METRIC SPACES 2023
EXERCISES 1

Q1: Show that for any $\alpha, \beta, \gamma, \delta \in \mathbb{R}$,

$$\max\{\alpha+\beta, \gamma+\delta\} \le \max\{\alpha, \gamma\} + \max\{\beta, \delta\} .$$

Q2: Let $S \subset \mathbb{R}$ be bounded, $c > 0$ and

$$cS = \{cx : x \in S\}$$

Prove that $\sup(cS) = c\sup(S)$ and $\inf(cS) = c\inf(S)$. What happens if $c < 0$?

Let $A, B \subseteq \mathbb{R}$ be non-empty sets and define

$$A \pm B = \{x \pm y : x \in A \text{ and } y \in B\}.$$

Prove that

$$\sup(A+B) = \sup(A) + \sup(B).$$

What about

$$\inf(A-B)?$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $A \subseteq \mathbb{R}$ be non-empty, we define $\sup_A(f)$ to be the set

$$\sup_A(f) = \sup\{f(x) : x \in A\},$$

with $\inf_A(f)$ defined in similar fashion. Suppose now that f and g are functions such that $f \le g$ on the set A, that is $f(x) \le g(x)$ for all $x \in A$. Prove that

$$\sup_A(f) \le \sup_A(g).$$

Q3: Suppose that X is a set equipped with some metric d. Show that the function

$$\widehat{d} : X \times X \to [0, \infty)$$

such that

$$(x, y) \mapsto \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

Set $X = \mathbb{R}$.

(a) Plot the graph of $y = \widehat{d}(x, 0)$ as x varies through $\mathbb{R}$,

(b) Suppose that $0 \le a < b < \infty$. Determine a formula for the 'diameter' ℓ of the interval (a, b) where

$$\ell = \sup\{\widehat{d}(t, t') : t, t' \in (a, b)\}.$$

Q4: Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d_{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

(a) Show that $(X \times Y, d_{X\times Y})$ is a metric space. (You may find Q1: helpful.)

(b) If d_X and d_Y are the discrete metrics on X and Y, then what is the metric $d_{X\times Y}$?
(c) If $X = \mathbb{R}$ with the standard metric, then what is the metric space $(X \times X, d_{X\times X})$?

Q5: Multi-step triangle inequalities: show that if (X, d) is a metric space and $x_1, x_2, \ldots, x_n \in X$ (where $n \geq 3$) then

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Suppose that $x, y, u, v \in X$. Use the three-step triangle inequality

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$$

to show that

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v).$$

Informally, this means that if x is close to u and y is close to v, then $d(x, y)$ is close to $d(u, v)$.

Q6: Consider the function d^* defined on $\mathbb{R}^2$ by

$$d^*(x, y) = \begin{cases} 0 & x = y \\ |x| + |y| + 2|x - y| & x \neq y \end{cases}$$

(a) Show that d^* is a metric.
(b) Determine which points belong to the open ball $B(0, r)$ in $(\mathbb{R}, d^*)$.

Q7: Show that for $x, y \in \mathbb{R}^N$:

$$d_\infty(x, y) \leq d_1(x, y) \leq N d_\infty(x, y)$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{N} d_\infty(x, y)$$

Hence find constants $A, B > 0$ such that:

$$A d_1(x, y) \leq d_2(x, y) \leq B d_1(x, y).$$

Q8: Equip the set $C[0, \pi]$ of all continuous real valued functions on the interval $[0, \pi]$ with the d_2 metric

$$d_2(f, g) = \left(\int_0^\pi (f(t) - g(t))^2 dt \right)^{1/2}$$

For $n \in \mathbb{N}$, let $f_n(t) = \pi^{-1/2} \sin(nt)$.

Derive a formula for $d_2(f_n, f_m)$ for all $n, m \in \mathbb{N}$.

Q9: Consider now the set of all sequences of all real numbers, that is $\mathbb{R}^{\mathbb{N}}$. Show that

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{|x_n - y_n|}{1 + |x_n - y_n|} \right)$$

is a metric on $\mathbb{R}^{\mathbb{N}}$. Your first step should be to justify the fact that the range of this function is a subset of $[0, \infty)$.

Q10: (a) Suppose d_1, d_2 are two metrics on a space X, and that d_3 is defined by

$$d_3(x, y) = \alpha d_1(x, y) + \beta d_2(x, y)$$

where α and β are non-negative and not both zero. Show that d_3 is also a metric on X.

(b) On $\mathbb{R}$, let d_1 be the trivial metric,

$$d_1(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

d_2 the standard metric,

$$d_2(x,y) = |x - y|$$

and

$$d_3(x,y) = d_1(x,y) + d_2(x,y).$$

In $(\mathbb{R}, d_3)$, find the elements of the open balls $B(0,1)$ and $B(0,2)$ and show that every subset of $\mathbb{R}$ is open. Which subsets of $\mathbb{R}$ are closed in this metric?