What we saw?

A bijection $f: \mathbb{N} \to \mathbb{Z}$
natural numbers (integers)

Finite Set: A set X is finite if there exists $r \in \mathbb{N}$ such that $|X| = r$. $\emptyset$, $\{1, 4\}$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Cardinality of a finite is the number of elements in that set.

Exercise: If A and B are two finite sets and $A \subseteq B$, then $|A| \leq |B|$.

Two sets A and B are equinumerous if there is a bijection $f: A \to B$.
$|A| = |B|$

How to prove that two infinite sets A and B are of same cardinality?

Define a bijection $f: A \to B$ or
" " " $g: B \to A$

$A \subseteq \mathbb{N}$ $A = \{x \in \mathbb{N} \mid 2 \mid x \text{ or } 3 \mid x\}$

Least element = 0
smallest

Injective: one-to-one

Surjective: onto

bijective: injective and surjective

Infinite Set: If a set is not finite, then it is called infinite set.
$\mathbb{N}$ = natural numbers
$\mathbb{Z}$ - integers.
$\mathbb{R}$. the set of real numbers.

Comparing the cardinality of two infinite sets:

Exercise: If A is a finite set, and $A \subseteq B$, then $|A| \leq |B|$,

Example: $|\mathbb{N}| = |\mathbb{Z}|$

Because there is a bijection $f: \mathbb{N} \to \mathbb{Z}$

one-to-one correspondance. 1004

$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

WELL-ORDERING PRINCIPLE:

Every nonempty subset of $\mathbb{N}$ has a least element.
smallest element

$B \subseteq \mathbb{N}$ $\quad B = \{x \in \mathbb{N} \mid x \text{ is odd and } x \geq 7\}$

$B = \{7, 9, 11, 13, 15, \ldots\}$

Least element is B.

Countable set: If a set X is finite or has the same cardinality as $\mathbb{N}$, then X is countable.

Infinite and has same cardinality as $\mathbb{N}$, then X is countably infinite.

Equivalently X is infinite or there exists a bijection $f: \mathbb{N} \to X$ or $f: X \to \mathbb{N}$

Example: $|\mathbb{Z}| = |\mathbb{N}|$ even though $\mathbb{N} \subsetneq \mathbb{Z}$

$\mathbb{Z}$ is countable.

Exercise: Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

$$\mathbb{N} \times \mathbb{N} = \{(0,0), (0,1), (0,2), \ldots\ldots, (1,0), (1,1), \ldots\ldots\}$$

Prove that $\mathbb{Z} \times \mathbb{Z}$ is countable

Exercise.

If there is an injective function $f: A \to B$, then $|A| \leq |B|$

(the cardinality of A is less than or equal to the cardinality of B)

Exercise: If there is a surjective function $f: A \to B$, then $|B| \leq |A|$.

Theorem: If X is a countable set and $B \subseteq X$, then B is a countable set.

Proof: Let X be a countable set.

Case-(i): X is finite set.

Then for any $B \subseteq X$, $|B| \leq |X|$

Case-(ii): X is countably infinite.

If B is finite, then B is countable.

If B is infinite, then we analyse the following.

As X is countably infinite,

hence, B is finite. Hence, B is countable.

As B is infinite, S is infinite.

$S \subseteq \mathbb{N}$ and $S \neq \emptyset$.

Due to well ordering principle, S has a smallest (least) element.

Let us order the elements of S as $a_0 < a_1 < a_2 < \ldots\ldots$

$S = \{a_0, a_1, a_2, \ldots\ldots\}$

there is a bijection $f: \mathbb{N} \to X$.

Define $S = \{n \in \mathbb{N} \mid f(n) \in B\}$

$a_0 \quad a_1 \quad a_2$

$\downarrow \quad \downarrow \quad \downarrow$

Define $g: \mathbb{N} \to B$ as follows

$g(k) = f(a_k)$

(intuition: find an ordering of the elements of B using the well-ordering principle)

First we justify that g is injective.

Consider $x \neq y$ such that $x, y \in \mathbb{N}$

$g(x) = f(a_x)$

$g(y) = f(a_y)$

Note that $a_x \neq a_y$ due to the ordering of elements in S.

As f is injective, hence $f(a_x) \neq f(a_y)$.

Therefore $g(x) \neq g(y)$.

Therefore, g is injective.

Now we justify that g is surjective.

Consider $x \in B$.

Then $x \in X$.

As f is surjective, there exists $r \in \mathbb{N}$ such that $f(r) = x$.

Then, $r = a_n$ for some $n \in S$

Then $g(n) = x$ such that $n \in \mathbb{N}$.

Hence, g is surjective.

As g is injective and surjective $\to$ hence g is bijection.

As the considered cases are exhaustive. this completes the proof.

COROLLARY: Every subset of a countable set is countable.

Exercise: If A is countable and $B \subseteq A$, then $A \setminus B$ $(A - B)$ is countable.

Uncountable set: If a set B is not countable, then B is uncountable.

B is neither finite nor countably infinite.

How to prove that a set X is uncountable?

Choose any subset $Y \subseteq X$

Assume that a bijection $f: \mathbb{N} \to X$ exists and justify that

(i) there is $a \in X$ such that $a \neq f(n)$ for any $n \in \mathbb{N}$, or

proof that f is not surjective

(ii) there are x, y such that $x \neq y$ but $f(x) = f(y)$

proof f is not injective

Previous class recap: $f: \mathbb{N} \to \mathbb{Z}$

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -\left(\frac{x+1}{2}\right) & \text{if } x \text{ is odd} \end{cases}$$

Consider $a \in \mathbb{Z}$

Case-(i): $a \geq 0$

Then $f(2a) = a$. and $2a \in \mathbb{N}$.

Case-(ii): $a < 0$.

Consider b such that $b = -a$ and $b > 0$.

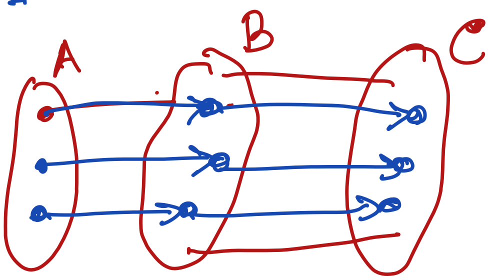
Consider $f(2b-1)$

$$= -\left(\frac{2b-1+1}{2}\right) = -b$$
$$= a$$

and $2b-1 > 0$. Hence $2b-1 \in \mathbb{N}$.

Hence f is surjective.

Exercise: If A and B are countably infinite sets. then define a precise bijection $g: \mathbb{N} \to A \cup B$.



Exercise: If $f: A \to B$ and $g: B \to C$, and

$$\boxed{\begin{aligned} &g \circ f(x) = g(f(x)) \\ &g \circ f: A \to C \end{aligned}}$$

Composition of functions.

If f and g are bijections then $g \circ f: A \to C$ is a bijection