Vector Calculus — Exercises 1

Please upload your solutions to Exercises 4d, 6, 11b, and 12d by 9AM on Wednesday 11 October.

(1) Find the angle between the vectors:

(a) $\underline{a} = (-1, 2), \quad \underline{b} = (\frac{1}{2}, -1)$.
(b) $\underline{a} = (1, 2, 3), \quad \underline{b} = (1, 1, -1)$.
(c) $\underline{a} = (1, -1, 0), \quad \underline{b} = (0, 1, 1)$.
(d) $\underline{a} = (1, 2, 1), \quad \underline{b} = (1, 1, 0)$.

(2) For two vectors $\underline{a}, \underline{b} \in \mathbb{R}^3$, with angle θ between them, what is $(\underline{a} \cdot \underline{b})^2 + \|\underline{a} \times \underline{b}\|^2$?

(3) For each of the following vector fields, $\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, determine whether it may be a gradient. If so, find a scalar field f such that $\nabla f = \underline{g}$.

(a) $\underline{g}(x, y) = (y, -x)$
(b) $\underline{g}(x, y) = (5x^4 + y, x - 12y^3)$
(c) $\underline{g}(x, y) = (3x^2 \cos y, -x^3 \sin y)$
(d) $\underline{g}(x, y) = (e^y \sec^2 x, e^y \tan x)$
(e) $\underline{g}(x, y) = (xy, -y)$
(f) $\underline{g}(x, y) = (x^2 + y^2, \cos y)$

(4) Make a sketch of the region $R \subset \mathbb{R}^2$ and evaluate the integral $\iint_R f \, dA$ for:

(a) $f(x, y) = 10 + 2x^2 + 2y^2$ and R is the triangle bounded by $y = x$, $x = 2y$, and $y = 2$.
(b) $f(x, y) = \dfrac{y}{x^2 + y^2}$ and R is the trapezoid bounded by $y = x$, $y = 2x$, $x = 1$, and $x = 2$.
(c) $f(x, y) = y$ and R is bounded by $y = 4 - x^2$ and $y = 4 - x$.
* (d) $f(x, y) = \dfrac{y}{1 + x}$ and R is bounded by $y = 0$, $y = \sqrt{x}$, and $x = 4$.

(5) Make a sketch of the region $R \subset \mathbb{R}^2$ and evaluate the integral $\iint_R f \, dA$ using polar coordinates for:

(a) $f(x, y) = x$ and R is the region in the first quadrant bounded by $x^2 + y^2 = 25$, $3x = 4y$, and $y = 0$.
(b) $f(x, y) = 1$ and R is the region in the first quadrant below the line $y = x$ and inside the circle of radius 1 centered at $(0, 1)$.

* **(6)** Compute and fully simplify:

$$\text{(a)} \quad \sum_{k=1}^{3} \frac{1}{k} \qquad\qquad \text{(b)} \quad \sum_{k=1}^{n} 1, \text{ where } n \in \mathbb{N}.$$

(7) For the maps and vector defined by

(i) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{f}(x, y, z) = (x^3 + y^2 - 2z,\ x - 2y^2 + z^3)$,
$\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{g}(u, v) = \big(e^{2u+v},\ e^{u-2v}\big)$,
$\underline{a} = (1, 1, 1)$

(ii) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{f}(x, y, z) = (x + yz, x - yz)$,
$\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{g}(u, v) = (u + v, uv)$,
$\underline{a} = (0, 1, 2)$

(iii) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^3$, $\underline{f}(x, y, z) = (xy^2z^2,\ z^2 \sin y,\ x^2 e^y)$,
$\underline{g} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{g}(u, v, w) = (u^2 + v + w,\ 2u + v + w^2)$,
$\underline{a} = (x, 0, z)$

do the following:

(a) Compute the matrices $D\underline{f}(x, y, z)$ and $D\underline{g}(u, v)$.
(b) Work out the composition $\underline{F}(x, y, z) = \underline{g}(\underline{f}(x, y, z))$.
(c) Compute $D\underline{F}(\underline{a})$.
(d) Compare this with the matrix product $D\underline{g}(\underline{b})\, D\underline{f}(\underline{a})$, where $\underline{b} = \underline{f}(\underline{a})$.

(8) Let $f_1, f_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ be any continuously differentiable functions. Let $g : \mathbb{R}^2 \to \mathbb{R}$, $g(u, v) = uv$. Define $\underline{f} = (f_1, f_2)$.

(a) Compute the matrix product $Dg(\underline{f}[\underline{x}])D\underline{f}(\underline{x})$.
(b) Compute the composition $F := g \circ \underline{f}$ and its derivative DF directly.

By the chain rule, these should give the same result.

(9) A scalar field f is *homogeneous of degree* c if it satisfies

$$f(\lambda\underline{x}) = \lambda^c f(\underline{x})$$

for all $\lambda > 0$ and $\underline{x} \in \mathbb{R}^n$.

Let f be such a scalar field. For any fixed $\underline{x} \in \mathbb{R}^n$, define $g(\lambda) = f(\lambda\underline{x})$, use the chain rule, and then set $\lambda = 1$ to show that

$$\underline{x} \cdot \nabla f(\underline{x}) = c\, f(\underline{x}). \tag{$*$}$$

(10) Verify that eq. $(*)$ holds for each of the following functions, and give the value of the degree c in each case.

(a) $f(x, y, z) = xyz$; (b) $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$; (c) $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

(11) Find a parametrization for each of the following curves:

(a) $C \subset \mathbb{R}^2$ is the part of the curve defined by $y^2 = x$ from $(0,0)$ to $(1,1)$.

* (b) $C \subset \mathbb{R}^2$ is the part of the ellipse defined by $x^2 + 4y^2 = 4$ anticlockwise from $(2,0)$ to $(0,1)$.

(c) $C \subset \mathbb{R}^2$ is the part of the graph $y = \cos x$ from $(\frac{\pi}{2}, 0)$ to $(-\frac{\pi}{2}, 0)$.

(d) $C \subset \mathbb{R}^3$ satisfying $z = x^2$ and $z = y^3$ from $(0,0,0)$ to $(1,1,1)$.

(e) $C \subset \mathbb{R}^3$ satisfying $x^2 + y^2 = 1$, $z = x^2$, and $y \geq 0$ from $(1,0,1)$ to $(-1,0,1)$.

(12) Evaluate the line integral $\int_C \underline{g}(\underline{x}) \cdot d\underline{x}$ for the following:

(a) $\underline{g}(x,y,z) = (x, y, xz - y)$ and C is the straight line from $(0,0,0)$ to $(1,2,4)$.

(b) $\underline{g}(x,y,z) = (x, y, xz - y)$ and C is parametrized by $\underline{p} : [0,1] \to \mathbb{R}^3$, $\underline{p}(t) = (t^2, 2t, 4t^3)$.

(c) $\underline{g}(x,y,z) = (y^2 - z^2, 2yz, -x^2)$ and C is parametrized by $\underline{p} : [0,1] \to \mathbb{R}^3$, $\underline{p}(t) = (t, t^2, t^3)$.

* (d) $\underline{g}(x,y,z) = (2xy, x^2 + z, y)$ and C is the line segment from $(1,0,2)$ to $(3,4,1)$.

(e) $\underline{g}(x,y,z) = (-x^2 y, x^3, y^2)$ and C is parametrized by $\underline{p} : [0,\pi] \to \mathbb{R}^3$, $\underline{p}(t) = (\cos t, \sin t, t)$