# Probability and Statistics Chapter 1 Questions

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## I. Textbook Problems

End of Chapter Problems 1.3.1, 1.3.4, 1.3.5, 1.4.1, 1.4.2, 1.4.3, 1.5.1, 1.5.3, 1.5.4, 1.6.4, 1.6.6, 1.7.5, 1.7.7 from the textbook by Roy Yates and David Goodman (2nd Ed).

## II. Se son l R ndomV

seasonal viral infection due to a virus named RandomV is prevalent in Delhi. bout 1 in 1000 people in Delhi are expected to be infected by RandomV. The virus spreads from an infected person to another person only when a sufficient amount of viral load is transmitted from the infected person to the other. test has been designed to detect the RandomV infection. mong the infected, the test gives a positive result in 90 of them. mong healthy people (that is those who are free of infection), the test is known to come up with a positive result in 10 of them. Given the above background about RandomV, solve the following questions.

**uestion 1.** Suppose you go to a gathering of 10 people who have come together from different parts of Delhi to celebrate Delhi's history. Derive the following probabilities.

(a) You meet a randomly chosen person in the gathering. What is the probability that the person is infected with RandomV?
(b) You meet a randomly chosen pair in the gathering. What is the probability that you end up meeting a pair of infected people?
(c) You meet a randomly chosen pair in the gathering. What is the probability that you end up meeting a pair in which one or more are infected?
(d) What is the probability that two or more in the gathering of 10 are infected?

**uestion 2.** Knowing the infectious nature of RandomV, a certain healthy person decides to limit his daily interaction with other people. However, given the person's work, he ends up meeting a total of two people on a certain day. He does not know whether the two people were infected. However, he has read that an infected person can with probability 0.1 transfer a viral load that is sufficient to infect a healthy person. lso, for a given healthy person, the outcomes of interactions with different infected people are independent of each other. The person is worried post his meetings with the two people and decides to get himself tested.

(a) Derive the probability that his test will give a positive result. [Hint: What is the probability that the person is infected?]
(b) Suppose his test gives a positive result. What is the probability that the person is infected with RandomV?

**uestion 3.** Consider the person in Question 2 and his meeting two people on his work day. Now suppose that a vaccine exists to protect against RandomV. The vaccine is known to have an efficacy of 95 , which is to say that the probability that an infected person transfers a viral load that is sufficient to infect a healthy *vaccinated* person is $1/20$ of the corresponding probability for a healthy unvaccinated person. It is also known that the behavior of the test is independent of whether a person is vaccinated or not. What is the probability that the above person is vaccinated, given that his test result is positive? [ *int* 1: Note that for events $A$, $B$, and $C$, $P[A, B|C] = P[A|B, C]P[B|C]$. *int* 2: The probability that the person is infected, given that he is vaccinated, maybe useful.].

## III. Others

**uestion 1.** Years of data driven research has led to a probabilistic model that explains the impact of student motivation on performance in exams and vice versa. To keep it simple enough, we will characterize motivation as either high or low and exam performance as being good, fair, or poor. t the beginning of examination season a student's motivation is high with probability $0.5$. student whose motivation is high demonstrates good performance in an exam with probability $0.6$ and fair performance with probability $0.3$. On the other hand, a student with low motivation demonstrates good performance with probability $0.1$ and fair performance with probability $0.3$.

We will consider an examination season that has two exams. Researchers have tried to model the impact of performance in the first exam on motivation levels of a student, which as described above impacts the student's performance in the second exam. Good performance in the first exam leads to a high level of student motivation to prepare for the next exam with probability $0.7$. Fair performance leads to high motivation with probability $0.5$ and poor performance in an exam leads to high motivation with probability $0.3$. nswer the following questions.

1) Clearly define events (use proper notation) that correspond to motivation before the first and the second exams and performance in the two exams. Draw a tree diagram that captures student motivation at the start of examination season, performance in the first exam, the resulting level of motivation, and the performance in the second exam. For each branch, clearly state the start and end event and the associated probability. You must state both the mathematical definition of the probability using the events and also the values. For example, you must say $P[A] = 0.2$ and not just either $P[A]$ or $0.5$.
2) Calculate the probability that the performance in the first exam is good.
3) Calculate the probability that the performance in the first exam is poor.
4) Calculate the probability that the performance in the second exam is good.
5) Calculate the probability that the performance in the second exam is poor.
6) student is known to have a performance of good in the second exam. What is your revised belief about the student's performance being poor in the first exam? Is performance of good in the second exam independent of poor performance in the first? Use the definition of independence of events to arrive at your answer.
7) Use the definition of independence of events to show whether a good performance in the second exam is independent of a high motivation at the start of the examination season.

**uestion 2.** Show from first principles that mutually exclusive events $A$ and $B$ are dependent. [Hint: Define mutually exclusive events and use the definition of independent events.]

**uestion 3.** The Institute of Statistics of a country named Uncertain Land has come out with a document that summarizes the jobs that employ recent graduates. In Uncertain Land $40$ of graduates have an engineering degree. nother $40$ have a degree in medicine. Graduates are employed in four kinds of jobs (a) Software, (b) Management consulting, (c) Healthcare, and (d) Government service. mong those with an engineering degree, $40$ are working in a software job, another $40$ are management consultants, and the rest are split between government service and healthcare. Those with a medicine degree work in a software job with probability $0.2$ and in healthcare with probability $0.7$. They never take up management consulting jobs, however. The graduates that have a degree neither in engineering nor in medicine, are equally split between software and government service. nswer the following.

1) Draw a tree diagram that summarizes all the given probabilities. For each probability that you state on the diagram you must not only write the number but also the probability it refers to. For example, it is not sufficient

to write $0.2$, you must write $P[E|F] = 0.4$, where $E$ and $F$ are any events and the probability that you are referring to is the probability of event $E$ conditioned on event $F$ having occurred.

2) What is the probability that a graduate has neither an engineering degree nor a degree in medicine?
3) What is the probability that a graduate is working in a software job? Use your tree diagram to support your answer.
4) What is the probability that a randomly chosen graduate has a medicine degree and is working in a software job? Use your tree diagram to support your answer.
5) What is the probability that a randomly chosen graduate working in a software job has a degree in engineering?
6) What is the probability that a randomly chosen graduate working in a software job has a degree other than medicine and engineering?
7) What degree occurs with the highest percentage among those employed in government service? What is this percentage?

**uestion 4.** Consider the three events $A$, $B$, and $C$. These events are related in following manner. We have $A \cap C =$ , $B \cap C =$ , and $S$ $A = B$, where $S$ is the sample space. Is the set $\{A, B\}$ an event space? Explain your answer.

## IV. STUDENTS ND PROB BILITY

The authors of a textbook on probability find that $30$ of students put in less than two hours of work per week outside class. The others put in more than $6$ hours per week outside class. Eighty percent of students who put in more than $6$ hours per week get an aggregate of more than $50$ marks out of $100$. Sixty percent of all students get more than $50$ marks. What percentage of students who work less than two hours per week get more than $50$ marks?

[Note: Define all events and probabilities of interest.]

## V. DRIVER W RNING SYSTEMS

The department of transportation of Delhi has decided to install multiple cameras on every bus. They hope that the cameras can warn a bus driver of pedestrians, animals, and vehicles in the vicinity of the bus, while ignoring other objects in the vicinity. While they plan to buy cameras, they need you to help with the Vision algorithms that will process the information captured by the cameras and warn the driver. Your algorithm must satisfy the following requirements.

1) It wrongly detects an animal where there is a pedestrian, with probability $0.01$,
2) n object in the vicinity that isn't one of pedestrians, animals, and vehicles, is detected as a pedestrian with probability $0.01$,
3) It wrongly detects a vehicle where there is a pedestrian, with probability of $0.05$,
4) It wrongly detects a pedestrian where there is an animal, with probability $0.01$,
5) It wrongly detects a pedestrian where there is a vehicle, with probability $0.02$,
6) The pedestrian must actually be in the vicinity *at least* $95$ of the times the algorithm gives a pedestrian warning.

What is the *minimum* probability with which your algorithm must correctly spot a pedestrian that is in the vicinity of the bus? Do the stated requirements satisfy the minimum probability? ssume that pedestrians, animals, vehicles, and others occur with equal probability.

[Note: You must define all events of interest. You must convert all requirements in to equivalent mathematical statements in terms of the defined events. You must also define the probability of interest in terms of the defined events. Using the above to answer the question]

## VI. Two Coin Tosses

You perform an experiment that involves tossing a coin twice and noting the outcome of each toss. coin toss leads to an outcome from the set $\{H, T\}$, where outcome $H$ denotes heads and outcome $T$ denotes tails. The coin tosses are independent and the probability that a toss gives heads is $p$. nswer the following questions.

(a) In terms of the given set of outcomes, write down the event that the first toss gives heads. Express the probability of the event in terms of probabilities of the outcomes it contains.
(b) Similar to above, write down the event that at least one toss gives heads. Express the probability of the event in terms of the probabilities of the outcomes it contains.
(c) You pick the outcome of any coin toss. What is the probability that the outcome is $H$?
(d) Express the probability of heads in the second toss conditioned on the knowledge that the first toss gave heads using standard notation. What is this probability as a function of $p$?
(e) Consider the probability of two heads conditioned on heads in the first toss. Express it using notation. What is this probability as a function of $p$?
(f) What is the probability that at least one heads was observed, given that the second toss gave heads?
(g) What is the probability of heads on the first toss, given that at least one heads was observed?

## VII. C tching the Copyc t

There are two coins $A$ and $C$, of which $C$ is a copycat. You are heading an effort to catch the copycat coin $C$. Coin $A$ has a mind of its own and tossing it gives heads with probability $2/3$. The behavior of coin $C$ is dependent on whether it is tossed first or after coin $A$. If coin $C$ is tossed first, the toss gives heads with probability $1/2$. On the other hand, if $C$ is tossed after coin $A$, coin $C$ tries to copy the outcome of the toss of coin $A$. Specifically, if the toss of $A$ gave heads, the toss of $C$ gives heads with probability $3/4$. If the toss of $A$ gave tails, the toss of $C$ gives heads with probability $1/3$.

You are scheduled to see the coins perform at an exhibition. The performance will proceed as follows. The two coins (identities unknown) will be brought in a jar. One of the coins will be picked randomly from the jar and tossed. Following this, the other coin in the jar will be tossed. You decide that you will choose the coin that is tossed first to be coin $C$ in case the outcomes of the two tosses are different. Else, you will choose the second coin to be tossed to be coin $C$. nswer the following questions. [Hint: Tree diagrams may come in handy.]

(a) What is the probability that coin $A$ is chosen for the first toss?
(b) What is the probability that the first coin toss gives heads and the second gives tails?
(c) What is the probability that the first coin toss gives tails and the second gives heads?
(d) What is the probability that both coin tosses give heads?
(e) What is the probability that both coin tosses give tails?
(f) What is the probability that you will choose the correct coin $C$?

## VIII. More Questions

**uestion 1.** There are 300 students in a section. The names of the students are placed in a bin. ssume all students have unique names. Each student is assigned a randomly chosen name from the bin.

ssume that a name chosen from the bin is placed back in it before the next assignment.

(a) Calculate the probability that any student is assigned the true name.
(b) Calculate the probability that every student is assigned the true name.

Now assume that a name chosen from the bin is removed from the bin before the next assignment.

(aa) Suppose a student is randomly chosen after all students have been assigned names chosen from the bin. Calculate the probability that the student was assigned the true name.
(bb) Suppose two students were randomly chosen post assignment of names to all. Calculate the probability that both were assigned their true names.
(cc) Is the event that a student is assigned the true name independent of the names assigned to other students? Explain your answer.
(dd) Calculate the probability that all students were assigned their true names.

**uestion 2.** randomly chosen person in a city claims to be healthy (event $H$ ) with probability $0.75$ and unwell (event $U$ ) otherwise. The city puts a person through the following treatment for a month. person who claims to be unwell is given a placebo (event $B$) with probability $0.25$ and a get-healthy drug (event $D$) with probability $0.75$. On the other hand, a person who claims to be healthy is given a placebo with probability $0.75$ and a get-healthy drug otherwise.

fter a month of treatment as described above, the person is subject to a diagnostic test and also asked whether the person feels healthy or unwell. Independent of the person's initial claim about being healthy or unwell, the test reports with probability $0.75$ a person who was on a placebo to be unwell (define $R_U$ as the event that test reports unwell) and reports with probability $0.75$ a person who was on the get-healthy drug to be healthy ($R$ is the event that the test reports healthy).

The diagnostic test has the following characteristics. It reports as unwell, a person who claims to be healthy post treatment (event $H_2$), with probability $0.1$. It reports as unwell, a person who claims to be unwell (event $U_2$), with probability $0.9$.

(a) Calculate the probability that a person is given the placebo.
(b) Calculate the probability that the test reports a person to be healthy.
(c) Calculate the probability that at the end of the treatment, a person whose test report says unwell, claims to be healthy.
(d) Calculate the probability that at the end of the treatment, a person whose test report says healthy, claims to be healthy.
(e) Calculate the probability that at the end of the treatment a person claims to be healthy.
(f) Suppose at the end of the treatment the person claims to be unwell. Calculate the probability that the person claimed to be healthy before the treatment?

**uestion 3.** Data suggests that $10$ of Indians have a B. Tech degree. We will call those with the degree engineers. mongst the engineers, about $10$ get a high-tech job. The kind of job an engineer gets is independent of jobs obtained by others. nswer the following questions.

(a) Derive the probability that there are at least two engineers in a room of $50$ people. Explain your steps.
(b) Derive the probability that there are exactly two engineers with a high-tech job in a room of $50$ people. Explain your steps.
(c) Derive the probability that in a randomly chosen pair of engineers, both the engineers have high-tech jobs.
(d) Suppose a room has $6$ engineers of which $3$ have high-tech jobs. We will create two pairs in the following manner. Randomly and without replacement choose an engineer as the first member of the first pair. If the chosen engineer has a high-tech job, choose without replacement the second member of the first pair to be one with a high-tech job with probability $0.8$. If the first member doesn't have a high-tech job, the second member is chosen randomly and without replacement from the unpaired engineers.

To create the second pair, choose the first member of the pair randomly (and without replacement) from those that remain. Then, in case both types of engineers still remain unpaired, choose without replacement the second member of the second pair to be one with a high-tech job with probability $0.6$ and one without a high-tech job otherwise. If one type of engineer had remained unpaired, randomly choose one of the remaining engineers as the second member of the second pair.

Calculate the probability that the two pairs together have three high-tech engineers. Draw a tree diagram to help you do so. Your tree diagram wants to record the type of engineer chosen at different stages of pairing.

**uestion 4.** random heart patient has a *serious* heart ailment with probability $0.2$ and a *moderate* heart ailment with probability $0.8$. We have two kinds of surgeons, those that are risk taking (RT) and those that are risk averse (R ). serious patient approaches a RT with probability $0.8$ and otherwise approaches a R . patient with a moderate ailment approaches with equal probability a R or a RT.

serious patient who approaches a RT is accepted by the RT with probability $0.8$ and with probability $0.2$ the patient remains without a surgeon. Once accepted by a RT, a serious patient has a probability $0.6$ of a successful surgery and $0.4$ of a failed surgery.

serious patient who approaches a R is accepted by the R with probability $0.3$ and with probability $0.7$ the patient remains without a surgeon. Once accepted by a R , a serious patient has a probability $0.6$ of a successful surgery and $0.4$ of a failed surgery.

patient with moderate ailment is accepted by whichever doctor the patient approaches with probability $1$. lso, once accepted by a RT, a moderately ill patient has a probability $0.9$ of a successful surgery and $0.1$ of a failed surgery. These probabilities remain the same in case the patient had approached a R instead. nswer the following questions.

(a) Draw the tree diagram.

(b) Derive the conditional probability that a patient is accepted by a RT, given that the patient is a serious heart patient and had a successful surgery.

(c) Derive the probability that a patient accepted by a RT has a successful surgery.

(d) Derive the probability that a patient accepted by a R has a successful surgery.