

29th October:

(1) How many ways can we select 3 students from a group of 6 students to stand in a line for a picture? ordering matters

$\{S_1, S_2, S_3, S_4, S_5, S_6\}$

$\boxed{S_4 S_2 S_5}$   $\boxed{S_1 S_6 S_4}$

changing the ordering gives different combinations  
ordering matters, different permutations

For each fixed choice of first and second student there are 4 ways to select the 3rd student.

Generalized Q1: How many ways can  $k$  students be selected from  $n$  students to stand in a line for a picture?

Similarly,  $k$ -permutation  
For every choice of first  $i$  students, the  $(i+1)$ -th student can be selected in  $(n-i)$  ways.  
 $(k-i)$

$$k! = 1, 2, 3, \dots, k$$

(2) How many different committees of 3-students can be formed from a group of 6 students? ordering does not matter

3 combination  
The first student can be selected in 6 ways.  
For each choice of selection of the first student, the second student can be selected in 5 ways.

Hence, there are total  $(6 \times 5 \times 4)$  ways to select 3 students to stand in a line for a picture.

Ans: The first student can be selected in  $n$  ways.  
For every choice of first student the second student can be selected in  $(n-1)$  ways.

Hence, the total number of ways to select  $k$  students in the line for a picture is  $n(n-1)(n-2)\dots(n-k+1)$

$$= \frac{n!}{(n-k)!} = {}^n P_k$$

$$0! = 1$$

If  $k = n$ .

$$n!$$

$$(n-k)!$$

Q2 Answer: Ordering does not matter. So,  $\frac{6 \times 5 \times 4}{3!}$  ways.

All possible subsets of exactly  $k$  elements from a set of exactly  $n$  elements

### $k$ -Combination

Given a set of  $n$  elements, how many ways can  $k$  distinct elements be selected?

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = \frac{n!}{k! (n-k)!}$$

Proof: To generate all  $k$ -permutations,

first choose a subset of  $k$ -elements.

then, enumerate all  $k$ -permutations of these chosen  $k$  elements.

$${}^n p_k = \binom{n}{k} k!$$

$$\text{ways} = \frac{6 \times 5 \times 4}{3!} = \binom{6}{3}$$

Ex3: There are 8 runners in

a race. 1st winner gets a gold medal. 2nd winner gets silver medal. 3rd winner gets bronze medal.

How many ways to award medals

when all possible outcomes occur and there are no ties.

No ties every runner finishes at distinct time stamp.

3-permutations from 8 runners.

$$(8 \times 7 \times 6) = 336$$

The number of subsets of  $k$  elements from a set of  $n$  elements is  $\binom{n}{k}$

Combinatorial Identity:  $\binom{n}{k} \geq 0$

Let  $n$  and  $k$  be nonnegative integers and  $k \leq n$ . Then

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{using combinatorial arguments}$$

$$\frac{n!}{k!(n-k)!}$$

COMBINATORIAL PROOF

Prove combinatorially:

If you prove by algebraic manipulation, then you will get zero marks.

Proof: Let  $S$  be a set of  $n$  elements.

$\mathcal{B}$  = all subsets of  $S$  with exactly  $k$  elements.

$\mathcal{A}$  = all subsets of  $S$  with exactly  $(n-k)$  elements.

Proof of injective: Let  $A_1 \neq A_2$

Then  $g(A_1) = S - A_1$  and

$g(A_2) = S - A_2$ .

There exists  $x \in A_1$  such that  $x \notin A_2$ .

Then the same element  $x \in S - A_2$  but  $x \notin S - A_1$

Hence  $S - A_1 \neq S - A_2$

implying that  $g(A_1) \neq g(A_2)$ .

Hence  $g$  is injective.

As  $g$  is both injective and

$$|\mathcal{B}| = \binom{n}{k}$$

$$|\mathcal{A}| = \binom{n}{n-k}$$

Define  $g: \mathcal{B} \rightarrow \mathcal{A}$  as

$$g(A) = S - A$$

$g$  maps  $A$  to its complement.

Proof of surjective:

consider any  $B \in \mathcal{A}$

$|B| = n-k$  and  $B \subseteq S$

$$\text{Then } g(S - B) = B$$

$$\text{and } |S - B| = n - (n - k) = k.$$

Therefore,  $g$  is surjective.

**BIJECTIVE PROOF**

surjective, therefore  $g$  is bijective. Hence,  $|B| = |\mathcal{D}|$

Implies that  $\binom{n}{k} = \binom{n}{n-k}$

one of the proof methods using combinatorial arguments.

ONE TYPE OF COMBINATORIAL PROOF.

PASCAL'S IDENTITY:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof:  $T$  be a set of  $n+1$  elements.

Consider any subset  $B \subseteq T$  with exactly  $k$  elements.

Choose any  $x \in T$ .

$B$  can be constructed in two different ways.

Second method: Choose a set  $B$  of  $k$  elements from  $T - \{x\}$ .

Since  $|T - \{x\}| = n$ , then the second method can be completed in  $\binom{n}{k}$  ways.

Crucial: the second method enforces that  $x \notin B$

Two ways of counting the same set of objects  
DOUBLE COUNTING.

clearly  $|T| = \binom{n+1}{k}$

First method: Choose  $x$  into  $B$ , and choose  $(k-1)$  elements from  $T - \{x\}$ .

$$|T - \{x\}| = n. \quad x \in B$$

Hence, there are  $\binom{n}{k-1}$  ways to complete the first method.

Since, the first method is completely disjoint from the second method.

(no set  $B$  of  $k$  elements chosen by first method

Additionally, first method and second method are mutually exhaustive cases.

{ is a set of  $k$  elements chosen by second method.)  
Hence, there are  $\binom{n}{k} + \binom{n}{k-1}$  ways to choose a set of  $k$  elements from  $T$ .

$$\text{Therefore. } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

DOUBLE COUNTING ARGUMENT