Discrete Structures-2025: Tutorial-5

Proofs, Sets and Functions

(1) Let p be a prime positive integer. Then, prove that for two positive integers n and r, n^r is divisible by p if and only if n is divisible by p.

(2) Determine with proof which of the following functions are bijections from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = 2x + 1$.

(b) $f(x) = x^2 + 1$.

(c) $f(x) = x^3$.

(d) $f(x) = \frac{x^2+1}{x^2+2}$.

(3) Prove that every positive integer n can be written as a sum of distinct powers of two. Equivalently, every positive integer can be expressed as a sum of a subset of the integers $2^0, 2^1, 2^2$, and so on.

(4) Prove that $X \subseteq Y$ if and only if $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. We use $\mathcal{P}(A)$ to denote the power-set of A.