

DSA - Tutorial 1 (Proof By Induction)

14-01-2025

Question 1: Factorial Identity

Problem: Prove by Mathematical Induction that:

$$n! = n \cdot (n - 1)!, \quad \text{for all } n \geq 1.$$

Solution

We will prove this statement by induction.

Base Case ($n = 1$)

For $n = 1$:

$$1! = 1 \quad \text{and} \quad 1 \cdot (1 - 1)! = 1 \cdot 0! = 1.$$

Since both sides are equal, the base case holds.

Inductive Hypothesis

Assume the statement is true for some $n = k$, i.e.,

$$k! = k \cdot (k - 1)!.$$

Inductive Step

We need to prove that the statement holds for $n = k + 1$, i.e.,

$$(k + 1)! = (k + 1) \cdot k!.$$

Using the inductive hypothesis, we know that:

$$(k + 1)! = (k + 1) \cdot k! = (k + 1) \cdot k \cdot (k - 1)!.$$

Thus, the statement holds for $n = k + 1$.

Conclusion

By the principle of mathematical induction, we have shown that:

$$n! = n \cdot (n - 1)! \quad \text{for all } n \geq 1.$$

Question 2: Sum of First n Odd Numbers

Problem: Prove by Mathematical Induction that:

$$T(n) : 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Solution

We will prove this statement by induction.

Base Case ($n = 1$)

For $n = 1$:

$$T(1) : 1 = 1^2.$$

So, the base case holds.

Inductive Hypothesis

Assume the statement is true for some $n = k$, i.e.,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

Inductive Step

We need to prove that the statement holds for $n = k + 1$, i.e.,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

Using the inductive hypothesis:

$$k^2 + (2k + 1) = (k + 1)^2.$$

Simplifying the right-hand side:

$$k^2 + 2k + 1 = (k + 1)^2.$$

Thus, the statement holds for $n = k + 1$.

Conclusion

By the principle of mathematical induction, we have shown that:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Question 3: Inequality $2^n > n^2$

Problem: Prove by Mathematical Induction that:

$$2^n > n^2, \quad \text{for all } n \geq 5.$$

Solution

We will prove this statement by induction.

Base Case ($n = 5$)

For $n = 5$:

$$2^5 = 32 \quad \text{and} \quad 5^2 = 25.$$

Since $32 > 25$, the base case holds.

Inductive Hypothesis

Assume the statement is true for some $n = k$, i.e.,

$$2^k > k^2.$$

Inductive Step

We need to prove that the statement holds for $n = k + 1$, i.e.,

$$2^{k+1} > (k + 1)^2.$$

We begin by using the inductive hypothesis:

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2.$$

Now, we need to prove that:

$$2 \cdot k^2 > (k + 1)^2.$$

Expanding both sides:

$$2k^2 > k^2 + 2k + 1.$$

Simplifying:

$$k^2 > 2k + 1.$$

For $k \geq 5$, this inequality holds true. Thus, the statement holds for $n = k + 1$.

Conclusion

By the principle of mathematical induction, we have shown that:

$$2^n > n^2 \quad \text{for all } n \geq 5.$$

Question 4: Sum of Squares of First n Natural Numbers

Problem: Prove by Mathematical Induction that the sum of the squares of the first n natural numbers is:

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

We will prove this statement by induction.

Base Case ($n = 1$)

For $n = 1$:

$$S(1) = 1^2 = 1 \quad \text{and} \quad \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1.$$

So, the base case holds.

Inductive Hypothesis

Assume the statement is true for some $n = k$, i.e.,

$$S(k) = 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Inductive Step

We need to prove that the statement holds for $n = k + 1$, i.e.,

$$S(k+1) = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Using the inductive hypothesis:

$$S(k+1) = \frac{k(k+1)(2k+1)}{6} + (k+1)^2.$$

Factor out $(k+1)$:

$$S(k+1) = \frac{(k+1)}{6} [k(2k+1) + 6(k+1)].$$

Simplifying the expression inside the brackets:

$$k(2k+1) + 6(k+1) = 2k^2 + k + 6k + 6 = 2k^2 + 7k + 6.$$

Thus, we have:

$$S(k+1) = \frac{(k+1)(2k^2 + 7k + 6)}{6}.$$

Factoring the quadratic expression:

$$2k^2 + 7k + 6 = (k+2)(2k+3).$$

Thus:

$$S(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Therefore, the statement holds for $n = k + 1$.

Conclusion

By the principle of mathematical induction, we have shown that:

$$S(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$