15th Sept: Strong Mathematical Induction:

Theorem 1: Every amount of postage of 12 cents or more can be formed using 4-cent and 5-cent stamps.

Proof: Mathematical Induction:

Basis Step: If $n \leq 11$, then the statement is vacuously true

$n = 12$, then 3 stamps each of 4-cents can be used to form a postage of 12 cents.

Induction Step: Consider $n = k+1$

As $P(k)$ is true, there exists a way to form k-cent postage using 4-cent and 5-cent stamp

Case (i): At least one 4-cent stamp is used to form k-cent stamp.

Then, remove one of the 4-cent stamps from k-cent postage. This gives a postage of $(k-4)$ cents.

12 cents: 3 stamps each of 4-cents.

13-cents: 2 stamps each 4 cents one stamp of 5-cent.

$P(n)$: For $n \geq 12$, every postage of n cents can be formed using 4-cents and 5-cent stamp.

$$\forall n \left((n \geq 12) \rightarrow \left(\exists k_1, \exists k_2 \ (n = 4k_1 + 5k_2) \right) \right)$$

$n, k_1, k_2 \in \mathbb{Z}$

Induction hypothesis: $P(k)$

For every n s.t. $12 \leq n \leq k$ a postage of k cents can be formed using 4-cent and 5-cent stamps.

$P(k)$ is true.

Then add a stamp of 5-cents into the postage of $(k-4)$-cent.

This forms a postage of $(k+1)$ cents.

$$k \begin{pmatrix} \text{4-cent} \\ + \boxed{k-4} \end{pmatrix} \rightarrow \text{Remove one 4-cent stamp.}$$

4-cent / 5-cent.

Add a new 5-cent stamp

Case-(ii): No 4-cent stamp is used to form the postage of k cents.

Only 5-cent stamps are used.

Then k is a multiple of 5.

As $k \geq 15$, at least 3 stamps are used and each are of 5-cents.

Remove these 3-stamps each having 5-cent.

This gives $(k-15)$ cent postage.

$k \geq 15$. why? because $k = 12, 13, 14$ not divisible by 5.

Add 4-stamps each of 4-cents.

This creates a postage of $(k+1)$ cents.

k only 5-cent at least 3 $\xrightarrow{\text{Remove 3-stamps}}$ $k-15$ cent postage

$k-15$ cent postage $\xrightarrow{\text{Add 4-stamps each of 4-cents}}$ $k+1$ cent postage

Strong Mathematical Induction:

Basis: $\boxed{n = 12}$ ✓ 4×3 4-cent stamps 3 times

$\boxed{n = 13}$ ✓ $(2 \times 4) + (1 \times 5)$

$\boxed{n = 14}$ ✓ 2 stamps each 5-cents and 1 stamp of 4-cent

$(2 \times 5) + (1 \times 4) = 14$

For every n s.t. $12 \leq n \leq k$ $P(n)$ is true.

$\boxed{n = 15}$ ✓ 3 stamps each having 5-cents.

Induction Step: Consider $n = k+1$

$n \geq 16$ As $P(k-3)$ is true,

$k - 3 \geq 13$. By induction hypothesis, a postage of $(k-3)$ cent can be formed using 4-cent and 5-cent stamps.

$\boxed{k-3 \geq 12}$

Add one 4-cent stamp into the postage of $(k-3)$-cents. This creates a postage of $(k+1)$ cents.

$k-5 \ngeq 11$ | $k-2$, $k-1$

induction hypothesis cannot be used.

$\boxed{k-4 \geq 12}$ $P(k-4)$ is true by induction hypothesis.

A postage of $k-4$ cents can be formed using 4-cent or 5-cent stamps.

Now, add 5-cent stamp into a postage of $(k-4)$ cents.

This gives a postage of $(k+1)$ cents.

BEZOUT'S THEOREM:

If a and b are two positive integers, then there exists integers x and y such that

$$\gcd(a,b) = xa + yb.$$

Proof: (beyond the scope)

$\gcd(x,y)$ = greatest the largest common divisor integer a s.t. of x and y. $a \mid x$ and $a \mid y$. $\gcd(x,y) = a$

Theorem: Let x, y, z be positive integers such that $x \mid yz$ and $\gcd(x,y) = 1$. Then $x \mid z$.

Proof: x, y, z are integers such that $x \mid yz$ and $\gcd(x,y) = 1$.

Then there is an integer k such that $yz = kx$.

Due to Bezout's Theorem there are integers s and t such

Multiplying by z on both sides,

$$z = sxz + tyz.$$

Since $yz = kx$, then

$$z = sxz + tkx$$

that $1 = \gcd(x, y) = sx + ty$

Exercise: If p is a prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some $i \in \{1, 2, \ldots, n\}$

Sets: $A \cup B$, $A \cap B$, $A \times B$

$$A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for some } i \in \{1, 2, \ldots n\}\}$$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i \text{ for } i \in \{1, 2, \ldots, n\}\}$$

A be a set. Then complement of A is $\overline{A}$

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

$\boxed{A \setminus B}$ $\boxed{A \text{ backslash } B}$

Complement of $\overline{A}$ is the set A itself. (needs a proof)

Exercise:

Then, $z = x(sz + tk)$

As s, z, t, k are integers, it follows that there exists an integer l such that

$$z = xl.$$

Hence, $x \mid z$.

$$A \times B = \{\underbrace{(a, b)}_{\text{ordered pairs}} \mid a \in A, b \in B\}$$

$$A_1 \cap \cdots \cap A_n = \{x \mid x \in A_i \text{ for all } i \in \{1, 2, \ldots n\}\}$$

$S \leftarrow$ a universal set (every other set is subset of S).

$$\overline{A} = \{x \mid x \notin A\}$$
$$= S - A$$

Exercise:

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$$

$$\overline{A_1 \cap \cdots \cap A_n} = \overline{A_1} \cup \cdots \cup \overline{A_n}$$

Hint: Let B = Complement of $\bar{A}$

$= \overline{\bar{A}}$

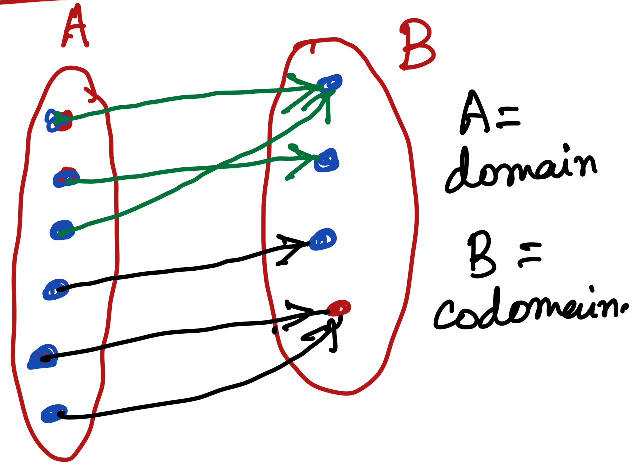
Then $B = A$

Two steps: Prove that if $x \in B$ then $x \in A$. Hence $B \subseteq A$.

Then prove that if $x \in A$, then $x \in B$

Hence $A \subseteq B$.

Function: Let A and B be two nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A. $f: A \to B$



Every element of A is mapped to a unique element of B.

$f: \mathbb{Z} \to \mathbb{N}$

$f(x) = x^2$

The followings must hold

Choose any $x \in \mathbb{Z}$, there must exist $y \in \mathbb{N}$ s.t. $f(x) = y$.

If $x \in \mathbb{Z}$, then $x^2 \geq 0$ and $x^2 \in \mathbb{Z}$

Hence, $x^2 \in \mathbb{N}$.

$f: \mathbb{N} \to \mathbb{N}$ $f(x) = \sqrt{x}$

Not a well-defined function

Because $\sqrt{3} \notin \mathbb{N}$

$f(x) = x^2$ $f: \mathbb{Z} \to \mathbb{N}$

Image of f is the set of

Consider a function $f: A \to B$ and

$\{ y \mid y = f(x) \text{ for some } x \in A \}$

$=$ Image of f.

all integers that are perfect squares.

$$\boxed{f(x) = \sqrt{x} \qquad f: \mathbb{R} \to \mathbb{R}}$$

Not well defind. Because

$\sqrt{-4}$ is not a real numbers

$\sqrt{x}$ can have positive and negative value.

Image of $S \subseteq A$. $S \neq \emptyset$.

$$f(S) = \{ y \in B \mid y = f(x) \text{ for some } x \in S \}$$