Discrete Structures-2025: Assignment-1

Due Date:

Marks: 10

(1) For every positive integer $n \geq 2$, $\sqrt[3]{n}$ is irrational if and only if n is not perfect cube.

Solution: We use this fact that was a question in mid-sem exam.

Fact: For a positive integer n and a prime number p, if n^3 is divisible by p, then n is divisible by p.

Using the fact above, we prove that $\sqrt[3]{n}$ is irrational if and only if n is not perfect cube.

There are two directions to prove in this statement.

($\Leftarrow$) First we prove that "if n is not a perfect cube, then $\sqrt[3]{n}$ is irrational".

We prove this by contradiction. Here it is. Hence, we assume that n is not a perfect cube but $\sqrt[3]{n}$ is rational.

Therefore, there exist integers a and b such that

$$\sqrt[3]{n} = \frac{a}{b}, \ \ where \ \ gcd(a,b) = 1 \ \ and \ \ b \neq 0$$

Then, we have

$$n = \frac{a^3}{b^3}$$

So,

$$n \cdot b^3 = a^3$$

Now by FUNDAMENTAL THEOREM OF ARITHMETIC, we can write n as a product of primes as follows:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

Thus, we get

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdot b^3 = a^3$$

$$\implies p_i \text{ divides } a^3 \quad \forall\ i = 1, 2, \ldots, k$$

So by using the fact of Question 1, we get

$$p_i \ \text{ divides } \ a \quad \forall\, i = 1, 2, ..., k$$

Hence, a can be written as:

$$a = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \cdot d$$

where d is an integer such that $gcd(d, p_i) = 1 \quad \forall \quad i = 1, 2, \ldots, k$ and each $x_i > 0$ are integers.

Therefore, we have

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdot b^3 = (p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \cdot d)^3$$

$$\implies p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdot b^3 = p_1^{3x_1} p_2^{3x_2} \cdots p_k^{3x_k} \cdot d^3$$

$$\implies b^3 = p_1^{3x_1 - a_1} p_2^{3x_2 - a_2} \cdots p_k^{3x_k - a_k} \cdot d^3$$

Now observe that, $3x_i - a_i \geq 0 \quad \forall \quad i = 1, 2, \ldots, k.$

(as b^3 is an integer and $gcd(d, p_i) = 1 \; \forall \; i = 1, 2, \ldots, k$)

$\boxed{\text{Case 1}}$ - If $3x_i - a_i > 0$ for some i

$$\implies p_i \text{ divides } b^3$$

And by using the fact of Question 1, we get

$$p_i \text{ divides } b$$

This contradicts the fact that $gcd(a, b) = 1$ (as p_i divides a as well)

$\boxed{\text{Case 2}}$ - If $3x_i - a_i = 0$ for all i

$$\implies b^3 = d^3$$

$$\implies b = d$$

By eq^n, we get

$$a = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \cdot b \quad \implies b \text{ divides } a$$

Therefore, $gcd(a, b) = b \quad \implies b = 1$ (as $gcd(a, b) = 1$)

Hence, we get

$$n = a^3 \qquad (as \quad n \cdot b^3 = a^3)$$

This contradicts the fact that n is not a perfect cube.

- Since we have reached a contradiction for both the cases which are exhaustive, we conclude that our assumption was wrong.

Hence, we conclude that if n is not a perfect cube, then $\sqrt[3]{n}$ is irrational.

($\Rightarrow$) Now we prove that "if $\sqrt[3]{n}$ is irrational, then n is not a perfect cube".

We prove this by contrapositive argument.

That is, we prove that "if n is a perfect cube, then $\sqrt[3]{n}$ is rational".

Assume that n is a perfect cube. Then, there exists an integer k such that $n = k^3$.

$\implies \quad \sqrt[3]{n} = k \quad$ which is a rational number $\qquad$ (as k is an integer)

Hence, we conclude that if $\sqrt[3]{n}$ is irrational, then n is not a perfect cube.

(2) Consider an $m \times n$ checkerboard with an even number of cells (mn is even) that has two squares missing. If one missing square is white and the other missing square is black, then this checkerboard can be perfectly covered using dominoes.

Proof: We will prove by strong induction on the number of dominoes, k, required for the tiling. Let the total number of squares on the board after removal be $N = mn - 2$. The number of dominoes needed is $k = \frac{N}{2} = \frac{mn-2}{2}$. (Note that k is a natural number since mn is even)

Let $P(k)$ be the proposition that any $m \times n$ checkerboard where mn is even, with one white and one black square removed, can be tiled by k dominoes.

Base Case (or Basis Step)

The smallest possible case for tiling is $k = 1$ domino. This implies the number of squares remaining on the board is $N = 2(1) = 2$.

Consider a 2×2 board ($mn = 4$) with one white and one black square removed. The two remaining squares will always be adjacent and of opposite colors, which can be perfectly covered by a single domino. Thus, the proposition $P(1)$ is true (Note that we will not consider $m \times 1$ or $n \times 1$ since in that case we can remove squares arbitrarily so $m, n \geq 2$.)

Inductive Hypothesis

Assume that $P(j)$ is true for all integers j such that $1 \leq j < k$. This means we assume that any such board (satisfying given requirements) requiring j dominoes (where $j < k$) can be successfully tiled.

Inductive Step

We now prove that $P(k)$ is true. Consider a board $\mathcal{B}$ that requires k dominoes, meaning it originally had $mn = 2k + 2$ squares.

Since the total number of cells mn is even, at least one of the dimensions, m or n, must be even. This allows us to always draw a line that splits the board $\mathcal{B}$ into two smaller sub-boards, $\mathcal{B}_1$ and $\mathcal{B}_2$, each containing an even number of cells.

Let the two missing squares be W (white) and B (black). There are two possible scenarios for the locations of these missing squares.

Scenario 1: Both missing squares are in the same sub-board. Suppose both W and B are located in sub-board $\mathcal{B}_1$.

- **Sub-board** $\mathcal{B}_1$: This is a board with an even number of cells, missing one white square (W) and one black square (B). The number of dominoes needed to tile it is less than k. By our Inductive Hypothesis, $\mathcal{B}_1$ can be perfectly tiled.
- **Sub-board** $\mathcal{B}_2$: This is a complete board with no missing squares and an even number of cells. Any such board can be tiled with dominoes.

Since both $\mathcal{B}_1$ and $\mathcal{B}_2$ can be tiled independently, the entire board $\mathcal{B}$ can be tiled.

Scenario 2: The missing squares are in different sub-boards. Suppose W is in sub-board $\mathcal{B}_1$ and B is in sub-board $\mathcal{B}_2$. We cannot directly apply our hypothesis because in each sub-board, a square of only one color is missing.

Then we will proceed in the following way:

1. Find two adjacent squares, w_x and b_x, that lie on the dividing line between $\mathcal{B}_1$ and $\mathcal{B}_2$. Let w_x be a white square in $\mathcal{B}_2$, and b_x be a black square in $\mathcal{B}_1$.
2. Now, we consider two different tiling problems:
   - Consider sub-board $\mathcal{B}_1$. It is missing its original white square W. We now consider it to be also missing the black square b_x. This modified $\mathcal{B}_1$ is a board with one white (W) and one black (b_x) square removed. Since it is smaller than the original board $\mathcal{B}$, it requires fewer than k dominoes. By the Inductive Hypothesis, this board can be tiled.
   - **Problem B**: Consider sub-board $\mathcal{B}_2$. It is missing its original black square B. We now consider it to be also missing the white square w_x. Similarly, this modified $\mathcal{B}_2$ has one white (w_x) and one black (B) square removed and can also be tiled by the Inductive Hypothesis.
3. We now combine these solutions. We tile the modified $\mathcal{B}_1$ and the modified $\mathcal{B}_2$ according to their respective valid tilings. The only squares left uncovered are w_x and b_x.
4. Since w_x and b_x are adjacent, we place a final domino across the dividing line to cover them both.

This completes the tiling for the entire board $\mathcal{B}$ in this scenario.

Conclusion

By principle of strong induction, the proposition holds for all $k \in \mathbb{N}$. □