Proof: PRINCIPLE OF MATHEMATICAL INDUCTION:

$P(x)$: a propositional function on x and $x \in \mathbb{N}$ or set of all positive $\{1, 2, \ldots\}$ integers.

Theorem 1: $1+3+5+\ldots+(2n-1) = n^2$

Proof: LHS = the sum of first n odd natural numbers.

BASIS STEP (Base Case): $n = 1$

LHS $= (2n-1) = 1$

RHS $= n^2 = 1$

Hence, LHS = RHS.

(Idea): Verify that $P(1)$ is true

INDUCTION STEP: Consider $n = k+1$.

LHS $= 1+3+5+\ldots + 2(k+1)-1$

$= 1+3+5+\ldots + (2k-1) + (2k+1)$

By induction hypothesis

$1+3+5+\ldots+(2k-1) = k^2$

Then, LHS $= k^2 + (2k+1)$

$= (k+1)^2 =$ RHS

Hence, $P(k+1)$ is true.

This completes the proof.

Last Class:

Let x and y be two integers.

If both xy and $(x+y)$ are even, then both x and y are even.

Case 1: x is odd and y is even.

Give a proof: (Both xy and $(x+y)$ are even) $\rightarrow$ (Both x and y are even)

Case 2: x is odd and y is odd

Case 3: x is even and y is odd.

$P(x)$: $1+3+5+\ldots+(2x-1) = x^2$

Induction Hypothesis:

Assume that $P(k)$ is true.

The proposition $P(n)$ is true when $n = k$.

Idea of induction step:

Assume induction hypothesis ($P(k)$ is true). Using this assumption prove that $P(k+1)$ is true.

Theorem 2: $P(n)$

$1+2+2^2+\ldots+2^n = 2^{n+1}-1$

$\sum_{i=0}^{n} 2^i$ the sum of first n powers of 2 and 1.

Proof (of Theorem 2):

Basis Step: $n = 0$

LHS = 1
RHS = $2^{0+1} - 1 = 1$

Hence, $P(0)$ is true.

Induction Step: Consider $n = k+1$.

$$LHS = 1 + 2 + 2^2 + \dots + 2^{k+1}$$
$$= \underbrace{1 + 2 + 2^2 + \dots + 2^k}_{} + 2^{k+1}$$

By induction hypothesis, $P(k)$ is true.

Hence, $1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$

Then, $LHS = 2^{k+1} - 1 + 2^{k+1}$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

Hence, $P(k+1)$ is true.

This completes the proof.

Theorem 3: If n is a composite integer, then there is an integer x such that $x \geq 2$ and $x \leq \sqrt{n}$.

Proof: Let n be a composite integer. Then, there exists $x \in \{2, 3, \dots, n-1\}$ such that $x \mid n$.

Then, there exists integer k

Induction hypothesis:

$P(k)$ is true, It means that
$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

$a \mid b$ a divides b $a, b \in \mathbb{Z}$

b is divisible by a

There exists an integer k such that $b = k \cdot a$

Prime: An integer $p > 1$ is prime if p is divisible by only p itself and 1. $1 \mid p$, $p \mid p$

but no other natural number divides p.

A positive integer $p > 1$ is composite if it is not prime. there other than 1 and p itself, there is another natural number x such that $x \mid p$.

Earlier class:

If $n = xy$ for some positive integers ~~an~~ x and y, then $x \leq \sqrt{n}$ or $y \leq \sqrt{n}$.

such that $\boxed{xk = n}$.

$n > 0$ and $x > 0$. Hence $k > 0$

Then, by (Theorem of earlier

Can it be the case that $k=1$?

If $k=1$, then $x=n$. That is not true. Hence $k>1$.

class $x \le \sqrt{n}$ or $k \le \sqrt{n}$.

$x \ne 1$ $\quad x>1$

Then, $x \le \sqrt{n}$ or $k \le \sqrt{n}$ and $x, k \ge 2$.

Hence, one integer y exists such that $y \ge 2$ and $y \le \sqrt{n}$.

$\underset{\le \sqrt{n}}{y=k}$ or $y = x \le \sqrt{n}$.

Theorem 4: $n^3 - n$ is divisible by 3 for all positive integers n.

$P(n)$: $n^3 - n$ is divisible by 3.

Proof: Proof by induction on n

Basis Step: $n=1$.

$n^3 - n = 1^3 - 1 = 0$

0 is divisible by 3. Hence, $P(1)$ is true.

Induction Step: Consider $n = k+1$.

INDUCTION HYPOTHESIS:

$P(k)$ is true.

$\boxed{k^3 - k}$ is divisible by 3.

$(k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= k^3 - k + 3k^2 + 3k$.

By induction hypothesis

$k^3 - k$ is divisible by 3.

Hence, there exists an integer x such that

$k^3 - k = 3x$.

Then, $(k+1)^3 - (k+1)$

$= (k^3 - k) + 3(k^2 + k)$

$= 3x + 3(k^2 + k)$

$= 3(x + k + k^2)$.

Therefore, $P(k+1)$ is true. as $(k+1)^3 - (k+1)$ is divisible by 3.

This completes the proof.

Theorem 5: $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all natural

$P(n)$: $7^{n+2} + 8^{2n+1}$ is divisible by 57. $n \in \mathbb{N}$

$\{0, 1, 2, \ldots\}$

numbers n.

Proof: Basis Step! $n=0$

$7^{n+2}+8^{2n+1}=7^2+8^1$

$=49+8=57$.

Hence, $7^{n+2}+8^{2n+1}$ is divisible by 57. It means $P(0)$ is true.

Induction Step: Consider $n=k+1$.

$7^{n+2}+8^{2n+1}=7^{k+3}+8^{2(k+1)+1}$

$=7^{k+2}\cdot 7+8^{2k+1}\cdot 64$

$=7^{k+2}\cdot 7+8^{2k+1}\cdot 7+8^{2k+1}\cdot 57$

$=7\left(7^{k+2}+8^{2k+1}\right)+57\cdot 8^{2k+1}$

Then, $7^{n+2}+8^{2n+1}=7^{k+3}+8^{2k+3}$

$=7\left(7^{k+2}+8^{2k+1}\right)+57\cdot 8^{2k+1}$

$=7\cdot 57x+57\cdot 8^{2k+1}$

$=57\left(7x+8^{2k+1}\right)$.

$\{0,1,2,\ldots\}$

Induction Hypothesis: $k>0$

$P(k)$ is true

$7^{k+2}+8^{2k+1}$ is divisible by 57.

$P(k+1)$

$8^{2(k+1)+1}$

$=8^{2k+1}\cdot 8^2$

$=8^{2k+1}\cdot 64$

By induction hypothesis, $7^{k+2}+8^{2k+1}$ is divisible by 57.

Hence, there is an integer x such that $7^{k+2}+8^{2k+1}=57x$

Since x is an integer, therefore $7^{k+3}+8^{2k+3}$ is divisible by 57.

Because $7x+8^{2k+1}$ is an integer.

Therefore $P(k+1)$ is true.

This completes the proof.

elements or members of the set.

Sets Terminologies: An unordered collection of distinct objects

Let S be a set. Then the power set of S $\mathcal{P}(S)$ is the set of all subsets of S.

$S = \emptyset$ $\quad \mathcal{P}(S) = \{\emptyset\}$

$S = \{1,3\}$ $\quad \mathcal{P}(S) = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$

Two sets X and Y are equal if X and Y both contain the same set of elements.

B = set of all even natural numbers $\{0, 2, 4, \ldots\ldots\}$

$= \{x \mid x \in \mathbb{N} \text{ and } x \text{ is even}\}$

A = the set of odd natural numbers less than 10.

$= \{1, 3, 5, 7, 9\}$

$= \{x \mid x \in \mathbb{N} \text{ and } 0 \leq x < 10\}$

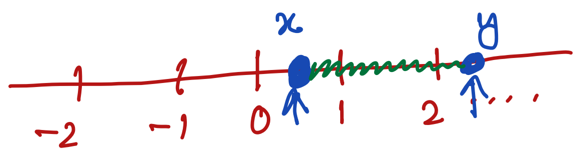
• $x, y \in \mathbb{R}$ (real numbers)

$[x,y] = \{a \mid a \in \mathbb{R} \text{ and } x \leq a \leq y\}$

closed interval ~~of~~ between x and y

$(x,y) = \{a \mid a \in \mathbb{R} \text{ and } x < a < y\}$

open interval between x and y.



$(x,y] = \{a \mid a \in \mathbb{R} \text{ and } x < a \leq y\}$

partially closed intervals

$(x,y] \subseteq [x,y]$

but $[x,y] \nsubseteq (x,y)$

Two sets A and B are equal

$\forall x \left(x \in A \leftrightarrow x \in B\right)$

$x \in A$ if and only if $x \in B$.

$\boxed{X \subseteq Y}$ $\quad \forall x \left(x \in X \rightarrow x \in Y\right)$

Lemma: Two sets $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Exercise: Prove this lemma.