3rd Nov:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Proof: (Double Counting Argument)

Let S_1 be a set of m elements and S_2 be a set of n elements such that $S_1 \cap S_2 = \emptyset$.

Hence, $|S_1 \cup S_2| = m+n$

We consider another way of select an r-sized subset from $S_1 \cup S_2$.

Hence it chooses in

$$\sum_{k=0}^{r} \binom{|S_1|}{k}\binom{|S_2|}{r-k} \text{ ways}$$

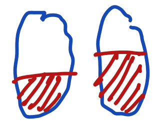
which is same as $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$ ways.

Not allowed

replace $\binom{n}{k} = \cancel{\frac{n!}{k!(n-k)!}}$ not allowed.

BIJECTIVE PROOF
DOUBLE COUNTING ARGUMENT

$|S_1 \cup S_2| = m+n$



$|S_1| = m$ $\quad$ $|S_2| = n$

LHS represents the collection of all subsets of $S_1 \cup S_2$ with exactly r elements.

- Pick k elements from S_1, and
- Pick $(r-k)$ elements from S_2.

such that $0 \le k \le r$. $\quad |S_1| = m$

$k \in \{0, 1, 2, 3, \dots, r\}$ $\quad |S_2| = n$

Therefore, RHS picks a total of r elements from $S_1 \cup S_2$.

Hence, LHS = RHS.

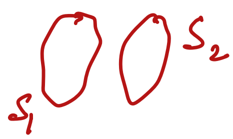
VANDERMONDE'S IDENTITY

$\binom{n}{1} = n$ or $\binom{n}{0} = 1$

$\binom{n}{n} = 1$

COROLLARY: $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Proof: Let S_1 and S_2 be pairwise disjoint sets each having exactly



$|S_1| = |S_2| = n$

LHS counts the collection of all subsets of $S_1 \cup S_2$ having exactly n elements.

n elements.

Another way to pick n elements from $S_1 \cup S_2$ can be performed by

- choose k elements from S_1 and
- choose $(n-k)$ elements from S_2.

such that $k \in \{0, 1, 2, \dots, n\}$.

Since $\binom{n}{n-k} = \binom{n}{k}$, the

$$RHS = \sum_{k=0}^{n} \binom{n}{k}^2$$

Therefore, $LHS = RHS$.

Therefore, RHS count provides us

$$\sum_{k=0}^{n} \binom{|S_1|}{k}\binom{|S_2|}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

$$LHS = \binom{2n}{n}$$

Using Vandermonde's Identity

$$\sum_{k=0}^{r} \binom{n_1}{k}\binom{n_2}{r-k} = \binom{n_1+n_2}{r}$$

$f: A \to B$ such that $|A| = n$ and $|B| = k$. How many possible functions can exist from A to B?

Any function $g: A \to B$ assigns every element of A to a unique element of B.

$|B| = k$ and $|A| = n$

Hence, total number of functions is k^n.

$$\left|\{f: A \to B \mid |A| = n, |B| = k\}\right| = ?$$

For every $x \in A$, there are $|B|$ many choices of $g(x)$.

Hence, the total number of functions is

$$\underbrace{\overset{k}{|B|} \cdot \overset{k}{|B|} \cdot \dots \cdot \overset{k}{|B|}}_{\substack{|A| \text{ times} \\ n}}$$

Binary strings: finite sequence of symbols $(a_1 a_2 a_3 \dots a_r) = r$

010011 length 6

How many binary strings of length n exists that have

$a_i \in \{0,1\}$ bits

$\underbrace{a_1 a_2 a_3 \ldots a_n}$ → k of these bits are one

rest are 0.

The number of ways to choose these k positions is $\binom{n}{k}$

length ... k-many ones?

Bit positions vary from $\{1,2,\ldots n\}$

k-of these bits positions are 1.

→ Answer to this question is $\boxed{\binom{n}{k}}$ n

THEOREM 4:

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof: LHS represents the number of binary strings of $(n+1)$ bits with exactly $(r+1)$ bits are one.

$(r+1)$ bits are 1, other bits are 0.

Equv, since the string has $(r+1)$ one's the rightmost 1 appears in $(r+1)$-th, or $(r+2)$-th, ..., or $(n+1)$th bit position

As a result, if the last one 1 appears to be k-th bit, then there must be r many 1's among the first $(k-1)$ bits.

Then we have that

$b_1 b_2 b_3 b_4 \ldots b_r \ldots b_q \ldots b_{n+1}$

$(r+1)$ bits are 1

the rightmost bit with value 1.

any bit from $\{b_{q+1}, \ldots b_{n+1}\}$ are 0.

$q \geq r+1$

First crucial observation:

The rightmost 1 must apper in q-th bit from the left so that $q \geq r+1$

$\underbrace{b_1 b_2 \ldots b_q}_{r \text{ many one's must appear here.}}$ $\underbrace{b_{q+1}}_{\text{Last 1.}}$ $\underbrace{\ldots b_n}_{\text{all } 0}$

→ There are $\binom{k-1}{r}$ many binary strings of length $(n+1)$ such that

⊛ each string has $(r+1)$ many one's

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

$$= RHS.$$

As we vary the position of rightmost one from $r+1$ to $n+1$, the

(*) Last one appears in k-th bit.

RHS counts all binary strings with $(n+1)$ bits that have exactly $(r+1)$ many 1's.

Hence, LHS = RHS

THEOREM: $$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$f: A \to \{0,1\}$

$|A| = n$

all possible subsets of A

power-set (A)

Proof: LHS represents the power set of A with n elements. Hence, LHS counts the number of all subsets of A that has n elements.

The number of subsets of A with k elements is $\binom{n}{k}$.

Equivalently, the size of any subset of A can vary between $0, 1, 2, \ldots, n$.

Since $k \in \{0, 1, 2, \ldots, n\}$

Therefore $RHS = \sum_{k=0}^{n} \binom{n}{k}$ counts the number of all possible subsets of A.

BINOMIAL THEOREM:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

Binomial coefficient.

$k=0$

THEOREM: $4^n = \sum_{k=0}^{n} \binom{n}{k} 3^k$

Proof: (Not using Binomial theorem)

LHS counts the number of functions from B to $\{1,2,3,4\}$ such that $|B| = n$.

Another way to count the number of all functions

$$f : B \to \{1,2,3,4\}$$

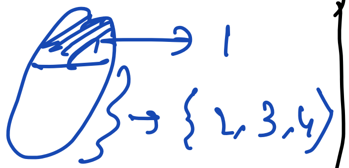
can be performed as follows.

(*) Fix a subset $D \subseteq B$ such that for every $x \in D$, $f(x) = 1$.

$$|B - D| = n - |D|$$

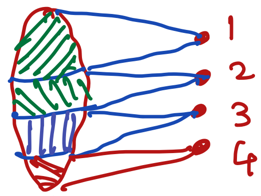
Since, $\boxed{\binom{n}{k} = \binom{n}{n-k}}$,

$\{1, 2, 3, 4\}$



$\boxed{\text{15th Nov}}$ → Class
→ Tutorial

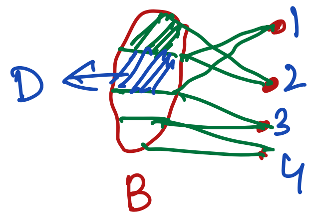
$|B|$



$\{1,2,3,4\}$

$|B| = n$

$f : B \to \{1,2,3,4\}$ $|B| = n$



Then for each such choice of $D \subseteq B$, count the number of functions $3^{n-|D|}$

$$\boxed{g : B - D \to \{2,3,4\}}$$

Since $0 \le |D| \le n$, the number of such functions is $\sum_{k=0}^{n} \binom{n}{k} 3^{n-k}$

$$= \sum_{k=0}^{n} \binom{n}{n-k} 3^{n-k}$$

$$= \sum_{r=0}^{n} \binom{n}{r} 3^{r} = \text{RHS}$$