EXERCISES FOR VECTOR CALCULUS 2023/24

Acknowlegments

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Exercises

(1) Find the angle between the vectors:

(a) $\underline{a} = (-1, 2), \quad \underline{b} = (\frac{1}{2}, -1)$.
(b) $\underline{a} = (1, 2, 3), \quad \underline{b} = (1, 1, -1)$.
(c) $\underline{a} = (1, -1, 0), \quad \underline{b} = (0, 1, 1)$.
(d) $\underline{a} = (1, 2, 1), \quad \underline{b} = (1, 1, 0)$.

(2) For two vectors $\underline{a}, \underline{b} \in \mathbb{R}^3$, with angle θ between them, what is $(\underline{a} \cdot \underline{b})^2 + \|\underline{a} \times \underline{b}\|^2$?

(3) For each of the following vector fields, $\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, determine whether it may be a gradient. If so, find a scalar field f such that $\nabla f = \underline{g}$.

(a) $\underline{g}(x, y) = (y, -x)$
(b) $\underline{g}(x, y) = (5x^4 + y, x - 12y^3)$
(c) $\underline{g}(x, y) = (3x^2 \cos y, -x^3 \sin y)$
(d) $\underline{g}(x, y) = (e^y \sec^2 x, e^y \tan x)$
(e) $\underline{g}(x, y) = (xy, -y)$
(f) $\underline{g}(x, y) = (x^2 + y^2, \cos y)$

(4) Make a sketch of the region $R \subset \mathbb{R}^2$ and evaluate the integral $\iint_R f \, dA$ for:

(a) $f(x, y) = 10 + 2x^2 + 2y^2$ and R is the triangle bounded by $y = x$, $x = 2y$, and $y = 2$.

(b) $f(x, y) = \dfrac{y}{x^2 + y^2}$ and R is the trapezoid bounded by $y = x$, $y = 2x$, $x = 1$, and $x = 2$.

(c) $f(x, y) = y$ and R is bounded by $y = 4 - x^2$ and $y = 4 - x$.

* (d) $f(x, y) = \dfrac{y}{1 + x}$ and R is bounded by $y = 0$, $y = \sqrt{x}$, and $x = 4$.

(5) Make a sketch of the region $R \subset \mathbb{R}^2$ and evaluate the integral $\iint_R f \, dA$ using polar coordinates for:

(a) $f(x, y) = x$ and R is the region in the first quadrant bounded by $x^2 + y^2 = 25$, $3x = 4y$, and $y = 0$.

(b) $f(x, y) = 1$ and R is the region in the first quadrant below the line $y = x$ and inside the circle of radius 1 centered at $(0, 1)$.

* **(6)** Compute and fully simplify:

(a) $\displaystyle\sum_{k=1}^{3} \frac{1}{k}$ (b) $\displaystyle\sum_{k=1}^{n} 1$, where $n \in \mathbb{N}$.

(7) For the maps and vector defined by

(i) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{f}(x, y, z) = (x^3 + y^2 - 2z,\ x - 2y^2 + z^3)$,
$\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{g}(u, v) = \left(e^{2u+v},\ e^{u-2v}\right)$,
$\underline{a} = (1, 1, 1)$

(ii) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{f}(x, y, z) = (x + yz, x - yz)$,
$\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{g}(u, v) = (u + v, uv)$,
$\underline{a} = (0, 1, 2)$

(iii) $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^3$, $\underline{f}(x, y, z) = (xy^2z^2,\ z^2 \sin y,\ x^2 e^y)$,
$\underline{g} : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{g}(u, v, w) = (u^2 + v + w,\ 2u + v + w^2)$,
$\underline{a} = (x, 0, z)$

do the following:

(a) Compute the matrices $D\underline{f}(x, y, z)$ and $D\underline{g}(u, v)$.
(b) Work out the composition $\underline{F}(x, y, z) = \underline{g}(\underline{f}(x, y, z))$.
(c) Compute $D\underline{F}(\underline{a})$.
(d) Compare this with the matrix product $D\underline{g}(\underline{b})\, D\underline{f}(\underline{a})$, where $\underline{b} = \underline{f}(\underline{a})$.

(8) Let $f_1, f_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ be any continuously differentiable functions. Let $g : \mathbb{R}^2 \to \mathbb{R}$, $g(u, v) = uv$. Define $\underline{f} = (f_1, f_2)$.

(a) Compute the matrix product $Dg(\underline{f}[\underline{x}])D\underline{f}(\underline{x})$.
(b) Compute the composition $F := g \circ \underline{f}$ and its derivative DF directly.

By the chain rule, these should give the same result.

(9) A scalar field f is *homogeneous of degree* c if it satisfies

$$f(\lambda\underline{x}) = \lambda^c f(\underline{x})$$

for all $\lambda > 0$ and $\underline{x} \in \mathbb{R}^n$.

Let f be such a scalar field. For any fixed $\underline{x} \in \mathbb{R}^n$, define $g(\lambda) = f(\lambda\underline{x})$, use the chain rule, and then set $\lambda = 1$ to show that

$$\underline{x} \cdot \nabla f(\underline{x}) = c\, f(\underline{x}). \tag{$*$}$$

(10) Verify that eq. $(*)$ holds for each of the following functions, and give the value of the degree c in each case.

(a) $f(x,y,z) = xyz$; (b) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$; (c) $f(x,y) = \dfrac{xy}{x^2+y^2}$.

(11) Find a parametrization for each of the following curves:

(a) $C \subset \mathbb{R}^2$ is the part of the curve defined by $y^2 = x$ from $(0,0)$ to $(1,1)$.

* (b) $C \subset \mathbb{R}^2$ is the part of the ellipse defined by $x^2 + 4y^2 = 4$ anticlockwise from $(2,0)$ to $(0,1)$.

(c) $C \subset \mathbb{R}^2$ is the part of the graph $y = \cos x$ from $(\frac{\pi}{2}, 0)$ to $(-\frac{\pi}{2}, 0)$.

(d) $C \subset \mathbb{R}^3$ satisfying $z = x^2$ and $z = y^3$ from $(0,0,0)$ to $(1,1,1)$.

(e) $C \subset \mathbb{R}^3$ satisfying $x^2 + y^2 = 1$, $z = x^2$, and $y \geq 0$ from $(1,0,1)$ to $(-1,0,1)$.

(12) Evaluate the line integral $\displaystyle\int_C \underline{g}(\underline{x}) \cdot d\underline{x}$ for the following:

(a) $\underline{g}(x,y,z) = (x, y, xz - y)$ and C is the straight line from $(0,0,0)$ to $(1,2,4)$.

(b) $\underline{g}(x,y,z) = (x, y, xz - y)$ and C is parametrized by $\underline{p} : [0,1] \to \mathbb{R}^3$, $\underline{p}(t) = (t^2, 2t, 4t^3)$.

(c) $\underline{g}(x,y,z) = (y^2 - z^2, 2yz, -x^2)$ and C is parametrized by $\underline{p} : [0,1] \to \mathbb{R}^3$, $\underline{p}(t) = (t, t^2, t^3)$.

* (d) $\underline{g}(x,y,z) = (2xy, x^2 + z, y)$ and C is the line segment from $(1,0,2)$ to $(3,4,1)$.

(e) $\underline{g}(x,y,z) = (-x^2 y, x^3, y^2)$ and C is parametrized by $\underline{p} : [0,\pi] \to \mathbb{R}^3$, $\underline{p}(t) = (\cos t, \sin t, t)$

(13) Use line integration to find a scalar field f such that $\underline{g} = \nabla f$ for each of the following:

(a) $\underline{g}(x,y,z) = (e^{y^2}, 2xye^{y^2} + \sin z, y\cos z)$

(b) $\underline{g}(x,y,z) = (ye^{xy+z}, xe^{xy+z}, e^{xy+z})$

(c) $\underline{g}(x, y, z) = (y + z, x + z, x + y)$

(d) $\underline{g}(x, y, z) = (y \sin z, x \sin z, xy \cos z)$

* (e) $\underline{g}(\underline{x}) = \dfrac{\underline{x}}{\|\underline{x}\|}$.

(f) $\underline{g}(\underline{x}) = \left(1 - \dfrac{x_1^2}{\|\underline{x}\|}, -\dfrac{x_1 x_2}{\|\underline{x}\|}, -\dfrac{x_1 x_3}{\|\underline{x}\|}\right) e^{-\|\underline{x}\|}$.

(14) For any $\underline{x} \in \mathbb{R}^n$ abbreviate $\|\underline{x}\| = r$. Then, for any $k \in \mathbb{R}$ let ϕ be the function defined by

$$\phi(\underline{x}) = \begin{cases} \dfrac{r^{k+2}}{k+2}, & \text{if } k \neq -2, \\ \log r, & \text{if } k = -2. \end{cases}$$

Compute $\nabla\phi$. Deduce that every vector field of the form $\underline{F}(\underline{x}) = r^k \underline{x}$ is a gradient.

(15) Newton's law of gravitation states that the force exerted on a particle of mass m located at $\underline{x} \in \mathbb{R}^3$ by a particle of mass M located at $\underline{0}$ is

$$\underline{F}(\underline{x}) = -\frac{GmM}{r^3}\,\underline{x},$$

where $G > 0$ is a constant and $r = \|\underline{x}\|$ again. For all $\underline{a}, \underline{b} \in \mathbb{R}^3$, use the result of Exercise 14 to find the work done by $\underline{F}$ in moving the particle from $\underline{a}$ to $\underline{b}$ along any curve C. How much potential energy is gained by the particle if moved from $\underline{a}$ to "infinity"?

(16) In each case, find the arc length of the curve C parametrized by the given $\underline{p}$. Also evaluate $\int_C f\, ds$, where $f(x, y, z) = z$.

(a) Conical spiral: $\underline{p} : [0, 2\pi] \to \mathbb{R}^3$, $\underline{p}(t) = (e^t \cos t, e^t \sin t, e^t)$.

(b) Tennis ball curve: $\underline{p} : [0, 2\pi] \to \mathbb{R}^3$,

$$\underline{p}(t) = \left(2\cos t - \tfrac{4}{3}\cos^3 t,\ \tfrac{4}{3}\sin^3 t - 2\sin t,\ \cos 2t\right).$$

* (c) Helix: $\underline{p} : [0, 2\pi] \to \mathbb{R}^3$, $\underline{p}(t) = (\cos t, \sin t, t)$.

(d) The conical helix: $\underline{p} : [0, 2\pi] \to \mathbb{R}^3$, $\underline{p}(t) = (t\cos t, t\sin t, t)$.

(e) The stretched helix: $\underline{p} : [0, 1] \to \mathbb{R}^3$, $\underline{p}(t) = (\cos t, \sin t, \frac{1}{2}t^2)$.

(17) In each case, use an appropriate substitution to integrate f over $R \subset \mathbb{R}^2$.

(a) $f(x, y) = \sin\sqrt{x^2 + y^2}$, R is the annulus bounded between the circles of radii $\pi/4$ and $3\pi/4$ centered at $\underline{0}$.

(b) $f(x, y) = e^{(x+2y)/(x+y)}$, R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

(c) $f(x, y) = \dfrac{x - y - 1}{x + y + 1}$, R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

* (d) $f(x, y) = \cos\left(\frac{\pi}{2}\frac{x-y}{x+y}\right)$, R is the triangle with vertices $(0, 0)$, $(\pi, 0)$, $(0, \pi)$.

(e) $f(x, y) = e^{1+x^2+2y^2}$, R is the region enclosed by the ellipse $x^2 + 2y^2 = 2$.

(f) $f(x, y) = x^2$, R is the region of the first quadrant bounded by the hyperbolae $xy = 1$, $xy = 2$, and the lines $2y = x$, $y = 2x$.

(g) $f(x, y) = (x - 2y)^2 \sin^2(x + 2y)$, R is the rhombus with vertices $(\pi, 0)$, $(2\pi, \pi/2)$, (π, π), and $(0, \pi/2)$.

(18) For each planar region $R \subset \mathbb{R}^2$, describe the boundary ∂R either by giving a parametrization or by giving a parametrization of each smooth component of ∂R.

(a) $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \le 4\}$.
(b) $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\}$.
(c) $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 9\}$.
(d) $R = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \le 1\}$.

(19) Consider a general region of the form

$$R = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\right\},$$

where $a < b$ and $\varphi_1 \le \varphi_2 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, and some function $Q \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$. Show that Green's theorem is true for the vector field $(0, Q)$ and this region, i.e., show

$$\oint_{\partial R} Q(x, y)\, dy = \iint_R \frac{\partial Q(x, y)}{\partial x}\, dx\, dy.$$

Hint: First, use the chain rule and the Fundamental Theorem of Calculus to compute

$$\frac{d}{dx}\int_{\varphi_1(x)}^{\varphi_2(x)} Q(x, y)\, dy.$$

(20) Use Green's theorem to integrate the vector field $\underline{F}$ around the closed curve C, oriented anticlockwise.

(a) $\underline{F}(x, y) = (x^2y, xy^2)$, C is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$.

* (b) $\underline{F}(x, y) = (x^2y, x^3)$, C is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.

(c) $\underline{F}(x, y) = (xy^2, xy)$, C is the triangle with vertices $(0, 0)$, $(1, 1)$, $(-1, 1)$.

(d) $\underline{F}(x, y) = (-y^3, x^3)$, C is the (rotated) square with vertices $(2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, -2)$.

(21) In each case, sketch the planar region R and use Green's theorem to find its area. (For each curve, $\underline{p} : [0, 2\pi] \to \mathbb{R}^2$. However, not all curves are oriented anticlockwise!)

(a) R is the region enclosed by the astroid, $\underline{p}(t) = (\cos^3 t, \sin^3 t)$.

(b) R is region enclosed by one arch of the cycloid, $\underline{p}(t) = (t - \sin t, 1 - \cos t)$, and the x-axis.

(c) R is the region enclosed by the cardioid, $\underline{p}(t) = (2\cos t - \cos 2t, 2\sin t - \sin 2t)$.

* (d) R is the region enclosed by the deltoid, $\underline{p}(t) = (2\cos t + \cos 2t, 2\sin t - \sin 2t)$.

Suggestion: To visualise R*, try Maple's parametric plotter:*

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> plot([x(t), y(t), t=0..2*Pi], scaling=constrained);
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(22) Check the identity $\varepsilon_{abe}\varepsilon_{cde} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ by direct calculation for the components:

(a) $a = 1, b = 2, c = 2, d = 1$.
(b) $a = 1, b = 2, c = 2, d = 3$.
(c) $a = b = c = d = 2$.

(23) Use the rules of index notation and the summation convention to simplify the following expressions (in 3 dimensions):

(a) δ_{aa} (b) $\delta_{ab}u_a v_b$ (c) $\varepsilon_{abc}\delta_{bc}$ (d) $\varepsilon_{abc}\varepsilon_{abd}$ (e) $\varepsilon_{abc}\varepsilon_{abc}$ * (f) $\varepsilon_{abc}\varepsilon_{cde}\varepsilon_{efa}$

(24) Use index notation to show that

$$(\underline{u} \times \underline{v}) \cdot (\underline{w} \times \underline{x}) = (\underline{u} \cdot \underline{w})(\underline{v} \cdot \underline{x}) - (\underline{u} \cdot \underline{x})(\underline{v} \cdot \underline{w})$$

for any $\underline{u}, \underline{v}, \underline{w}, \underline{x} \in \mathbb{R}^3$.

(25) For $\underline{u}, \underline{v} \in \mathbb{R}^3$, express

$$(\underline{u} \cdot \underline{v})^2 + \|\underline{u} \times \underline{v}\|^2$$

in index notation, simplify it, and express the result in vector notation. You should get the same identity as in Exercise 1.2. *Hint:* Remember that $\|\underline{w}\|^2 = \underline{w} \cdot \underline{w}$.

(26) Let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^3$. Use the identity for $\underline{u} \times (\underline{v} \times \underline{w})$ from the lectures to show that

$$\underline{0} = \underline{u} \times (\underline{v} \times \underline{w}) + \underline{v} \times (\underline{w} \times \underline{u}) + \underline{w} \times (\underline{u} \times \underline{v}).$$

(This is the *Jacobi identity*. It means that $\mathbb{R}^3$ with the operation $\times$ is a *Lie algebra*. These are very important in algebra and particle physics.)

(27) Let A and B be $n \times n$ matrices with components A_{ab} and B_{ab}. Write down the component C_{ab} of $C = AB$. The *trace* of an $n \times n$ matrix A is defined by $\operatorname{Tr} A = A_{aa}$. Use index notation to show that $\operatorname{Tr} AB = \operatorname{Tr} BA$.

Use this result to prove that if H and J are similar $n \times n$ matrices, i.e., $H = PJP^{-1}$ for some invertible $n \times n$ matrix P, then $\operatorname{Tr} H = \operatorname{Tr} J$.

(28) Let A be an antisymmetric 3×3 matrix, i.e., $A^T = -A$, where T denotes the transpose. Express this condition in index notation.

For the vector $\underline{\omega}$ defined by

$$\omega_a = -\tfrac{1}{2}\varepsilon_{abc}A_{bc},$$

show that $A\underline{x} = \underline{\omega} \times \underline{x}$, for all vectors $\underline{x}$.

(29) Continuing from Exercise 28, if $\underline{\omega} = \theta\, \underline{n}$, where $\|\underline{n}\| = 1$, show that the matrix exponential of A, $\exp A = \mathbb{I} + \sum_{r=1}^{\infty} \frac{A^r}{r!}$ (where $\mathbb{I}$ is the identity matrix) has components

$$[\exp A]_{ab} = \delta_{ab} \cos\theta + n_a n_b (1 - \cos\theta) - \varepsilon_{abc} n_c \sin\theta.$$

Hint: Note that $\Omega(t) = \exp(At)$ is the unique solution to the matrix differential equation $\dot{\Omega}(t) = A\Omega(t)$ with $\Omega(0) = \mathbb{I}$.

* **(30)** Let ψ be the scalar field,

$$\psi(x, y, z) = x^2 e^y \sin z,$$

and $\underline{F}$ the vector field

$$\underline{F}(x, y, z) = (z \cos y, x^2 y, z).$$

Compute $\nabla\psi$, $\nabla^2\psi$, $\nabla \cdot \underline{F}$, and $\nabla \times \underline{F}$.

(31) For $\phi, \psi \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ and $\underline{F}, \underline{G} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$, use index notation to show that

(a) $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$

* (b) $\nabla \times (\phi\underline{F}) = (\nabla\phi) \times \underline{F} + \phi\, (\nabla \times \underline{F})$

(c) $\nabla \times (\underline{F} \times \underline{G}) = (\underline{G} \cdot \nabla)\underline{F} - (\underline{F} \cdot \nabla)\underline{G} + \underline{F}\, (\nabla \cdot \underline{G}) - (\nabla \cdot \underline{F})\, \underline{G}$

(32) If $g \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R})$, then at any point $\underline{x} \in \mathbb{R}^3$, the gradient $\nabla g(\underline{x})$ is normal at $\underline{x}$ to the level surface of g passing through $\underline{x}$. If another vector field $\underline{F} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ is also normal to the level surfaces of g, then it must be parallel to ∇g at every point. This implies that

$$\underline{F} = f\, \nabla g$$

for some scalar field $f \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$. Show that this implies

$$0 = \underline{F} \cdot (\nabla \times \underline{F}).$$

(You will need various identities from the lectures and exercises.)

(33) The *Lie bracket* of two vector fields $\underline{F}, \underline{G} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ is another vector field defined as

$$[\underline{F}, \underline{G}] := (\underline{F} \cdot \nabla)\underline{G} - (\underline{G} \cdot \nabla)\underline{F}.$$

(a) Express this in index notation.

(b) Use index notation to show that

$$[\underline{F}, \underline{G}] = (\nabla \cdot \underline{F})\, \underline{G} - (\nabla \cdot \underline{G})\, \underline{F} - \nabla \times (\underline{F} \times \underline{G}).$$

(c) Use part (a) to show that for $\underline{F}, \underline{G}, \underline{H} \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$ this satisfies the Jacobi identity (Exercise 26)

$$\underline{0} = [\underline{F}, [\underline{G}, \underline{H}]] + [\underline{G}, [\underline{H}, \underline{F}]] + [\underline{H}, [\underline{F}, \underline{G}]].$$

(34) In each case, evaluate the triple integral

$$I = \iiint_E f\, dV$$

of the given function $f : \mathbb{R}^3 \to \mathbb{R}$ over the given solid region $E \subset \mathbb{R}^3$. Use Cartesian, cylindrical, or spherical coordinates as appropriate..

* (a) $f(x, y, z) = z$ and E is the tetrahedron
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le y \le z \le 1\}.$$

(b) $f(x, y, z) = x^2 + y^2 + z$ and E is the cylinder
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 1,\ x^2 + y^2 \le a^2\}.$$

(c) $f(x, y, z) = y$ and
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le z \le x,\ 0 \le y\},$$

(d) $f(x, y, z) = xy + zy$ and E is the triangular prism of height $h > 0$ with vertices $\underline{0}$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, h)$, $(1, 0, h)$ and $(0, 1, h)$.

* (e) $f(x, y, z) = x^2$ and E is the solid cone
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le z^2,\ 0 \le z \le 1\}.$$

(f) $f(x, y, z) = z$ and E is bounded below by the cone where $x^2 + y^2 = \frac{1}{3}z^2$ and $z \ge 0$, and above by the sphere of radius 1 centered at the origin.

(g) $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$ and E is the region inside the sphere of radius 1 in the 1st octant,
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \ge 0,\ x^2 + y^2 + z^2 \le 1\}.$$

(35) Find the volume of the intersection $C_1 \cap C_2$ of the cylinders
$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le a^2\}, \text{ and } C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \le a^2\}$$
by triple integration.

(36) Suppose $0 < b < a$. Let $E \subset \mathbb{R}^3$ be the solid torus,
$$E = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 \le b^2\right\}.$$
To integrate over E, make the change of variables:
$$x = (a + r\cos\phi)\cos\theta, \quad y = (a + r\cos\phi)\sin\theta, \quad z = r\sin\phi,$$
where $0 \le r \le b$ and $0 \le \theta, \phi \le 2\pi$. (This is the toroidal version of spherical polar coordinates.)

(a) Show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r(a + r\cos\phi)$.
(b) Hence show that the volume of E is $2\pi^2 ab^2$.
(c) Suppose E is filled with material of uniform density. Show that moment of inertia I_z about the z-axis is:
$$I_z = (a^2 + \tfrac{3}{4}b^2)M,$$

where M is the total mass. Find also I_x (or I_y; they are the same). Which is greater? Is this what you expect?

(37) Let $f\colon [a,b] \to \mathbb{R}$ be continuous and positive, and let $E \subset \mathbb{R}^3$ be the solid of revolution enclosed by rotating the graph of f around the z-axis:

$$E = \{(x, y, z) \mid x^2 + y^2 \leq f(z)^2\}.$$

(a) Describe E in cylindrical coordinates. That is, write down a region W in r-θ-z-space that corresponds to E.
(b) Use cylindrical coordinates to show that the volume of E is

$$\text{Volume}(E) = \pi \int_a^b f(z)^2 dz.$$

(38) Let

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\},$$

where $\varphi_1, \varphi_2\colon [a,b] \to \mathbb{R}$ are continuous positive functions with $\varphi_1 \leq \varphi_2$. Let $\bar{y}$ be the y-coordinate of the centroid of R, defined by

$$\bar{y} = \frac{1}{\text{Area}(R)} \iint_R y \, dA.$$

If $E \subset \mathbb{R}^3$ is the solid region obtained by rotating R about the x-axis, use Exercise 37 to derive *Pappus' formula:*

$$\text{Volume}(E) = 2\pi \bar{y} \,\text{Area}(R).$$

Check that this gives the correct formula for the volume of a torus. [See Exercise 36.]

(39) For each surface, evaluate the flux integrals of the vector fields defined by

$$\underline{f}(x, y, z) = (1, 1, 1),$$
$$\underline{g}(x, y, z) = (xz, yz, z^2),$$
$$\underline{h}(x, y, z) = (0, e^z, 0).$$

(a) The rectangle Σ_1 with vertices $(1, 0, -1)$, $(0, 1, -1)$, $(-1, 0, 1)$ and $(0, -1, 1)$, parametrized by $\underline{p} : [0, 1] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (0, -1, 1) + u(1, 1, -2) + v(-1, 1, 0).$$

(b) The triangle Σ_2 with vertices $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, parametrized by $\underline{p} : [0, 1] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (1 - u, u[1 - v], uv).$$

(c) The half-cylinder Σ_3, parametrized by $\underline{p} : [0, \pi] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (\cos u, \sin u, v).$$

* (d) The surface Σ_4 parametrized by $\underline{p} : [0, 1] \times [-1, 1] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (u, uv, v^2).$$

(40) For a region $R \subset \mathbb{R}^2$ and a function $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$, show that the area of the graph

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in R,\ z = f(x, y)\}$$

is

$$\mathrm{Area}(\Sigma) = \iint_R \sqrt{1 + (D_1 f)^2 + (D_2 f)^2}\, dA.$$

(41) Suppose that $a < b$ and $f \in \mathcal{C}^1([a, b], \mathbb{R})$ with $f \geq 0$. This determines a *surface of revolution*

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = f^2(z)\}.$$

(a) Describe Σ in cylindrical coordinates.
(b) Use this to parametrize Σ.
(c) Compute the surface area of Σ.

(42) In each case, compute the area of the give surface, Σ. Then compute the surface integral

$$I = \iint_\Sigma f\, dS$$

for the given function $f : \mathbb{R}^3 \to \mathbb{R}$.

(a) Σ is the cylinder parametrized by $\underline{p} : [0, 2\pi] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(\theta, z) = (\cos\theta, \sin\theta, z)$$

and $f(x, y, z) = x^2 z$.

* (b) Σ is the helicoid parametrized by $\underline{p} : [0, 1] \times [0, \pi] \to \mathbb{R}^3$,

$$\underline{p}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

and $f(x, y, z) = y$.

(c) For $0 < b < a$, Σ is the torus is parametrized by $\underline{p} : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = ([a + b\cos u]\sin v, [a + b\cos u]\cos v, b\sin u)$$

and

$$f(x, y, z) = \frac{|z|}{x^2 + y^2}$$

(43) Compute the flux integral of $\underline{g} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\iint_{\partial E} \underline{g} \cdot d\underline{S}$$

both directly and by using Gauss' Theorem.

(a) The cube $E = [0, 1] \times [0, 1] \times [0, 1]$ and $\underline{g}(x, y, z) = (x^2, y^2, z^2)$.
(b) $E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1 - x^2 - y^2\}$ and $\underline{g}(x, y, z) = (x, y, 0)$.

(c) The cylinder $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}$ and $\underline{g}(x, y, z) = (x, y, z^2)$.

(d) The ball (solid sphere) $E = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| \leq 1\}$ and $\underline{g}(x, y, z) = z\sqrt{x^2 + y^2}\,\underline{e}_3$.

(44) Use Gauss' Theorem to evaluate the flux $\iint_\Sigma \underline{F} \cdot d\underline{S}$ for each of the following:

(a) $\underline{F}(x, y, z) = (3xy^2, 3x^2y, z^3)$ and $\Sigma = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| = 1\}$ with the outward orientation.

(b) $\underline{F}(x, y, z) = (xye^z, xy^2z^3, -ye^z)$ and Σ is the boundary of the box bounded by the planes where $x = 3$, $y = 2$, and $z = 1$, and the coordinate planes.

(c) $\underline{F}(x, y, z) = (3xy^2, xe^z, z^3)$ and Σ is the boundary of the solid region

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 2,\ y^2 + z^2 \leq 1\}.$$

(d) $\underline{F}(x, y, z) = (x^2 \sin y, x \cos y, -xz \sin y)$ and $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^8 + y^8 + z^8 = 8\}$ with the outward orientation.

(e) $\underline{F}(x, y, z) = (x^4, -x^3z^2, 4xy^2z)$ and Σ is the boundary of the solid bounded by $x^2 + y^2 = 1$, $z = x + 2$, and $z = 0$.

(45) Let $E \subset \mathbb{R}^3$ be a solid region, $f \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ a scalar field, and $\underline{F}, \underline{G} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ vector fields. Use Gauss' Theorem and the identities

$$\nabla \cdot (f\,\underline{G}) = (\nabla f) \cdot \underline{G} + f\,\nabla \cdot \underline{G}$$
$$\nabla \cdot (\underline{F} \times \underline{G}) = (\nabla \times \underline{F}) \cdot \underline{G} - \underline{F} \cdot (\nabla \times \underline{G})$$

to derive the following formulae for "integration by parts" for triple integrals:

(a) $\iiint_E f\,\nabla \cdot \underline{G}\,dV = \iint_{\partial E} f\,\underline{G} \cdot d\underline{S} - \iiint_E (\nabla f) \cdot \underline{G}\,dV$

(b) $\iiint_E \underline{G} \cdot (\nabla \times \underline{F})dV = \iint_{\partial E} (\underline{F} \times \underline{G}) \cdot d\underline{S} + \iiint_E \underline{F} \cdot (\nabla \times \underline{G})dV$

(46) Supposing that $\Sigma \subset \mathbb{R}^3$ is the boundary of a solid region and $\underline{F} \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$, use Gauss' Theorem to show that

$$\iint_\Sigma (\nabla \times \underline{F}) \cdot d\underline{S} = 0.$$

(47) For each surface, Σ, describe the boundary, $\partial\Sigma$, precisely.

(a) For $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, Σ is parametrized by $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(x, y) = (x, y, x^2).$$

(b) Σ is parametrized by $\underline{p} : [0, 1] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(x, y) = (x, y, xy).$$

(c) Σ is the part of the unit sphere in the first octant, parametrized by $\underline{p} : [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \to \mathbb{R}^3$,

$$\underline{p}(\theta, \phi) = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\phi).$$

(48) Let $\underline{F} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ be a vector field and Σ a surface with parametrization $\underline{p} : R \to \mathbb{R}^3$. Define a 2-dimensional vector field $\underline{g} : R \to \mathbb{R}^2$ by

$$\underline{g}(\underline{u}) = (D_1 p_a[\underline{u}], D_2 p_a[\underline{u}])\, F_a(\underline{p}[\underline{u}]).$$

(a) If $\underline{q} : [a, b] \to \mathbb{R}^2$ is a parametrization of ∂R, then $\underline{p} \circ \underline{q}$ is a parametrization of $\partial\Sigma$. Use this to show that

$$\oint_{\partial\Sigma} \underline{F}(\underline{x}) \cdot d\underline{x} = \oint_{\partial R} \underline{g}(\underline{u}) \cdot d\underline{u}\,.$$

(b) Compute (i.e., simplify the expression for) curl $\underline{g}$, the planar curl.

(c) Use index notation to compute the triple product

$$(\nabla \times \underline{F})(\underline{p}[\underline{u}]) \cdot (D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}])$$

and show that this equals curl $\underline{g}(\underline{u})$.

(d) Use Green's Theorem to show that Stokes' Theorem is true for $\underline{F}$ and Σ.

(49) Let Σ be the portion of the surface defined by $2z = x^2 + y^2$ below the plane $z = 2$, and oriented by the upward unit normal vector field. Sketch Σ and give a parametrization of this surface. If $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\underline{F}(x, y, z) = (3y, -xz, yz^2),$$

compute $\iint_\Sigma (\nabla \times \underline{F}) \cdot d\underline{S}$ both directly and using Stokes' theorem.

(50) If $\Sigma \subset \mathbb{R}^3$ is a surface, and $\underline{b} \in \mathbb{R}^3$ is a fixed vector, show that

$$2\iint_\Sigma \underline{b} \cdot d\underline{S} = \oint_{\partial\Sigma} (\underline{b} \times \underline{x}) \cdot d\underline{x}.$$

(51) By calculating both of the integrals appearing in Stokes' formula, verify that Stokes' Theorem holds for each of the vector fields defined by

$$\underline{F}(x, y, z) = (z, x, y), \qquad \underline{G}(x, y, z) = (e^z, y, z), \qquad \underline{H}(x, y, z) = (x^2, z^2, y^2),$$

and each of the surfaces

(a) The rectangle Σ_1 with vertices $(1, 0, -1)$, $(0, 1, -1)$, $(-1, 0, 1)$ and $(0, -1, 1)$, parametrized by $\underline{p} : [0, 1] \times [0, 1] \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (0, -1, 1) + u(1, 1, -2) + v(-1, 1, 0).$$

(b) The triangle Σ_2 with vertices $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, parametrized by $\underline{p} : [0,1] \times [0,1] \to \mathbb{R}^3$,

$$\underline{p}(u,v) = (1-u, u[1-v], uv).$$

(c) The half-cylinder Σ_3, parametrized by $\underline{p} : [0,\pi] \times [0,1] \to \mathbb{R}^3$,

$$\underline{p}(u,v) = (\cos u, \sin u, v).$$

(d) The surface Σ_4 parametrized by $\underline{p} : [0,1] \times [-1,1] \to \mathbb{R}^3$,

$$\underline{p}(u,v) = (u, uv, v^2).$$