Hedetniemi's Conjecture

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1 Abstract

Recently, Shitov provided a counterexample to a conjecture of Hedetniemi that had been unsolved for fifty years. While not elementary, this counterexample is considered relatively simple, and thus provides a prime beginning for investigation of deep graph-theoretic results. In this paper, we will first introduce terminology and explain Hedetniemi's conjecture. We will then explain, in high-level terms, Shitov's counterexample. Finally, we will look at partial resolutions of Hedetniemi's conjecture, such as the affirmative resolutions for graphs of chromatic number 4 and fractional coloring systems, and the negative resolutions for digraphs or graphs with infinite chromatic number.

2 Hedetniemi's Conjecture

Let A_1 and A_2 be graphs defined as (V_1, E_1) and (V_2, E_2), respectively, where V_i are their vertex sets and E_i are their edge sets.

Definition 1 *We define the **tensor product** of A_1 and A_2, denoted as $A_1 \otimes A_2$, as having vertex set $V_1 \times V_2$, and an edge between (x_1, y_1) and (x_2, y_2) for $x_i \in V_1$ and $y_i \in V_2$ iff $x_1x_2 \in E_1$ and $y_1y_2 \in E_2$.*

Definition 2 *We define the **chromatic number** of a graph A, $\chi(A)$, to be the least positive integer c for which A is c-colorable but not $(c-1)$-colorable.*

Lemma 1 $\chi(A_1 \otimes A_2) \leq \min(\chi(A_1), \chi(A_2))$

WLOG let $\chi(A_1) \leq \chi(A_2)$. We provide a $\chi(A_1)$-coloring of $\chi(A_1 \otimes A_2)$. Color all vertices of the form (v_1, v_2), where $v_1 \in A_1$ and $v_2 \in A_2$, in the color of v_1 in a $\chi(A_1)$-coloring of A_1. Since (v_1, v_2) is only connected to vertices that are of the form (x_1, x_2), where v_1x_1 is an edge in A_1, (v_1, v_2) will have a distinct color from its neighbors, so this is a valid coloring. Furthermore, we only use the colors involved in a $\chi(A_1)$-coloring of A_1, meaning that only $\chi(A_1)$ colors are used.

Hedetniemi asserted that this bound was strict, or equivalently:

Conjecture 1 $\chi(A_1 \otimes A_2) = \min(\chi(A_1), \chi(A_2))$

He was, however, unable to prove this. Much mathematical attention focused on proving the conjecture; however, in 2019, Shitov found a rather large, but reasonably simple counterexample to the conjecture.

We'll also define the Poljak-Rodl function here, as it provides for easy extension of the conjecture.

Definition 3 *We define the **Poljak-Rodl function** $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = \min \chi(G \otimes H)$ where G and H both have chromatic number of x.*

Obviously, Hedetniemi's conjecture can then be restated as follows:

Conjecture 2 *For all n, $f(n) = n$, where f is the Poljak-Rodl function.*

It's worth noting that there exists a weaker version of the conjecture, which simply states that the Poljak-Rodl function is unbounded.

The conjecture derives its importance from the simplicity of both its statement and resolution, even when expressed in terms of the more-complicated Poljak-Rodl function, and the difficulty of finding both. This unique combination allows the amateur mathematician to gain insight into actively-researched topics while not encountering particularly advanced mathematics.

3 Shitov's Counterexample, Further Progress

We introduce the notion of an exponential graph.

Definition 4 *Let G be a graph with n vertices and let c be a positive integer. We define the exponential graph $\mathcal{E}_c(G)$ as follows: The vertex set consists of the c-colorings $\varphi : V(G) \to \{1, 2, \dots, c\}$. Two vertices φ and ψ are adjacent iff $\varphi(x) \neq \psi(y)$ for all $xy \in E(G)$.*

Note that the c-colorings do not have to be proper (that is, adjacent vertices can be the same color). Thus, if $|V(G)| = n$, then $\mathcal{E}_c(G)$ has c^n vertices. Furthermore, consider two copies of $V(G)$, say V_1 and V_2. Color one with φ and the other with ψ. Connect each vertex of V_1 to its neighbors in V_2. Then the colorings φ and ψ are adjacent if and only if this graph is properly colored.

The exponential graph as defined allows for loops. In particular, a vertex φ is adjacent to itself when it is a proper coloring, since each vertex will be a different color from its neighbors. Thus, the graph has no loops iff $\chi(G) > c$. Consider the following conjecture.

Conjecture 3 *$\chi(G) > n$ implies that $\chi(\mathcal{E}_n(G)) = n$.*

The relevance of the exponential graph in the Hedetniemi conjecture was shown in a 1985 paper by El-Zahar and Sauer [1]. They proved that the two conjectures stated above are equivalent. Indeed, let G, H be two graphs with chromatic number $n + 1$ whose product $G \otimes H$ has chromatic number n, making it a counter-example to the Hedetniemi conjecture. Suppose $f : G \otimes H \to \{1, \dots, n\}$

is a proper coloring. Then for every vertex $v \in H$, we have a map f_v which is an n-coloring of G by letting $f_v(x) = f(x, v)$. That is, $f_v \in \mathcal{E}_n(G)$. The map $\phi : H \to \mathcal{E}_n(G)$ sending v to f_v preserves edges. Indeed, if (g, h) is adjacent to (g', h') in $G \otimes H$, then $f_h(g) = f(g, h) \neq f(g', h') = f_{h'}(g)$. Thus, we have $n < \chi(H) \leq \chi(\mathcal{E}_n(G)$. The converse is that if G is an $n+1$-chromatic graph with $\chi(\mathcal{E}_n(G)) > n$, then $G \otimes \mathcal{E}_n(G)$ is a counter-example to the Hedetniemi conjecture.

Shitov uses this fact to construct a such a graph G along with exploiting sufficiently large n. His proof introduces one other important notion.

Definition 5 *The* ***strong product*** *of graphs G, H is denoted $G \boxtimes H$ and has vertex set $G \times H$. The vertices (g, h) and $(g', h'$ are adjacent if and only if*

1. *$g = g'$ and h is adjacent to h'*

2. *$h = h'$ and g is adjacent to g'*

3. *g is adjacent to g' and h is adjacent to h'*

Effectively, it can be considered an extension of the tensor product defined earlier. Shitov shows that by letting G be a graph of girth ≥ 6, then for sufficiently large q we have $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ where $c = \lceil 3.1q \rceil$. We will not go into the full mechanics of the proof here, but it is interesting that the proof, at its core, boils down to an application of the pigeonhole principle [2]. However, it's reasonable to see that the graph itself grows massively. Shitov estimates that G has at least 4^{100} vertices and that the exponential graph involved has at least 4^{10000} vertices.

The next logical question to ask is what the smallest counter-examples are. Recent work in this year has shown that there exists a counter-example for graphs G with 10,952 vertices and H with 33,377 vertices [3]. Even more interestingly, Claude Tardif has recently claimed that there exists a counterexample with G and H having chromatic number 14, and their tensor product having chromatic number 13. He further claims that, if the chromatic numbers of graphs with 13965 and 49416 vertices can be bounded, then the chromatic number of the tensor product can be even lower, 11 and 12 respectively. To do this, he uses two notions, wide colorings (as defined by Simonyi and Tardos) and the generalized Mycelskian. These are out of the scope of our paper, however.

Another interesting result comes from He and Wigderson's recent proof that Hedetniemi's conjecture is "asymptotically false," meaning that there exists some positive constant δ such that for sufficiently large c, graphs G and H with chromatic number not less than $(1+\delta)c$ exist such that their tensor product has chromatic number C. In terms of the Poljak-Rodl function defined earlier, this states that there exists a constant $\epsilon > 1$ such that $f(n) < \frac{n}{\epsilon}$. This result is rather notable, because it dispels earlier conjectures that, say, $f(n) - n \leq 9$.

This result does not, however, help resolve the weak Hedetniemi conjecture. A result of Poljak and Rodl asserts that if the Poljak-Rodl function is bounded above, that bound would be at most 9. Then, disproving the weak Hedetniemi conjecture would simply involve proving that Hedetniemi's conjecture is true for 10-chromatic graphs.

4 Variations on Hedetniemi

Unlike many other conjectures, there was no consensus as to whether the Hedetniemi conjecture was true or not. The conjecture was proven false for several families of graphs such as directed graphs and graphs with infinite chromatic number. However, nobody could prove or disprove it in the setting of finite, undirected graphs. It's interesting to consider the cases for which the conjecture is proven to be true until Shitov's counter-example. One can see that it holds for graphs G and H that are 1-colorable. Indeed, suppose neither G nor H are 1-colorable. Then they both have at least 1 edge, say gg' and hh' respectively. Then (g, h) is adjacent to (g', h') in $G \otimes H$ so it is not 1-colorable. The converse is that if $G \otimes H$ is 1-colorable, then one of G, H is as well.

We will also show the proof for 2-colorable graphs. Suppose graphs G, H are not 2-colorable. Then they are not bipartite (or their vertex sets could be split in two by color and no two vertices in the same set would be adjacent). Equivalently, G and H both contain odd cycles, say $(g_1, g_2, \ldots, g_m$ and $(h_1, h_2, \ldots, h_n$ respectively. Then the product $G \otimes H$ contains the cycle $((g_1, h_1), \ldots, (g_i, h_j), \ldots, (g_m, h_n)$ of length $\text{lcm}(m, n)$. Thus, it also is not 2-colorable. Again, the converse is that if a graph $G \otimes H$ is 2-colorable, then at least one of G or H is also 2-colorable.

The El-Zahar and Sauer paper mentioned earlier showed that if $G \otimes H$ is 3-colorable, then one of G or H must also be, though the proof is much more involved than those above. A corollary is that the conjecture holds for G or H being 4-colorable because the inequality in $\chi(G \otimes H) \leq min(\chi(G), \chi(H))$ is strict only when $\chi(G \otimes H)$ is 3, in which case the right side of the inequality is also 3.

Attempts to prove the conjecture for higher chromatic numbers have not been particularly fruitful. The case for 5-colorable graphs is only shown to be true for highly specific constraints.

Another interesting case is the disproof of Hedetniemi's conjecture for graphs with infinite chromatic number, which has been known since 1985. The paper of Hajnal that shows this is rather set-theoretic in nature and is thus beyond the scope of this work, but the gist is that there exist two graphs with chromatic number $\aleph_1$ whose tensor product has chromatic number $\aleph_0$, at least within the frame of ZFC. In ZFC, it's known that $\aleph_1 > \aleph_0$, so this implies the falsehood of Hedetniemi's conjecture for infinite graphs. It's notable that Hajnal's graphs G and H are uncountable. It is a theorem of Hajnal that there do not exist graphs G and H countable such that their tensor product is finite, but we again will not show the proof here, as it involve concepts like filters that are out of the scope of this paper.

We will, however, note an interesting consequence. If there did exist graphs G and H countable such that their tensor product is uncountable, this would actually imply the incorrectness of both Hedetniemi and Weak Hedetniemi. For the first part, we will show that Hedetniemi implies the nonexistence of two countable graphs G and H such that their tensor product is finite.

Assume $\chi(G \otimes H) = k$ for k finite. Then, for any finite subgraphs G_i, H_i

of G, H respectively, $\chi(G_i \otimes H_i) \leq k$, since $G_i \otimes H_i$ is a subgraph of $G \otimes H$. Now, we know that $\chi(G) > k$, because $\chi(G) = \aleph_0$. Therefore, as per a result in infinite graph theory, there must be a finite subgraph G_1 of G that has chromatic number greater than k. Assuming Hedetniemi's conjecture then tells us that $\chi(G_1 \otimes H_i)$ is the minimum of $\chi(G_1)$ and $\chi(H_i)$. But we know that $\chi(G_1) > k$ and $\chi(G_1 \otimes H_1) \leq k$, so we must have $\chi(H_i) \leq k$ for every finite subgraph H_i of H, meaning that $\chi(H) < k$ by another result in infinite graph theory. Therefore, Hedetniemi implies the falsehood of this conjecture, so the truth of this conjecture would require the falsehood of Hedetniemi. While we now know Hedetniemi to be false, it became apparent with the theorem of Hajnal we discussed above that countable graphs with a tensor product with finite chromatic number could not be used to disprove it.

Also, since the existence of two graphs of countable chromatic number whose tensor product is finite would imply that the Poljak-Rodl function f is bounded (because even as the input goes to infinity, the output stays finite), the finding of two such graphs would imply the falsehood of the weak form of Hedetniemi's conjecture. Again, the theorem of Hajnal seems to have cut off many of the avenues for exploring Hedetniemi's conjecture via countably infinite graphs.

5 Conclusion

Graph theory is a field rich with open problems that are relatively simple to state yet exceedingly difficult to answer. Shitov's relatively simple counter-example to the long standing Hedetniemi conjecture is a fascinating solution that showcases the magnitude that constructions in proofs can reach. Although the conjecture has been proven false, there are several problems that have yet to be solved, including finding the smallest counter-example and understanding more about why the conjecture fails. Shitov's work has laid the ground for a more fruitful exploration into this conjecture and several related problems.

References

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[2] Yaroslav Shitov. Counterexamples to hedetniemi's conjecture. *Annals of Mathematics*, 190(2):663–667, 2019.

[3] Xuding Zhu. Relatively small counterexamples to hedetniemi's conjecture. *arXiv preprint arXiv:2004.09028*, 2020.