17th Nov: $G=(V,E)$

Theorem: An undirected graph has an even number of vertices of odd degree. $G=(V,E)$

Proof:

$$\begin{cases} V = V_1 \cup V_2 \\ V_1 = \{v \in V \mid \deg(v) \text{ is odd}\} \\ V_2 = \{v \in V \mid \deg(v) \text{ is even}\} \end{cases}$$

Clearly $V_1 \cap V_2 = \phi$.

Consider $\sum_{x\in V} \deg(x)$

$$= \sum_{x\in V_1} \deg(x) + \sum_{y\in V_2} \deg(y)$$

What is crucial is that $\sum_{y\in V_2} \deg(y)$ is even

because for every $y \in V_2$ $\deg(y)$ is even

For every $x \in V_1$ $\deg(x)$ is odd.

Since $\sum_{x\in V_1} \deg(x)$ is even

therefore $|V_1|$ must be even.

Hand Shaking Theorem:

Let G be a graph with n vertices and m edges.

Then $\sum_{u\in V} \deg(u) = 2m$

$|V| = n$
$|E| = m$

Due to handshaking theorem

$$\sum_{x\in V} \deg(x) = 2m$$

Then, $\left[\sum_{x\in V_1} \deg(x) + \sum_{y\in V_2} \deg(y)\right]$ is even.

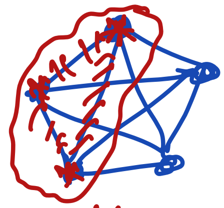
Hence $\sum_{y\in V_2} \deg(y)$ is even.

Then $\sum_{x\in V_1} \deg(x) = 2m - \sum_{y\in V_2} \deg(y)$

Hence $\sum_{x\in V_1} \deg(x)$ is even.

Hence, the statement is proved.

Complete graph: A complete graph on n vertices, denoted by K_n, is a simple undirected graph that contains exactly one edge between each pair of vertices.



K_5

K_n has $\binom{n}{2}$ edges exactly.

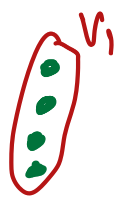
Clique

All cliques $\doteq$ a graph class

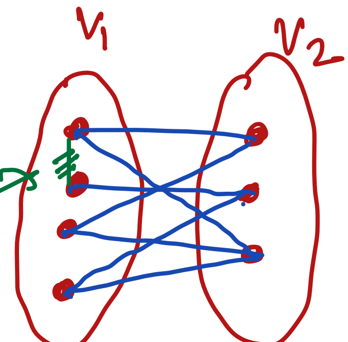
Exercise: If a graph G is a complete graph, then prove that every induced subgraph of G is a complete graph.

Exercise: Prove or disprove: If G is a complete graph, then every subgraph of G is a complete graph

Bipartite Graph: A simple graph G is bipartite if its vertex set V can be partitioned into V_1 and V_2 such that every edge (x, y) in the graph connects a vertex of V_1 and a vertex of V_2.



edge must not be two.



V_1 = red V_2 = blue

Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned

Proof: Let G be a simple graph. $G = (V, E)$

($\Rightarrow$) Assume that G is bipartite.

Then there exists A and B

the same color.

We assign red to the vertices of A and blue to the vertices of B.

Consider any edge $(x, y) \in E$

Since G is bipartite $x \in A$ if and only if $y \in B$.

Hence. x and y are assigned diffent colors.

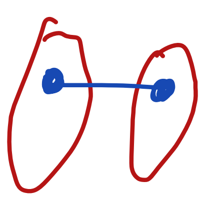
($\Leftarrow$) Suppose that there exists a coloring $f: V \rightarrow \{1, 2\}$ such that for every edge $(x, y) \in E$

$f(x) \neq f(y)$.

Additionally, $x \in V_2$ implies that $y \in V_1$.

Hence, every edge connects a vertex in V_1 with a vertex in V_2.

Therefore. G is bipartite.



such that $A \cap B = \emptyset$ and $A \cup B = V$ and every edge in E connects a vertex in A to a vertex in B.

Then, x is assigned red if and only if y is assigned blue.

Then, consider V_1 and V_2

$$V_1 = \{x \in V \mid f(x) = 1\}$$

$$V_2 = \{x \in V \mid f(x) = 2\}$$

Since for every $(u, v) \in E$

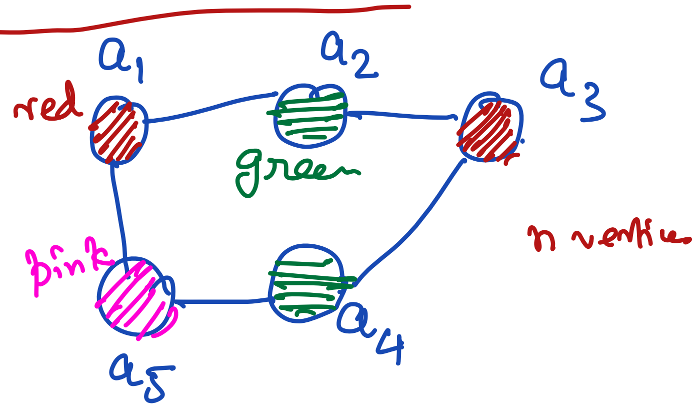
$f(x) \neq f(y)$, it means that $x \in V_1$ implies that $y \in V_2$.

Therefore, (V_1, V_2) is a bipartition that satisfies the conditions for a graph to be bipartite.

Graph Coloring: Let $G=(V,E)$ be an undirected graph. If there exists $f: V \to \{1, 2, \ldots, k\}$ such that for every edge $(x,y) \in E$, $f(x) \neq f(y)$, then f is a proper coloring of G with k colors.

This graph has a proper coloring with 3 colors.

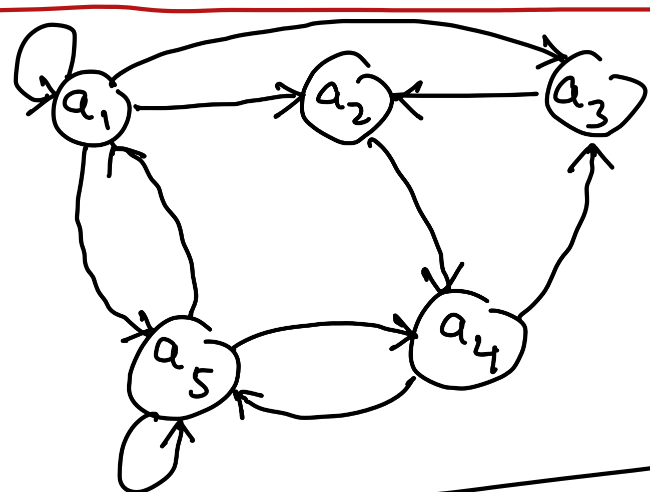
But has no proper coloring with 2 colors.



As a consequence of the previous theorem, a bipartite graph admits a proper coloring with 2 colors.

$deg^-(u)$ = the number of edges with u as their terminal vertices
in-degree

$deg^+(u)$ = the number of edges with u as their initial vertex.
out-degree



Theorem:

$$\sum_{x \in V} deg^-(x) = \sum_{x \in V} deg^+(x)$$

Edge (x,y) has an initial vertex x and a terminal vertex y.

Proof: Consider every edge (x,y) $\quad x \longrightarrow y$

Hence edge (x,y) is counted once for $deg^+(x)$

Therefore every edge is counted exactly once for in-degree and exactly once for out-degree.

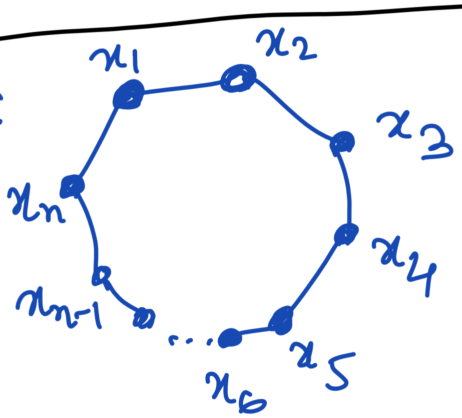
and counted once for $\deg^-(y)$.

Hence $\text{LHS} = \sum_{x \in V} \deg^+(x) = |E| = m.$

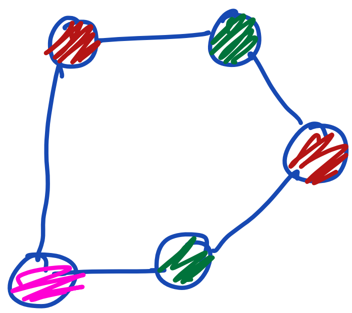
and $\text{RHS} = \sum_{x \in V} \deg^-(x) = |E| = m.$

Therefore, LHS = RHS.

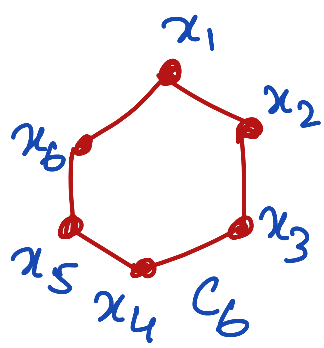
<u>Cycle graph</u>:



A cycle C_n with $n \geq 3$ contains n vertices $x_1, x_2, \ldots, x_n$ and edges $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)$



C_5



Is C_5 bipartite? No

Is C_6 bipartite