Worksheet # 4

10 Feb 2025

In the previous lecture, we studied joint distribution for random variables X and Y. It is defined as follows:
Joint CDF:

$$F(x, y) := P(X \leq x, Y \leq y) \ quad(\textbf{Discrete})$$

Joint PDF:

$$f(x, y) \quad \text{s.t.} \quad P((x, y) \in B) = \iint_B f(x, y)\, dx\, dy \quad (\textbf{Continuous})$$

Problem 1

The joint Moment Generating Function is defined as:
For random variables X, Y, it is given by

$$M(t_1, t_2) = E(e^{t_1 X + t_2 Y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x, y)\, dx\, dy$$

1. Find $E(X)$ in terms of $M(t_1, t_2)$.
2. Find $E(Y)$ in terms of $M(t_1, t_2)$.
3. Find $E(X^2)$ in terms of $M(t_1, t_2)$.
4. Find the joint MGF for X and Y if the joint PDF is given by:

$$f(x, y) = \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}, \quad x > 0, y > 0$$

5. Use (iv) to find (i), (ii), and (iii).

Problem 2

Let X and Y be two independent random variables with respective moment generating functions:

$$M_X(t) = \frac{1}{1 - 5t}, \quad t < \frac{1}{5}$$

$$M_Y(t) = \frac{1}{(1 - 5t)^2}, \quad t < \frac{1}{5}$$

Find $E(X + Y^2)$.

Problem 3

If $X \sim \text{Exp}(\lambda_x)$ and $Y \sim \text{Exp}(\lambda_y)$, then does $X + Y \sim \text{Exp}(\lambda_x + \lambda_y)$? Is it true or false? Justify.

Problem 4

Using the change of variables, show that if X, Y are independent random variables with $N(0, 1)$ distribution, then $X^2 + Y^2$ has $\text{Exp}(1/2)$ distribution.

Problem 5

Let the joint PDF of random variables X and Y be:

$$f(x, y) = \frac{\sqrt{ab - c^2}}{2\pi} e^{-\frac{1}{2}(ax^2 + by^2 + 2cxy)}$$

1. What are the conditions on a, b, c for this to be a valid PDF?
2. For what values of the parameters are X and Y independent?