12th Nov!

Big-Oh: Let $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x)$ is $O(g(x))$ if there exist constants a and k such that for every $x > k$,

$$|f(x)| \leq a|g(x)|$$

$f(x)$ is Big-Oh of $g(x)$

$$\exists a \exists k \forall x \left((x > k) \Rightarrow (|f(x)| \leq a|g(x)|)\right)$$

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Negation:

$$\forall a \forall k \exists x \left((x > k) \wedge (|f(x)| > a|g(x)|)\right)$$

(i) If the input is an array of n numbers, then bubble sort takes $O(n^2)$-steps.

(ii) If the input is array of n numbers, then merge sort takes $O(n \log n)$ steps. Big-Oh

$f(x)$ is not $O(g(x))$?

No matter the values of a and k, there is x such that $(x > k) \wedge |f(x)| > a|g(x)|$

Ex 1: $f(x) = x^2 + 2x + 1$

For any $x \geq 1$, $x \leq x^2$

and $1 \leq x^2$

Choose $a = 4$.

Choosing $a = 4$ and $k = 1$

Then, for all $x > k$,

$$f(x) \leq 4x^2 \approx ax^2.$$

Exam time: Proof must be using properties of inequalities.

x^2 grows as fast as $(x+1)^2$.

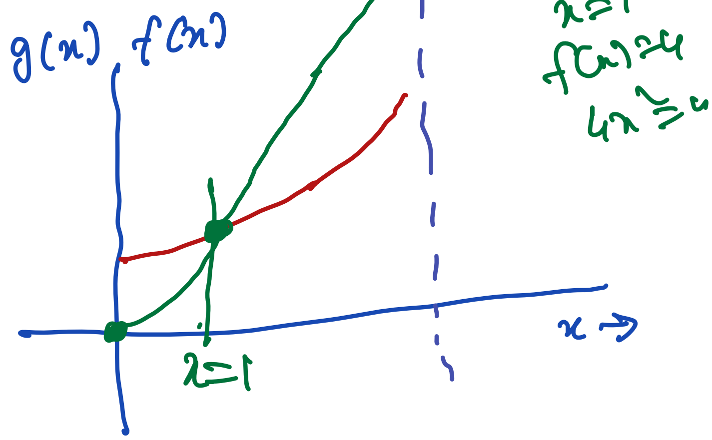
Then choose constant $k = 1$

Observe that

$$f(x) = x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2$$

Hence, $f(x)$ is $O(x^2)$

$$\boxed{x^2 = g(x)}$$



Ex-2: n^3 is $O(2^n)$ → Theorem

Proof: If we prove that

$\boxed{\text{for all } n \geq 10, \; n^3 \leq 2^n}$

then we are done. Prove it for integer.

Base Case: $n = 10$

$LHS = 10^3$ and $RHS = 1024$

Hence, $LHS \leq RHS$.

Induction hypothesis:

Let the statement $n^3 \leq 2^n$ be true for all $n = 10, 11, \ldots, k$.

$LHS = k^3 + 3k^2 + 3k + 1$

$\leq k^3 + 3k^3 + 3k^3 + k^3$

$= 8k^3$.

$\boxed{(k+1)^3 \overset{?}{\leq} 4 \cdot 2^{k+1}}$

What we have proved is

For every $n \geq 10$, there exists constants b such that

$n^3 \leq b \cdot 2^n$

Choose constant $k = 10$

Lemma $f, g : \mathbb{Z} \to \mathbb{Z}$

	n^3	2^n
$n=1$	1	2
$n=2$	8	4
$n=3$	27	8
⋮	729	512
$n=10$	1000	1024

1 then we have proved by induction that $n^3 \leq b \cdot 2^n$

Induction Step:

Choose $n = k+1$

$LHS = (k+1)^3 = k^3 + 3k^2 + 3k + 1$

$RHS = 2^{k+1} = 2 \cdot 2^k$

Induction hypothesis provides us that $8k^3 \leq 8 \cdot 2^k$

$\boxed{= 4 \cdot 2^{k+1}}$

$= 2^{k+3}$

Lemma: Choose a constant p such that for every $n \geq p$, there exists constant b such that $n^3 \leq b \cdot 2^n$

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n^3 is a polynomial function

2^n is an exponential function

2^n grows as fast as a polynomial

n^3 is $O(2^n)$

n^c c is a constant independent of n.

Exercise

Ex-3: $1+2+3+\dots+n$ is $O(n^2)$

Theorem: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

function

$a_0, a_1, a_2, \dots, a_n$ constants.

$f(x)$ is $O(x^n)$

Proof: $|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0|$

x and y $\quad x+y \leq |x|+|y|$

$x \leq |x|$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \dots + |a_1| x + |a_0|$$

Choose $k=1$

for any $x \geq k$ $\quad x^{n-1} \leq x^n$

$$\leq |a_n| x^n + |a_{n-1}| x^n + \dots + |a_1| x^n + |a_0| x^n$$

$$= \left(\sum_{i=0}^{n} |a_i|\right) x^n$$

$$f(x) \leq A \cdot x^n$$

$$A = \sum_{i=0}^{n} |a_i|$$

Hence, $f(x)$ is $O(x^n)$

Highest power of x dominates the lower powers.

Ex-4: Prove that n^2 is not $O(n)$.

Proof: Suppose for the sake of contradiction that there are constants b and k such that for all $n > k$, $n^2 \leq bn$

Then $n \leq b$

Disprove that n^2 is $O(n)$

Prove or disprove that n^2 is $O(n)$

Can we say that $n \leq b$?

$$\boxed{(b+1)^2 \leq b^2 + b} \quad \times$$

positive

$n = b+1$

But it can not hold true that for all $n > k$, $n \leq b$

Choose a value of k

Then $n > \max\{b, k\}$

Hence, n^2 is not $O(n)$

Choose $n = bk + b > k$

$n \leq b$ is false.

$k = a+2$ then it does not hold true.

Recurrence Relation:

If an object is defined in terms of itself, then it is called a RECURSION.

Base Case: $f(0) = 3$

$f(n+1) = 2f(n) + 3 \leftarrow$ for all $n \geq 0$

$f(1) = 2f(0) + 3$

$$f(k) = 2f(k-1) + 3$$
$$= 2(2f(k-2) + 3) + 3$$
$$= 2^2 f(k-2) + 2 \cdot 3 + 3$$
$$= 2^2(2f(k-3) + 3) + 2 \cdot 3 + 3$$
$$= 2^3 f(k-3) + 2^2 \cdot 3 + 2 \cdot 3 + 3$$
$$\vdots$$
$$= 2^k \cdot f(0) + 3(2^{k-1} + 2^{k-2} + \dots + 2^0)$$

$\rightarrow$ geometric

A function that calls itself in its body is called recursion

Factorial(n) If(n<0) return 0

If $n = 0$ then return 1;

Else return $n * f(n-1)$.

$\rightarrow T(n) = 2T(n-1) + 5$

provides a function that is $O(2^n)$

CIA

GENERATING FUNCTION

MASTER'S THEOREM

ADA

$O(2^k)$

$= 2^k \cdot 3 + 3\left(2^{k-1} + 2^{k-2} + \dots + 2^0\right)$

$\leq 2^k \cdot 3 + 3 \cdot \left(\underbrace{2^k + 2^k + \dots + 2^k}_{k \text{ times}}\right)$

$3k \cdot 2^k$

$\leq 2^k \cdot 3 + 3k \cdot 2^k$

$\leq 6k \cdot 2^k$

$\approx O(k \cdot 2^k)$

Ex. 6: $f(n) = f(n-1) + 2$

for all $n \geq 1$ $\quad T(n) = T(n-1) + 3$

$\boxed{f(0) = 1} \rightarrow$ Base Case.

Linear function $O(n)$

$f(n) = f(n-1) + 2$

$= f(n-2) + 2 + 2$

$\vdots$

$= f(n-k) + \underbrace{(2 + 2 + \dots + 2)}_{k \text{ times}}$

$\vdots$

$= f(0) + \underbrace{(2 + \dots + 2)}_{n \text{ times}}$

$= 1 + 2n \leq 3n$

Ex. 7: $T(n) = T(n-1) + n$

Base Case: $T(1) = 1$

$T(n) = T(n-1) + n$

$= T(n-2) + (n-1) + n$

$= T(n-3) + (n-2) + (n-1) + n$

$\vdots$

$= T(1) + \underbrace{2 + 3 + \dots + n}$

$= 1 + 2 + \dots + n$

$\approx \frac{n(n+1)}{2} \approx O(n^2)$

Quadratic function

$T(n) = T(n-1) + \alpha \cdot n$

α is a constant

$O(n^2)$ quadratic function