Discrete Structures-2025: Quiz-4

Sets, Functions, and Relations

Full Marks: 10
Time: 30 minutes

October 13, 2025

(1) If $g_1 : A_1 \to A_2$ and $g_2 : A_2 \to A_3$, then $g_2 \circ g_1 : A_1 \to A_3$ is defined as for every $x \in A_1$, $g_2 \circ g_1(x) = g_2(g_1(x))$. This is called *composition* of g_1 with g_2.

Let A be a countably infinite set, $f : A \to B$ be a surjective function, and B be an infinite set. Then prove that there exists a set $X \subseteq \mathbb{N}$ such that there is a bijective function between X and B.

(Hint: use composition of functions concept and well-ordering principle)

N.B. – Your answer must provide the formal proof that an explicit set $X \subseteq \mathbb{N}$ exists. Vague/handwaving reasonings will cause marks deductions.

Solution: Our assumption only says that A is a countably infinite set, and B is an infinite set such that $f : A \to B$ is a surjective function.

Since A is countably infinite, there exists $g : \mathbb{N} \to A$ such that g is a bijection. We use this description of the bijection $g : \mathbb{N} \to A$ and $f : A \to B$.

For every $y \in B$, consider $A(y) = \{x \in A \mid f(x) = y\}$.

Since f is surjective, every $y \in B$ has a pre-image. It implies that $A(y) \neq \emptyset$.

As $g : \mathbb{N} \to A$ is a bijection, for every $x \in A$, there exists a unique $a \in \mathbb{N}$ such that $g(a) = x$. Hence, we consider $D(y) = \{a \in \mathbb{N} \mid f(g(a)) = y\}$. Precisely, $D(y) \subseteq \mathbb{N}$ is the set of elements $a \in \mathbb{N}$ such that $f \circ g(a) = y$.

Since $A(y) \neq \emptyset$, it follows that $D(y) \neq \emptyset$.

Since $D(y) \subseteq \mathbb{N}$ and $D(y) \neq \emptyset$, due to WELL-ORDERING PRINCIPLE, there exists a smallest (least) element $b_y \in D(y)$ such that $f(g(b_y)) = y$. We put b_y into X.

Following this principle, we define

$$X = \{b_y \in \mathbb{N} \mid b_y \text{ is the smallest/least element of } D(y) \text{ such that } f(g(b_y)) = y\}$$

Subsequently, we define $h : X \to B$ such that

$$h(a) = f(g(a))$$

Consider any two distinct $y_1, y_2 \in B$. Since $y_1 \neq y_2$ and f is a WELL-DEFINED function, $A(y_1) \cap A(y_2) = \emptyset$. As $A(y_1) \cap A(y_2) = \emptyset$, it follows that $D(y_1) \cap D(y_2) = \emptyset$. Hence, the least element $b_{y_1} \in D(y_1)$ must be different from the least element $b_{y_2} \in D(y_2)$.

It remains to prove that h is a bijection.

Injectivity: Consider $a, b \in X$ such that $a \neq b$.

Then, $h(a) = f(g(a))$ and $h(b) = f(g(b))$. Since $a \neq b$, $g(a) \neq g(b)$.

Now, what is crucial is that by construction, X contains exactly one element from $D(y)$ for every $y \in B$. Hence, for every $a, b \in X$ if that $a \neq b$, then both a and b cannot appear in $D(y)$ for some $y \in B$. Since $a \in D(y)$ and $b \in D(y')$ such that $y \neq y'$, it follows that $D(y) \cap D(y') = \emptyset$. Hence, $f(g(a)) \neq f(g(b))$. Therefore, h is injective.

Surjectivity: Consider $y \in B$. Then, $A(y), D(y) \neq \emptyset$, as we have already argued. As X contains a unique element (the least element) from $D(y)$, it follows that there exists an element $b \in X$ such that $h(b) = y$. Hence, h is surjective.

Since h is both injective and surjective, it follows that h is bijection.

This completes the proof.