24th Nov:

CONNECTIVITY:

$u(=x_0), x_1, x_2, x_3, x_4, x_2, x_5,$

$x_6, x_7, x_8, x_6, x_9, x_{10}$
$(=v)$

sequence means order matter

also called walk where vertices

or edges can repeat.

A simple path is a path where no vertex is repeated.

6 edges in a simple path $P_{x_0, x_{10}}$ between x_0 and x_{10}.

$x_0, x_1, x_2, x_5, x_6, x_9, x_{10}$

CONNECTED GRAPH: An

undirected graph is connected if there exists a path between every pair of vertices

$V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

$E = \{(x_1, x_2), (x_2, x_3), (x_1, x_3),$
$(x_4, x_5), (x_6, x_7)\}$

Exercise: Let G be a simple graph with n vertices. Then, there exists two vertices that are of same degree.

A path of length n from u to v is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence

$x_0(=u), x_1, x_2, \ldots, x_{n-1}, x_n(=v)$
$\underbrace{\quad}_{e_1} \underbrace{\quad}_{e_2}$

of vertices such that

e_k is (x_{k-1}, x_k).

Length of $P_{x_0, x_{10}}$ is 6

THEOREM: An undirected graph is connected if and only if there is a simple path between every pair of vertices.

C_1 every pair

C_2

C_3
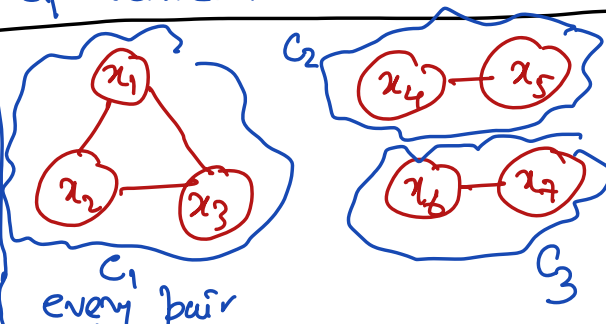
A connected component C of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph.

of vertices in C_1, there is a path

A maximal connected subgraph of a graph.

A connected graph has only one connected compon

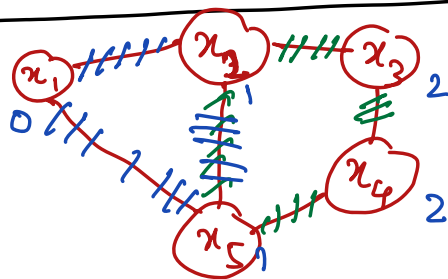Between x_1 and x_2, there is are paths $[x_1, x_2]$ Length 1

$[x_1, x_5, x_2]$ $[x_1, x_5, x_4, x_3, x_2]$

Length 2 Length 4

Shortest path between x_2 and x_4 has length 2.

Shortest path

A longest path between u and v in G is a simple path that uses $_n$ the maximum number of edges.
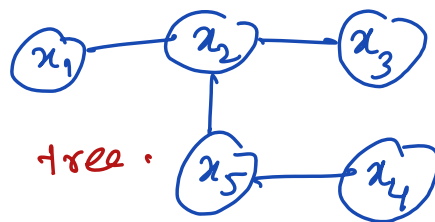

Shortest path between x_1 and x_2 has length 1.

A path between u and v is shortest in G if it is a simple path and uses smallest number of edges.

Cyle of a graph: A cyde (circuit) is a path that starts and ends with the same vertex.

A cyde is a simple cycle if it has no repeated internal vertex

$x_0, x_1, x_2, \ldots, x_{n-1}, x_n$

x_0

TREE:
A tree is an undirected connected graph with no cycle.

tree.

Lemma:
If a connected graph G has a pendant vertex x, then $G - \{x\}$ is a connected graph.

Proof: Let G be a connected graph and x be a pendant vertex of G. Suppose that y is the unique neighbor of x in G. Let $H = G - \{x\}$.
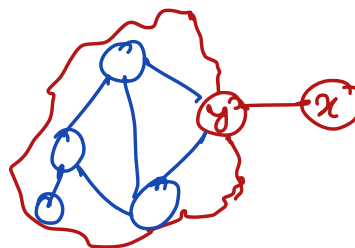
If P_1 does not contain x, then we are done.

Why? There is a path between w and s in $G - \{x\}$ itself.

Remove every two consecutive occurrences of the edge (x, y) and construct a sequence P_2.

This path P_2 is in $G - \{x\}$.

Hence, $G - \{x\}$ is connected.

Pendant vertex:

Consider any two vertices w and s in $G - \{x\}$.

As G is connected, there is a path P_1 between w and s in G.

Otherwise P_1 uses x. Then P_2 uses the edge (x, y) two consecutive times, y, x, y

This sequence P_2 is a path between w and s

$$(x_1, x_2, \ldots, x_n, x_1)$$

Theorem: A undirected graph is a tree if and only if there is a unique simple path between every pair of vertices.

Proof: ($\Leftarrow$) If there is a unique simple path between every pair of vertices, then trivially there is a path between every pair of vertices. Hence G is connected.

Suppose that G has a cycle. Choose a cycle with smallest number of edges.

Therefore G is a connected graph with no cycle. Hence
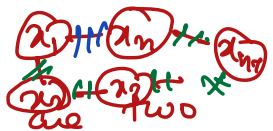
($\Rightarrow$) Let G be a tree.

Suppose for the sake of contradiction, there are two vertices x and y between which there are two distinct

The cycle starts from x_1, the penultimate vertex is x_n. It has length at least 3. $(x_1, x_2, \ldots, x_{n-1}, x_n, x_1)$

Then between x_1 to x_n, there is one simple path P_1 is $x_1, x_2, \ldots, x_{n-1}, x_n$

Another simple path P_2 is x_1, x_n.

Hence, there are two distinct paths between a pair of vertices.

This is a contradiction. Hence G has no cycle.

G is a tree.

simple paths. P_1 and P_2.

$x, w_1, w_2, \ldots, w_k, y$

$x, v_1, v_2, \ldots, v_l, y$.

Consider the sequence

$x, w_1, w_2, \ldots, w_k, y, v_l,$

$v_{l-1}, v_{l-2}, \ldots, v_1, x$

This sequence is a cycle. Contradiction to the fact that G is a tree.

Lemma : A tree with n vertices has $(n-1)$ edges.

Proof : Induction on n

Base Case : $n=1$

Trivially a tree with one vertex has no edge. Hence the statement is true.

Let x be the vertex of a tree T with $(k+1)$ vertices. Then $(T-\{x\})$ is a connected graph.

Hence, $(T-\{x\})$ is a tree with k vertices.

Due to induction hypothesis, $(T-\{x\})$ has $(k-1)$ edges.

$(x_1) \!-\! (x_2) \qquad (x_1)$

Induction Hypothesis : For every $2 \le n \le k$, a tree with n vertices has $(n-1)$ edges.

Induction Step : $n = k+1$.

A tree has a leaf that is a pendat vertex.

Subsequently, $T-\{x\}$ cannot have a cycle. Otherwise that cycle is present in T itself.

Then T has all edges of $(T-\{x\})$ and an extra edge.

Therefore, T has k edges.