Classical Dynamics - (teaching weeks 1-8) - Semester 1 2023

A selection of questions to try

Assignments will be set from questions on this sheet, which will be updated as the semester progresses (there will be four sections for this part of the module, I, II, III, IV).

- **Assignment 1:**

  Questions I 2, I 3 (ii, iii), I 4(vi), II 1, II 3

  Submit (upload to Moodle) by: 11.00 on Monday 9th October 2023.

- **Assignment 2:**

  Questions II 7, II 8, II 11, II 13.

  Submit (upload to Moodle) by: 11.00 on Monday 23rd October 2022.

- **Assignment 3:**

  To be announced

  Submit (upload to Moodle) by: 11.00 on Monday 13th November 2023.

- **Assignment 4:**

  To be announced

  Submit (upload to Moodle) by: 11.00 on Monday 27th November 2022.

Note: The assignments and due dates will be repeated on Moodle.

Note: Besides the feedback provided on submitted written work you have the opportunity during Seminars to discuss these questions (both those assigned and already returned and the others on the sheet), and to raise any other queries you might have with the module.

Note: Seminars are scheduled on Wednesdays, Thursdays and Fridays in teaching weeks 3,5,7,9. If your work is handed in on time (ie before 11.00 on Monday of the same week) it will be marked and the mark returned to you via Moodle before the seminar.

Section I

I1. Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are a set of mutually orthogonal unit vectors (as usual), calculate:

$$\text{(i)}\ \mathbf{a}\cdot\mathbf{b}, \quad \text{(ii)}\ \mathbf{a}\times\mathbf{b}.$$

(iii) What is the angle between the two vectors $\mathbf{a}$, $\mathbf{b}$?
(iv) What is the area of a triangle two of whose sides are represented by $\mathbf{a}$, $\mathbf{b}$?

If $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ calculate

$$\text{(v)}\ \mathbf{a}\cdot\mathbf{b}\times\mathbf{c}, \quad \text{(vi)}\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}), \quad \text{(vii)}\ (\mathbf{a}\times\mathbf{b})\times\mathbf{c}.$$

Using the result from (vi), verify the formula: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$.

I2. The vector $\mathbf{d}(t)$ is time-dependent but of constant length (ie $|\mathbf{d}|$ is constant). Show that the time derivative of $\mathbf{d}$ is orthogonal to $\mathbf{d}$. Illustrate this fact using the vectors

(i) $\mathbf{r}(t) = \cos(\theta)\ \mathbf{i} + \sin(\theta)\ \mathbf{j}$;
(ii) $\mathbf{s}(t) = \sin(\theta)\cos(\phi)\ \mathbf{i} + \sin(\theta)\sin(\phi)\ \mathbf{j} + \cos(\theta)\ \mathbf{k}$,

where the angles θ, ϕ depend upon time.

I3. Given $\mathbf{a} = t^2\,\mathbf{i} + (2t+1)\,\mathbf{j} + t\,\mathbf{k}$, $\mathbf{b} = (t-1)\,\mathbf{i} - t\,\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{j}$, find

$$\text{(i)}\ \frac{d}{dt}(\mathbf{a}+\mathbf{b}) \qquad \text{(ii)}\ \frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) \qquad \text{(iii)}\ \frac{d}{dt}(\mathbf{a}\times\mathbf{b}) \qquad \text{(iv)}\ \frac{d}{dt}(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}).$$

In each of cases (ii-iv) verify that the 'derivative of a product' rule holds.

I4. Show (using the summation convention)

(i) $\delta_{ab}\delta_{bc} = \delta_{ac}$, (ii) $\delta_{aa} = 3$, (iii) $\epsilon_{aab} = 0$, (iv) $\epsilon_{abc}x_b x_c = 0$, $a, b, c = 1, 2, 3$.

(v) If M is a symmetric matrix with components M_{ab}, $a, b = 1, 2, 3$, show that

$$\text{(a)}\ \delta_{ab}M_{ab} = \mathrm{Tr}(M); \quad \text{(b)}\ M_{ab}\epsilon_{abc} = 0,\ c = 1, 2, 3.$$

(vi) The matrix N is defined in terms of the components n_a, $a = 1, 2, 3$ of a unit vector $\mathbf{n}$ by $N_{ab} = (\delta_{ab} - \epsilon_{abc}n_c)$.

(a) Show that $N_{ab}n_b = n_a$, and (b) simplify $N_{ad}N_{bd}$.

I5. A rotation is represented (using the summation condition in the third term) by

$$R_{ab}(\theta), \mathbf{n}) = \cos\theta\delta_{ab} + (1-\cos\theta)\,n_a n_b + \sin\theta\epsilon_{abc}n_c,$$

where n_a, $a = 1, 2, 3$ are the components of a unit vector $\mathbf{n}$. Verify that

$$R_{ac}R_{bc} = \delta_{ab}, \quad R_{ab}n_b = n_a, \quad \text{and} \quad \det(R) = 1.$$

If you find this daunting, you could try special cases first. For example:

(i) $\theta = \pi$ for which $R_{ab} = -\delta_{ab} + 2n_a n_b$, or (ii) $\theta = \pi/2$ for which $R_{ab} = n_a n_b + \epsilon_{abc}n_c$.

Section II

II 1. The position vector of a projectile moving near the Earth's surface is given by

$$\mathbf{r} = ut\cos\theta\ \mathbf{i} + (ut\sin\theta - (1/2)gt^2)\ \mathbf{k}$$

where u, θ, g, $\mathbf{i}$, $\mathbf{k}$ are constant with $\mathbf{i}\cdot\mathbf{k} = 0$.

(a) Find the vectors representing its velocity and acceleration.

(b) Find its maximum height and horizontal distance travelled (range), and show that if the horizontal range is less than u^2/g there are two possible angles θ to achieve the same range for a given u.

(c) Show that the trajectory is a parabola.

(Note, $\mathbf{i}$ and $\mathbf{k}$ are unit vectors in a horizontal and the vertical directions, respectively.)

II 2. A particle is moving in a straight line through the point A (represented relative to the origin O by the vector $\mathbf{a}$) with a constant velocity $\mathbf{v}$, where $\mathbf{a}$, $\mathbf{v}$ are not proportional to each other (ie their directions are different). If d is the distance from the particle to the origin when it is closest to the origin, show that

$$d^2 = |\,\mathbf{a}|^2 - \frac{(\mathbf{a}\cdot\mathbf{v})^2}{|\,\mathbf{v}|^2}.$$

II 3. The position vector of a particle is given by

$$\mathbf{r} \;=\; a\cos\omega t\ \sin\Omega t\ \mathbf{i} \;+\; a\sin\omega t\ \sin\Omega t\ \mathbf{j} +\; a\cos\Omega t\ \mathbf{k},$$

where a, ω and Ω are positive constants. Show that the particle moves on a sphere of radius a.

(a) Find the velocity vector of the particle and show that its speed is given by

$$a(\Omega^2 + \omega^2\sin^2\Omega t)^{1/2}.$$

(b) Determine times at which the maximum and minimum speeds are achieved?

(c) Where on the sphere does the particle achieve its maximum speed and where does it achieve its minimum speed?

II 4. A particle of mass m has an equation of motion

$$m\ddot{\mathbf{r}} = -\kappa\,\mathbf{r},$$

where $\kappa > 0$ is constant. Consider the time derivative of the particle's kinetic energy (ie $\frac{1}{2}m|\dot{\mathbf{r}}|^2$) and hence determine an expression for the total conserved energy. Verify that the angular momentum vector $\mathbf{J} = m\,\mathbf{r}\times\dot{\mathbf{r}}$ is also conserved.

II 5. A particle of mass m is moving on the x-axis with equation of motion

$$m\ddot{x} = \lambda\sin(\kappa x),$$

where λ, κ are constants. Find an expression for the total conserved energy of the particle.

II 6. Using the expression for the Lorentz force, a charged particle of mass m and charge q moving in a magnetic field $\mathbf{B}$ is described by the equation of motion:

$$m\ddot{\mathbf{r}} = q\ \dot{\mathbf{r}}\times\mathbf{B}.$$

Show that the kinetic energy of the particle is conserved.

II 7. Let $\mathbf{r}(t)$ be the position vector of a particle with respect to the origin O and let $r = |\,\mathbf{r}|$. Show that

$$\frac{dr}{dt} = \frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}.$$

A particle of mass m moves in such a way that

$$m\ddot{\mathbf{r}} = \kappa\,\frac{\dot{\mathbf{r}}\times\mathbf{r}}{r^3}, \quad r \neq 0,$$

where κ is a constant. Show that its speed must be constant but its angular momentum $\mathbf{J} = m\,\mathbf{r}\times\dot{\mathbf{r}}$ is not a constant vector. Show that for a suitable choice of a constant ρ an adjusted angular momentum given by $\mathbf{L} = \mathbf{J} + (\rho\,\mathbf{r}/r)$ is conserved (ie $\dot{\mathbf{L}} = 0$).

II 8. A molecule consisting of two atoms of mass m, position vectors $\mathbf{r}_1$ and $\mathbf{r}_3$, and one atom of mass $M > m$, with position vector $\mathbf{r}_2$, is modelled by the equations of motion:

$$m\ddot{\mathbf{r}}_1 = \kappa|\,\mathbf{r}_2 - \mathbf{r}_1|^2(\mathbf{r}_2 - \mathbf{r}_1),\ m\ddot{\mathbf{r}}_3 = \kappa|\,\mathbf{r}_2 - \mathbf{r}_3|^2(\mathbf{r}_2 - \mathbf{r}_3),$$

and

$$M\ddot{\mathbf{r}}_2 = -\kappa|\,\mathbf{r}_2 - \mathbf{r}_1|^2(\mathbf{r}_2 - \mathbf{r}_1)\ -\kappa|\,\mathbf{r}_2 - \mathbf{r}_3|^2(\mathbf{r}_2 - \mathbf{r}_3),$$

where κ is a positive constant. Define the centre of mass of the molecule and show that it moves with constant velocity. Define the total angular momentum of the molecule and show it is conserved. Find an expression for the total conserved energy of the molecule.

II 9. A molecule with four atoms (masses $m_i,\ i = 1, 2, 3, 4$) is modelled by imagining that the atoms are connected to each other (ie each is connected to the other three) by identical light springs. Then, the force between a pair of atoms (located at $\mathbf{r}_i$, $\mathbf{r}_j$, with respect to the origin O) is directed along the line joining them and given by

$$\mathbf{F}_{ij} = -\lambda\,(|\,\mathbf{r}_i - \mathbf{r}_j| - a)\,\frac{\mathbf{r}_i - \mathbf{r}_j}{|\,\mathbf{r}_i - \mathbf{r}_j|},$$

where $\lambda > 0$ is a constant and a is the undisturbed length of a spring. Write down the equation of motion for each atom and find the equation of motion for the centre of mass of the molecule. Find expressions for the total conserved energy (kinetic energy plus potential energy) and the total angular momentum about O of the molecule.

[Hint: if you find this tricky try first for two (one spring) or three particles (three springs).]

II 10. A particle moves in a circular orbit in a plane under the influence of a central force directed towards the centre of the circle. In plane polar coordinates, the orbit is described by $(r, \theta) = (R, \omega t)$ and the potential for the force is $V(r)$. Examine the motion when the particle is disturbed slightly from the circular orbit (but remains in the same plane) by expressing the equation of motion in terms of small quantities $r = R + \epsilon$ and $\theta = \omega t + \eta$ and keeping terms linear in ϵ, η. Find a condition on the potential that ensures the orbit is stable and verify that it is satisfied if the force is gravitational (ie 'inverse square law').

II 11. Assuming that the eccentricity of a planet's elliptic orbit is small $(0 < e \ll 1)$, show that the ratio of the times taken by the planet to travel over the two parts of its orbit separated by the minor axis is approximately $1 + (4e/\pi)$.

Hint: when the orbit is described in polar coordinates the quantity $h = r^2\dot{\theta}$ is a constant. Consider how this can be related to the area of a portion of the orbit and then consider the areas of the two parts of the orbit separated by the minor axis.

II 12. Halley's comet moves in a very eccentric orbit around the Sun with $e = 0.97$ and an orbital period of 76 years (on average). Using Kepler's third law, together with the corresponding data for the earth's orbit, find (i) the semi-major axis of the orbit, and (ii) the distance of closest approach to the sun.

II 13. A satellite moves with a speed V in a circular orbit around the moon. The radius of the orbit is twice the radius of the moon and the centre of the orbit coincides with the moon's centre. Its speed is suddenly reduced to λV where $0 < \lambda < 1$; find the value of λ so that the new orbit just touches the surface of the moon. (Assume the change can be achieved in a short time and the motion of the satellite remains in the same plane).

II 14. The Apollo Programme's Command Module had a circular orbit around the Moon with a speed V at a height of about 300 km above the Moon's surface. In order to return to Earth after its mission its speed must be increased to λV in order to escape the Moon. Calculate the least value of λ to ensure this (assume the change can be achieved in a short time and the motion of the satellite remains in the same plane).
[Consult: https://www.nasa.gov/mission_ pages/apollo/missions/index.html]