

Functions: Let  $A, B \neq \emptyset$ . A function  $f: A \rightarrow B$  is an assignment of exactly one element of  $B$  to each element of  $A$ .

For all  $x \in A$  there exists unique  $y \in B$  such that  $f(x) = y$ .

### One-to-one function (injective function)

A function  $f: A \rightarrow B$  is injective if for all  $x, y \in A$ , if  $f(x) = f(y)$  then  $x = y$ .

How to prove a function injective?

Choose  $x, y \in A$  and assume that  $x \neq y$ . Then prove that  $f(x) \neq f(y)$ .

A function is real-valued if its codomain is  $\mathbb{R}$ .

A function is integer-valued if its codomain is  $\mathbb{Z}$ .

$$f_1: A \rightarrow \mathbb{R}$$

$$f_2: A \rightarrow \mathbb{R}$$

$$\text{Then } (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

Mid-sem: Topics will be until the class of 15th Sept. Includes strong mathematical induction. Basic concepts on sets.

$A = \text{domain}$  Range of  $f$  is  
 $B = \text{codomain}$   $\{f(x) \mid x \in A\}$   
 $A(x, y)$

$$\forall x \in A \forall y \in A (f(x) = f(y) \Rightarrow (x = y)) B(x, y)$$

Equivalent interpretation

$$\forall x \in A \forall y \in A ((x \neq y) \Rightarrow (f(x) \neq f(y)))$$

$$f: \mathbb{N} \rightarrow \mathbb{N} \quad f(x) = x^2$$

clearly if  $x \neq y$ , then  $x^2 \neq y^2$ . Hence  $f(x) \neq f(y)$ .  
 Hence injective

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2$$

$$x = -2, y = 2 \quad f(x) = f(y) \\ = 4$$

$$((x \neq y) \Rightarrow (f(x) \neq f(y))) \text{ is}$$

violated for  $x = -2, y = 2$

Hence, not injective.

$A \rightarrow B$   
f maps from A to B

$$(f_1, f_2)(x) = f_1(x), f_2(x)$$

SURJECTIVE: A function

$f: A \rightarrow B$  is surjective if

for every  $y \in B$  there exists  $x \in A$  such that  $f(x) = y$ .

$f(x) = y$ , then  $y$  is the image of  $x$  under  $f$ .  
 $x$  is a preimage of  $y$  under  $f$ .

$$g: \mathbb{R} \rightarrow \mathbb{R}^+ \quad g(x) = x^2$$

Is this surjective?

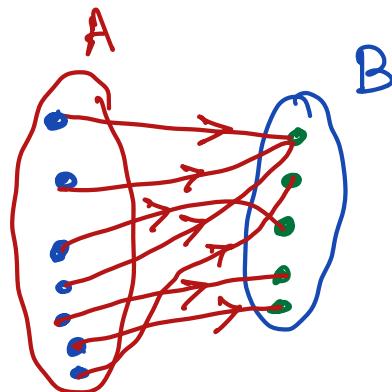
Consider any  $y \in \mathbb{R}^+$ . Then  $y \geq 0$ . Then  $\sqrt{y}$  exists and  $\sqrt{y} \in \mathbb{R}$ . Hence  $g: \mathbb{R} \rightarrow \mathbb{R}^+$  is surjective.

How to prove that a function  $g: A \rightarrow B$  is surjective?

Idea: Choose any arbitrary element  $y \in B$ .

Justify by arguments that

$$\forall y \in B \exists x \in A [f(x) = y]$$



Every element of  $B$  has a pre-image.

$$\mathbb{R}^+ = \{x \mid x \in \mathbb{R}, x \geq 0\}$$

nonnegative reals.

$f: \mathbb{N} \rightarrow \mathbb{N}$  Is  $f$  surjective?

NO. why?

Choose  $3 \in \mathbb{N}$ .

$$\sqrt{3} \notin \mathbb{N}$$

Hence, 3 has no pre-image under  $f$  in  $\mathbb{N}$ .

So.  $f: \mathbb{N} \rightarrow \mathbb{N}$  is not surjective.

How to prove not surjective?

Choose an element  $y \in B$ .

there exists  $x \in A$  s.t.  $f(x) = y$

justify that  $\forall x \in A$

$f(x) \neq y$ .

A function  $g: A \rightarrow B$  is a bijection (bijective) if  $g$  is injective and surjective.

$$f: \mathbb{N} \rightarrow \mathbb{N}$$

$$f(x) = x$$

$x \neq y, f(x) \neq f(y)$ .  
a bijective function identity mapping

$$f: \mathbb{N} \rightarrow \mathbb{Z}$$

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{(x+1)}{2} & \text{if } x \text{ is odd} \end{cases}$$

How to prove that  $f$  is bijective

Proof that  $f$  is injective:

Consider  $x, y \in \mathbb{N}$  and assume that  $x \neq y$ .

Case (i):  $x$  and  $y$  are even  
 $x \neq y$  means  $x = 2a$  and  $y = 2b$  such that  $a \neq b$ , and  $a, b \in \mathbb{N}$ .

Then  $f(x) = a$       }      clearly  
                 $f(y) = b$       }       $a \neq b$ .

Hence,  $f(x) \neq f(y)$ .

Hence,  $f$  is injective.

Case (ii):  $x$  is even but  $y$  is odd

$$\text{Then } f(x) \geq 0$$

$$\text{and } f(y) < 0$$

$$\text{Hence, } f(x) \neq f(y).$$

Case (iii):  $x$  and  $y$  are odd.  
 $\boxed{x \neq y}$

$$f(x) = -\frac{x+1}{2}$$

$$f(y) = -\frac{y+1}{2}$$

$$\text{As } x \neq y, -\frac{x+1}{2} \neq -\frac{y+1}{2}.$$

$$\text{Hence, } f(x) \neq f(y).$$

As the case analysis is complete,  
 $f$  is injective.

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Proof that  $f$  is surjective:

Choose  $y \in \mathbb{Z}$

Case (i):  $y = 0$ .

Then  $f(0) = 0$  and  $y$  has a pre-image under  $f$ .

Case (ii):  $y < 0$ .

$$f((2y+1)) = -\frac{(2y+1)-1}{2} \\ = y.$$

There exists an odd positive integer  $x$  such that

$$y = -\frac{x+1}{2}.$$

Hence,  $f$  is surjective

As  $f$  is injective and surjective both, therefore  $f$  is bijective.

Inverse of a function: Let

$f: A \rightarrow B$  be a bijective function

Then  $f^{-1}: B \rightarrow A$  such that

$f^{-1}(b) = a$  if and only if

$$f(a) = b.$$



Case (ii):  $y > 0$

Then  $f(2y) = y$

$2y \in \mathbb{N}$ . Hence,

pre-image of  $y$  exists under  $f$ .

$$y = -\frac{x+1}{2}$$

Check this part

$$2y = -x - 1$$

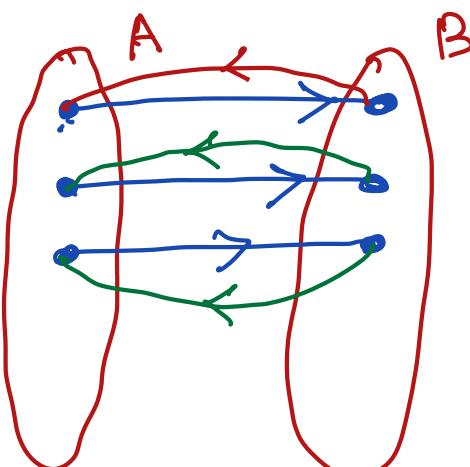
$$2y + 1 = -x$$

$$x = -(1+2y)$$

As the case analysis is exhaustive,  $y$  has a pre-image under  $f$ .

BIJECTIVE: ONE-TO-ONE

CORRESPONDANCE



If  $f$  is bijection, then  $f^{-1}$  is also bijection.

Two sets A and B are equinumerous if there exists a function  $f: A \rightarrow B$  such that  $f$  is bijective.  $f^{-1}: \mathbb{Z} \rightarrow \mathbb{N}$

$$\begin{array}{l} 1 \rightarrow -1 \\ 3 \rightarrow -2 \quad (-2)(-2) - 1 = -2 \\ 5 \rightarrow -3 \quad (-2)(-3) - 1 = 5 \end{array}$$

Two sets are of same cardinality if they are equinumerous.

There is a bijection between  $\mathbb{N}$  and  $\mathbb{Z}$

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\left(\frac{x+1}{2}\right) & \text{if } x \text{ is odd} \end{cases}$$

$$f^{-1}(a) = \begin{cases} 2a & \text{if } a \geq 0 \\ -2a - 1 & \text{if } a < 0 \end{cases}$$

$\mathbb{N}$  and  $\mathbb{Z}$  are equinumerous

### Finite set

### Infinite set

$X$  has distinct elements.

If there exists  $k \in \mathbb{N}$  such that  $X$  has exactly  $k$  elements then  $|X| = k$ . Cardinality of  $X$  is  $k$ .

Then  $X$  is called finite set

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$

are all infinite sets.

If a set is not finite then it is infinite

How to prove that  $X$  is infinite?

Assume that  $X$  has  $k$  elements  $x_1, x_2, \dots, x_k$  for some  $k \in \mathbb{N}$ .

Then justify that there exists  $y \in X$  such that  $y \neq x_i$  for any  $1 \leq i \leq k$ .

Mid-sem: 2 hours

60 marks.