Discrete Structures-2025: Mid-Sem (Model Solution)

Total Marks: 60

September 27, 2025

(1) Prove that $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$ is a tautology. **(8 Marks)**

Solution:

We construct a truth table involving variables p, q, and r:

p	q	r	$p \vee q$	$\neg p$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	F	T

Conclusion: For all truth assignments, truth value of given formula is true as shown in last column.

Therefore, the formula $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$ is a **tautology**.

(2) For each of the statements, determine whether they are true or false. Justify your answer for each the statements.

(a) The domain for x and y is the set of all real numbers.

$\exists x\, \forall y\, ((y \neq 0) \ \to \ (xy = 1))$ **(3 Marks)**

Solution:

Given statement: There exists a real number x such that for all non-zero real number y, $xy = 1$.

Claim: The statement is **False**.

Let if possible, there exists a number $a \in \mathbb{R}$ where $a = x$ such that

$$ay = 1 \quad \forall\, y \in \mathbb{R}\backslash\{0\} \qquad ----(\star)$$

Now suppose $y = \frac{1}{2}$, then from $(\star)$, we have

$$a \cdot \tfrac{1}{2} = 1$$

$$\implies \quad a = 2. \qquad ----(\star\star)$$

Again if we take another value say $y = 5$, then from $(\star)$, we have

$$a \cdot 5 = 1 \implies \quad a = \tfrac{1}{5}.$$

This is a contradiction to $(\star\star)$.

$\implies$ There is no unique value of x which works for all y.

Hence, the given statement is **False**.

(b) The domain for x and y is the set of all real numbers.

$\exists x\, \forall y\, (x \geq y^2)$ **(3 Marks)**

Solution:

Given statement: There exists a real number x such that for all real number y, $x \geq y^2$.

Claim: The statement is **False**.

Let if possible, there exists $k \in \mathbb{R}$ such that

$$k \geq y^2 \quad \forall y \in \mathbb{R} \qquad ----(\star)$$

Clearly, $\quad y^2 \geq 0 \quad \forall y \in \mathbb{R} \quad \implies k \geq 0$

Now if we choose $y = \sqrt{k} + 1$, then from $(\star)$, we have

$$k \geq (\sqrt{k}+1)^2$$

$$\implies \quad k \geq k + 1 + 2\sqrt{k}$$

$$\implies \quad 0 \geq 1 + 2\sqrt{k} \qquad ----(\star\star)$$

Now since $k \geq 0$, we have $\sqrt{k} \geq 0$.

$$\implies 2\sqrt{k} \geq 0$$

$$\implies 1 + 2\sqrt{k} \geq 1$$

This is a contradiction to $(\star\star)$ since there is no real number which is less than 0 and greater than 1.

Hence, the given statement is **False**.

(c) The domain for x and y is the set of all integers.

$\exists x\, \forall y\, (x \leq y^2)$ **(3 Marks)**

Solution:

Given statement: There exists an integer x such that for all integers y, $x \leq y^2$.

Claim: The statement is **True**.

We know that for any integer y, $\quad y^2 \geq 0$.

So any integer x such that $x \leq 0$ will satisfy the given statement.

For example, if we take $x = 0$, then for all integers y,

$$0 \leq y^2$$

Hence, the given statement is **True**.

(d) The domain for x, y and z is the set of all integers.

$\forall x\ \exists y\ \forall z\ (x + z = y)$ **(3 Marks)**

Solution:

Given statement: For every integer x, there exists an integer y such that for all integers z,

$$x + z = y.$$

Claim: The statement is **False**.

To prove this statement is false, we will use a counterexample.

So for any pair x, y we can show that $x + z = y$ cannot be satisfied for all z.

Suppose $x = 2$ then

$$2 + z = y \quad \forall\ z \in \mathbb{Z}$$

If we take $z = 1$, then $y = 3$.

If we take $z = 2$, then $y = 4$.

So a single value of y cannot satisfy the equation $2 + z = y$ for all z.

Hence, the given statement is **False**.

(3) For every positive prime number p, prove the statement: "If n^3 is divisible by p, then n is divisible by p."

(**6 Marks**)

Solution:

We prove this statement by 'contrapositive argument'.

Let p be a prime and n is not divisible by p.

By **FUNDAMENTAL THEOREM OF ARITHMETIC**, there exist prime numbers $p_1, p_2, \ldots, p_k$ and positive integers $a_1, \ldots, a_k > 0$ such that

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

As n is not divisible by p, it follows that for every $i \in \{1, 2, \ldots, k\}$,

$$p_i \neq p$$

.

Then, observe that

$$n^3 = p_1^{3a_1} p_2^{3a_2} \cdots p_k^{3a_k}$$

Note that for every $i \in \{1, 2, \ldots, k\}$, $p \neq p_i$.

Therefore, for every $i \in \{1, \ldots, k\}$, $p_i^{3a_i}$ is not divisible by p.

Hece, n^3 is not divisible by p.

(4) Prove that $\sqrt[3]{25}$ is irrational. **(8 Marks)**

Solution:

We will prove that $\sqrt[3]{25}$ is irrational by using **proof by contradiction**.

Suppose $\sqrt[3]{25}$ is rational. Then it can be expressed as:

$$\sqrt[3]{25} = \frac{p}{q} \qquad ----(\star)$$

where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$ (i.e., p and q are co-prime).

By cubing $(\star)$, we get:

$$\left(\frac{p}{q}\right)^3 = 25 \quad \Rightarrow \frac{p^3}{q^3} = 25 \quad \Rightarrow p^3 = 25q^3 \qquad ----(\star\star)$$

This implies that p^3 is divisible by 5.

Since 5 is a prime number, therefore by using the statement of Q3

p is also divisible by 5.

So we can write, $p = 5k$ for some natural number $k \geq 1$. Then

$$p^3 = (5k)^3 = 125k^3$$

Substitute back into the eq^n $(\star\star)$:

$$125k^3 = 25q^3 \quad \Rightarrow 5k^3 = q^3$$

This implies q^3 is divisible by 5, so q is divisible by 5 as well.

We have shown that both p and q are divisible by 5, which contradicts the fact that $\gcd(p,q) = 1$.

Hence our assumption that $\sqrt[3]{25}$ is rational was wrong and therefore

$$\boxed{\sqrt[3]{25} \text{ is irrational.}}$$

(5) If A, B, D, and E are sets, then prove that

$A \times (B \cup (D \cap E)) = (A \times (B \cup D)) \cap (A \times (B \cup E))$. **(8 Marks)**

Solution:

$\boxed{METHOD\ 1}$

Let A,X and Y be any sets.

We use two standard facts:

- Distributivity: $A \cup (X \cap Y) = (A \cup X) \cap (A \cup Y)$.
- Cartesian product distributes over intersection:$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$.

Then

$$\begin{aligned} A \times \big(B \cup (D \cap E)\big) &= A \times \big((B \cup D) \cap (B \cup E)\big) \\ &= \big(A \times (B \cup D)\big) \cap \big(A \times (B \cup E)\big). \end{aligned}$$

$\boxed{METHOD\ 2}$

Two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Using this, we proceed as follows:

To show: (i) $A \times (B \cup (D \cap E)) \subseteq (A \times (B \cup D)) \cap (A \times (B \cup E))$

Let $(a, x) \in A \times (B \cup (D \cap E))$.

Then $a \in A$ and $x \in B \cup (D \cap E)$.

This means that $x \in B$ or $x \in D \cap E$.

If $x \in B$, then $(a, x) \in A \times (B \cup D)$ and $(a, x) \in A \times (B \cup E)$.

If $x \in D \cap E$, then $x \in D$ and $x \in E$.

Thus, $(a, x) \in A \times (B \cup D)$ and $(a, x) \in A \times (B \cup E)$.

In both cases, $(a, x) \in (A \times (B \cup D)) \cap (A \times (B \cup E))$.

Since (a, x) is an arbitrary element of $A \times (B \cup (D \cap E))$ this holds for all elements of $A \times (B \cup (D \cap E))$.

Therefore, $\quad A \times (B \cup (D \cap E)) \subseteq (A \times (B \cup D)) \cap (A \times (B \cup E))$ - - - - (1)

(ii) $(A \times (B \cup D)) \cap (A \times (B \cup E)) \subseteq A \times (B \cup (D \cap E))$

Let $(a, x) \in (A \times (B \cup D)) \cap (A \times (B \cup E))$.

Then $(a, x) \in A \times (B \cup D)$ and $(a, x) \in A \times (B \cup E)$.

This means that $a \in A$, $x \in B \cup D$, and $x \in B \cup E$.

If $x \in B$, then $(a, x) \in A \times (B \cup (D \cap E))$.

If $x \notin B$, then $x \in D$ and $x \in E$, which means $x \in D \cap E$.

Thus, $(a, x) \in A \times (B \cup (D \cap E))$.

Since (a, x) is an arbitrary element of $(A \times (B \cup D)) \cap (A \times (B \cup E))$ this holds for all elements of $(A \times (B \cup D)) \cap (A \times (B \cup E))$.

Therefore, $\quad (A \times (B \cup D)) \cap (A \times (B \cup E)) \subset A \times (B \cup (D \cap E))$ - - - - (2)

Hence by $eq^n\ (1)\ and\ (2)$,

$$\boxed{A \times (B \cup (D \cap E)) = (A \times (B \cup D)) \cap (A \times (B \cup E))}$$

(6) Prove that for every positive integer n, $16^{n+1} + 17^{2n-1}$ is divisible by 273. (**8 Marks**)

Solution:

We prove this statement using the principle of mathematical induction on n.

Step 1: Base Case ($n = 1$)

$$16^{1+1} + 17^{2(1)-1} = 16^2 + 17^1 = 256 + 17 = 273$$

(which is clearly divisible by 273)

Hence, the base case holds.

Step 2: Inductive Hypothesis

Assume the statement holds for $n = k$. This means that $16^{k+1} + 17^{2k-1}$ is divisible by 273.

i.e.,

$$16^{k+1} + 17^{2k-1} = 273\,t \qquad ----(\star)$$

where t is a natural number.

Step 3: Inductive Step

We want to prove that the statement holds for $n = k+1$, i.e.,

$$16^{(k+1)+1} + 17^{2(k+1)-1} = 16^{k+2} + 17^{2k+1} \;\; \textit{is divisible by } 273.$$

Now,

$$\begin{aligned}
16^{k+2} + 17^{2k+1} &= 16 \cdot 16^{k+1} + 17^2 \cdot 17^{2k-1} \\
&= 16 \cdot 16^{k+1} + 16 \cdot 17^{2k-1} + (17^2 - 16) \cdot 17^{2k-1} \\
&= 16 \cdot (16^{k+1} + 17^{2k-1}) + 273 \cdot 17^{2k-1} \\
&= 16 \cdot 273\,t + 273 \cdot 17^{2k-1} \qquad (\textit{by } eq^n\ (\star)) \\
&= 273 \cdot (16\,t + 17^{2k-1})
\end{aligned}$$

Therefore, $16^{k+2} + 17^{2k+1}$ is divisible by 273

Step 4: Conclusion

Hence by the principle of mathematical induction,

$$\boxed{16^{n+1} + 17^{2n-1} \text{ is divisible by 273 for every positive integer } n.}$$

(7) Suppose that all we have are 3-cent and 10-cent stamps. Then, prove that we can make any postage of 18-cents or more.

(10 Marks)

Solution:

We prove this using strong mathematical induction on n.

Let $P(n)$ be the proposition that postage of n cents can be formed using only 3-cent and 10-cent stamps.

Step 1: Base Cases

We verify $P(18)$, $P(19)$, and $P(20)$ are true:

- $P(18) : 6 \times 3 = 18$
- $P(19) : 3 \times 3 + 1 \times 10 = 19$
- $P(20) : 2 \times 10 = 20$

Step 2: Inductive Hypothesis

Assume that for some $k \geq 20$, the proposition $P(n)$ holds for all n such that $18 \leq n \leq k$.

Step 3: Inductive Step

We want to show that $P(k+1)$ is also true.

Since $k + 1 \geq 21$, (as $k \geq 20$), we have $k - 2 \geq 18$.

Hence we have, $18 \leq k - 2 \leq k$

Therefore, by the inductive hypothesis, $P(k-2)$ is true

i.e., we can form postage of $k - 2$ cents using 3-cent and 10-cent stamps.

Then, by adding one more 3-cent stamp to the combination for $k - 2$, we get:

$$k + 1 = (k - 2) + 3$$

So $P(k+1)$ is also true.

Step 4: Conclusion

By the principle of strong mathematical induction, $P(n)$ is true for all $n \geq 18$.

That is,

$$\boxed{\text{Any postage of 18 cents or more can be formed using only 3-cent and 10-cent stamps.}}$$