Math 3012 Final Exam Part B

Akash Narayanan

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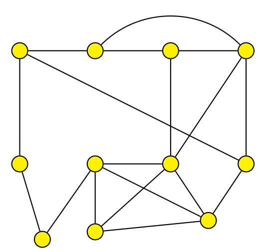
1. (20 points) Consider the graph $\mathbf{G}$ shown below:

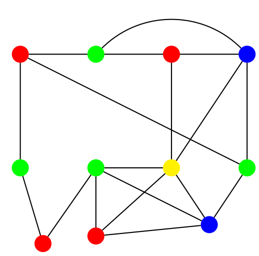


   (a) (10 points) Find $\omega(\mathbf{G})$ for this graph. Show that $\chi(\mathbf{G}) = \omega(\mathbf{G})$ by providing a proper coloring of $\mathbf{G}$. You may indicate your coloring by writing directly on the figure.

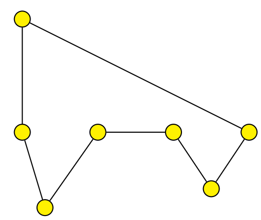
   (b) (10 points) Explain why the graph $\mathbf{G}$ is not perfect.

Solution:

(a) $\omega(\mathbf{G})$ is the maximum clique size, and the largest clique in $\mathbf{G}$ has size 4. It is trivial that $\chi(\mathbf{G}) \geq \omega(\mathbf{G})$, but we can see that the equality holds with the coloring shown below.



(b) A perfect graph is one in which $\chi(\mathbf{H}) = \omega(\mathbf{H})$ for all induced subgraphs $\mathbf{H}$. The graph $\mathbf{G}$ contains the following 7-cycle as an induced subgraph:



Certainly for any odd cycle $\mathbf{C}$ larger than 3 we have $\chi(\mathbf{C}) = 3$ but $\omega(\mathbf{C}) = 2$. Therefore, $\mathbf{G}$ is not perfect because it contains an induced subgraph whose chromatic number is not equal to its maximum clique size.

2. (20 points) Using induction, prove that for every positive integer n,

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Solution: We begin with the base case of $n = 1$. Certainly

$$1 = \frac{1(1+1)}{2}$$

Now suppose that the statement holds for some $k \geq 1$. Then we have

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

That is, if the statement holds for k then it also holds for $k+1$. By the Principle of Induction, the statement holds for all integers $n \geq 1$.

3. (20 points)

   (a) (10 points) Find the general solution to the advancement operator equation:

$$A^4(A-5+4i)^3(A-1)^2(A+8)(A-9)f=0$$

   (b) (10 points) Find the solution to the advancement operator equation:

$$(A^2-11A+28)f(n)=0, f(0)=-2, \text{ and } f(1)=1$$

Solution:

(a) For the first equation, we use the fact that it is factored. We know that a basis for the solution space is given by functions of the form $n^i r^n$ where $r \neq 0$ is a root of the advancement polynomial and $0 \leq m$ where m is the multiplicity of r. Note that we exclude the root 0 because the basis function formed is simply 0 which is not linearly independent of the other functions. Using this, the general form of solutions to the recurrence equation is:

$$\begin{aligned} f(n) &= c_1(5+4i)^n + c_2 n(5+4i)^n + c_3 n^2(5+4i)^n \\ &\quad + c_4 + c_5 n + c_6(-8)^n + c_7 9^n \end{aligned}$$

(b) We start by factoring the advancement operator equation:

$$(A^2-11A+28)f(n)=(A-7)(A-4)f(n)$$

Then using the general theorem we know that the general form of solutions to the equation is:

$$f(n)=c_1 7^n + c_2 4^n$$

Now we use the initial conditions to set up a system of equations.

$$\begin{aligned} c_1 + c_2 &= -2 \\ 7c_1 + 4c_2 &= 1 \end{aligned}$$

Using elimination, we find that

$$\begin{aligned} 3c_1 &= 9 \\ \implies c_1 &= 3 \end{aligned}$$

Substituting this back into the top equation yields $c_2 = -5$. Therefore, the solution to the advancement operator equation is

$$f(n) = 3 \cdot 7^n - 5 \cdot 4^n$$

4. (20 points) In a Bernoulli trial set up, there are three outcomes ξ_1, ξ_2, and ξ_3 with probabilities p_1, p_2, and p_3 respectively. The trial is repeated until either ξ_1 occurs (this is a win) or ξ_2 occurs (this is a loss). As long is ξ_3 occurs, the trial is repeated. What is the probability of a win?

Solution: There are an infinite number of independent events resulting in a win so the probability of a win P is the sum of the probabilities of these events.

$$\begin{aligned} P &= p_1 + p_3 \cdot p_1 + p_3^2 \cdot p_1 + p_3^3 \cdot p_1 + \cdots \\ &= p_1 \left(\frac{1}{1 - p_3} \right) \\ &= \frac{p_1}{p_1 + p_2} \end{aligned}$$

where we use the infinite geometric series from the first to second line (since $p_3 < 1$ it lies within the radius of convergence).

5. (20 points)

   (a) (10 points) Write the inclusion/exclusion formula for the number of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

   (b) (10 points) Evaluate your formula when $n = 6$ and $m = 3$.

Solution:

(a) The number of surjections from a set of size n to a set of size m is given by the formula

$$S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

(b) Using the formula, we find

$$\begin{aligned} S(6, 3) &= \sum_{k=0}^{3} (-1)^k \binom{3}{k} (3-k)^6 \\ &= 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 - 0^6 \\ &= 540 \end{aligned}$$

That is, given a set A of size 6 and a set B of size 3, there are 540 surjections from A to B.