6th October:

How to prove that a given set A is countable?

Is there a set that is uncountable?

Theorem: $\mathbb{R}$ = set of all real numbers is uncountable.

Proof: To prove that there cannot exist any bijection between $\mathbb{R}$ and $\mathbb{N}$, the crucial idea is this

Step 1: Assume that a bijection $g: \mathbb{N} \rightarrow \mathbb{R}$ exists.

Step 2: Then infer some fact(s) which leads to a contradiction.

As every subset of a countable set is countable, it is sufficient to choose a subset $(0,1) \subseteq \mathbb{R}$ and prove that $(0,1)$ is uncountable.

Assume that $(0,1)$ is countable. Then consider any arbitrary

Trick-1: Provide an explicit bijection $g: \mathbb{N} \rightarrow A$ or an explicit bijection $g: A \rightarrow \mathbb{N}$

Trick-2: Find a superset $B \supseteq A$ and prove that B is countable.

Trick-3: Choose a different favorite set of yours, $\mathbb{Z}$ or $\mathbb{N} \times \mathbb{N}$ and provide a bijection between $\mathbb{Z}$ to A or $\mathbb{N} \times \mathbb{N}$ to A.

Trick-4: Provide explicit injective functions $f: \mathbb{N} \rightarrow A$ and an injective function $g: A \rightarrow \mathbb{N}$. (Shrouder Bernstein's Theorem)

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$$

we will prove now that $(0,1)$ is uncountable.

every $x \in (0,1)$ has a decimal representation

$0 \cdot d_1 d_2 d_3 \cdots$, $d_i \in \{0, 1, 2, \ldots 9\}$

bijection $g: \mathbb{N} \to (0,1)$.

We list the numbers $g(0), g(1)$
$g(2), \ldots$

$g(0)$ | $0.\boxed{a_{00}}a_{01}a_{02}a_{03}\ldots\ldots$

$g(1)$ | $0.a_{10}\boxed{a_{11}}a_{12}a_{13}\ldots\ldots$

$g(2)$ | $0.a_{20}a_{21}\boxed{a_{22}}a_{23}\ldots\ldots$

$g(k)$ | $0.a_{k,0}a_{k,1}a_{k,2}a_{k,3}\ldots\boxed{a_{k,k}}\ldots$

change a_{00} of $g(0)$. If $a_{00}=4$ then $b_0 = 5$

change a_{11} of $g(1)$

$\vdots$

change $a_{k,k}$ of $g(k)$ If $a_{k,k}=4$ set $b_k=5$

If $a_{k,k} \neq 4$, then set $b_k=4$.

Crucial observation:

y differs from $g(k)$ at one digit.

Hence, $y \neq g(k)$.

This is true for every $k = 0, 1, 2, 3, 4, \ldots$

$0.11\underline{9999}\ldots$ is the same as $0.1\underline{2000}\ldots$

$a_{i,j} \in \{0, 1, 2, \ldots, 9\}$

Since g is a bijection, for all $i \neq j$, $g(i) \neq g(j)$.

Construct a number

$y = 0.b_0 b_1 b_2 b_3 b_4 \ldots\ldots$

For every $i = 0, 1, 2, \ldots$

change the $(i+1)$-th digit after decimal point.

For every $i = 0, 1, 2, \ldots$

If $a_{i,i} = 4$, then set $b_i = 5$

If $a_{i,i} \neq 4$, then set $b_i = 4$.

$\boxed{y = 0.b_0 b_1 b_2 b_3 \ldots}$

Hence, for every $k \in \mathbb{N}$

$y \neq g(k)$

but $y \in (0,1)$.

Therefore $g : \mathbb{N} \to (0,1)$ is not surjective which contradicts the assumption that g is a bijection.

Therefore $(0,1)$ is uncountable.

As $(0,1) \subseteq \mathbb{R}$ and

DIAGONALIZATION
(modify diagonal entries)

$(0,1)$ is uncountable, hence $\mathbb{R}$ is uncountable.