Predicate Logic

If the assumption is $(\exists x\, P(x)) \wedge (\exists x\, Q(x))$ is true, then we cannot conclude that $\exists x\,(P(x) \wedge Q(x))$ is true.

Then, take an arbitrary p in the domain.
$A(p) \vee B(p)$ is true.
But, it can happen that $A(p)$ is true but $B(p)$ is false.

Suppose we assume that $\forall x\,(A(x) \vee B(x))$ is true.
Can we conclude that $\boxed{\forall x\, A(x) \vee \forall x\, B(x)}$ is true? $\boxed{\text{NO}}$

$A(x)$
$B(x)$

By assumption, we have $\forall x\,(A(x) \vee B(x))$ is true.

Take another element q in the domain such that $A(q)$ is false but $B(q)$ is true. Then also $A(q) \vee B(q)$ is true.

There is an element q such that $A(q)$ is false. Hence, $\forall x\, A(x)$ is false.

There is an element p such that $B(p)$ is false. Hence $\forall x\, B(x)$ is false.

Then $\boxed{\forall x\, A(x) \vee \forall x\, B(x)}$ is false.

Hence, we cannot conclude this.

$A(x) =$ x is an even integer
$B(x) =$ x is an odd integer.

$\forall x\, B(x)$ is false.
"Every integer is odd" is false

$\forall x\,(A(x) \vee B(x))$ is true ~~the~~ domain integers
Every integer is odd or even.
But $(\forall x\, A(x))$ is false
Every integer is even – is false

Suppose $\forall x\, A(x) \vee \forall x\, B(x)$ is true
Then can we conclude that $\forall x\,(A(x) \vee B(x))$ is true? YES

Why? $\forall x\, A(x)$ is true or $\forall x\, B(x)$ is true
Then $\forall x\,(A(x) \vee B(x))$ is true.

Exercise: 1) Assume that $\forall x\,(A(x) \wedge B(x))$ is true. Then, is $\forall x\, A(x) \wedge \forall x\, B(x)$ true?

2) Assume that $\forall x\,(A(x) \to B(x))$ is true. Can we conclude that $(\forall x\, A(x)) \to (\forall x\, B(x))$ is true?

Predicates over two variables:

$P(x,y)$: $y = x+7$

Domain = set of all real numbers.

Quantification is possible on x and y.

① For all real numbers x and y, $x = y+7$. $\boxed{\forall x \forall y \; P(x,y)}$

Is it true? Choose $x=3$ and $y=2$

Then $P(3,2)$ is false.

Hence, $\forall x \forall y \; P(x,y)$ is false.

② For all real numbers x, there exists a real number y such that $x = y+7$. $\boxed{\forall x \exists y \; P(x,y)}$

For any $x \in \mathbb{R}$ (real numbers) $x-7 \in \mathbb{R}$. Hence $\forall x \exists y \; P(x,y)$ is true

NESTED QUANTIFIER

$\boxed{\exists y \forall x \; P(x,y)}$ → Not a correct sentence for ②.

↓

There is a real number y such that for every real number x, $x = y+7$.

Fixes a real number y. Then it says for all real numbers x, $x = y+7$

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Example-1 Revisit: For all real numbers x, y; $x = y+7$. $\boxed{\text{FALSE}}$

$\forall x \forall y \; P(x,y)$.

Negation of $\forall x \forall y \; P(x,y)$.

There are real numbers x and y such that $\boxed{\neg P(x,y)}$

$\neg P(x,y)$: $x \neq y+7$

Then $\exists x \exists y \neg P(x,y)$ is true

$x = y \in \mathbb{R}$

③ $\forall x \forall y \left((x<0) \wedge (y<0) \rightarrow (xy>0) \right)$

How to write it in English?

Domain: set of all real numbers.

The product of any two negative real numbers is positive. SIMPLIFIED

For all real numbers x, y, if $x, y < 0$, then $xy > 0$

How to understand the truth value

of a nested quantifier? $A(x,y)$

$\forall x \forall y\, A(x,y)$: is true when $A(x,y)$ is true for every pair of elements x and y.

$\forall x \exists y\, A(x,y)$

$\exists x \forall y\, A(x,y)$

$\exists x \exists y\, A(x,y)$

$\forall x \exists y\, A(x,y)$ is true when for every element x, there is an element y such that $A(x,y)$ is true.

$\forall x \forall y\, A(x,y)$ is false when there is a pair of elements x, y for which $A(x,y)$ is false.

$\exists x \exists y\, \neg A(x,y)$ is true.

$\neg \forall x \forall y\, A(x,y) \equiv \exists x \exists y\, \neg P(x,y)$

$\exists x \forall y\, A(x,y)$ is true when there is an element x such that for every element y $A(x,y)$ is true.

$\forall x \exists y\, A(x,y)$ is false when there is an element x such that for all elements y $A(x,y)$ is false

$\exists x \forall y\, A(x,y)$ is false when for every element x there is an element y such that $A(x,y)$ is false.

$\neg \exists x \forall y\, A(x,y) \equiv \forall x \exists y\, \neg A(x,y)$

$\neg \forall x \exists y\, A(x,y) \equiv \exists x \forall y\, \neg A(x,y)$

$\exists x \exists y\, A(x,y)$ is true when $A(x,y)$ is true for some pair of elements x, y

$\exists x \exists y\, A(x,y)$ is false when $A(x,y)$ is false for every pair of elements x and y

$\neg \exists x \exists y\, A(x,y) \equiv \forall x \forall y\, \neg A(x,y)$.

Example: A negative real number does not have a square root that is a real number.

$A(x): x < 0$

$B(x,y): y \neq \sqrt{x}$.

Then this sentence

$$\forall x\, ((x < 0) \to (\forall y\, (y \neq \sqrt{x})))$$

$$\equiv \forall x\, (A(x) \to (\forall y\, B(x,y)))$$

Domain: $\mathbb{R}$ = real numbers.

If $x < 0$, then $\sqrt{x}$ does not exists in $\mathbb{R}$. $\sqrt{x} \notin \mathbb{R}$.

$\boxed{\text{choose any } y \in \mathbb{R},\ y \neq \sqrt{x}}$

$\boxed{\forall y\ y \neq \sqrt{x}}$ $\quad \forall y\, B(x,y)$

How about its negation

There is a negative real number x s.t. $\sqrt{x} \in \mathbb{R}$.

Negation of above $\Rightarrow$ $\exists x \left(A(x) \wedge \neg (\forall y\, B(x,y)) \right)$ statement is false.

$$\equiv \exists x \left(A(x) \wedge \exists y \neg B(x,y) \right)$$

Example: The difference between two positive integers is not necessarily positive.

Negation of $\forall x \forall y \left(B(x) \wedge B(y) \rightarrow A(x,y) \right)$

$$\exists x \exists y \left(\neg \left(\underbrace{B(x) \wedge B(y)}_{p} \rightarrow \underbrace{A(x,y)}_{q} \right) \right)$$

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

$$\exists x \exists y \left(B(x) \wedge B(y) \wedge \neg A(x,y) \right)$$

$$\exists x \exists y \left((x > 0) \wedge (y > 0) \wedge (x - y < 0) \right)$$

Negation: The difference between two positive integers is always positive. For all integers x, y if $x > 0$ and $y > 0$ $\boxed{x - y > 0}$

$A(x,y): x - y > 0$

$B(x): x > 0$ $\quad$ $B(y): y > 0$

Negation:

$$\forall x \forall y \left(B(x) \wedge B(y) \rightarrow A(x,y) \right)$$

Domain: integers.

Every nonnegative real number has a square root that is a real number

Domain: all real numbers.

$A(x): x \geq 0$

$B(x,y): y = \sqrt{x}$.

For every real number x, if $x \geq 0$ then there is a real number y such that $B(x,y)$ is true.

$$\forall x \left((x \geq 0) \rightarrow (\exists y\; B(x,y)) \right)$$

Section 1.4 Rosen's

Section 1.5 of Rosen's Book