Discrete Structures-2025: Quiz-2 (Model Solution)

Method of Proofs

Total Marks: 20

September 15, 2025

(1) For each of the following statements, first express them using predicates, quantifiers, and logical connectives. Subsequently, write whether the statement is true or false.

(i) For every integer x, $x^3 \geq x$. **(2 Marks)**

Solution:

Let the predicate be $A(x): \ x^3 \geq x.$

The statement is $\forall x \ A(x)$

Statement is **False.**
Counter example: For $x = -2$, we have $(-2)^3 = -8 < -2$.

(ii) For all integers x, y, if $x > y$, then $x^2 > y^2$. **(2 Marks)**

Solution:

Let the predicates be

$$A(x,y): \ x > y \qquad \textit{and} \qquad B(x,y) : x^2 > y^2$$

The statement is

$$\forall x \, \forall y \ \left(\ A(x,y) \rightarrow B(x,y) \ \right)$$

Statement is **False.**
Counter example: $x = -1, \ y = -2$. Then $-1 > -2$ but $(-1)^2 = 1 \leq 4 = (-2)^2$.

(iii) For all natural numbers x, if there exists a natural number y such that $x = y^2$, then $x \geq y$.
(2 Marks)

Solution:

Let the predicates be

$$A(x,y):\ x=y^2 \qquad \textit{and} \qquad B(x,y): x \geq y$$

The statement is

$$\forall x\, \forall y\ (\ A(x,y) \to B(x,y)\ )$$

Statement is **True**

Proof: Assume $x = y^2$ for some $y \in \mathbb{N}$. Then

$$x - y = y^2 - y = y(y-1).$$

Since $y(y-1) \geq 0$ for all $y \in \mathbb{N}$, we have $x - y \geq 0$, i.e. $x \geq y$.

(2) **Prove that:** For every positive integer n, n^2 is even if and only if $3n+4$ is even. **(8 Marks)**

Solution:

We prove both directions separately.

($\Rightarrow$) Suppose n^2 is even.

Claim: n is even

On the contrary assume that n is odd then $n = 2t+1$ for some positive integer t

$$n^2 = (2t+1)^2 = 4t^2 + 4t + 1 = 2(2t^2 + 2t) + 1,$$

which is odd, a contradiction.

So $n = 2m$ for some positive integer m. Then

$$3n + 4 = 3(2m) + 4 = 6m + 4 = 2(3m+2),$$

which is even.

($\Leftarrow$) Suppose $3n+4$ is even.

Then $3n + 4 = 2k$ for some integer k.

$$3n = 2k - 4 = 2(k-2),$$

which is even.

Since $3n$ is even, and 3 is odd, it follows that n must be even.

So $n = 2m$ for some integer m. Then

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2),$$

which is even.

Therefore, both directions hold:

$$n^2 \text{ is even} \iff 3n+4 \text{ is even.}$$

(3) **Prove that:** For every positive integer n, $9^n + 3$ is divisible by 4. **(6 Marks)**

Solution:

(By Mathematical Induction):

Step 1. Base case ($n = 1$):

$$9^1 + 3 = 9 + 3 = 12,$$

which is divisible by 4. So the statement is true for $n = 1$.

Step 2. Inductive hypothesis:

Assume that for some $k \geq 1$,

$$9^k + 3 \text{ is divisible by } 4.$$

That means, there exists an integer m such that

$$9^k + 3 = 4m.$$

Step 3. Inductive step ($n = k + 1$):

Consider

$$9^{k+1} + 3.$$

We can write

$$9^{k+1} + 3 = 9 \cdot 9^k + 3.$$

Now split it using the hypothesis:

$$= 9(9^k + 3) - 27 + 3 = 9(9^k + 3) - 24.$$

From the hypothesis, $9^k + 3 = 4m$. Substituting:

$$9^{k+1} + 3 = 9(4m) - 24 = 36m - 24 = 4(9m - 6).$$

Thus $9^{k+1} + 3$ is divisible by 4.

Step 4. Conclusion: By the principle of mathematical induction,

$$9^n + 3 \text{ is divisible by 4 for all positive integers } n.$$