MTH100 LINEAR ALGEBRA

Worksheet 1

Question 1. Reduce the following matrix to an RREF matrix using elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 1 \end{bmatrix}.$$

Question 2. Reduce the following matrix to an RREF matrix using elementary row operations.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

Question 3. Explicitly describe all non-zero 2×2 RREF matrices. You may also try to do this for 2×3 and 3×3 RREF matrices.

Question 4. Define a relation T on the real number system $\mathbf{R}$ by xTy if $y - x \in \mathbf{Z}$, the set of integers. Is T an equivalence relation? Justify your answer. If yes, can you find a special representative in each equivalence class, just as we could do for row equivalence of matrices?

Question 5. Prove that row-reduction is an equivalence relation on the set $\mathbf{R}^{m \times n}$ of all m by n matrices with real entries.

Question 6. Show that if E is an equivalence relation on a set X, then any two distinct equivalence classes must be disjoint.Also show that every element of X has to belong to an equivalence class.

The equivalence class of any element $a \in X$ is the set of all elements of X which are related to a, the formal definition is:

$[a] = \{x \in X : xEa, i.e.x \text{ is related to a under the ralation } E\}$

Question 7. Show that if P is a partition of a set X, then there exists an equivalence relation E on X such that the equivalence classes correspond to the parts of the given partition P.($Q.7$ *is the converse of* $Q.6$)

Question 8. Find the solution set in the vector form for the homogeneous system $Ax = 0$ given A below. NB: A must be row-reduced to an RREF matrix in order to give the solution in standard form.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

MTH 100 : Worksheet 2

1. (a) Row reduce the augmented matrix of the system given below to an RREF matrix using elementary row operations.

$$\begin{aligned} 3x + 2y + 7z + 9w &= 7 \\ 6x + 14y + 22z + 15w &= 13 \\ x + 4y + 5z + 2w &= 2 \end{aligned}$$

   (b) Is the system consistent or inconsistent? If consistent, express the solution in the form of a vector $\mathbf{u}$ which is a solution of the non-homogeneous system plus scalar multiples of vector(s) which are solutions of the associated homogeneous syatem.

2. (a) Row reduce the augmented matrix of the system given below to an RREF matrix using elementary row operations.

$$\begin{aligned} x + 5y - 3z &= -4 \\ -x - 4y + z &= 3 \\ -2x - 7y &= a \end{aligned}$$

   (b) For what values of a is the above system consistent and for what values of a is it inconsistent? Justify your answer.

3. For what values of a and b, the system

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + 4z &= 9 \\ x - y + az &= b \end{aligned}$$

   has
   (a) Unuque solution
   (b) Infinitely many solutions
   (c) No solution

4. Is it possible for a non-homogeneous system $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, to be inconsistent when the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution (i.e. only the trivial solution)? Answer YES or NO, and justify your answer. If YES, construct an example and verify. If NO, explain with reference to suitable propositions and theorems.

5. (a) Find the values of x for which the following matrix is an augmented matrix corresponding to a consistent system.

$$A = \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

(b) Find the RREF of the matrix formed by replacing x in A by π.

6. Prove that : If the matrix B has been obtained from the matrix A by an elementary row operation, then the vector $\mathbf{v}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{v}$ is a solution of the homogeneous system $B\mathbf{x} = \mathbf{0}$
(Note: Think about a similar version for a non-homogeneous system and prove it.)

Worksheet 3

1. Determine the inverse of the given matrix A using row reduction.
$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

2. TRUE OR FALSE?

   (a) The sum of two invertible matrices(square matrices of same order) is always invertible.

   (b) If matrices A and B commute ,i.e. $AB = BA$, then invertibility of A implies invertibility of B.

   Justify your answer. Prove if TRUE or give counter-example if FALSE.

3. Suppose $AB = AC$,where B and C are $n \times k$ matrices and A is invertible. Show that $B = C$. Is this true in general when A is not invertible? Justify your answer (prove if true, give counter-example if false).

4. (a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

   (b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

   (c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

5. For each of the following clearly state True or False (prove if true, counter example if false)

   - For any square matrix A, if A^k is invertible for some positive integer $k > 1$, then A itself is invertible.
   - If a 3×3 square matrix A satisfies $A^3 = 0$, then $A = 0$.(Here 0 indicates the zero matrix.)

6. Check whether A is invertible and find A^{-1} if it exists. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

7. Suppose A is 2×1 matrix and B is 1×2 matrix.Prove that $C = AB$ is not invertible.

8. Prove the following generalization of previous problem.

   If A is $m \times n$ matrix and B is $n \times m$ matrix and $n < m$ then prove that AB is not invertible.

9. Let A be an $n \times n$ (square) matrix.Prove the following statements:

   - If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.
   - If A is not invertible then there exists an $n \times n$ matrix B such that $AB = 0$, but $B \neq 0$

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Prove using elementary row operations that A is invertible iff $ad - bc \neq 0$

11. Prove that an upper triangular(square) matrix is invertible iff every entry on the main diagonal is different from zero.(An $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ is called upper triangular if $a_{ij} = 0$ for $i > j$,i.e. every entry below the main diagonal is zero.)

12. Given an $m \times n$ matrix A and $n \times k$ matrix B, the product $AB = \begin{bmatrix} Av_1 & Av_2 & .... & Av_k \end{bmatrix}$ in column form where $B = \begin{bmatrix} v_1 & v_2 & .... & v_k \end{bmatrix}$ is in column form. Construct an example to illustrate this rule. The matrix A in your example should be atleast 3×3 and B should be atleast 3×2.

13. Prove the following proposition in general case,i.e. for any row operation e and any matrix A.

    Proposition If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A, $e(A) = EA$.

    (NB: the three cases of scaling, replacement and interchange require separate proofs.)

Worksheet 4

1. (a) Obtain a LU decomposition of the matrix A given below.

   (b) Solve the non-homogeneous system $Ax = b$ where b is given below (using the LU decomposition obtained in previous part)

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

2. Do the same problem with $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3. Compute L and U for the symmetric matrix A

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

   Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

4. Find L and U for: $A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$

   Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

5. Solve $Lc = b$ to find c. Then solve $Ux = c$ to find x.

   What was A? ($A = LU$)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

6. Factor the following tridiagonal matrices $A = LU$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}$$

7. Find L and U for $T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

8. (a) Find an LU factorization of the following matrix $A = \begin{bmatrix} 7 & -1 & 0 \\ 14 & 0 & 1 \\ 7 & -3 & 3 \end{bmatrix}$

   (b) Using the LU-factorization method solve the linear system $Ax = b$ where A is the matrix given in the previous part and b is the vector given below $b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

9. Obtain a LU decomposition of the matrix A given below.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

MTH 100 : Worksheet 5

1. Let U and W be two subspaces of the vector space V. Show that $U \cap V$ is also a subspace of V.

2. In the following is W a subspace of V? (Base field is $\mathbf{R}$ in all.) Justify your answer.

   (a) $V = \mathbf{R_n}[t]$ = vector space of all polynomials of degree $\leq n$,$W = \{p(t) \in V$:deg p(t)$= n\} \cup \{\mathbf{0(t)}\}$.Here $\mathbf{0(t)}$ indicates the zero polynomial.

   (b) $V = \mathbf{R^3}$,$W = \{(x, y, z) : x, y, z \in \mathbf{Q}\}$

   (c) $V = \mathbf{R^3}$,$W = \{(x, y, z) : xy = 0\}$

   (d) $V = \mathbf{R^3}$,$W = \{(x, y, z) : x^2 + y^4 + z^6 = 0\}$

3. Consider the space V of all 2×2 matrices over $\mathbf{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.

   (a) All symmetric matrices (Definition: For any $m \times n$ matrix $A = [a_{ij}]$, its transpose is the $n \times m$ matrix $B = [b_{ij}]$,given by $b_{ij} = a_{ji}$. The standard notion for the transpose of A is A^T. A matrix is symmetric if $A = A^T$)
   (b) All A such that $AB = BA$ where B is some fixed matrix in V.
   (c) All A such that $BA = 0$ where B is some fixed matrix in V.
   (d) Would the above results hold for all $n \times n$ matrices where n is a general positive integer?

4. Consider the space V of all $n \times n$ matrices over $\mathbf{R}$ and let W be the subset consisting of all upper triangular matrices.

   (a) Show that W is a subspace of V.

   (b) Show further that W satisfies closure with regard to products and multiplicative inverses,i.e. if $A, B \in W$,then $AB \in W$, and if $A \in W$ happens to be invertible, then $A^{-1} \in W$.

5. Let V be a vector space. Prove the following:

   (a) The additive inverse vector of any vector $\mathbf{u}$ is unique; we use the notation **-u** for the inverse vector.

   (b) $0\mathbf{u} = \mathbf{0}$ for every vector u.

   (c) $c\mathbf{0} = \mathbf{0}$ for every scalar c.

(d) Cancellation Law,i.e. show that if **u**+**v** = **u**+**w**, for **u**,**v**,**w** $\in V$, then **v**=**w**.

6. Give an example of a set X and an operation involving elements of X, which does not satisfy the cancellation law. Briefly justify your answer.

7. Show that the set $\mathbf{Q}[\sqrt{\mathbf{2}}] = \{a + b\sqrt{2}, a, b \in \mathbf{Q}\}$ is a field.

   Remark: Note that $\mathbf{Q}[\sqrt{\mathbf{2}}]$ is a subset of $\mathbf{R}$; the wording for this situation is :$\mathbf{Q}[\sqrt{\mathbf{2}}]$ is a subfield of $\mathbf{R}$. (Hint: The key step is to show that nonzero elements of $\mathbf{Q}[\sqrt{\mathbf{2}}]$ have multiplicative inverses in $\mathbf{Q}[\sqrt{\mathbf{2}}]$.)

8. (a) Is $\mathbf{R}$ a vector space over $\mathbf{Q}$? Justify your answer in brief.

   (b) Is $\mathbf{C}$ a vector space over $\mathbf{R}$? Justify your answer in brief.

   (c) Can you generalize the answers to 1) and 2) above to a statement about fields and vector spaces? Explain briefly.

9. Modular arithmetic and fields: Let n be a fixed but arbitrary positive integer, $n \geq 2$. Put $Z_n = \{0, 1, 2, ..., n-1\}$. Define the operations of modular addition and modular multiplication on Z_n by $x \bigoplus y = (x + y)$ (mod n) and $x \bigotimes y = xy$ (mod n)

   NB: Recall that z (mod n)= remainder after the division of z by n for all z $\in \mathbf{Z}$. Note that we have $0 \leq$ remainder $< n$,i.e., z(mod n)$\in Z_n$ for all z $\in \mathbf{Z}$.

   (a) Show that if $x \in Z_n$, then x has an inverse in Z_n with regard to the operation $\bigoplus$(i.e. additive inverse.)

   (b) ***We have already shown in class that $Z_2$is a field.*** Now show that Z_3 and Z_5 are fields.***(Hint: you may assume that $\bigoplus$ and $\bigotimes$ satisfy closure, associativity, commutativity and distributivity on Z_n. This is straightforward but a little lengthy.Also see the hint of question 8)***

   (c) Are Z_4 and Z_6 fields? Justify your answer briefly.

   (d) Can you generalize the above to state a condition for Z_n not to be a field? Briefly justify your statement.

10. Consider the system $R^{3\times 3}$ of 3×3 (square) matrices with real entries. A non-zero matrix A is said to be a zero-divisor if there exists some non-zero matrix B such that $AB = 0$, the zero matrix.

    (a) If A is invertible, then it cannot be a zero divisor.TRUE OR FALSE? Justify your answer.

    (b) If A is not invertible, then it must be a zero divisor.TRUE OR FALSE? Justify your answer.

11. (a) Obtain an LU decomposition of the matrix A given below.

(b) Solve the non-homogeneous system $Ax = b$, for b_1 and b_2 given below, using the LU decomposition obtained in first part. Take b_1 and b_2 as column vectors. Explain the difference in the answers for these two vectors b_1 and b_2.

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix}$ $b_1 = (1, 4, 5)$ $b_2 = (3, 7, 15)$

12. (a) Obtain an LU decomposition of the matrix A given below.

    (b) Solve the non-homogeneous system $Ax = b$, for b given below, using the LU decomposition obtained in first part. Take b as column vector.

    $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix}$ $b = (4, 9, 14)$

13. Is $\mathbf{R^2}$ a subspace of $\mathbf{R^3}$?(YES/NO) Justify your answer briefly.

14. Let $V = \{x \in \mathbf{R} : x > 0\}$. Define the addition for V by $x \bigoplus y = xy$ and scalar multiplication by any $\alpha \in \mathbf{R}$ by $\alpha * x = x^\alpha$

    (a) Verify the closure axioms, the commutative, zero and inverse properties for addition and the property $1 * x = x$ for all $x \in V$

    (*Remark: V is in fact a vector space over the field* $\mathbf{R}$.However, you need not verify the other properties of a vector space.)

    (b) Is V a subspace of $\mathbf{R}$ regarded as a vector space over itself?(YES/NO) Justify your answer clearly.

    (*This question was given as an exam problem for a previous batch.*)

MTH 100 : Worksheet 6

1. Let $F = \mathbf{Z_2}$ and consider the vector space $V = F^4$, the space of all ordered 4-tuples with entries from F.

   (a) Suppose $v \in V$, $v \neq 0$. What can you say about the additive inverse of v?

   (b) Consider the vectors $v_1 = (1,0,1,0), V_2 = (1,1,0,0)$ and $v_3 = (0,0,1,1)$. Determine Span $\{v_1, v_2\}$ and Span$\{v_1, v_2, v_3\}$.

   (c) Construct subspaces U and W of V which have 3 and 5 vectors, respectively.

   (d) Apply what you have learned from (a), (b) and (c) to state and prove a result about the possible orders of subspaces of V. **Note:** For any finite set X, the **order** of X is the number of elements in X, notation $|X|$.

   (e) Generalize your result in (d) to subspaces of F^n for any arbitrary positive integer n.

2. Prove the following : If $S = \{v_1, v_2, ...v_p\}$ is the set of vectors in a vector space V, then Span S=Span$\{v_1, v_2, ...v_p\}$ is the smallest subspace which contains S, i.e.if W is a subspace such that $S \subseteq W$, then $S \subseteq$ Span S$\subseteq W$.

3. Let U and W be the subspaces of the vector space V. Then the sum of U and W is defined as $U + W = \{u + w : u \in U, w \in W\}$. Show that $U + W$ is a subspace of V. Show further that $U + W$ is the smallest subspace of V containing both U and W.

4. Let U and W be subspaces of the vector space V. Show by means of a suitable counterexample that $U \cup W$(set theoretic union) need not be a subspace of V. Then prove that $U \cup W$ is a subspace if and only if either $U \subset W$ or $W \subset U$.**(NB: This results holds whether V is finite dimensional or infinite dimensional. Hence, you can't use any propositions related to basis or dimension in the proof.)**

5. Let W be a real vector space. Let V be a non-empty set and let $f : V \to W$ be a bijection (i.e. an injective and surjective function). For $u, v \in V$, and $c \in \mathbf{R}$, define $u \oplus v = f^{-1}(f(u) + f(v))$ and $c * v = f^{-1}(cf(v)$. Show that V is a real vector space under the operations $\oplus$ and $*$. Why is it necessary for f to be a bijection?

6. Given the following vectors in $\mathbf{R^3}$: $\mathbf{u} = (1, 3, 5)$, $\mathbf{v} = (1, 4, 6)$, $\mathbf{w} = (2, -1, 3)$ and $\mathbf{b} = (6, 5, 17)$

   (a) Does $b \in W = \text{Span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$?

   (b) If the answer to first part is yes, express $\mathbf{b}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$

7. Prove Remark related to linear dependence/independence): Any list which contains a linearly dependent list is linearly dependent.

8. Prove Remark related to linear dependence/independence: Any subset of linearly independent set is linearly independent.

9. Determine whether the given matrices in the vector space $\mathbf{R^{2\times 2}}$ are linearly dependent or linearly independent. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

10. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e.functions) are linearly dependent or linearly independent:

    $f_1(x) = 1,\ f_2(x) = \sin(x),\ f_3(x) = \sin(2x)$

    (You must justify your answer)

MTH 100 : Worksheet 7

1. Show that $B = \{v_1, v_2, ...v_p\}$ is a basis of the vector space V if and only if every vector $v \in V$ is uniquely expressible as a linear combination of the elements of B.

2. Find a basis for the vector space V of all 2×2 matrices over $\mathbf{R}$. Generalize the idea to find a basis for the vector space of all $m \times n$ matrices over $\mathbf{R}$.

3. Given the standard basis $\mathbf{B} = \{e_1, e_2, e_3\}$ of $\mathbf{R^3}$ and the linearly independent vectors $v_1 = (0, 1, 1)$ and $v_2 = (1, 1, 1)$, apply the method of Steinitz Exchange Lemma to exchange two of the vectors in B and obtain a basis C which includes v_1 and v_2. Show your calculations in detail.

4. Expand the linearly independent set $S = \{u, w\}$ to a basis of $\mathbf{R^3}$, using the propositions proved in the class. Here $u = (3, 3, 7)$ and $w = (10, 9, 21)$. Justify your answer.

5. Show that any $m \times m$ matrix A with real entries is invertible iff its columns form a basis for $\mathbf{R^m}$

6. Given any $m \times m$ (square) matrix A and any polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n (a_n \neq 0)$ of degree n, we say that A **satisfies the polynomial p(t)** if $p(A) = a_0 I_m + a_1 A + a_2 A^2 + \cdots + a_n A^n = \mathbf{0}$,i.e.the zero matrix. Show that any non-zero $m \times m$ matrix A must satisfy atleast one (non-zero) polynomial of degree $\leq m^2$.

7. Consider the space $\mathbf{C}$ of complex numbers as a vector space over the field $\mathbf{R}$ of real numbers.

   (a) Is $\mathbf{C}$ finite dimensional (YES/NO)? If YES, determine the dimension of $\mathbf{C}$

   (b) Prove or disprove: There exists a field F lying strictly between $\mathbf{R}$ and $\mathbf{C}$,i.e there is a field F such that $\mathbf{R} \subseteq F$ but $\mathbf{R} \neq F$, and $F \subseteq \mathbf{C}$ but $F \neq \mathbf{C}$.

8. (a) Show that if the vectors v_1, v_2,..., v_n are linearly independent, then so are the vectors $v_1 - v_2$, $v_2 - v_3$,..., $v_{n-1} - v_n$, v_n, which have been obtained by subtracting from each vector the following vector (except the last one).

   (b) Let V be a vector space over a field F. Suppose $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in V, and $w \in V$. Show that if $v_1 + w, v_2 + w, .., v_n + w$ are linearly dependent in V,then $w \in$ Span(S).

9. Consider the vector space $V = C[a, b]$, the space of all continuous real valued functions with domain closed interval $[a, b]$. Is V finite dimensional (YES/NO)? Justify your answer. If YES, construct a basis for V and determine its dimension.

10. Let $V = \mathbf{R}^\infty$, $W = \{< a_n >$: only finitely many of the terms in $< a_n >$ are non-zero$\}$(NB: *in future, we will use notation* p_∞ for the subspace W defined here)

    (a) Show that W is a subspace of V

    (b) Is W finite dimensional? Justify your answer.

    (c) Is V finite dimensional? Justify your answer.

    (d) Consider c, the vector space of all convergent sequences in $\mathbf{R}^\infty$, is c finite dimensional? Justify your answer.

11. Let $F = Z_2$ and consider the vector space $V = F^n$, the space of all ordered n-tuples with entries from F, where n is an arbitrary but fixed positive integer, Recall that for any finite set X, the order of X is the number of elements in X,notation $|X|$. Let W be any non-zero subspace of V. Determine the possible values of $|W|$. Justify your answer.

12. Expand the linearly independent set $S = \{u, w\}$ to a basis of $\mathbf{R^3}$ using the approach of propositions proved in class. Here $u = (1, 2, 3)$ and $w = (2, 4, 5)$.

    Justify your answer.

MTH 100 : Worksheet 8

1. V is a vector space with dim(V)=n. W_1 and W_2 are subspaces of V such that dim(W_1)=dim(W_2)=n-1 and $W_1 \cap W_2 = \{0\}$. Find n.

2. Given the vector space $\mathbf{R^3}$, let W_1 be the set of vectors of the form $(x, y, 0)$ and let W_2 be the set of vectors of the form $(0, a, b)$

   (a) Show that W_1 and W_2 are subspaces of $\mathbf{R^3}$.

   (b) Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

   (c) Find two distinct subspaces U_1 and U_2 of $\mathbf{R^3}$ such that $\mathbf{R^3} = W_1 \bigoplus U_1 = W_1 \bigoplus U_2$ i.e. find two distinct complements of W_1. Justify your answer.

3. Given the matrix A below:

   (a) Find a basis for each of the spaces Nul A, Col A and Row A.

   (b) Find a basis for Row A, consisting of rows of the given matrix A, different from the one in previous part.

   (c) Is A invertible? Justify. $A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$

4. Given the matrices A and B below.

   (a) Find a basis for the row space of A a basis for the row space of B, showing your calculations.

   (b) Let $U =$ Span $\{(1, 2, -1, 3), (2, 4, -1, 2), (3, 6, 3, -7)\}$ and Let $W =$ Span $\{(1, 2, -4, 11), (2, 4, -5, 14)\}$ Is $U = W$? Justify your answer. $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$

5. Given any $m \times n$ matrix A, show that rank A$\leq \min\{m, n\}$.

   Given a non-trivial example in which equality is achieved and a non-trivial example in which strict inequality holds.

MTH 100 : Worksheet 9

1. Given any two $m \times n$ matrices A and B, prove that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$. Give a non-trivial example in which equality is achieved and a non-trivial example in which strict inequality holds.

2. Determine whether the following are linear transformations(Yes or No) Justify your answers.

   (a) $T : \mathbf{R^3} \to \mathbf{R^2}$ given by $T(x, y, z) = (x + y, x - z)$

   (b) $T : \mathbf{R^3} \to \mathbf{R^2}$ given by $T(x, y, z) = (x + y, z^2)$

   (c) $U : \mathbf{R^{n \times n}} \to \mathbf{R^{n \times n}}$ given by $U(A) = A^T$ where A^T indicates the transpose of the matrix A.

   (d) $M : \mathbf{R}[t] \to \mathbf{R}[t]$ given by $M(p(t)) = tp(t)$ for all polynomials $p(t) \in \mathbf{R}[t]$.

3. Determine all linear transformations $T : \mathbf{R^1} \to \mathbf{R^1}$

   (N.B.:$\mathbf{R^1}$ is the vector space consisting of all 1-tuples with real entries; it is essentially the same as $\mathbf{R}$, however regarded as only a vector space rather than a field.)

4. Consider the space $V = C[\mathbf{R}]$ and consider the mapping $D_\epsilon : V \to V$ given by $D_\epsilon(f) = f_\epsilon$ where $f_\epsilon(x) = f(x + \epsilon)$ for all x.

   Here ϵ is an arbitrary but fixed real number. Is D_ϵ a linear transformation? Justify your answer.

5. Prove that there does not exist a linear transformation $T : \mathbf{R^5} \to \mathbf{R^2}$ such that Ker T= $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R^5}, x_1 = 3x_2, x_3 = x_4 = x_5\}$

6. Consider the field $\mathbf{C}$ of complex numbers as a vector space over the field $\mathbf{R}$. Show that the function

   $\phi : \mathbf{C} \to \mathbf{C}$ given by $\phi(z) = \bar{z}$ is a linear transformation. Here $\bar{z}$ indicates the complex conjugate of z i.e. if $z = a + ib$, then $\bar{z} = a - ib$. Show that complex conjugation is actually a multiplicative function i.e. if $w, z \in \mathbf{C}$, then $\phi(wz) = \phi(w)\phi(z)$. Finally show that ϕ is the only multiplicative linear transformation from $\mathbf{C}$ to $\mathbf{C}$ other than the zero and identity transformations.

7. Applying a proposition proved in class, construct three linear transformations, T_1, T_2, T_3 with domain $\mathbf{R^2}$ and codomain $\mathbf{R^3}$ such that rank $(T_i) = i$ for $i = 1, 2, 3$.

8. Let V be an n-dimensional space and let T be a linear operator V such that Range(T) = Kernel (T)

   Show that n must be even.

   Give an example of such an operator. (Note: A linear operator T on V is a linear transformation $T : V \to V$ i.e. the codomain is the same as domain.)

MTH 100 : Worksheet 10

1. (a) Find the coordinates of the vectors $v_1 = (2, 3, 4)$ and $v_2 = (1, -1, 2)$ with respect to the ordered basis $\beta = \{(1, 1, 1), (1, 2, 3), (1, 3, 6)\}$
   (NB: the vectors have been written as 3-tuples, but should be regarded as column vectors.)

   (b) If $[v]_\beta = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_\beta$, find $[v]_S$ where S is the standard basis for $\mathbf{R^3}$.

2. Find the matrix relative to the standard basis of the linear operator T on $\mathbf{R^3}$ given by:

   $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$

3. Let $T : \mathbf{R^3} \to \mathbf{R^2}$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$

   (a) Find the matrix of T with respect to standard basis for $\mathbf{R^3}$ and $\mathbf{R^2}$

   (b) Verify that $\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ is a basis for $\mathbf{R^3}$

   (c) Now, determine the matrix of T with respect to the ordered bases β and $\beta' = \{(0, 1), (1, 0)\}$ for $\mathbf{R^3}$ and $\mathbf{R^2}$ respectively.

4. (a) Find the matrix relative to the standard basis of the linear operator T on $\mathbf{R^3}$ given by:
   $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$

   (b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1, 1, 1), (1, 2, 3), (1, 3, 6)\}$
   [**NB: the change of basis matrix $P_{S \to \beta}$ for this basis has been calculated in Question 1.**]

5. (a) Prove that similarity is an equivalence relation on the set $R^{n \times n}$ of square $n \times n$ matrices (n≥ 2).

   (b) Prove or disprove: There exist square matrices (atleast 2×2) A and B such that B is row-equivalent to A, but B is not similar to A.

   (c) Prove or disprove: There exist square matrices (atleast 2×2) A and B such that B is similar to A, but B is not row-equivalent to A.

6. Let $V = R^{2\times 2}$ = vector space of 2×2 matrices with real entries and consider the function $U : V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T is the transpose of A.

   (a) Show that U is a linear operator.

   (b) Determine the matrix of U with regard to any suitable ordered basis β of V.

   (c) Determine a basis for Ker U and determine a basis for Range U.

   (d) Determine the dimension of $\mathrm{Sym}_n(R)$, the space of symmetric $n \times n$ matrices with real entries. Briefly explain your answer.

7. Show that a linear transformation $T : V \to W$, where V and W are finite dimensional with dim V = dim W, is injective iff it is surjective.

8. Let $V = F^{n\times n}$ for a fixed $n \geq 2$, and let $P \in V$ be a fixed but arbitrary invertible matrix. Then the mapping $S_P : V \to V$ given by $S_P(A) = PAP^{-1}$ is known as similarity transformation induced by P. Show that S_P is an isomorphism. Further, show that S_P is a multiplicative transformation, i.e. $S_P(AB) = S_P(A)S_P(B)$ for all $A, B \in V$

9. Let $V = \mathbf{R^2}$ and consider the ordered bases $\alpha = \{u_1, u_2\}$ and $\beta = \{v_1, v_2\}$, where the vectors are as given below.(NB: regard all vectors as column vectors in V.)

   (a) Find the change of basis matrix $P_{\alpha\to\beta}$

   (b) Hence find $[\mathbf{v}]_\beta$ given that $[\mathbf{v}]_\alpha = (10, 20)$.

   (c) Is there some way to check your answer to 2)? Explain your method and use it to check your answer.

   $u_1 = (3, 1), u_2 = (11, 4), v_1 = (3, 2), v_2 = (7, 5)$

MTH 100 : Worksheet 11

1. (a) Let $T : V \to W$ and $U : W \to Z$ be linear transformations, where V, W, Z are finite dimensional vector spaces. Show that rank$(UT) \leq$ min { rank(T), rank(U)}.

   (b) Using 1) or otherwise, show that given any $m \times n$ matrix A and any $n \times k$ matrix B, rank$(AB) \leq$ min { rank(A), rank(B)}.

   Give a non-trivial example (i.e. the matrices A, B should be of non-zero, non-identity and should be of minimum size 2×2) in which equality is achieved, and an example in which strict inequality holds.

2. Let $T : V \to W$ be a bijective linear transformation. Since T is bijective, T is an invertible function i.e. the inverse function of T, $T^{-1} : W \to V$ is well-defined. Prove that $T^{-1} : W \to V$ is also a linear transformation.

   Remark: This holds for finite dimensional as well as infinite dimensional spaces. Thus the proof cannot make use of bases or results for dimension.

3. A linear transformation T from V into W is said to be non-singular if Ker $T = \{0\}$ Prove

   (a) T is non-singular iff T is injective.

   (b) T is non-singular iff T carries every linear independent subset of V into a linear independent subet of W.

   (c) If V and W are finite dimensional with dim$V$ = dim$W$, then T is non-singular iff T is invertible.

4. Show that a linear transformation $T : V \to W$ where V and W are finite dimensional with dim$V$ = dim W, is injective iff it is surjective.

5. Give an example of a vector space V and two operators $T, U : V \to V$ such that T is surjective but not injective and U is injective but not surjective.

6. Show that if A^2 is the zero matrix, then 0 is the only eigen value of A.

7. Show that λ is an eigen value of A iff λ is an eigen value of A^T.

8. Suppose A is an $n \times n$ square matrix such that all the row sums equal the some scalar s. Show that s is an eigen value of A.

9. Suppose that A is an $n \times n$ square matrix and $\text{Rank}(A) = k$.

   Show that A can have atmost $(k+1)$ distinct eigen values.

10. (a) Find the characteristic polynomial $q(\lambda)$ of the matrix A given below, and verify that A satisfies its characteristic polynomial.

    (b) Show that both the polynomials $p(\lambda) = \lambda^2 - 3\lambda + 2$ and $r(\lambda) = \lambda^2 - 4\lambda + 4$ are divisors of $q(\lambda)$.

    Does A satisfy either $p(\lambda)$ or $r(\lambda)$?

    (c) What conclusion can you derive from (2)? Explain briefly.

    (d) Verify that $\lambda = 1$ and $\lambda = 2$ are both eigenvalues of A, and determine atleast three linearly independent eigen vectors of A.

    $$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

MTH 100 : Worksheet 12

1. Find the eigen values and corresponding eigenvectors for the matrix A given below. Is A diagonalizable? Justify your answer.

   $A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$

2. For each matrix find all eigenvalues and a basis of each eigenspace. Which matrix can be diagonalized and why? If yes, indicate the diagonal matrix D and the invertible matrix P such that $A = PDP^{-1}$. [*Hint:* $\lambda = 4$ *is an eigenvalue.*]

   (a) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

3. A 7×7 matrix A has three eigenvalues. One eigenspace is $2-$ dimensional and one of the others is $3-$ dimensional. Is it possible for A to be not diagonalizable? Justify your answer.

4. (a) If A is row-equivalent to the identity matrix, then A must be diagonalizable. Is this statement TRUE or FALSE?

   (b) Justify your answer to 1). Give a proof if TRUE or a concrete counter-example if FALSE. In the second case, you should verify that your counter-example is row equivalent to identity matrix but not diagonalizable.

5. For the given matrix A, find the invertible matrix P and the matrix B which has the form given below such that $A = PBP^{-1}$. In other words, find the values a and b. Finally, express B as a rotation followed by a scaling. $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

6. For the matrix A given in previous question diagonalize it over a complex field.

7. Let $V = C^{\infty}[\mathbf{R}]$, the vector space of real functions having continuous derivatives of all orders. Let D be the differentiation operator on V. Determine the eigenvalues and corresponding eigenvectors of D.

8. Recall the interpolation inner product on $R_n[t]$:

   Let $t_0, t_1, t_2, ..., t_n$ be distinct real numbers.(Note that there are $(n+1)$ numbers.)

   For any two polynomials p and q in $R_n[t]$, we define:

   $< p, q >= p(t_0)q(t_0) + p(t_1)q(t_1) + ... + p(t_n)q(t_n)$

   Verify the inner product axioms for this example and explain why we need $(n+1)$ distinct numbers.

9. Let $V = C[a, b]$. Verify the inner product properties for the inner product given by:

   $< f, g >= \int_a^b f(t)g(t)dt$

10. Use the Gram-Schmidt process to find an orthonormal basis given the basis $\{x_1 = (2, 1, 2), x_2 = (4, 1, 0), x_3 = (3, 1, -1)\}$ for $\mathbf{R^3}$.

11. Let V be the vector space $R_2[t]$ of polynomials of degree ≤ 2 with real coefficients with the inner product $< p, q >= p(0)q(0) + p(-2)q(-2) + p(2)q(2)$, i.e. the interpolation inner product.

    (a) Find an orthogonal basis for V starting from the standard basis $\{1, t, t^2\}$ using the Gram-Schimdt process.

    (b) Find the coordinates of $p(t) = 1 + 2t + 3t^2$ with respect to the orthogonal basis found in previous part.

12. Let W be the subspace of $\mathbf{R^3}$ spanned by the vector $v = (1, 2, 3)$. Find the orthogonal bases for W and $W^{\perp}$ respectively. Is the union of these two bases a basis for $\mathbf{R^3}$?

13. Let S be a finite subset of an inner product vector space V, and define $S^{\perp} = \{v \in V :< v, u >= 0 \text{ for every } u \in S\}$,i.e. $S^{\perp}$ is the set of vectors orthogonal to S. Show that in fact $S^{\perp}$ is a subspace of V. If $W =$ Span S, what is the relationship between $S^{\perp}$ and $W^{\perp}$? Justify your answer.

14. Let $A \in R^{m \times n}$,i.e. A is an $m \times n$ matrix with real entries. Show that Nul A is the orthogonal complement of Row A.

15. (a) Let V be a complex inner product space,i.e. the usual symmetry property is replaced by the property:$< u, v >= \overline{< v, u >}$. Show that $< u, cv >= \bar{c} < u, v >$. Here the bar indicates the complex conjugate.

    (b) Suggest a suitable inner product for $\mathbf{C^n}$ regarded as a vector space over C and verify the inner product properties.

MTH 100 : Worksheet 13

1. Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$ for the following matrix :
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

2. Find a singular value decomposition (SVD) for the matrix A given below.
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

3. Find a singular value decomposition (SVD) for the matrix A given below.
$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

4. Let A be an $n \times n$ invertible square matrix.
(a) Show that the eigen values of A^{-1} are the reciprocals of the eigen values of A.
(b) Find a singular value decomposition of A^{-1}, assuming that you already have an SVD of A.

5. Let U be an $m \times n$ matrix with orthonormal columns, and suppose $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{R^n}$. Show that :
(a) $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$
(b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$
(c) $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$