1st October:

If there exists an injective function $f: A \to B$, then $|A| \leq |B|$.

$\mathbb{N} = \{0, 1, 2, \ldots\}$ = natural numbers

Consider $\mathbb{N} \times \mathbb{N}$
all ordered pairs of natural numbers.

How do we prove that
$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$?

First, let us prove that there exists an injective function
$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

Proof: $(x, y) \to \mathbb{Z}$
and $(x_1, y_1) \neq (x_2, y_2)$
implies $f(x_1, y_1) \neq f(x_2, y_2)$

Consider $(x_1, y_1) \neq (x_2, y_2)$

Case-(i): $p_1 = p_2$. Then, as $(x_1, y_1) \neq (x_2, y_2)$, it must be that $y_1 \neq y_2$.

(DIRECT PROOF)

$$f(x_1, y_1) - f(x_2, y_2)$$
$$= \frac{\cancel{p_1^2} + \cancel{p_1}}{\cancel{2}} + y_1 - \frac{\cancel{p_2^2} + \cancel{p_2}}{2} - y_2$$
$$= y_1 - y_2 \neq 0$$
because $y_1 \neq y_2$.

Last Class:

Let A be a countably infinite. Then for any $B \subseteq A$, B is countable.

Two sets A and B are called equinumerous if there exists a bijection between A and B.
equipotent

We have seen that
$|\mathbb{Z}| = |\mathbb{N}|$

desirable characteristics.

$$f(x, y) = \frac{(x+y)(x+y+1)}{2} + y$$

from hypothesis $x, y \geq 0$ as $(x, y) \in \mathbb{N} \times \mathbb{N}$

$\boxed{x_1 + y_1 = p_1}$, $\boxed{x_2 + y_2 = p_2}$

$$f(x_1, y_1) = \frac{(x_1+y_1)^2 + (x_1+y_1)}{2} + y_1$$
$$= \frac{p_1^2 + p_1}{2} + y_1$$

$$f(x_2, y_2) = \frac{p_2^2 + p_2}{2} + y_2$$

Therefore, $f(x_1,y_1) \neq f(x_2,y_2)$

Case-(ii): $p_1 \neq p_2$

Assume for the sake of contradiction that $f(x_1,y_1) = f(x_2,y_2)$

$$\frac{p_1^2+p_1}{2}+y_1 = \frac{p_2^2+p_2}{2}+y_2$$

$$\frac{p_1^2-p_2^2+p_1-p_2}{2} = y_2-y_1$$

$$\Rightarrow \frac{(p_1+p_2)(p_1-p_2)+(p_1-p_2)}{2} = y_2-y_1$$

$p_1 > p_2$ or $p_1 < p_2$

Without loss of generality, assume that $\boxed{p_1 > p_2}$. (the other case $p_1 < p_2$ is symmetric/similar)

$$\Rightarrow \frac{(p_1-p_2)(p_1+p_2+1)}{2} = y_2-y_1$$

Analyzing LHS, $p_1-p_2 > 0$

$p_1+p_2+1 > 2p_2+1 > 2p_2$

As $p_1-p_2 > 0$, and $p_1, p_2 \in \mathbb{N}$

hence $p_1-p_2 \geq 1$

Then LHS $> \frac{1 \cdot 2p_2}{2} = p_2$

$p_1-p_2 \geq 1$

and $p_1+p_2+1 > 2p_2$

as LHS = RHS, then

RHS $> p_2 = x_2+y_2$

$y_2-y_1 > x_2+y_2$

Then $x_2 < -y_1$

As $y_1, x_2 \geq 0$, hence $x_2 < -y_1$ leads to a contradiction

Hence $f(x_1,y_1) \neq f(x_2,y_2)$

This completes the proof that f is an injective function.

Hence, there exists an injective function $f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$.

As an implication, we have $|\mathbb{N}\times\mathbb{N}| \leq |\mathbb{N}|$

How do we prove that there exists an injective function $g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Hence. $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$

$g(x) = (2, x)$

$x \neq y$, implies $(2, x) \neq (2, y)$.

SCHROUDER-BERNSTEIN THEOREM :

If there are two sets X and Y such that there is an injective function $f: X \to Y$ and an injective function $g: Y \to X$, then there exists a bijective function between X and Y.

$|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$

How to prove that a given infinite set B is countable ?

Trick-1: Provide an explicit bijection $f: \mathbb{N} \to B$ or $f: B \to \mathbb{N}$

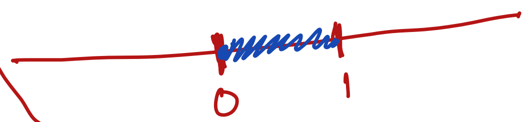
Trick-2: Provide an explicit injective function, $f: B \to \mathbb{N}$ and an explicit injective function $g: \mathbb{N} \to B$.

Theorem-2: Define two sets.

$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ → closed interval

$[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ → partially open interval.

$\mathbb{R}$ = real numbers



Then $|[0, 1]| = |[0, 1)|$

$(0, 1)$ → open interval

Proof: Injective function $f: [0, 1) \to [0, 1]$

Observe that $[0,1) \subset [0,1]$

Then $f(x) = x$ is an injective function $f: [0,1) \to [0,1]$.

Hence, there exists an injective function from $[0,1)$ to $[0,1]$

Using Schroder Bernstein's Theorem, there is a bijection between $[0,1]$ and $[0,1)$.

Hence, $|[0,1)| = |[0,1]|$.

Exercise: Prove that

(i) $|(0,1)| = |(0,1]|$

(ii) $|[0,1]| = |(0,1)|$

(iii) $|(0,1)| = |[0,1)|$

(iv) $|[0,1]| = |(0,1]|$

(v) $|[0,1)| = |(0,1]|$

How do we provide an injective function $g: [0,1] \to [0,1)$

$g(x) = \frac{x}{2}$

$x \neq y$ implies $\frac{x}{2} \neq \frac{y}{2}$.

Hence there exists an injective function from $[0,1]$ to $[0,1)$.

$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$

$(0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

$[0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$

$[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$