Metric Spaces Problems

1) $\max\{\alpha+\beta, \gamma+\delta\} \leq \max\{\alpha,\gamma\} + \max\{\beta,\delta\}$

<u>proof:</u>

First observe that

$$\alpha \leq \max\{\alpha,\gamma\} \text{ and } \beta \leq \max\{\beta,\delta\}$$

$$\Rightarrow \quad \alpha+\beta \leq \max\{\alpha,\gamma\} + \max\{\beta,\delta\} \qquad (*1)$$

Similarly

$$\gamma \leq \max\{\alpha,\gamma\} \text{ and } \delta \leq \max\{\beta,\delta\}$$

$$\Rightarrow \quad \gamma+\delta \leq \max\{\alpha,\gamma) + \max\{\beta,\gamma\} \qquad (*2)$$

Both (*1) and (*2) are true for all $\alpha, \beta, \gamma, \delta$

Hence

$$\max\{\alpha+\beta, \gamma+\delta\} \leq \max\{\alpha,\gamma\} + \max\{\beta,\delta\}$$

∎

2) $S \subset \mathbb{R}$ be bounded, $c > 0$

$$cS = \{cx : x \in S\}$$

$$\sup(cS) = c\sup(S)$$

proof:

Since S is bounded, $\sup(S)$ has to exist.

Let B be the set of all upperbounds. By defn of least upperbound,

$$\sup(S) = \min(B)$$

Note: if B has a minimal element, b, then cb is the minimal element of cB.

To see this suppose $b = \min(B)$. By defn of min;

$$b \leq x \quad \forall x \in B \implies cb \leq cx \quad \forall cx \in cB$$

Therefore cb is a lowerbound for cB

$b \in B \implies cb \in cB \implies cb$ is an element of cB.

cb is a lowerbound and an element of cB

$\implies cb = \min(cB) \implies c\min(B) = \min(cB)$

Since B is the set of all upperbounds of B, cB is the set of all upperbounds of cB

(As shown,
$x \leq b \;\forall x \in S, bc \Rightarrow cx \leq cb \;\forall cx \in cS$
So any cb is an upperbound and $b \in B \Rightarrow cb \in cB$)

Now by definition of sup,
$$\sup(cS) = \min(cB) = c\min(B) = c\sup(S)$$
$$\Rightarrow \sup(cS) = c\sup(S)$$

Similarly since S is bounded $\inf(S)$ has to exist.

Let A be the set of all lowerbounds of S. By defn of greatest lowerbound,
$$\max(A) = \inf(S)$$

Note: if a is a maximal element of A, then cA is a maximal element of cA.

To see this, suppose $a = \max(A)$. By defn of max,

$$x \leq a \ \forall x \in A \Rightarrow cx \leq ca \ \forall cx \in cA.$$

Therefore ca is an upperbound for cA.

$a \in A \Rightarrow ca \in cA \Rightarrow ca$ is an element of cA.

ca is an upperbound and an element of cA

$\Rightarrow ca = \max(cA) \Rightarrow c\max(A) = \max(cA)$

Since A is the set of all lowerbounds of S, cA is the set of all lowerbounds of cS.

As shown:

$a \leq x$ for all $x \in S$, $a \in A \Rightarrow ca \leq cx \ \forall cx \in cS$ & $ca \in cA$.

So any ca is a lowerbound & $a \in A \Rightarrow ca \in cA$

By defn of inf

$$\inf(cS) = \max(cA) = c\max(A) = c\inf(S)$$

$$\Rightarrow \inf(cS) = c\inf(S)$$

If $c<0$, we would get

$$\inf(cS) = c\sup(S) \text{ and}$$
$$\sup(cS) = c\inf(S)$$

6) $$d^*(x,y) = \begin{cases} 0 & x=y \\ |x|+|y|+2|x-y| & x \neq y \end{cases}$$

Showing d^* is a metric,

M1) <u>For $x \neq y$</u>

since $|x| \geq 0$, $|y| \geq 0$ and $2|x-y| \geq 0$,

$|x|+|y|+2|x-y| \geq 0 \Rightarrow d^*(x,y) \geq 0$

For $x=y$, trivially true

$d^*(x,y) = 0 \Rightarrow d(x,y) \geq 0$

M2) if $x=y$, then $d(x,y)=0$

if $x \neq y$, $|x-y| \neq 0$. Therefore

$|x|+|y|+2|x-y| > 0 \Rightarrow d^*(x,y) > 0$

$\Rightarrow d^*(x,y) \neq 0$

M3) for $x = y$,

$$d^*(x,y) = 0 = d^*(y,x)$$

For $x \neq y$,

$$d^*(x,y) = |x| + |y| + 2|x-y|$$
$$= |y| + |x| + 2|y-x|$$
$$= d^*(y,x)$$

M4) Showing that $d^*(x,z) \leq d^*(x,y) + d^*(y,z)$

$$d^*(x,z) = |x| + |z| + 2|x-z|$$
$$\leq |x| + |z| + 2|y| + 2|x-z| \quad (|y| \geq 0)$$
$$= |x| + |z| + 2|y| + 2|(x-y) + (y-z)|$$
$$\leq |x| + |z| + 2|y| + 2|x-y| + 2|y-z|$$
$$= (|x| + |y| + 2|x-y|) + (|y| + |z| + 2|y-z|)$$
$$= d^*(x,y) + d^*(y,z)$$

$$\Rightarrow d^*(x,z) \leq d^*(x,y) + d^*(y,z)$$

(b) Open ball $B(0,r)$

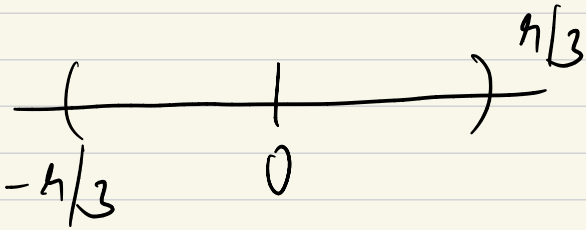
$$B(0,r) = \{x \in \mathbb{R} \mid d(x,0) < r\}$$

$$d(x,0) < r \Rightarrow |x| + |0| + 2|x-0| < r$$

$$\Rightarrow 3|x| < r$$

$$\Rightarrow |x| < \frac{r}{3}$$

$$\Rightarrow -\frac{r}{3} < x < r/3$$



Q7) $x, y \in \mathbb{R}^N$

$$d_\infty(x,y) = \max\{|x_i - y_i| : i \in \{1, \dots, N\}\}$$

$$d_1(x,y) = \sum_{i=1}^{N} |x_i - y_i|$$

Let the max occur at index j, i.e.

$$d_\infty(x,y) = \max\{|x_i - y_i| : i \in \{1, \dots, N\}\}$$

$$= |x_j - y_j|$$

$$\leq |x_j - y_j| + \sum_{\substack{i=1 \\ j \neq i}}^{N} |x_i - y_i|$$

$$\leq \sum_{i=1}^{N} |x_i - y_i|$$

$$= d_1(x,y) \Rightarrow d_\infty(x,y) \leq d_1(x,y)$$

$$d_1(x,y) = \sum_{i=1}^{N} |x_i - y_i|$$

$$\leq \sum_{i=1}^{N} \max\{|x_i - y_i| : i \in \{1, \ldots, N\}\}$$

$$= \sum_{i=1}^{N} |x_j - y_j|$$

$$= N|x_j - y_j|$$

$$= N d_\infty(x,y)$$

$$\Rightarrow d_1(x,y) \leq N d_\infty(x,y)$$

Therefore

$$d_\infty(x,y) \leq d_1(x,y) \leq N d_\infty(x,y)$$

$$|x_i - y_i| \leq |x_i - y_i|^2 \quad \forall i \in \{1, \dots, N\}$$

$$\leq \sum_{i=1}^{N} |x_i - y_i|^2 \quad \forall i \in \{1, \dots, N\}$$

$$\Rightarrow \quad |x_i - y_i|^2 \leq \sum_{i=1}^{N} |x_i - y_i|^2 \quad \forall i \in \{1, \dots, N\}$$

$$\Rightarrow \quad |x_i - y_i| \leq \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \quad \forall i \in \{1, \dots, N\}$$

$$\Rightarrow \quad \max\{|x_i - y_i|\} \leq \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2}$$

$$\Rightarrow \quad d_\infty(x, y) \leq d_2(x, y)$$

$$d_2(x,y) = \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2}$$

$$\leq \sqrt{\sum_{i=1}^{N} \left(\max\{|x_i - y_i|\}\right)^2}$$

$$= \sqrt{\sum_{i=1}^{N} d_\infty^2(x,y)}$$

$$= \sqrt{N}\, d_\infty(x,y)$$

$\Rightarrow$

$$d(x,y) \leq \sqrt{N}\, d_\infty(x,y)$$

Therefore

$$d_\infty(x,y) \leq d_2(x,y) \leq \sqrt{N}\, d_\infty(x,y)$$

Multiplying $(*2)$ b N, we get,

$$d_\infty(x,y) \le d_1(x,y) \le N d_\infty(x,y) \le N d_2(x,y) \le \sqrt{N} d_\infty(x,y)$$

In particular,

$$d_1(x,y) \le N d_2(x,y) \Rightarrow \frac{1}{N} d_1(x,y) \le d_2(x,y)$$

$$\Rightarrow A = \frac{1}{N}$$

Multiplying $(*1)$ by $\sqrt{N}$,

$$d_\infty(x,y) \le d_2(x,y) \le \sqrt{N} d_\infty(x,y) \le \sqrt{N} d_1(x,y), N\sqrt{N} d_1(x,y)$$

In particular

$$d_2(x,y) \le \sqrt{N} d_1(x,y) \Rightarrow B = \sqrt{N}$$

Q8) $C[0,\pi]$, the set of all continuous functions on interval $[0,\pi]$

d_2 metric with

$$d_2(f,g) = \left(\int_0^{\pi} (f(t) - g(t))^2 \, dt \right)^{1/2}$$

$$d_2^2(f,g) = \int_0^{\pi} \left(\frac{1}{\sqrt{\pi}} \sin nt - \frac{1}{\sqrt{\pi}} \sin mt \right)^2 dt$$

$$= \int_0^{\pi} \frac{1}{\pi} \sin^2 nt \, dt + \int_0^{\pi} \frac{1}{\pi} \sin mt \, dt - \frac{2}{\pi} \int_0^{\pi} \sin nt \sin mt \, dt$$

Solving

$$\int_0^{\pi} \frac{1}{\pi} \sin^2 nt \, dt = \int_0^{\pi} \frac{1 - \cos 2nt}{2\pi} \, dt$$

$$= \left[\frac{t}{2\pi} - \frac{\sin(2nt)}{4\pi n} \right]_0^{\pi}$$

$$= \frac{1}{2} - \frac{1}{4\pi n} \sin(2\pi n)$$

Solving

$$\int_0^{\pi} \frac{2}{\pi} \sin nt \sin mt \, dt = \int_0^{\pi} \frac{\cancel{2}}{\pi} \frac{\cos((n-m)t) - \cos((n+m)t)}{\cancel{2}} dt$$

$$= \frac{1}{\pi} \left[\frac{\sin(n-m)t}{n-m} - \frac{\sin(n+m)t}{n+m} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left(\left(\frac{\sin(n-m)\pi}{n-m} - \frac{\sin(n+m)\pi}{n+m} \right) - 0 \right)$$

$$= 0$$

$\int_0^{\pi} \frac{1}{\pi} \sin^2 mt$ is similar to the first one

So

$$d_2^2(f,g) = \frac{1}{2} - \frac{1}{4\pi n} \sin(2\pi n) + \frac{1}{2} - \frac{1}{4\pi m} \sin(2\pi m)$$

$\Rightarrow$

$$d_2(f,g) = \left(1 - \frac{1}{4\pi n}\sin(2\pi n) - \frac{1}{4\pi m}\sin(2\pi m)\right)^{1/2}$$