8th Oct :

Last class: $(0,1)$ is uncountable.

Assume a bijection $g : \mathbb{N} \to (0,1)$

$g(0), g(1), \ldots\ldots, g(n).$

We constructed $y = 0.b_1 b_2 b_3 b_4 \ldots$

$y \neq g(n)$ for any $n \in \mathbb{N}$.

$y \in (0,1)$. g is not surjective

The digits that we change to construct y are diagonal entries.

DIAGONALIZATION

$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$

$g(n) = 0.a_{n,1} a_{n,2} a_{n,3} \ldots$

$b_k = 4$ if the $(k+1)$-th digit $a_{k,k} \neq 4$

$b_k = 5$ if $a_{k,k} = 4$

$g(0)$	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$\ldots$
$g(1)$	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$\ldots$
$g(2)$				
$\vdots$				
$g(k)$	$a_{k,0}$	$a_{k,1}$		$a_{k,k}$
$\vdots$				

Theorem: For any set S,

$$|\text{power-set}(S)| > |S|$$

Proof: $\mathcal{P}(S) = \text{power-set}(S)$.

$2^S = $ power-set of S.

First, we justify that there exists an injective function $f : S \to \mathcal{P}(S)$ when $S \neq \emptyset$.

For every $x \in S$ define

$$\boxed{f(x) = \{x\}}.$$

There cannot exist a bijection between S and $\text{powerset}(S)$

but there is an injective function $f : S \to \text{power-set}(S)$

If $S = \emptyset$, then $\mathcal{P}(S) = \{\emptyset\}$

$|\mathcal{P}(S)| = 1$

Clearly if $x \neq y$ and $x, y \in S$

then $\{x\} \neq \{y\}$.

Hence, $f(x) \neq f(y)$.

Therefore, f is an injective function.

Now, we prove that a bijection $g: S \to \mathcal{P}(S)$ cannot exist.

(Proof by contradiction)

Assume that a bijection $g: S \to \mathcal{P}(S)$ exists.

For every $x \in S$, there are two cases. $x \in g(x)$ or $x \notin g(x)$

We construct B such that

for every $x \in S$,

if $x \in g(x)$, we do not add x into B
$x \notin B$.

if $x \notin g(x)$, then add x into B.

Formally,

$$B = \{x \in S \mid x \notin g(x)\}$$

Clearly, $B \subseteq S$. (from definition on B).

$B \in \mathcal{P}(S)$.

Trivially $|S| < |\mathcal{P}(S)|$

But, observe that $\{x, y\} \neq f(a)$ for any $a \in S$.

Hence, f is not surjective

Therefore, f is not bijection

$x \to g(x)$ $\quad \boxed{g(x) \subseteq S}$

$g(x) \in \mathcal{P}(S)$

Since g is a bijection $\boxed{B = g(y)}$ for some $y \in S$.

(because g is surjective function)

Consider $y \in S$.

If $y \in B$, then $y \notin g(y)$.

Hence, $B \neq g(y)$, contradicting that $B = g(y)$

If $y \notin B$, then $y \in g(y)$.

Then $B \neq g(y)$, leading

Therefore g is not a surjective function.

Hence, g is not a bijection.

Since an injective function $f: S \to \mathcal{P}(S)$ exists.

therefore $\boxed{|S| \leq |\mathcal{P}(S)|}$

to a contradiction that $B = g(y)$.

Since a bijection between S and $\mathcal{P}(S)$ cannot exist, therefore $\boxed{|S| \neq |\mathcal{P}(S)|}$.

Combining these two facts, we conclude that $|S| < |\mathcal{P}(S)|$.

Where does diagonalization come?

$g: S \to \mathcal{P}(S)$

$S = \{x_1, x_2, x_3, x_4, \ldots\}$

Does $x_i \in g(x_i)$ yes? or no?

$x_1 \notin B$

$x_2 \notin B$

x_3 $x_4 \in B$

$x_n \notin B$

$x_m \in B$.

DIAGONALIZATION

	$g(x_1)$	$g(x_2)$	$g(x_3)$	$\ldots$	$g(x_n)$	$\ldots$	$g(x_m)$
x_1	yes						
x_2		yes					
x_3			no				
x_4				no			
$\vdots$							
x_n					yes		
$\vdots$							
x_m							no

Construction of B involved modification of diagonal entries.

Relations: A relation on a set B is a subset of $B \times B$.

(relation of a set B to B itself)

$B = \{1, 2, 3, 4\}$

$R_1, R_2 \subseteq B \times B$

$R_1 = \{(1,1), (1,2), (2,2), (2,3), (2,4), (3,3)\}$

not reflexive

$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

reflexive

→ relation on B

relation of B to B.

$B = \{1, 2, 3, 4\}$

$S_1 = \{(a,b) \mid a, b \in \mathbb{Z} \text{ and } a \leq b\}$

$\mathbb{Q}$ $\mathbb{R}$

relation on $\mathbb{Z}$

$\mathbb{R}$ $\mathbb{Q}$

$(0,0), (0,1), (0,2), \ldots$

$(1,1), (1,2), \ldots$

$\boxed{(1,0) \notin S_1}$

$S_2 = \{(a,b) \mid a, b \in \mathbb{Z} \text{ and } a \text{ divides } b\}$

A relation S on a set B is **reflexive** if for every $x \in S$, $(x,x) \in S$.

A relation S on a set B is **symmetric** if for every $x, y \in B$ if $(x,y) \in S$, then $(y,x) \in S$.

R_2 is symmetric

A relation S on a set B is **antisymmetric** when for every $x, y \in B$, if $(x,y), (y,x) \in S$ then $x = y$.

equivalently

if $x \neq y$, then $(x,y) \notin S$ or $(y,x) \notin S$.

(x, y) (y, w)

A relation S on a set B is **transitive** if for every $x, y, w \in B$, if $(x,y), (y,w) \in B$ then $(x,w) \in B$.

$S_1 = \{(a,b) \mid a, b \in \mathbb{R}, a \leq b\}$

Is S_1 transitive?

$a \leq b$ and $b \leq c$ implies $a \leq c$.

A relation S on a set B is an **equivalence relation** if S is reflexive, symmetric and transitive.

S_1 is a partial order

S_1 is not an equivalence relation

A relation S on a set B is **partial order** if S is reflexive, antisymmetric and transitive

Example: $S = \{(a,b) \mid a, b \in \mathbb{R}$ and $a - b \in \mathbb{Z}\}$.

Then S is an equivalence relation.

Proof:

Reflexivity: Consider $a \in \mathbb{R}$. Then $a - a = 0 \in \mathbb{Z}$. Hence, $(a,a) \in S$. Hence, S is reflexive.

Symmetricity: Consider any $(a,b) \in S$. Then $a - b \in \mathbb{Z}$. Note that $b - a \in \mathbb{Z}$. Then, $(b,a) \in S$. Hence, S is symmetric.

Transitivity: Let $a, b, c \in \mathbb{R}$ and $(a,b), (b,c) \in S$. Then, $a-b, b-c \in \mathbb{Z}$. $(a-b) + (b-c) = a - c$. Since the sum of two integers is an integer, $a-c \in \mathbb{Z}$.

Hence, $(a,c) \in S$. Therefore, S is transitive.

Since S is reflexive, symmetric, and transitive, therefore, S is an equivalence relation

Prove or disprove: $T = \{(a,b) \mid a,b \in \mathbb{Z} \text{ and } a \text{ divides } b\}$. is an equivalence relation

Answer: This statement is false.

We have to justify that T is not reflexive or not symmetric or not transitive.

It is sufficient to explain that there exists $x, y \in \mathbb{Z}$ such that $(x,y) \in T$ but $(y,x) \notin T$.

Choose $x = 1$ and $y = 3$.

Clearly 1 divides 3 but 3 does not divide 1.

Hence, $(1,3) \in S$ but $(3,1) \notin S$.

Therefore, T is not symmetric.

Hence, T is not an equivalence relation.

Exercise: $T = \{(a,b) \mid a,b \in \mathbb{Z} \text{ and } a \text{ divides } b\}$.

is a partial order.

Reflexivity: fill up argument

Antisymmetric: fill up argument

Transitive: fill up argument.

Note: When you are disproving a statement, avoid using the word "may not be"