Vector and Complex Calculus L3

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2.3 Vector line Integral

- $C = C_1 \cup C_2 \Rightarrow \int_C = \int_{C_1} + \int_{C_2}$

- $\underline{f} \approx \underline{g} \Rightarrow \int \underline{f} \approx \int \underline{g}$ and

constant vector field $\underline{v}$: $\int_{\underline{a}}^{\underline{b}} \underline{v} \cdot d\underline{x} = \underline{v}(\underline{b} - \underline{a})$

Defn 2.3.2: The line integral of a vector field $\underline{f}$ along a oriented curve C is defined as

$$\int_C g(\underline{x}) \cdot d\underline{x} = \int \underline{g}(\underline{p}(t)) \cdot \dot{\underline{p}}(t)\, dt$$

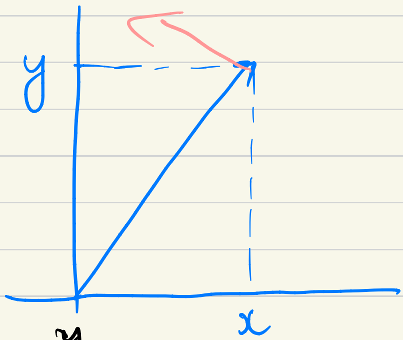
scalar product

$p(\underline{t})$ is a parametrization of oriented curve C

$t \in [a, b]$

Example 2.3.3:

Vector field $\underline{g}(x,y) = (-y, x)$



$y = x^2$

Integrate vector field $\underline{g}$ on parabola C

$\underline{P}(t) : [0,1] \to \mathbb{R}^2$

$$P(t) = (\underbrace{t}_{x}, \underbrace{t^2}_{y = x^2})$$

$$\underline{g}(\underline{p}(t)) = (-t^2, t)$$

$$\mathring{p}(t) = (1, 2t)$$

Scalar product $\underline{g}(\underline{p}(t)) \cdot \mathring{p}(t)$

$$\int_0^1 \underbrace{(-t^2, t)(1, 2t)}_{\in \mathbb{R}} \, dt = \int_0^1 -t^2 + 2t^2 \, dt$$

$$= \int_0^1 t^2 \, dt = \frac{1}{3}$$

Remark: We have chosen parametrization (t, t^2)
But any oriented parametric work should work
as they are equivalence classes and related

$p \, R \, q$

Reparametrizing

let $\quad t = h(\tau) \qquad h : [a,b] \to [a',b'] \quad \underline{\underline{h' \geq 0}}$

h must be increasing

$$\underbrace{\underline{q}(\tau)}_{\in [a',b']} = \underbrace{\underline{p}(h(\tau))}_{\in [a,b]}$$

integration by substitution

$$\int_a^b \underline{g}(\underline{p}(t)) \cdot \underline{\dot{p}}(t)\, dt$$

$$= \int_{a'}^{b'} \underline{g}\big(\underbrace{\underline{\dot{p}}(h(\tau))}_{q(\tau)}\big) \cdot \underbrace{\underline{\dot{p}}(h(\tau))\dot{h}(\tau)}_{\dot{q}(\tau)}\, d\tau$$

$$= \int_{a'}^{b'} \underline{g}(q(\tau)) \cdot \dot{q}(\tau)\, d\tau$$

↳ line integral

$\Rightarrow$ hence line integral does not change by repatesizing

Example:

$$\underline{g}(x,y) = (x,x)$$

C is the oriented line segment from $(0,0)$ to $(1,2)$

parametric curve $\underline{P}(t)$ is given by

$$\underline{P}(t) = (t, 2t) = t(1,2)$$

$$\underline{P} : [0,1] \to \mathbb{R}^2, \quad \underline{P}(0) = (0,0)$$

$$\underline{P}(1) = (1,2)$$

$$\int_0^1 (t,t) \cdot (1,2)\, dt = 3/2$$

Trying when $h' \leq 0$ (wrong method)

Let $h(\tau) = 1-\tau$

$\underline{g}(x,y) = (x,x)$

$q(\tau) = \underline{p}(h(\tau))$

$= (1-\tau, 2-2\tau)$

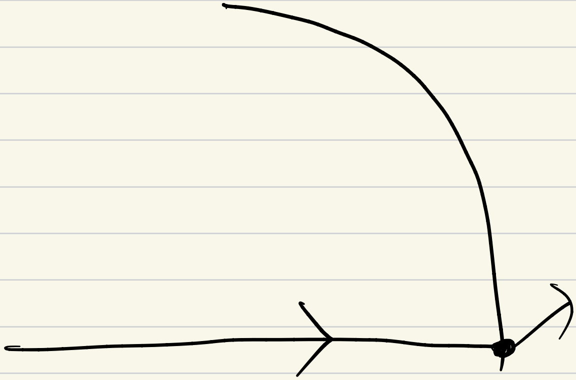
$\dot{q} = (-1, -2) = -1(1,2)$

$$\int_0^1 (1-\tau, 1-\tau)\cdot(-1,-2)\, d\tau = -3/2$$

So we get negative of original answer

Examples:

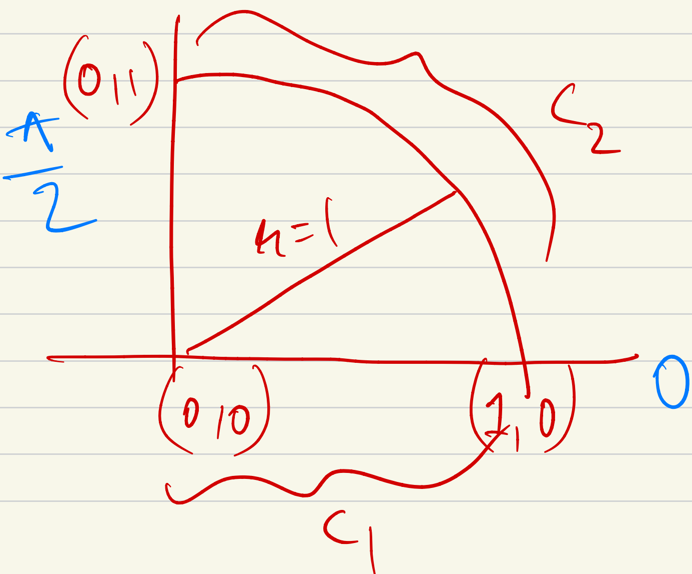
$g(x+y,0)$ but $g(x,y)$ is not a smooth curve



no tangent.

We use Reimann method (lecture 2)

Split into smooth subsections



$$\int_C = \int_{C_1} + \int_{C_2}$$

C_1 parameterised by $(t, 0)$, $P_1(t) = (t, 0)$

$$\int_{C_1} g(\underline{x}) \cdot d\underline{x} = \int_0^1 (x \cdot dx = 1/2$$

C_2 parameterized by

$$C_2 = (\cos\theta \quad \sin\theta)$$

$$P_2(t) = (\cos t \quad \sin t)$$

Integrating from $0 \to \pi/2$.

$$= \int_0^{\pi/2} (\cos\theta \quad t \sin\theta)(-\sin t \quad \cos t)\, dt$$

$$= \pi/2 \quad 0$$

Using formula

$\frac{1}{2}\sin(2t) = \sin t \cos t$

$$\int_0^{\pi/2} (\cos t + \sin t, 0)(-\sin t, \cos t)\, dt$$

$$\int_0^{\pi} (-\sin t\cos t - \sin^2 t)\, dt$$

$$= -\int_0^{\pi} \frac{1}{2}\sin 2t + \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= \frac{-1}{2}\int_0^{\pi} \sin 2t + 1 - \cos 2t$$

$$= \frac{-1}{2}\left[-\frac{1}{2}\cos 2t + t - \frac{1}{2}\sin 2t\right]$$

$$= -\frac{1}{2}\left(\left(-\frac{1}{2}+\pi+0\right)-\left(-1/2\right)\right)$$

$$= -\frac{1}{2}(\pi) = \boxed{-\frac{\pi}{2}}$$

2.4 Fundemental Thm of Calculus

Single Variable Case

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Multivariable Case

Let $f : \mathbb{R}^m \to \mathbb{R}$ then $\underline{\nabla} f : \mathbb{R}^m \to \mathbb{R}^m$

↳ vector feild

$$\int_C \underline{\nabla} f(\underline{x}) \cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$$

$\underline{a} \overset{C}{\rightsquigarrow} \underline{b}$ $\quad \underline{p}(a) = \underline{a}$

$p(b$

Proof:

continuously differentiable

Let $f \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R})$ & $\underline{p} \in \mathcal{C}^1([a,b], \mathbb{R}^m)$

parametric

$$\frac{d}{dt} f(\underline{p}(t)) = Df(\underline{p}(t))\, D(\underline{p}(t)) \in \mathbb{R}$$

(chain rule)

$1 \times m$ matrix $\quad m \times 1$ matrix

$\Rightarrow$ 1×1 matrix $\in \mathbb{R}$

$$\nabla f(\underline{p}(t)) \cdot \dot{\underline{p}}(t)$$

$$\int_C \nabla f(\underline{x}) \cdot d\underline{x} = \int_a^b \underline{\nabla} f(\underline{p}(t)) \cdot \dot{\underline{p}}(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left(f(\underline{p}(t))\right) dt$$

$$= f(\underline{p}(b)) - f(\underline{p}(a))$$

$$= f(\underline{b}) - f(\underline{a}) \quad \blacksquare$$

<u>Example</u>:

$$g(x,y) = (x, x+y)$$

happens to be a gradient.

Find scalar field $f \in C^1(\mathbb{R}^2, \mathbb{R})$ such that

$$\nabla f = g \quad \& \quad f(\underline{0}) = 0$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x+y)$$

$$\Rightarrow \frac{\partial f}{\partial x} = y \quad \& \quad \frac{\partial f}{\partial y} = x+y$$

$$\frac{\partial f}{\partial x} = y \quad \Rightarrow \int \partial f = \int y \, \partial x$$

$$= f = xy + g(y)$$

$$\frac{\partial f}{\partial y} = \cancel{x} + g'(y) = \cancel{x} + y$$

$$\Rightarrow g'(y) = y$$

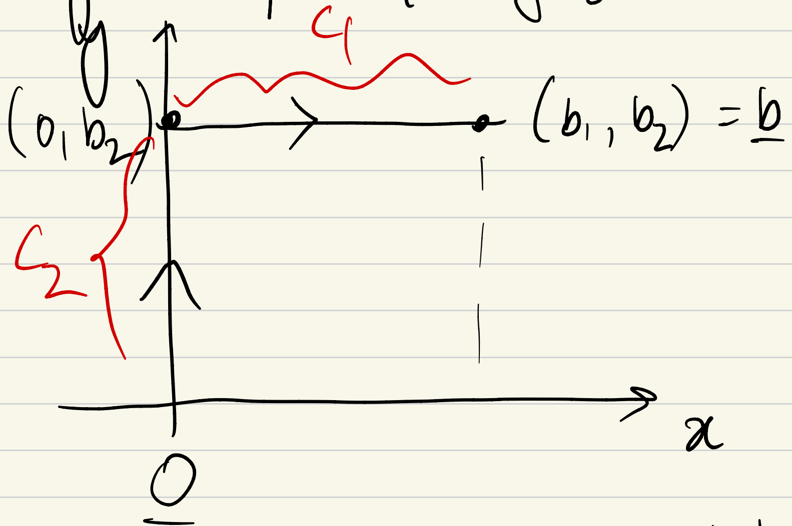
$$\Rightarrow g(y) = \frac{y^2}{2} + c$$

$$f = xy + y^2 + c$$

$$f(\underline{0}) = 0 + c \Rightarrow c = 0$$

$$f(x,y) = xy + \frac{y^2}{2}$$

Solving same problem using fundamental thm:



$$\underline{\nabla} f(x,y) = (y, x+y)$$

let $\underline{a} = \underline{0}$
↳ initial condition

$$f(\underline{b}) - f(\underline{a})$$
$$= f(\underline{b}) = \int_{0\,C_1}^{b_2} y\,dy + \int_{0\,C_2}^{b_2} b_2\,dx$$

by using parametrization
fill details in

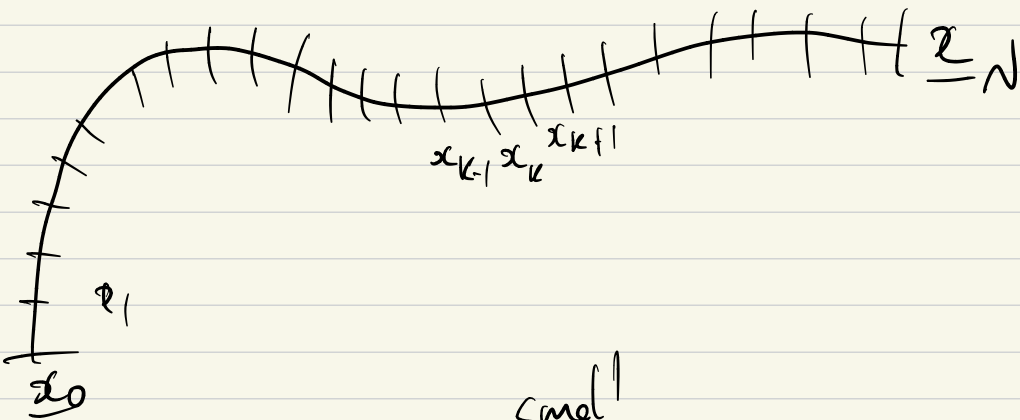
$$= \frac{1}{2} b_2^2 + b_1 b_2$$

Since $b = (b_1, b_2)$ is generic, replace with (x, y)

$$f(\underline{b}) = f(b_1, b_2) = b_1 b_2 + \frac{1}{2} b_2^2$$

$$\Rightarrow f(x, y) = xy + \frac{1}{2} y^2$$

2.5 Arc Length



split curve into small segments

$$\text{Length } C = \sum_{k=1}^{N} \text{length } c_k \approx \sum_{k=1}^{N} \|x_k - x_{k-1}\|$$

if splits are small enough

basically linearization

$$\approx \sum \|\dot{p}(t_k)\|(t_{k-1}-t_k)$$

looks like riemann sum

$$\approx \int_a^b \| \underline{p}(t) \| \, dt$$