

Functions: Let $A, B \neq \emptyset$. A function $f: A \rightarrow B$ is an assignment of exactly one element of B to each element of A .

For all $x \in A$ there exists unique $y \in B$ such that $f(x) = y$.

One-to-one function (injective function)

A function $f: A \rightarrow B$ is injective if for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

How to prove a function injective?

Choose $x, y \in A$ and assume that $x \neq y$. Then prove that $f(x) \neq f(y)$.

A function is real-valued if its codomain is \mathbb{R} .

A function is integer-valued if its codomain is \mathbb{Z} .

$$\begin{array}{ll} f_1: A \rightarrow \mathbb{R} & f_1: A \rightarrow \mathbb{Z} \\ f_2: A \rightarrow \mathbb{R} & f_2: A \rightarrow \mathbb{Z} \end{array}$$

Then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Mid-sem: Topics will be until the class of 15th Sept. Includes strong mathematical induction. Basic concepts on sets.

A = domain Range of f is $\{f(x) \mid x \in A\}$
 B = codomain $A(x, y)$

$$\forall x \in A \forall y \in A \left(\underbrace{f(x) = f(y)}_{A(x, y)} \Rightarrow \underbrace{(x = y)}_{B(x, y)} \right)$$

Equivalent interpretation

$$\forall x \in A \forall y \in A \left(\underbrace{(x \neq y) \Rightarrow (f(x) \neq f(y))}_{\text{injective}} \right)$$

$f: \mathbb{N} \rightarrow \mathbb{N}$ $f(x) = x^2$

clearly if $x \neq y$, then

$x^2 \neq y^2$. Hence $f(x) \neq f(y)$.
Hence injective

$f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = x^2$

$x = -2, y = 2$ $f(x) = f(y) = 4$

$((x \neq y) \Rightarrow (f(x) \neq f(y)))$ is violated for $x = -2, y = 2$

Hence, not injective.

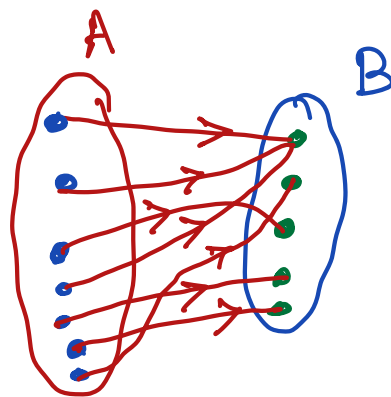
$A \rightarrow B$
 f maps from A to B

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x)$$

SURJECTIVE: A function $f: A \rightarrow B$ is **surjective** if for every $y \in B$ there exists $x \in A$ such that $f(x) = y$.

$f(x) = y$, then y is the image of x under f .
 x is a preimage of y under f .

$$\forall y \in B \exists x \in A (f(x) = y)$$



Every element of B has a pre-image.

$$g: \mathbb{R} \rightarrow \mathbb{R}^+ \quad g(x) = x^2$$

Is this **surjective**?

Consider any $y \in \mathbb{R}^+$. Then $y \geq 0$. Then \sqrt{y} exists and $\sqrt{y} \in \mathbb{R}$. Hence $g: \mathbb{R} \rightarrow \mathbb{R}^+$ is surjective.

How to prove that a function $g: A \rightarrow B$ is surjective?

Idea: Choose any arbitrary element $y \in B$.

Justify by arguments that

$$\mathbb{R}^+ = \{x \mid x \in \mathbb{R}, x \geq 0\}$$

nonnegative reals.

$f: \mathbb{N} \rightarrow \mathbb{N}$ Is f surjective?
 NO. why?

Choose $3 \in \mathbb{N}$.

$$\sqrt{3} \notin \mathbb{N}$$

Hence, 3 has no pre-image under f in \mathbb{N} .

So, $f: \mathbb{N} \rightarrow \mathbb{N}$ is not surjective.

How to prove not surjective?

Choose an element $y \in B$.

there exists $x \in A$ s.t. $f(x) = y$

Justify that $\forall x \in A$
 $f(x) \neq y$.

A function $g: A \rightarrow B$ is a bijection (bijective) if g is injective and surjective.

$f: \mathbb{N} \rightarrow \mathbb{N}$ $f(x) = x$
 $x \neq y, f(x) \neq f(y)$. \downarrow
a bijective function identity mapping

$f: \mathbb{N} \rightarrow \mathbb{Z}$

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{(x+1)}{2} & \text{if } x \text{ is odd} \end{cases}$$

How to prove that f is bijective

Proof that f is injective:

Consider $x, y \in \mathbb{N}$ and assume that $x \neq y$.

Case (i): x and y are even
 $x \neq y$ means $x = 2a$ and $y = 2b$ such that $a \neq b$ and $a, b \in \mathbb{N}$.

Then $f(x) = a$
 $f(y) = b$ } clearly $a \neq b$.

Hence, $f(x) \neq f(y)$.

Hence, f is injective.

Case (ii): x is even but y is odd

Then $f(x) \geq 0$
and $f(y) < 0$

Hence, $f(x) \neq f(y)$.

Case (iii): x and y are odd.
 $x \neq y$

$$f(x) = -\frac{x+1}{2}$$

$$f(y) = -\frac{y+1}{2}$$

As $x \neq y$, $-\frac{x+1}{2} \neq -\frac{y+1}{2}$.

Hence, $f(x) \neq f(y)$.

As the case analysis is exhaustive, f is injective.

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Proof that f is surjective:

Choose $y \in \mathbb{Z}$

Case (i): $y = 0$.

Then $f(0) = 0$ and y has a pre-image under f .

Case - (ii): $y < 0$.

$$f((2y+1)) = -\frac{-(2y+1)-1}{2} = y.$$

There exists an odd positive integer x such that

$$y = -\frac{x+1}{2}.$$

Hence, f is surjective.

As f is injective and surjective both, therefore f is bijective.

Inverse of a function: Let

$f: A \rightarrow B$ be a bijective function

Then $f^{-1}: B \rightarrow A$ such that

$f^{-1}(b) = a$ if and only if $f(a) = b$.



Case (ii): $y > 0$

$$\text{Then } f(2y) = y$$

$2y \in \mathbb{N}$. Hence, pre-image of y exists under f .

Check this part

$$y = -\frac{x+1}{2}$$

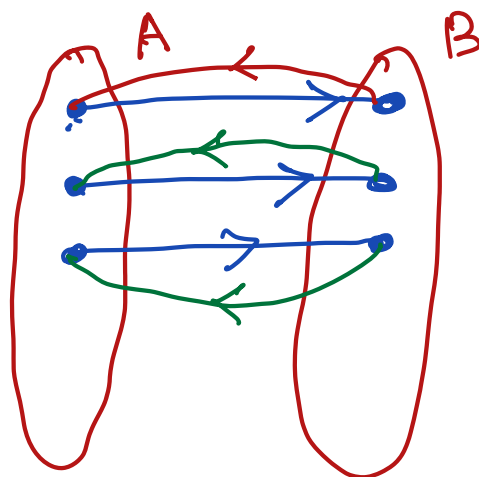
$$2y = -x-1$$

$$2y+1 = -x$$

$$x = -(1+2y)$$

As the case analysis is exhaustive, y has a pre-image under f .

BIJECTIVE: ONE-TO-ONE CORRESPONDANCE



If f is bijection, then f^{-1} is also bijection.

Two sets A and B are equinumerous if there exists a function $f: A \rightarrow B$ such that f is bijective. $f^{-1}: \mathbb{Z} \rightarrow \mathbb{N}$

$$1 \rightarrow -1$$

$$3 \rightarrow -2$$

$$5 \rightarrow -3$$

$$(-2)(-2) - 1 = -2$$

$$(-2)(-3) - 1 = 5$$

There is a bijection between \mathbb{N} and \mathbb{Z}

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -\left(\frac{x+1}{2}\right) & \text{if } x \text{ is odd} \end{cases}$$

$$f^{-1}(a) = \begin{cases} 2a & \text{if } a \geq 0 \\ -2a-1 & \text{if } a < 0 \end{cases}$$

Two sets are of same cardinality if they are equinumerous

\mathbb{N} and \mathbb{Z} are equinumerous

Finite set

Infinite set

X has distinct elements.

If there exists $k \in \mathbb{N}$ such that X has exactly k elements then $|X| = k$

Cardinality of X is k .

Then X is called finite set

\mathbb{N} , \mathbb{Z} , \mathbb{R} , \mathbb{Q}

are all infinite sets.

If a set is not finite then it is infinite

How to prove that X is infinite?

Assume that X has k elements x_1, x_2, \dots, x_k for some $k \in \mathbb{N}$.

Then justify that there exists $y \in X$ such that $y \neq x_i$ for any $1 \leq i \leq k$.

Mid-sem: 2 hours
60 marks.