Discrete Structures-2025: Tutorial-9

Combinatorics-I: Basic Counting Techniques, Pigeon Hole Principle, Permutation, Combinations, and Combinatorial Proofs

(1) Show that if there are 30 students in a class, then at least 2 of them have their last name starting with the same letter.

(2) Let n be a positive integer. Prove that: among any set of n consecutive integers, exactly one is divisible by n.

(3) Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. Prove that for any set of 8 numbers from A, there are two numbers whose sum adds up to 15.

(4) Let $n_1, n_2, \ldots, n_t$ be positive integers. Prove that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some $i \in \{1, 2, \ldots, t\}$, the i-th box contains at least n_i objects.

(5) Prove the following identities using combinatorial arguments.

(a) If n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(b) If n is a positive integer, then $\binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+2}{n+1}/2$.

(6) Prove that: if any 5 points are chosen within a square of side length 2 units, then there are two points such that the distance between them is at most $\sqrt{2}$ units.

(7) How many bit strings (each bit is 0 or 1) of length 12 have

(a) at most three 1s.

(b) at least four 1s.

(c) exactly five 1s.