Discrete Structures-2025: Quiz-3

Sets and Functions

Full Marks: 20
Time: 40 minutes

October 6, 2025

(1) Let A and $D_1, \dots, D_n$ denote a collection of sets. Then prove that

$$A \times (\bigcup_{i=1}^{n} D_i) = \bigcup_{i=1}^{n} (A \times D_i)$$

(8 Marks)

Solution:

$\boxed{METHOD\ 1}$ **(By Subset Inclusion):**

We prove the equality

$$A \times \left(\bigcup_{i=1}^{n} D_i \right) = \bigcup_{i=1}^{n} (A \times D_i)$$

by showing mutual inclusion.

(i) Proof that $A \times (\bigcup_{i=1}^{n} D_i) \subseteq \bigcup_{i=1}^{n} (A \times D_i)$:

Let $(a, b) \in A \times (\bigcup_{i=1}^{n} D_i)$.

Then $a \in A$ and $b \in \bigcup_{i=1}^{n} D_i$.

Hence, there exists some k with $1 \leq k \leq n$ such that $b \in D_k$.

Therefore, $(a, b) \in A \times D_k \subseteq \bigcup_{i=1}^{n} (A \times D_i)$.

Thus,

$$A \times \left(\bigcup_{i=1}^{n} D_i \right) \subseteq \bigcup_{i=1}^{n} (A \times D_i).$$

(ii) Proof that $\bigcup_{i=1}^{n} (A \times D_i) \subseteq A \times (\bigcup_{i=1}^{n} D_i)$:

Let $(a, b) \in \bigcup_{i=1}^{n} (A \times D_i)$.

Then there exists some k with $1 \leq k \leq n$ such that $(a, b) \in A \times D_k$.

Hence, $a \in A$ and $b \in D_k \subseteq \bigcup_{i=1}^{n} D_i$.

Therefore, $(a,b) \in A \times (\bigcup_{i=1}^{n} D_i)$.

Thus,

$$\bigcup_{i=1}^{n}(A \times D_i) \subseteq A \times \Big(\bigcup_{i=1}^{n} D_i\Big).$$

From (i) and (ii), the two sets are equal:

$$A \times \Big(\bigcup_{i=1}^{n} D_i\Big) = \bigcup_{i=1}^{n}(A \times D_i).$$

$\boxed{METHOD\ 2}$ **(By Mathematical Induction on n):**

Base Case: For $n = 1$, we have

$$A \times D_1 = A \times D_1$$

which is trivially true.

Inductive Hypothesis: Assume that for some $k \geq 1$,

$$A \times \Big(\bigcup_{i=1}^{k} D_i\Big) = \bigcup_{i=1}^{k}(A \times D_i).$$

Inductive Step: Consider $n = k + 1$. Then

$$\begin{aligned}
A \times \Big(\bigcup_{i=1}^{k+1} D_i\Big) &= A \times \Big(\Big(\bigcup_{i=1}^{k} D_i\Big) \cup D_{k+1}\Big) \\
&= (A \times (\bigcup_{i=1}^{k} D_i)) \cup (A \times D_{k+1}) && \big(as\ A \times (B \cup C) = (A \times B) \cup (A \times C)\big) \\
&= \Big(\bigcup_{i=1}^{k}(A \times D_i)\Big) \cup (A \times D_{k+1}) && (by\ using\ induction\ hypothesis) \\
&= \bigcup_{i=1}^{k+1}(A \times D_i).
\end{aligned}$$

Hence, by the principle of mathematical induction, the result holds for all $n \in \mathbb{N}$.

$$\boxed{A \times \Big(\bigcup_{i=1}^{n} D_i\Big) = \bigcup_{i=1}^{n}(A \times D_i)}$$

(2) A function $f : \mathbb{Z} \to \mathbb{Z}$ is *strictly increasing* if for every $a, b \in \mathbb{Z}$, if $a < b$, then $f(a) < f(b)$. Then prove that: if a function $g : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing, then g is an injective function.

(6 Marks)

Solution:

We prove injectivity by contradiction.

Suppose g is strictly increasing but not injective.

Then there exist distinct integers $x, y \in \mathbb{Z}$ with

$$x \neq y \qquad \text{and} \qquad g(x) = g(y).$$

Without loss of generality assume $x < y$ (if $y < x$ the same argument applies with the roles of x and y interchanged).

Since g is strictly increasing and $x < y$, we must have

$$g(x) < g(y).$$

This contradicts the assumption $g(x) = g(y)$. Therefore no two distinct integers can have the same image under g, so g is injective.

(3) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 2m - n$. **Prove or disprove that:** f is a surjective function. **(6 Marks)**

Solution:

To prove that f is surjective, we must show that for every integer $k \in \mathbb{Z}$, there exists a pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$f(m, n) = k.$$

That is, we must find integers m and n satisfying

$$2m - n = k.$$

Rewriting, we obtain

$$n = 2m - k.$$

Now, for any fixed $k \in \mathbb{Z}$, we may choose $m = 0$ (for simplicity), which gives

$$n = 2(0) - k = -k.$$

Then

$$f(0, -k) = 2(0) - (-k) = k.$$

Hence, for every $k \in \mathbb{Z}$, there exists at least one pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ (i.e., $(0, -k)$) such that $f(m, n) = k$.

Therefore, f is surjective.

$\boxed{NOTE}$: Any other method for showing surjectivity is also valid.