15th October: COMBINATORICS

Analytical method of counting some objective from a collection of finite sets. X (informal)

Example: A new company has 3 employees, Sanchez, Patel, and Protik. The company rents a floor of building with 16 offices. How many ways are there to assign offices to these 3 employees?

Solution:

First assign an to Sanchez. There are 16 ways to assign. (because there are 16 offices) and no office is allocated to anyone.

For each possible office assignment to Sanchez, there are 15 ways (16−1 ways) to assign office to Patel.

Hence, there are 16 × 15 ways to assign two offices to Sanchez and Patel.

For each such possible assignment of offices to Sanchez and Patel, there are 14 ways (16 − 2) to assign an office to Protik.

Hence, total number of ways to assign offices to Sanchez, Patel, and Protik is $16 \times (16-1) \times (16-2)$

PRODUCT RULE: A procedure is broken into d tasks. If there are n_1 ways to do Task-1, n_2 ways to do Task-2, ... for every $i \in \{1, 2, \ldots, d\}$, n_i ways to do Task-i, then there are $n_1 n_2 \ldots n_d$ ways to complete the procedure

Example 2: How many bit strings (a string with 0 and 1) are

0101 → bit string of size 4

10110 → bit string of size 5.

possible of size n)

There are n tasks in this procedure. Each task can be done in 2 ways.

$\boxed{\text{Task is assigning } k\text{-th bit to } 0 \text{ or } 1.}$

Hence, the total number of ways to construct an n bit string is

$$\underbrace{2 \cdot 2 \cdot 2 \cdots \cdots 2}_{n \text{ times}} = 2^n$$

Tasks: Every element $a_i \in B$ or $a_i \notin B$. $\rightarrow$ 2 ways to do ~~to~~ this task.

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There are n tasks. (why?)
There are n elements.

Example: How many positive integers between 5 and 31 are divisible by 3 or 4?

Solution: $A = \{x \in \mathbb{N} \mid 5 \leq x \leq 31 \text{ and } 3 \mid x\}$.

$$|A| = \frac{|X|}{3} = 9$$

$$B = \{y \in \mathbb{N} \mid 5 \leq y \leq 31 \text{ and } 4 \mid y\}$$

$$|B| = \left\lfloor \frac{|X|}{4} \right\rfloor = 6$$

(number of possible truth values of a collection of n proposition)

How many subsets of a set S is possible?

S has n elements. $\{a_1, a_2, \ldots, a_n\} = S$

Procedure: construct a subset $B \subseteq S$.

Therefore, the number of ways to construct $B \subseteq S$ is

$$\underbrace{2 \cdot 2 \cdot 2 \cdots \cdots 2}_{n \text{ times}} = 2^n.$$

$A = \{6, 9, 12, 15, 18, 21, 24, 27, 30\}$

$\left.\begin{array}{l} 3 \times 2 \\ 3 \times 10 \end{array}\right\}$ 9 and

$X = \{x \in \mathbb{N} \mid 5 \leq x \leq 31\}$ 30

$|X| = 31 - 5 + 1 = 27$, $\cancel{29}$ $\cancel{26}$

$$|B| = \frac{27}{4}$$

$B = \{8, 12, 16, 20, 24, 28\}$

$A \cap B = \{x \in \mathbb{N} \mid 5 \leq x \leq 31 \text{ and}$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 9 + 6 - 2 = 13$$

$$|A \cap B| = \left\lfloor \frac{|X|}{12} \right\rfloor = 2 \qquad 12 \mid x$$

double counting

If there are $(k+1)$ balls that are put into k boxes, then there is a box that contains at least 2 balls.

$B_1 \quad B_2 \quad \ldots\ldots \quad B_k$

Proof: (proof by contraposition)

Suppose that every box contains at most one ball.

Then the total number of balls $\leq$ (the number of boxes) $* 1$ $\quad 3.7$

$= k \cdot 1 = k.$

Leads to contradiction that there are $(k+1)$ balls.

PIGEON HOLE PRINCIPLE.

n balls are put into m boxes, then one box will have $\lceil n/m \rceil$ balls.

$\lceil n/m \rceil < \frac{n}{m} + 1$

$(\lceil n/m \rceil - 1) \cdot m =$ total number of balls

$< \left(\frac{n}{m} + 1 - 1\right) m = n$

Example: If there are 5 integers chosen from $\{1,2,3,4,5,6,7,8\}$ then there must be a pair from these 5 integers whose sum is equal to 9.

5 balls $\quad X \subseteq \{1,2,3,4,5,6,7,8\}$

$|X| = 5.$

When 5 numbers are chosen

Answer: Consider the boxes

$(1,8) \quad (2,7) \quad (3,6) \quad (4,5)$

There are 4 boxes.

each having exactly 2 numbers.

from $\{1,2,3,4,5,6,7,8\}$ → from these 4 boxes, due to pigeon hole principle, there will be a box from which $\lceil 5/4 \rceil$ numbers will be chosen.

For each of the 4 boxes, the numbers sum is equal to 9.

Therefore, there will be two integers chosen whose sum is equal to 9.

Exercise: If $(k+1)$ numbers are chosen from $\{1,2,3,\ldots,2k\}$ then there are two numbers whose sum is equal to $(k+1)$.

→ Prove the statement.

Example: Let d be a positive integer. Then, among any set of $(d+1)$ consecutive positive integers, there are two integers with exactly same remainder when divided by d.

Answer: Let the numbers are $\{a_1, \underbrace{a_1+1}_{a_2}, \ldots, \underbrace{a_1+d}_{a_{d+1}}\}$

For every $k \in \{1,2,\ldots,d+1\}$ consider the remainder b_k when a_k is divided by d.

$$a_k = d \cdot x_k + b_k$$

The list of possible values of the remainders are $\{0,1,2,\ldots,d-1\}$.

Hence, there are d possible remainders.

There are $(d+1)$ possible remainders $\{b_1, b_2, \ldots, b_{d+1}\}$

Then there are $\left\lceil \frac{d+1}{d} \right\rceil$ many numbers from

Here, there are b_i, b_j with $i \neq j$ such that $b_i = b_j$.

$\{b_1, b_2, \ldots, b_{d+1}\}$ that have the same value.

Therefore there are two numbers a_i and a_j $(i \neq j)$ such that both have the same remainder when divided by d.

Book - exercise:

How many numbers you must choose so that there are two numbers whose sum is 11?

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

if you choose 6 numbers

then —

(1, 10) (2, 9) (3, 8) (4, 7) (5, 6)