $m$ employees $E$.

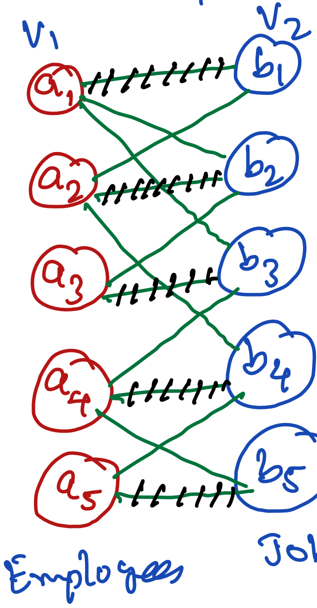
Every employee $x$ is trained to a set $B_x \subseteq J$ jobs.

**Objective:** Assign every job to exactly one employee such that no two jobs are assigned to the same employee and every employee is assigned a job which he/she is trained to do

**MATCHING:** A matching $M$ in a simple graph is a subset of edges of the graph such that no two edges are incident with the same vertex.

### Bipartite graph:

$G = (V, E)$ such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ and for every edge $(x, y) \in E$, $x \in V_1$ if and only if $V_2$.

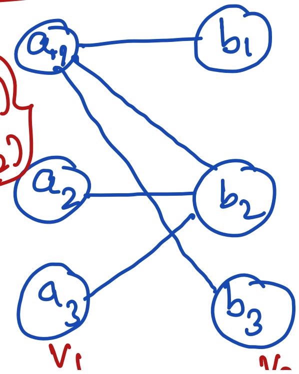



$(a_1, b_1)$
$(a_2, b_2)$
$(a_3, b_3)$
$(a_4, b_4)$
$(a_5, b_5)$

if $(s, t)$ and $(u, v)$ are edges, then all $s, t, u, v$ are distinct vertices.

A vertex $x$ is matched by a matching $M$ if there exist an edge in $M$ that is incident to $x$.

$\{(a_1, b_1), (a_3, b_2)\}$




Has no matching for which all vertices are matched

$\{(a_1,b_1),(a_2,b_2)\}$ $\{(a_1,b_2)\}$ $\{(a_1,b_3),(a_3,b_2)\}$

Maximal matchings

Maximum matching is a maximal matching with highest possible cardinality.

For a bipartite graph $G=(V,E)$ with bipartition $(V_1,V_2)$ has a complete matching from $V_1$ to $V_2$ if there is a matching $M$ such that $|M|=|V_1|$ such that $|M|=|V_1|$

**HALL'S THEOREM:**

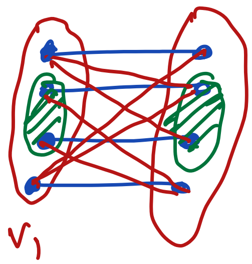
A bipartite graph $G=(V,E)$ with bipartition $(V_1,V_2)$ has a complete matching from $V_1$ to $V_2$ if and only if for every $A \subseteq V_1$, $|N(A)| \geq |A|$.

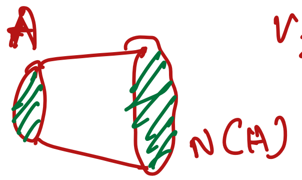
**Proof:** Let $G=(V,E)$ be a bipartite graph with bipartition $(V_1,V_2)$.

$(\Rightarrow)$ Forward direction.

Let $G$ has a complete matching from $V_1$ to $V_2$.

Then, consider any $A \subseteq V_1$.




$\forall A \subseteq V_1 \quad |N(A)| \geq |A|$

$G$ has a complete matching from $V_1$ to $V_2$

Let $V_1 = \{a_1, a_2, \dots a_n\}$ and a complete matching $M$ from $V_1$ to $V_2$ is $\{(a_1,x_1),(a_2,x_2),\dots,(a_n,x_n)\} = M$.

$N(A) = \{y \in V_2 \mid (a_i, y) \in E\}$

Note that $\{x_i \mid (a_i, x_i) \in M \text{ and } a_i \in A\} \subseteq N(A)$

Then $\underbrace{|\{x_i \mid (a_i, x_i) \in M \text{ and } a_i \in A\}|}_{= |A|} \leq |N(A)|$ since $M$ is a matching

Hence $|N(A)| \geq |A|$. This proves the forward direction

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($\Leftarrow$) Backward direction

For every $A \subseteq V_1$, $|N(A)| \geq |A|$

Base Case: $|V_1| = 1$.

Then choose $A \subseteq V_1$, $|A| = |V_1|$

$N(A)$ has one vertex.

Hence $|N(A)| \geq 1 = |A|$.

Then for $x \in A$, choose $y \in N(A)$

Then $\{(x, y)\} = M$.

For all $|V_1|$ with values

$1, 2, \ldots, k$.

Induction Step: Consider $|V_1| = k+1$.

By assumption, for all $B \subseteq V_1$,

$|N(B)| \geq |B|$.

$|A| \leq k$

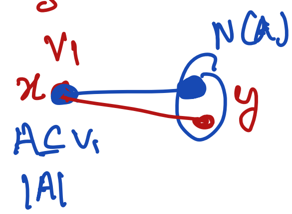
$|N(A)| \geq |A| + 1$

Case (i): For all $A \subseteq V_1$ s.t. $|A| \leq k$

$|N(A)| \geq |A| + 1$.

Proof by strong induction.

on $|V_1|$.

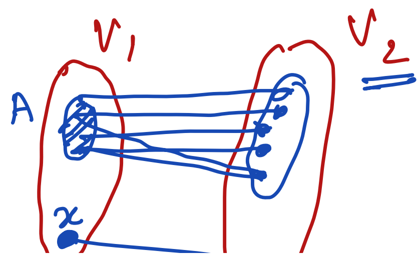



Therefore $\{(x, y)\}$ is a complete matching from $V_1$ to $V_2$.

Induction Hypothesis:

Let $k$ be a positive integer and $G$ be a bipartite graph with bipartition $(V_1, V_2)$ and $1 \leq |V_1| = j \leq k$.

if for all $A \subseteq V_1$, $|N(A)| \geq |A|$, then there exists a complete matching from $V_1$ to $V_2$.



$|N(A)| \geq | \ldots$

Case (ii): There exists $A \subseteq V_1$ s.t. $|A| = k$ and $|N(A)| = |A|$.

Case-(i): Choose $x \in V_1$.

Consider $W = V_1 - \{x\}$.

Then $|W| = k$.

$|N(W)| \geq |W| + 1$ because of Case-(i)

$x$ has a neighbor $y \in V_2$

Consider the subgraph

$$\boxed{(V_1 - \{x\}, V_2 - \{y\})}$$

Due to induction hypothesis there is a complete matching $M'$ from $V_1 - \{x\}$ to $V_2 - \{y\}$.

Neither $x$ nor $y$ is matched by $M'$.

Then $M = \{(x, y)\} \cup M'$ is a complete matching from $V_1$ to $V_2$.

Case-(ii): There exists $B \subseteq V_1$ of size at most $k$ such that $|B| = |N(B)|$.

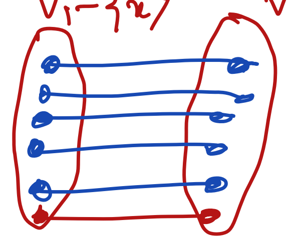

Then for any $A \subseteq V_1 - \{x\}$ $N(A)$ contains a set of $|A|+1$ vertices.

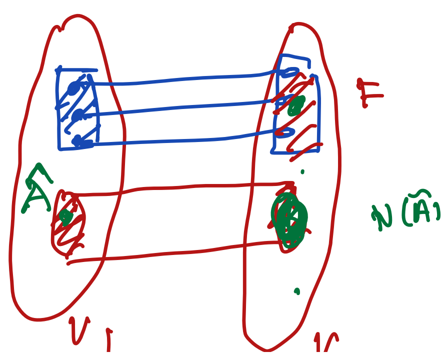
Hence, $N(A) \cap (V_2 - \{y\})$ contains a set of $|A|$ vertices.

Then, in the subgraph $(V_1 - \{x\}, V_2 - \{y\})$

For every $A \subseteq V_1 - \{x\}$

$$|N(A) \cap (V_2 - \{y\})| \geq |A|.$$



Let $F = N(B) \cap V_2$

Since $|B| = |N(B)|$, $|F| = |B|$

For all $A \subseteq V_1 - B$, $|N(A) \cap (V_2 - F)| \geq |A|$

Suppose not.

Then, there exists $\hat{A} \subseteq V_1 - B$ such that

$$|N(\hat{A}) \cap (V_2 - F)| < |\hat{A}|$$

$N(B) \cap V_2$

Then $N(\hat{A} \cup B) \subseteq (N(\hat{A}) \cap (V_2 - F)) \cup F$

since $F \cap N(\hat{A})$

Then $|N(\hat{A} \cup B)| = |\underbrace{N(\hat{A})}| + |F|$

$$< |\hat{A}| + |F| = |\hat{A}| + |B|$$

$$= |\hat{A} \cup B|$$

This contradicts our assumption for $\hat{A} \cup B$.

Due to inductive hypothesis there exists a complete matching

$M'$ from $B$ to $F$

and a complete matching $M''$ from $V_1 - B$ to $V_2 - F$.

$M' \cup M''$ is a complete matching from $V_1$ to $V_2$.