Probability and Statistics: Quiz 1 Solution

January 27, 2025

Question (1):

Mary takes a multiple-choice exam in her PS101 class. The exam consists of 15 questions, and each question has 5 possible answers, only one of which is correct. Mary did not study for the exam, so she guesses independently on every question. Let X denote the number of questions that Mary gets right.

(a) What is a possible distribution model for this problem?

(b) What is the expected number of questions that Mary would get right?

(c) What is the probability that Mary answers exactly 3 questions correctly?

(d) What is the probability that Mary would get atmost 5 questions right?

(e) What is the probability that Mary would get more than half of the questions right?

Solution:

(a) The distribution model for this problem is $X \sim \text{Binomial}(n = 15, p = 1/5 = 0.2)$.

(b) The expected number of questions that Mary gets right is:

$$E(X) = n \cdot p = 15 \cdot 0.2 = 3.$$

(c) The probability that Mary answers exactly 3 questions correctly is:

$$P(X = 3) = \binom{15}{3}(0.2)^3(0.8)^{12}.$$

Computing this:

$$P(X = 3) = \binom{15}{3} \cdot (0.008) \cdot (0.068719476736) \approx 0.2501389.$$

(d) The probability that Mary answers *at most* 5 questions correctly is:

$$P(X \leq 5) = \sum_{x=0}^{5} P(X = x).$$

$$P(X \leq 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5).$$

$$P(X \leq 5) \approx 0.868.$$

(e) The probability that Mary answers *more than half of* questions correctly is:

$$P(X \geq 8) = \sum_{x=8}^{15} P(X = x).$$

$$P(X \geq 8) = 1 - P(X \leq 7).$$

$$P(X \leq 7) \approx 0.999.$$

Thus:

$$P(X \geq 8) = 1 - 0.999 = 0.001.$$

Question (2):

Consider the probability mass function (p.m.f.)

$$f(x) = c \cdot \left(\frac{1}{4}\right)^x, \quad x = 1, 2, 3, 4, \ldots$$

(a) Find c such that $f(x)$ satisfies the conditions of being a p.m.f. for a random variable X.

(b) Find $P(X \text{ is odd})$.

Solution:

(a) For $f(x)$ to be a valid p.m.f., the sum of all probabilities must equal 1:

$$\sum_{x=1}^{\infty} f(x) = 1.$$

Substituting $f(x) = c \cdot \left(\frac{1}{4}\right)^x$:

$$\sum_{x=1}^{\infty} c \cdot \left(\frac{1}{4}\right)^x = 1.$$

Factor out c:

$$c \cdot \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = 1.$$

The series $\sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x$ is a geometric series with first term $a = \frac{1}{4}$ and common ratio $r = \frac{1}{4}$:

$$\sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Thus:

$$c \cdot \frac{1}{3} = 1 \quad \Rightarrow \quad c = 3.$$

(b) To find $P(X \text{ is odd})$, sum over all odd values of x:

$$P(X \text{ is odd}) = f(1) + f(3) + f(5) + \ldots.$$

Substituting $f(x) = 3 \cdot \left(\frac{1}{4}\right)^x$:

$$P(X \text{ is odd}) = 3 \cdot \left[\left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^5 + \dots \right].$$

This is a geometric series with first term $a = \frac{1}{4}$ and common ratio $r = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$:

$$P(X \text{ is odd}) = 3 \cdot \frac{\frac{1}{4}}{1 - \frac{1}{16}} = 3 \cdot \frac{\frac{1}{4}}{\frac{15}{16}} = 3 \cdot \frac{1}{4} \cdot \frac{16}{15} = 3 \cdot \frac{4}{15} = \frac{12}{15} = 0.8.$$