Inequalities

Theorem

For any $x, y \in \mathbb{R}$,

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Proof: Without loss of generality, let $x \geq 0$, $y \geq 0$

$$\frac{|x+y|}{1+|x+y|} = \frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y}$$

$$\leq \frac{x}{1+x} + \frac{y}{1+y}$$

$$= \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

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Theorem Hölder's Inequality

Let $x_i \geq 0$ and $y_i \geq 0$ $\forall\, i = 1, 2, \ldots, n$ and suppose that $p > 1$ and $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

When $p = q = 2$, we get Cauchy-Schwartz inequality

$$\boxed{\sum_{i-1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}}$$

Proof:

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Theorem: Minkowski inequality

Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1,2,\dots,n$ and suppose that $p \geq 1$. Then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}$$