Classical Dynamics Assignment

I2) $\underline{d}(t)$ is a time dependant vector

$\underline{d}(t)$ has a constant magnitude $\Rightarrow |\underline{d}(t)| = \text{constant}$

Let $|\underline{d}(t)| = K \in \mathbb{R} \Rightarrow |\underline{d}(t)|^2 = K^2$

$\left(|\underline{d}(t)|^2 = \underline{d}(t) \cdot \underline{d}(t)\right)$ scalar product

$$\Rightarrow \sum_{i=1}^{3} (d_i(t))^2 = K^2$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{i=1}^{3} (d_i(t))^2\right) = \frac{d}{dt} K^2$$

$$\Rightarrow \sum_{i=1}^{3} \frac{d}{dt}\left(d_i(t)^2\right) = 0$$

$$\Rightarrow \sum_{i=1}^{3} 2 d_i(t) \frac{d_i(t)}{dt} = 0$$

$$\Rightarrow 2 \sum_{i=1}^{3} d_i \dot{d}_i = 0$$

$$\Rightarrow \sum_{i=1}^{3} d_i \dot{d_i} = 0$$

$$\Rightarrow \underline{d}(t) \cdot \dot{\underline{d}}(t) = 0$$

$$\Rightarrow \theta = 90^\circ$$

Hence $\underline{d}(t)$ and $\dot{d}(t)$ are orthogonal.

(i) $\underline{r}(t) = \cos(\theta)\underline{i} + \sin\theta \underline{j}$

$$\dot{\underline{r}}(t) = -\dot{\theta}\sin\theta\underline{i} + \dot{\theta}\cos\theta\underline{j}$$

$$\underline{r}(t) \cdot \dot{\underline{r}}(t) = -\cancel{\dot{\theta}\sin\theta\cos\theta} + \cancel{\dot{\theta}\sin\theta\cos\theta}$$

$$= 0$$

(ii) $\underline{s}(t) = \sin\theta\cos(\phi)\,\underline{i} + \sin\theta\sin(\phi)\,\underline{j} + \cos\theta\,\underline{k}$

$$\dot{\underline{s}}(t) = \frac{d}{dt}\left(\sin\theta\cos\phi\right)\underline{i} + \frac{d}{dt}\left(\sin\theta\sin\phi\right)\underline{j}$$

$$+\frac{d}{dt}\cos\theta\,\underline{k}$$

$$= \left[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\phi\sin\theta\right]\underline{i} +$$

$$\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\cos\phi\sin\theta\right]\underline{j} - \dot{\theta}\sin\theta\,\underline{k}$$

$$\underline{s}(t)\cdot\dot{\underline{s}}(t) = (\sin\theta\cos\theta)(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\phi\sin\theta)$$
$$+ (\sin\theta\sin\phi)(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\cos\phi\sin\theta)$$
$$- \dot{\theta}\sin\theta\cos\theta$$

$$= \dot{\theta} \sin\theta \cos\theta \cos^2\phi - \dot{\phi} \sin^2\theta \sin\phi \cos\phi$$

$$+ \dot{\theta} \sin\theta \cos\theta \sin^2\phi + \dot{\phi} \sin^2\theta \sin\theta \cos\theta$$

$$- \dot{\theta} \sin\theta \cos\theta$$

$$= \dot{\theta}\left(\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\cos^2\phi - \sin\theta\cos\theta\right)$$

$$+ \dot{\phi}\left(\cancel{\sin^2\theta \sin\theta \cos\theta} - \cancel{\sin^2\theta \sin\theta \cos\theta}\right)$$

$$= \dot{\theta}\left(\sin\theta\cos\theta\left(\underbrace{\cos^2\phi + \sin^2\phi}_{=1}\right) - \sin\theta\cos\theta\right)$$

$$= \dot{\theta}\left(\cancel{\sin\theta\cos\theta} - \cancel{\sin\theta\cos\theta}\right)$$

$$= 0$$

$$\Rightarrow \underline{s}(t) \cdot \underline{\dot{s}}(t) = 0$$

I3) $\underline{a} = t^2\underline{i} + (2t+1)\underline{j} + t\underline{k}$

$\underline{b} = (t-1)\underline{i} - t\underline{j} + \underline{k}$

$\underline{c} = 0\underline{i} + \underline{j} + 0\underline{k}$

$\underline{a}\cdot\underline{b} = t^2(t-1) + (2t+1)(-t) + t(1)$

$= t^3 - t^2 - 2t^2 - t + t$

$\Rightarrow t^3 - 3t^2$

$\frac{d}{dt}(\underline{a}.\underline{b}) = \frac{d}{dt}(t^3 - 3t^2) = 3t^2 - 6t$

$\frac{d}{dt}(\underline{a}) = 2t\underline{i} + 2\underline{j} + \underline{k}$

$\frac{d}{dt}(\underline{b}) = 1\underline{i} - 1\underline{j} + 0\underline{k}$

$\underline{a}\cdot\frac{d}{dt}(\underline{b}) = t^2 - 2t - 1$

$\underline{b}\cdot\frac{d}{dt}(\underline{a}) = 2t^2 - 2t - 2t + 1$

$$\underline{a} \cdot \frac{d}{dt}\underline{b} + \underline{b} \cdot \frac{d}{dt}\underline{a} = 3t^2 - 6t$$

$$\Rightarrow \frac{d}{dt}(\underline{a} \cdot \underline{b}) = \underline{b} \cdot \frac{d}{dt}(\underline{a}) + \underline{a} \cdot \frac{d}{dt}(\underline{b})$$

(iii) $(\underline{a} \times \underline{b})_m = \varepsilon_{klm} a_k b_l$

$$(\underline{a} \times \underline{b})_1 = \varepsilon_{kl1} a_k b_l = \varepsilon_{231} a_2 b_3 + \varepsilon_{321} a_3 b_2$$

$$= a_2 b_3 - a_3 b_2$$

$$= (2t+1)1 - t(-t)$$

$$= t^2 + 2t + 1$$

$$(\underline{a} \times \underline{b})_2 = \varepsilon_{kl2} a_k b_l = \varepsilon_{312} a_3 b_1 + \varepsilon_{132} a_1 b_3$$

$$= a_3 b_1 - a_1 b_3$$

$$= t(t-1) - t^2$$

$$= -t$$

$$(\underline{a}\times\underline{b})_3 = \varepsilon_{kl3}\, a_k b_l = \varepsilon_{123}\, a_1 b_2 + \varepsilon_{213}\, a_2 b_1$$

$$= a_1 b_2 - a_2 b_1$$

$$= -t^3 - (2t+1)(t-1)$$

$$= -t^3 - (2t^2 + t - 2t - 1)$$

$$= -t^3 - 2t^2 + t + 1$$

$\Rightarrow$

$$(\underline{a}\times\underline{b}) = (t^2+2t+1)\,\underline{i} - t\underline{j} + (-t^3-2t^2+t+1)\,\underline{k}$$

$$\frac{d}{dt}(\underline{a}\times\underline{b}) = (2t+2)\,\underline{i} - \underline{j} + (-3t^2-4t+1)\,\underline{k}$$

$$\left(\frac{d\underline{a}}{dt}\right) \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2t & 2 & 1 \\ t-1 & -t & 1 \end{vmatrix}$$

$$\underline{i}\begin{vmatrix} 2 & 1 \\ -t & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2t & 1 \\ t-1 & 1 \end{vmatrix}$$

$$+ \underline{k}\begin{vmatrix} 2t & 2 \\ t-1 & -t \end{vmatrix}$$

$$= \underline{i}(2+t) - \underline{j}(2t-t+1)$$

$$+ \underline{k}(-2t^2 - 2t + 2)$$

$$= \underline{i}(2+t) - \underline{j}(t+1) + \underline{k}(-2t^2 \cancel{-2t} + 2)$$

$$\underline{a} \times \frac{d\underline{b}}{dt}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ t^2 & 2t+1 & t \\ 1 & -1 & 0 \end{vmatrix} =$$

$$\underline{i} \begin{vmatrix} 2t+1 & t \\ -1 & 0 \end{vmatrix} - \underline{j} \begin{vmatrix} t^2 & t \\ 1 & 0 \end{vmatrix} + \underline{k} \begin{vmatrix} t^2 & 2t+1 \\ 1 & -1 \end{vmatrix}$$

$$= \underline{i}\, t \quad - \underline{j}(-t) + \underline{k}\left(-t^2 - 2t - 1\right)$$

$$\frac{d\underline{a}}{dt} \times \underline{b} + \underline{a} \times \frac{d\underline{b}}{dt}$$

$$= (2t+2)\underline{i} + 1\underline{j} + \underline{k}\left(-3t^2 - 4t + 1\right)$$

$$\Rightarrow \frac{d}{dt}(\underline{a} \times \underline{b}) = \frac{d}{dt}(\underline{a}) \times \underline{b} + \underline{a} \times \frac{d}{dt}\underline{b}$$

I 4) (vi)

(a) $N_{ab}n_b = (\delta_{ab} - \varepsilon_{abc}n_c)n_b$

$$= \underbrace{\delta_{ab}n_b}_{n_a \text{ as } \delta_{ab} = \begin{cases} 1 & b=a \\ 0 & b\neq a \end{cases}} - \varepsilon_{abc}n_b n_c$$

$$= n_a - \varepsilon_{abc}n_b n_c$$

$$= n_a - \begin{pmatrix} \cancel{\varepsilon_{a11}n_1n_1} + \cancel{\varepsilon_{a22}n_2n_2} + \cancel{\varepsilon_{a33}n_3n_3} \\ + \cancel{\varepsilon_{a21}n_2n_1} + \cancel{\varepsilon_{a12}n_1n_2} + \cancel{\varepsilon_{a31}n_3n_1} \\ + \cancel{\varepsilon_{a13}n_1n_3} + \cancel{\varepsilon_{a23}n_2n_3} + \cancel{\varepsilon_{a32}n_3n_2} \end{pmatrix}$$

(as $\varepsilon_{abc} = -\varepsilon_{acb}$ as changing order of 1 element changes parity of the permutation)

$= n_a$

(b) $N_{ab} = (\delta_{ab} - \varepsilon_{abc} n_c)$

$$N_{ad} = (\delta_{ad} - \varepsilon_{adc} n_c)$$

$$N_{bd} = (\delta_{bd} - \varepsilon_{bdc} n_c)$$

$$N_{ad} N_{bd} = (\delta_{ad} - \varepsilon_{adc} n_c)(\delta_{bd} - \varepsilon_{bdc} n_c)$$

$$= \left(\delta_{ad}\delta_{bd} - \varepsilon_{adc}\delta_{bd} n_c - \varepsilon_{bdc}\delta_{ad} n_c - \varepsilon_{adc}\varepsilon_{bdc} \underbrace{n_c n_c}_{\substack{\text{scalar product} \\ =1 \text{ as unit} \\ \text{vector}}}\right)$$

$$= \delta_{ab} - n_c(\varepsilon_{adc}\delta_{bd} - \varepsilon_{bdc}\delta_{bd}) - \varepsilon_{adc}\varepsilon_{bdc}$$

$$= \delta_{ab} - n_c \underbrace{\begin{pmatrix} \delta_{a1}\varepsilon_{1bc} + \delta_{a2}\varepsilon_{2bc} + \varepsilon_{a3}\varepsilon_{2bc} \\ + \delta_{b2}\varepsilon_{1bc} + \delta_{b2}\varepsilon_{2bc} + \delta_{b3}\varepsilon_{2bc} \end{pmatrix}}_{= \begin{cases} 0 & \text{if } a \neq b \\ 0 & \text{if } a = b \text{ as } \varepsilon_{aac} \text{ or } \varepsilon_{bbc} = 0 \end{cases}}$$

$$+ \varepsilon_{adc}\varepsilon_{bdc}$$

$$= \delta_{ab} + \varepsilon_{adc}\varepsilon_{bdc} \qquad (*)$$

$$= \delta_{ab} + \begin{pmatrix} \cancel{\varepsilon_{a11}\varepsilon_{b11}} + \cancel{\varepsilon_{a22}\varepsilon_{b22}} + \cancel{\varepsilon_{a33}\varepsilon_{b33}} + \\ \varepsilon_{a21}\varepsilon_{b21} + \varepsilon_{a12}\varepsilon_{b12} + \varepsilon_{a31}\varepsilon_{b21} + \\ \varepsilon_{a13}\varepsilon_{b13} + \varepsilon_{a23}\varepsilon_{b23} + \varepsilon_{a32}\varepsilon_{a23} \end{pmatrix}$$

$$= \delta_{ab} + \begin{pmatrix} 2\varepsilon_{a21}\varepsilon_{b21} + 2\varepsilon_{a31}\varepsilon_{b31} \\ + 2\varepsilon_{a23}\varepsilon_{b23} \end{pmatrix}$$

$$\left(\text{as } \varepsilon_{adc}\varepsilon_{bdc} = (-\varepsilon_{acd})(-\varepsilon_{bcd}) = \varepsilon_{acd}\varepsilon_{bcd} \right)$$

$$= \delta_{ab} + 2\underbrace{\left(\varepsilon_{a21}\varepsilon_{b21} + \varepsilon_{a31}\varepsilon_{b31} + \varepsilon_{a23}\varepsilon_{b23} \right)}_{= \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a\neq b \end{cases} = \delta_{ab}}$$

$$= \delta_{ab} + 2\delta_{ab}$$

$$= 3\delta_{ab}$$

II 1) $\underline{r} = ut\cos\theta\,\underline{i} + \left(ut\sin\theta - \tfrac{1}{2}gt^2\right)\underline{k}$

$\dot{\underline{r}}(t) = u\cos\theta\,\underline{i} + (u\sin\theta - gt)\underline{k}$

$\ddot{\underline{r}}(t) = -g\underline{k}$

(b) Maximum height when vertical velocity is 0

$\dot{\underline{r}}(t_1) = u\cos\theta\,\underline{i} + (u\sin\theta - gt_1)\underline{k}$

$u\sin\theta - gt_1 = 0 \Rightarrow t_1 = \dfrac{u\sin\theta}{g}$

Max height $= \underline{r}(t_1) = \dfrac{u^2\sin\theta\cos\theta}{g}\underline{i} + \left(\dfrac{u^2\sin^2\theta}{g} + \dfrac{1}{2}g\cdot\dfrac{u^2\sin^2\theta}{g^2}\right)\underline{k}$

$\Rightarrow \underline{r}_1(t_1) = \dfrac{u^2}{2g}\left(\sin2\theta\,\underline{i} \quad \sin^2\theta\,\underline{k}\right)$

range when vertical component is 0 (at $t = t_2$)

$$\Rightarrow u t_2 \sin\theta - \frac{g t_2^2}{2} = 0$$

$$\Rightarrow t_2 \left(u\sin\theta - g t_2/2 \right) = 0$$

$$\Rightarrow t_2 = 0 \quad \text{or} \quad t_2 = \frac{2u\sin\theta}{g}$$

$$\Rightarrow t_2 = \frac{2u\sin\theta}{g}$$

range D is

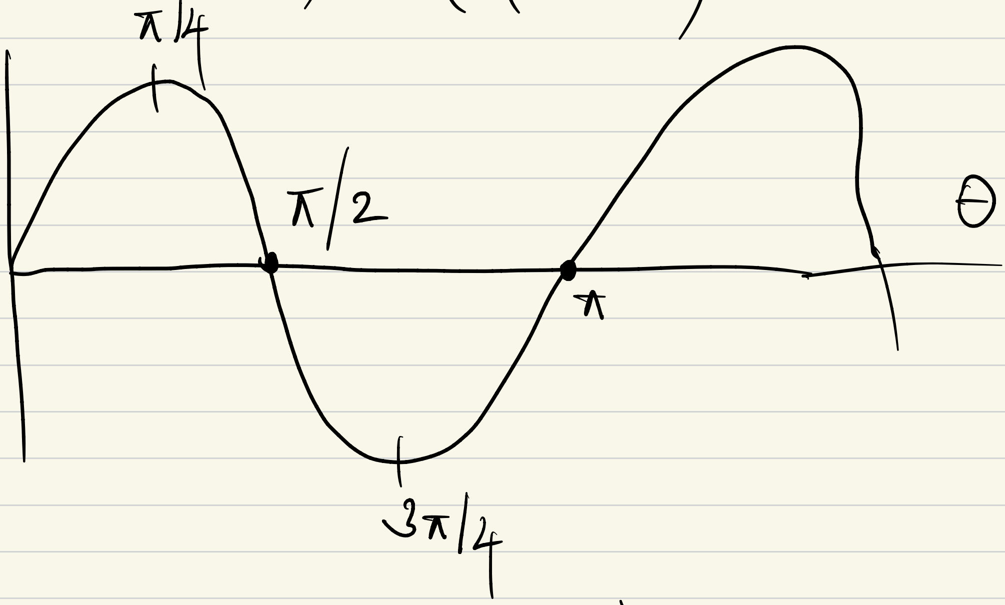
$$D = \underline{r}(t_2) = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{u^2}{g} \sin 2\theta$$

$$\frac{u^2}{g}\sin 2\theta < \frac{u^2}{g} \Rightarrow \sin 2\theta < 1$$

Now, $\sin 2\theta = 1 \Rightarrow \theta = \pi/4$.

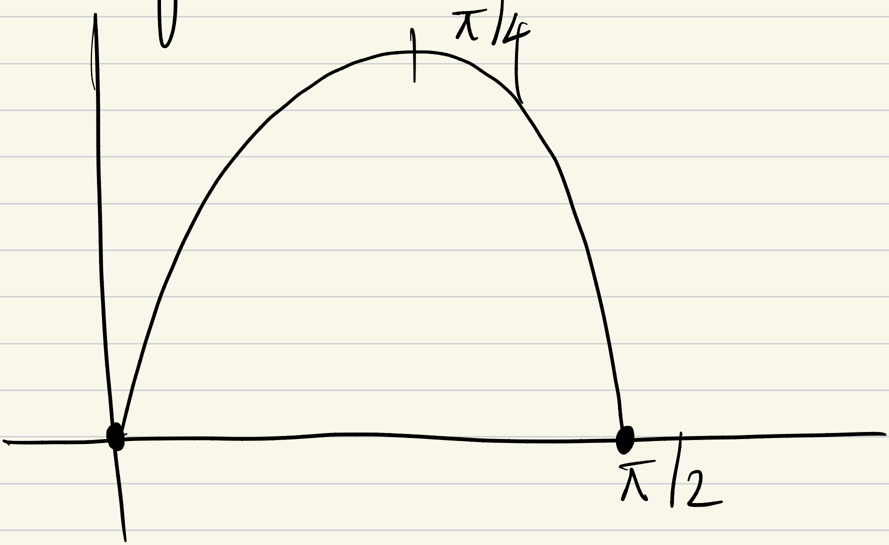
And $\sin(2\theta) = \sin(2\theta + 2k\pi)$

$$\Rightarrow \sin(2(\theta + k\pi))$$



$\sin 2\theta < 1$ when $\theta = \mathbb{R} \setminus \{\dots, -\frac{3\pi}{4}, \frac{\pi}{4}, \dots\}$

Taking $\theta \in [0, \pi/2]$



$\sin 2\theta < 1$ when $\theta \in [0, \pi/4) \cup (\pi/4, \pi/2)$

As we can see, for any given u, we have 2 values of θ for which $\sin 2\theta < 1$

(c) $\underline{r}(t) = ut\cos\theta\,\underline{i} + \left(ut\sin\theta - gt^2/2\right)\underline{k}$

let $x = ut\cos\theta \Rightarrow \dfrac{x}{u\cos\theta}$

let $y = ut\sin\theta - \dfrac{gt^2}{2}$

$$= \frac{\cancel{u}\sin\theta\, x}{\cancel{u}\cos\theta} - \frac{g}{2}\cdot\frac{x^2}{u^2\cos^2\theta}$$

$$= \tan\theta\, x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

g, θ and u are constant

let $\tan\theta = A$ $\quad \dfrac{g}{2u^2\cos^2\theta} = B$

$\Rightarrow y = Ax - Bx^2 \Rightarrow$ quadratic eqn

$\Rightarrow$ eqn of parabola

III 3)

let $x = a\cos\omega t \sin\Omega t$

$y = a\sin\omega t \sin\Omega t$

$z = a\cos\Omega t$

$$x^2+y^2+z^2 = a^2\cos^2\omega t\sin^2\Omega t + a^2\sin^2\omega t\sin^2\Omega t + a^2\cos^2\Omega t$$

$$= a^2\left(\cos^2\omega t\sin^2\Omega t \; \sin^2\omega t\sin^2\Omega t + \cos^2\Omega t\right)$$

$$= a^2\left(\sin^2\Omega t\left(\cos^2\omega t + \sin^2\omega t\right) + \cos^2\Omega t\right)$$

$$= a^2\left(\sin^2\Omega t + \cos^2\Omega t\right)$$

$$\Rightarrow x^2+y^2+z^2 = a^2 = |\underline{a}|^2$$

↳ equation of spheres radius $= a$

center $= (0,0,0)$

III 3)

a) $\underline{r} = a\cos\omega t\sin\Omega t\,\underline{i} + a\sin\omega t\sin\Omega t\,\underline{j} + a\cos\Omega t\,\underline{k}$

$$\dot{\underline{r}}(t) = \left(-a\omega\sin\omega t\sin\Omega t + a\Omega\cos\omega t\cos\Omega t\right)\underline{i}$$

$$+\left(a\omega\cos\omega t\sin\Omega t + a\Omega\sin\omega t\cos\Omega t\right)\underline{j}$$

$$- a\Omega\sin\Omega t\,\underline{k}$$

speed $= |\dot{\underline{r}}(t)|$

$$|\dot{\underline{r}}|^2 = a^2\omega^2\sin^2\omega t\sin^2\Omega t + a^2\Omega^2\cos^2\omega t\cos^2\Omega t$$

$$\cancel{-2a^2\omega\Omega\sin\omega t\sin\Omega t\cos\omega t\cos\Omega t}$$

$$+a^2\omega^2\cos^2\omega t\sin^2\Omega t + a^2\Omega^2\sin^2\omega t\cos^2\Omega t$$

$$\cancel{+2a^2\omega\Omega\cos\omega t\sin\Omega t\sin\omega t\cos\Omega t}$$

$$+a^2\Omega^2\sin^2\Omega t$$

$$= a^2\omega^2\sin^2\Omega t\left(\sin^2\omega t + \cos^2\omega t\right) +$$
$$a^2\Omega^2\cos^2\Omega t\left(\sin^2\omega t + \cos^2\omega t\right)$$
$$+ a^2\Omega^2\sin^2\Omega t$$

$$= a^2\omega^2\sin^2\Omega t + a^2\Omega^2\left(\sin^2\Omega t + \cos^2\Omega t\right)$$

$$= a^2\left(\omega^2\sin^2\Omega t + \Omega^2\right)$$

$$\Rightarrow |\dot{\underline{r}}|^2 = a^2\left(\omega^2\sin^2\Omega t + \Omega^2\right)$$

$$\Rightarrow |\dot{r}| = a\left(\Omega^2 + \omega^2\sin^2\Omega t\right)^{1/2}$$

(b) max/min speed

$$\frac{d}{dt}|\dot{x}| = 0 \Rightarrow \frac{d}{dt} a\left(\Omega^2 + w^2\sin^2\Omega t\right)^{1/2} = 0$$

$$\Rightarrow a\frac{d}{dt}\left(\Omega^2 + w^2\sin^2\Omega t\right)^{1/2} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\Omega^2 + w^2\sin^2\Omega t\right)^{1/2} = 0$$

$$\Rightarrow \frac{1}{2}\left(\Omega^2 + \sin^2\Omega t\right)^{-1/2} \cdot w^2\left(2\sin\Omega t\right) \cdot \Omega\cos\Omega t = 0$$

$$\Rightarrow 2w^2\Omega\sin\Omega t\cos\Omega t = 0$$

$$\Rightarrow \sin 2\Omega t = 0$$

$$\Rightarrow 2\Omega t = 0, \pi$$

$$\Rightarrow t = 0, \frac{\pi}{2\Omega}$$

Max when $t = \dfrac{\pi}{2\Omega}$

Min when $t=0$

(c) when $t=0$,

$x=0$, $y=0$, $z=a$

So at point $(0,0,a)$ is when speed is min

when $t = \dfrac{\pi}{2\pi}$

$x = a\cos\left(\dfrac{\omega}{\Omega}\dfrac{\pi}{2}\right)$ $y = a\sin\dfrac{\omega}{\Omega}\dfrac{\pi}{2}$, $z=0$.

at point $\left(a\cos\left(\dfrac{\omega}{\Omega}\dfrac{\pi}{2}\right), a\sin\left(\dfrac{\omega}{\Omega}\dfrac{\pi}{2}\right), 0\right)$

when speed is max