Discrete Structures-2025: Quiz-5

Sets and Functions

Full Marks: 30
Time: 45 minutes

October 27, 2025

(1) Let S be a relation on a set A. Prove or disprove: if S is symmetric and transitive, then S is reflexive. (**10 Marks**)

Solution:

[Disprove by counter-example]

We give an explicit counterexample. Let A be any nonempty set; for concreteness take

$$A = \{1\}.$$

Define the relation $S \subseteq A \times A$ by

$$S = \varnothing.$$

We verify the three properties.

Symmetry. Suppose $(x, y) \in S$. There is no such pair because $S = \varnothing$. Hence the implication "if $(x, y) \in S$ then $(y, x) \in S$" holds vacuously for all $x, y \in A$. Therefore S is symmetric.

Transitivity. Suppose $(x, y) \in S$ and $(y, z) \in S$. Again, no such x, y, z exist, so the implication "if $(x, y) \in S$ and $(y, z) \in S$ then $(x, z) \in S$" is vacuously true for all $x, y, z \in A$. Thus S is transitive.

Reflexivity (fails). Reflexivity would require $(1, 1) \in S$. But $S = \varnothing$, so $(1, 1) \notin S$. Consequently S is not reflexive.

Hence S is symmetric and transitive but not reflexive, which disproves the claim.

$\boxed{\mathbb{NOTE}}$: The example above is one of many possible counterexamples. Any empty relation on a nonempty set is always symmetric and transitive but fails to be reflexive.

(2) Let $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$ and $A = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation S on A as follows. We say that for $(a, b), (c, d) \in A$, $((a, b), (c, d)) \in S$ if either (i) $a + b < c + d$ or (ii) $a + b = c + d$ and $a \leq c$.

(a) Prove that S is a partial order.

(b) Prove or disprove that S is a total order. **(7 Marks + 3 Marks)**

Solution:

(a) Prove that S is a partial order

We must prove reflexivity, antisymmetry, and transitivity.

1. Reflexivity

For any $(a, b) \in A$, we have $a + b = a + b$ and $a \leq a$. This satisfies condition (ii).
Thus, $(a, b) \preceq (a, b)$. The relation is reflexive.

2. Antisymmetry

Assume $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$.

- $(a, b) \preceq (c, d) \implies a + b < c + d$ **or** ($a + b = c + d$ and $a \leq c$).
- $(c, d) \preceq (a, b) \implies c + d < a + b$ **or** ($c + d = a + b$ and $c \leq a$).

The cases $a + b < c + d$ and $c + d < a + b$ cannot occur.
Therefore, the only possibility is that $a + b = c + d$ and $a \leq c$, **AND** $c + d = a + b$ and $c \leq a$.
From $a \leq c$ and $c \leq a$, we must have $a = c$.
Substituting $a = c$ into $a + b = c + d$ gives $a + b = a + d$, which implies $b = d$. Thus, $(a, b) = (c, d)$.
The relation is antisymmetric.

3. Transitivity

Assume $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$. We analyze the sums $S_1 = a + b$, $S_2 = c + d$, $S_3 = e + f$.

- **Case 1:** $S_1 < S_2$ **or** $S_2 < S_3$**.**
  If $S_1 < S_2$ and $S_2 \leq S_3$ (either $S_2 < S_3$ or $S_2 = S_3$), then $S_1 < S_3$.
  If $S_1 \leq S_2$ (either $S_1 < S_2$ or $S_1 = S_2$) and $S_2 < S_3$, then $S_1 < S_3$.
  If $S_1 < S_2$ and $S_2 < S_3$, then $S_1 < S_3$.
  In all these sub-cases, $a + b < e + f$, which implies $(a, b) \preceq (e, f)$ by condition (i).
- **Case 2:** $S_1 = S_2$ **and** $S_2 = S_3$**.**
  This implies $S_1 = S_3$, so $a + b = e + f$.
  From $(a, b) \preceq (c, d)$ and $a + b = c + d$, we have $a \leq c$.
  From $(c, d) \preceq (e, f)$ and $c + d = e + f$, we have $c \leq e$.
  Therefore, $a \leq c$ and $c \leq e$ implies $a \leq e$.
  Thus, we have $a + b = e + f$ and $a \leq e$, which implies $(a, b) \preceq (e, f)$ by condition (ii).

In all cases, $(a, b) \preceq (e, f)$. The relation is transitive.

Since S is reflexive, antisymmetric, and transitive, it is a **partial order**.

(b) Prove or disprove that S is a total order

We prove that S **is** a total order. We must show that for any two distinct elements $(a, b), (c, d) \in A$, they are comparable. That is, $(a, b) \preceq (c, d)$ or $(c, d) \preceq (a, b)$.

Let $(a, b), (c, d) \in A$.

- **Case 1:** $a + b < c + d$**.** By condition (i), $(a, b) \preceq (c, d)$.
- **Case 2:** $a + b > c + d$**.** (i.e., $c + d < a + b$) By condition (i), $(c, d) \preceq (a, b)$.
- **Case 3:** $a + b = c + d$**.**
  - If $a < c$, then $a \leq c$. By condition (ii), $(a, b) \preceq (c, d)$.
  - If $a > c$, then $c < a$, so $c \leq a$. By condition (ii), $(c, d) \preceq (a, b)$.
  - If $a = c$, then $a \leq c$. By condition (ii), $(a, b) \preceq (c, d)$. (This also implies $b = d$, so they are the same element).

In all possible cases, the elements are comparable. Therefore, S satisfies the totality property and is a total order.

(3) Let S be a reflexive relation on A. We define $S^2 = \{(a, b) \mid a, b \in A$ and there exists $d \in A$ such that $(a, d), (d, b) \in S\}$. Prove or disprove that $S^2 = S$. **(10 Marks)**

Solution:

[Disprove by counter-example]

Let $A = \{1, 2, 3\}$ and define

$$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}.$$

Then S is reflexive as $(a, a) \in S \;\; \forall a \in A$.

Now as $(1, 2) \in S$ and $(2, 3) \in S$, so there exists $d = 2$ with $(1, d), (d, 3) \in S$ implies $(1, 3) \in S^2$. But $(1, 3) \notin S$. Therefore $S^2 \neq S$.

$\boxed{\mathbb{NOTE}}$: The counterexample above is not unique: any reflexive relation that is not transitive works as a counterexample.