10th Nov:

$x_1 + x_2 + x_3 + x_4 = 10$.

How many integral solutions are possible for this equation such that every $\boxed{x_i > 0}$ (positive integer)

There are 9 places at which you can put the bar.

3 places have to be chosen

It is equivalent to choosing a set of 3 elements from a set of 9 elements.

Choose $B \subseteq A$ s.t. $|B| = 3$

There are $\binom{9}{3}$ ways.

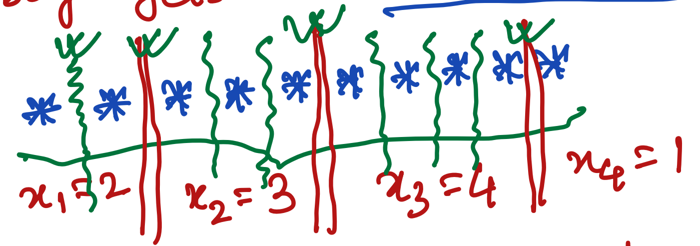
THEOREM: How many positive integral solutions are possible to the equation

$$x_1 + x_2 + x_3 + \dots + x_k = n$$

Example: How many integers are there between 1000 and 9999 (inclusive) that are divisible by 6 or 7?

$A \cap B$ = the numbers in X 42

How to distribute 10 chocolates to 4 boys such that every boy gets at least one chocolate



How many places can you put the bar?

$*$ | $*$ $*$ | $*$ $*$ $*$ $*$ $*$ | $*$ $*$

$x_1 = 1$ | $x_2 = 2$ | $x_3 = 5$ | $x_4 = 2$

9 places are distinct

A = set of 9 places (distinct)

b_1 b_2 ... b_{n-1} b_n

$*$ $*$ $*$ ... $*$ $*$

$n-1$ places to put a bar.

out of which $(k-1)$ bars can be chosen.

The number of possible solutions is $\binom{n-1}{k-1}$

Universal set $= X$

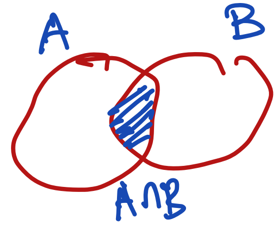
$|X| = 9000$

A = the numbers in X divisible by 6

B = the numbers in X divisible

divisible by 6 and 7.

by 7.



Compute $|A \cup B| = |A| + |B| - |A \cap B|$

For three sets A, B, and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$n = 3$

How to do it for n finite sets

$A_1, A_2, A_3, A_4, \ldots, A_n$ finite sets.

$(-1)^{3+1} |A \cap B \cap C|$

Theorem:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

k-sets intersection

$$(-1)^k |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| + \ldots\ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

PRINCIPLE OF INCLUSION - EXCLUSION

Proof: An element of $A_1 \cup \ldots \cup A_n$ is counted exactly once by LHS. (follows from definition).

Let $x \in A_1 \cup A_2 \cup \ldots \cup A_n$.

How many times is the element x counted by RHS?

How many times is x counted in $\sum_{i=1}^{n} |A_i|$? $\binom{r}{1}$ times.

x appears in exactly one set

x appears in exactly 2 sets

⋮

x appears in r of these n sets. $1 \le r \le n$

Assume that x appears in exactly r of the sets. $1 \le r \le n$

How many times is x counted in $\sum_{1 \le i < j \le n} |A_i \cap A_j|$? $\binom{r}{2}$ times.

In general, x is counted $\binom{r}{k}$ times in the summation involving k-sets

$$\boxed{\binom{r}{1}-\binom{r}{2}+\binom{r}{3}+\dots+(-1)^{r+1}\binom{r}{r}}$$

$= \binom{r}{0}$

$= ?$

$$\binom{r}{0}-\binom{r}{1}+\binom{r}{2}+\dots+(-1)^{r+1}\binom{r}{r}$$

$= 0$

x is counted exactly 1 time by the RHS.

$$\boxed{\sum_{1\le a_1<a_2<\dots<a_k\le n}\left|A_{a_1}\cap A_{a_2}\cap\dots\cap A_{a_k}\right|}$$

x is counted $\binom{r}{k}$ times.

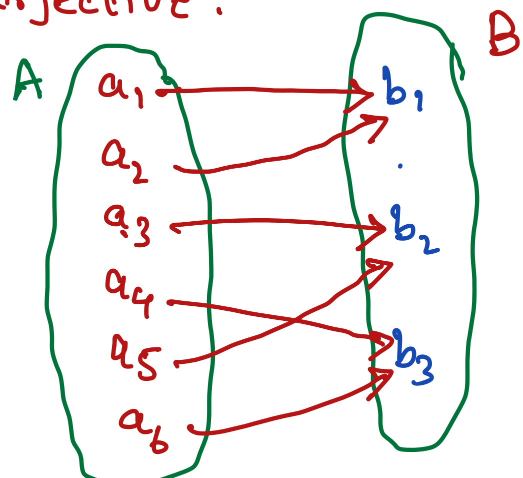
$$(1+1)^r=\sum_{k=0}^{r}\binom{r}{k}$$

$$\underbrace{(1-1)^r}_{0^r}=\boxed{\sum_{k=0}^{r}(-1)^k\binom{r}{k}}=0$$

$$=\binom{r}{0}+\sum_{k=1}^{r}(-1)^k\binom{r}{k}$$

Let A be a set of 6 elements and B be a set of 3 elements. How many surjective functions are possible from A to B?

$g: A \to B$ such that g is surjective.



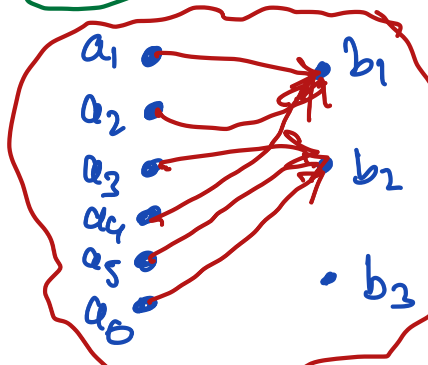
Number of all functions $= |B|^{|A|}$

$= m(A,B) = 3^6$

Count the number of functions that are not allowed. $= p(A,B)$

Objective: $m(A,B) - \boxed{p(A,B)}$

What we want to do now?



Among all functions but not allowed.

We want to count the number of fuctions that are not allowed.

P_1 = the set of functions from A to B that do not have b_1 in the range.

P_3 = the set of functions from A to B that do not have b_3 in the range.

$$P_3 = \{g: A \to B \mid b_3 \neq g(x) \text{ for any } x \in A\}$$

$\boxed{P_1 \cap P_2}$ = the set of functions that do not have b_1 and b_2 in the range.

$\boxed{P_1 \cap P_3}$ = the set of functions that do not have b_1 and b_3 in the range.

$|P_3|$ = the number of functions from A to $\{b_1, b_2\}$

$$= |\{b_1, b_2\}|^{|A|} = 2^6 = |P_3|$$

$P_1 \cup P_2 \cup P_3$ = the set of functions from A to B such that b_1 or b_2 or b_3

$$P_1 = \{g: A \to B \mid b_1 \neq g(x) \text{ for any } x \in A\}$$

P_2 = the set of functions from A to B that do not have b_2 in the range.

$$P_2 = \{g: A \to B \mid b_2 \neq g(x) \text{ for any } x \in A\}$$

$|P_1| = ?$ $|P_2| = ?$ $|P_3| = ?$

$\boxed{P_2 \cap P_3}$ = the set of functions that do not have b_2 and b_3 in the range. (every element is mapped to b_1 only)

$$|P_2 \cap P_3| = |\{b_1\}|^{|A|} = 1^6.$$

$$|P_1 \cap P_2| = 1^6 = |P_1 \cap P_3|$$

Similarly, $|P_1| = 2^6$

and $|P_3| = 2^6$.

the set of functions from A to B that are not surjective.

does not appear in the range

$|P_1 \cap P_2 \cap P_3| = ?\ 0$ because no such fnct exists.

$$|P_1 \cup P_2 \cup P_3| = |P_1| + |P_2| + |P_3| - |P_1 \cap P_2| - |P_2 \cap P_3| - |P_3 \cap P_1| + |P_1 \cap P_2 \cap P_3|$$

$$= 2^6 + 2^6 + 2^6 - (1+1+1) + 0$$

$$= 3.2^6 - 3$$

Hence. $p(A,B) = 3.2^6 - 3$

$m(A,B) = 3^6$.

Hence, answer is

$$m(A,B) - p(A,B)$$

$$= 3^6 - 3.2^6 + 3$$

If $|A| = m$ and $|B| = n$, then the number of surjective functions from A to B is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \dots + (-1)^{n-1}\binom{n}{n-1}$$

Suggestion: Avoid using complicated formula. → Mistake implies less credit

Provide counting argument from first principle. → partially correct implies better partial credit.