Probability and Statistics: Worksheet 3

February 3, 2025

1. Uniform Distribution Transformations (5 Points)

Let $U \sim \text{Uniform}[0,1]$. Find the density and distribution functions of:

(a) U^p where $p > 0$.

(b) $\frac{U}{1-U}$.

(c) $\log(\frac{1}{U})$.

(d) $\frac{2}{\pi}\sin^{-1}(U)$.

2. Transformations of Normal Distribution (4 Points)

Let $X \sim N(0,1)$. Find the density of:

(a) $aX + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

(b) X^2.

(c) X^3.

(d) e^X.

3. Memoryless Property (5 Points)

In class, we learned about the memoryless property: For any $t, s > 0$

$$P(X > s + t | X > t) = P(X > s)$$

We showed that if $X \sim \text{Exp}(\lambda)$, then X has the memoryless property. Let us now show the converse, i.e., if a non-negative random variable Y has the memoryless property, then Y must have an exponential distribution.

Hint:

1. Write $G(x) = P(X > x)$ in terms of the CDF of X.
2. Express the memoryless property in terms of $G(x)$.
3. Find $G(nt)$ for any integer n using step 2.

4. Find $G\left(\frac{t}{n}\right)$ for any integer $n > 0$.
5. Combine steps 3 and 4 to find $G\left(\frac{mt}{n}\right)$.
6. Any real number can be written as the limit of rational numbers. Use this to find $G(xt)$ for any $x \in \mathbb{R}$.
7. Plug in $t = 1$ in step 6 and find $G(x)$.

4. Coupon Problem (5 Points)

A box contains n coupons with 1 no. on each coupon.We do not know the numbers but we know that they are distinct. Coupons are drawn one after another from the box, without replacement. If the kth number drawn is larger than all the previous numbers, what is the probability that it is the largest of the n numbers?