Probability and Statistics: Worksheet 2 Solution

January 27, 2025

Q(1)

Solution:
(a)

$$\sum P(X) = 0.3 + 0.4 + 0.1 = 0.8$$

Since $\sum P(X) \neq 1$, this is **not** a valid probability distribution.

(b)

$$\sum P(X) = \frac{5}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{10}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \neq 1$$

Since $\sum P(X) \neq 1$ and , this **is** not a valid probability distribution.

(c)

$$\sum P(X) = \frac{1}{10} + \frac{3}{10} + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{10}{10} = 1$$

Since $\sum P(X) = 1$ and $P(X) \geq 0$, this **is** a valid probability distribution.

Q (2)
Consider a random variable Y with the probability mass function

$$f(y) = c \cdot \frac{2^y}{y!}, \quad y = 2, 3, 4, 5, \ldots$$

where $c = \frac{1}{e^2 - 3}$. We want to calculate the expected value of Y.

The expected value $E[Y]$ is given by:

$$E[Y] = \sum_{y=2}^{\infty} y \cdot f(y) = \sum_{y=2}^{\infty} y \cdot c \cdot \frac{2^y}{y!}$$

Substituting for c:

$$E[Y] = c \sum_{y=2}^{\infty} y \cdot \frac{2^y}{y!} = c \sum_{y=2}^{\infty} 2^y \cdot \frac{y}{y!}$$

$$E[Y] = c \cdot 2 \sum_{y=2}^{\infty} 2^{y-1} \cdot \frac{1}{(y-1)!}$$

$$E[Y] = c \cdot 2 \sum_{z=1}^{\infty} 2^z \cdot \frac{1}{z!}$$

$$E[Y] = c \cdot 2 \sum_{z=0}^{\infty} 2^z \cdot \frac{1}{z!} - 1$$

$$E[Y] = c \cdot 2(e^2 - 1)$$

Substituting back for $c = \frac{1}{e^2-3}$:

$$E[Y] = 2(e^2-1)c = 2(e^2-1) \cdot \frac{1}{e^2-3}$$

Thus, we have:

$$E[Y] = \frac{2(e^2-1)}{e^2-3}$$

Q (3) **Proof:**

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k}$$

Using the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$, we rewrite the sum:

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1}$$

Factor out n from the summation:

$$n\sum_{k=1}^{n} \binom{n-1}{k-1}$$

Change the index of summation. Let $j = k - 1$, so when $k = 1$, $j = 0$, and when $k = n$, $j = n - 1$:

$$n\sum_{j=0}^{n-1} \binom{n-1}{j}$$

Using the identity $\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$:

$$n \cdot 2^{n-1}$$

Thus, we have shown that:

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$