

Probability and Statistics: Worksheet 1

Solution

January 20, 2025

Q (1) Three events A , B , and C are called **independent** if all of the following four conditions hold:

1. $P(A \cap B) = P(A) \cdot P(B)$
2. $P(A \cap C) = P(A) \cdot P(C)$
3. $P(B \cap C) = P(B) \cdot P(C)$
4. $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

These conditions imply both *pairwise independence* and the joint independence of the three events.

Solution:

Consider drawing a single card from a standard deck of 52 cards. Define the following events:

- A : Drawing a red king (hearts or diamonds).
- B : Drawing a black king (spades or clubs).
- C : Drawing any queen (hearts, diamonds, spades, or clubs).

These events are mutually exclusive because

- we cannot draw a red king (event A) and a black king (event B) at the same time.
- we cannot draw a red king (event A) and a queen (event C) at the same time.
- we cannot draw a black king (event B) and a queen (event C) at the same time.

Thus, $A \cap B = \emptyset$, $A \cap C = \emptyset$, and $B \cap C = \emptyset$.

However, these events are not ****independent****:

- $P(A) = \frac{2}{52} = \frac{1}{26}$, $P(B) = \frac{2}{52} = \frac{1}{26}$, and $P(C) = \frac{4}{52} = \frac{1}{13}$.
- Since $A \cap B \cap C = \emptyset$, $P(A \cap B \cap C) = 0$, but $P(A) \cdot P(B) \cdot P(C) > 0$.

Hence, A , B , and C are mutually exclusive but not independent.

Q (2) : To ensure that this is a valid probability model, we need to make sure that the sum of the probabilities over all possible outcomes in the sample space $S = \{1, 2, 3, 4, \dots\}$ equals 1, i.e.,

$$\sum_{k=1}^{\infty} P(k) = 1.$$

$$\sum_{k=2}^{\infty} \frac{(\ln 3)^k}{k!} = e^{\ln 3} - \frac{(\ln 3)^0}{0!} - \frac{(\ln 3)^1}{1!}.$$

Since $e^{\ln 3} = 3$, we get:

$$\sum_{k=2}^{\infty} \frac{(\ln 3)^k}{k!} = 3 - 1 - \ln 3 = 2 - \ln 3.$$

Thus, the total probability equation becomes:

$$p + (2 - \ln 3) = 1.$$

Solving for p :

$$p = 1 - (2 - \ln 3) = \ln 3 - 1.$$

Q (3) Suppose an individual is randomly selected from the population of adult males. Let A be the event that the selected individual is over 6ft in height, and B be the event that the individual is a professional basketball player.

Which do you think is larger: $P(A | B)$ or $P(B | A)$? Explain why.

Solution: $P(A | B)$ represents the probability that an individual is over 6ft in height given that he is a professional basketball player. $P(B | A)$ represents the probability that an individual is a professional basketball player given that he is over 6ft in height.

Since professional basketball players are generally much taller than average, the event of being over 6ft is very likely for them. Thus, $P(A | B)$ is high, approaching 1.

On the other hand, not all individuals over 6ft in height are professional basketball players, as there are many tall people who do not play basketball professionally. Therefore, $P(B | A)$ is relatively low.

thus

$$P(A | B) > P(B | A).$$

Q (4) An automobile insurance company classifies each driver as a good risk, a medium risk, or a poor risk. Of those currently insured, 30% are good risks, 50% are medium risks, and 20% are poor risks. In any given year, the probability that a driver will have a traffic accident is: - 0.1 for a good risk, - 0.3 for a medium risk, - 0.5 for a poor risk.

(a) What is the probability that a randomly selected driver insured by this company will have a traffic accident during 2023?

(b) If a randomly selected driver insured by this company did not have a traffic accident during 2023, what is the probability that the driver is a good risk?

(c) If a randomly selected driver insured by this company had a traffic accident during 2023, what is the probability that the driver is a poor risk?

(d) Suppose a driver insured by this company is not a poor risk. What is the probability that the driver had a traffic accident during 2023?

(e) The company announced that it will raise insurance premiums for drivers who are either poor risks or had a traffic accident during 2023 or both. What proportion of customers would have their premiums raised?

(f) Are the events "a randomly selected driver is a medium risk" and "a randomly selected driver

had a traffic accident during 2023” independent?

(g) Are the events ”a randomly selected driver is a medium risk” and ”a randomly selected driver had a traffic accident during 2023” mutually exclusive?

Solution:

Let G , M , and P represent the events that a driver is a good, medium, and poor risk, respectively. Let A represent the event of a traffic accident. The probabilities are:

$$P(G) = 0.30, \quad P(M) = 0.50, \quad P(P) = 0.20,$$

$$P(A | G) = 0.1, \quad P(A | M) = 0.3, \quad P(A | P) = 0.5.$$

(a) The total probability that a randomly selected driver has a traffic accident is:

$$P(A) = P(A | G)P(G) + P(A | M)P(M) + P(A | P)P(P).$$

Substituting the values:

$$P(A) = (0.1 \times 0.30) + (0.3 \times 0.50) + (0.5 \times 0.20) = 0.03 + 0.15 + 0.10 = 0.28.$$

(b) The probability that a randomly selected driver did not have a traffic accident is $P(A^c) = 1 - P(A) = 0.72$. The probability that the driver is a good risk given no accident:

$$P(G | A^c) = \frac{P(A^c | G)P(G)}{P(A^c)},$$

where $P(A^c | G) = 1 - P(A | G) = 0.9$.

$$P(G | A^c) = \frac{0.9 \times 0.30}{0.72} \approx 0.375.$$

(c) The probability that the driver is a poor risk given a traffic accident:

$$P(P | A) = \frac{P(A | P)P(P)}{P(A)} = \frac{0.5 \times 0.20}{0.28} \approx 0.357.$$

(d) The probability that the driver is not a poor risk is $P(G \cup M) = 1 - P(P) = 0.80$. The probability of a traffic accident for a driver who is not a poor risk:

$$P(A | G \cup M) = \frac{P(A \cap (G \cup M))}{P(G \cup M)} = \frac{(0.03 + 0.15)}{0.80} = 0.225.$$

(e) The proportion of customers whose premiums would be raised is:

$$P(P \cup A) = P(P) + P(A) - P(P \cap A) = 0.20 + 0.28 - (0.20 \times 0.50) = 0.38.$$

(f) To check independence between M and A :

$$P(M \cap A) = P(A | M)P(M) = 0.3 \times 0.50 = 0.15.$$

Since $P(M \cap A) \neq P(M)P(A)$, the events are **not independent**.

(g) To check mutual exclusivity: Since $P(M \cap A) > 0$, the events are **not mutually exclusive**

Q (5) We know that,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Since $P(E \cup F) \leq 1$ then

$$P(E \cup F) \leq 1.$$

so we have

$$P(E) + P(F) - P(E \cap F) \leq 1.$$

$P(E \cap F)$:

$$-P(E \cap F) \leq 1 - P(E) - P(F).$$

Multiply both sides of the inequality by -1 so

$$P(E \cap F) \geq P(E) + P(F) - 1.$$

hence

$$P(E \cap F) \geq P(E) + P(F) - 1.$$