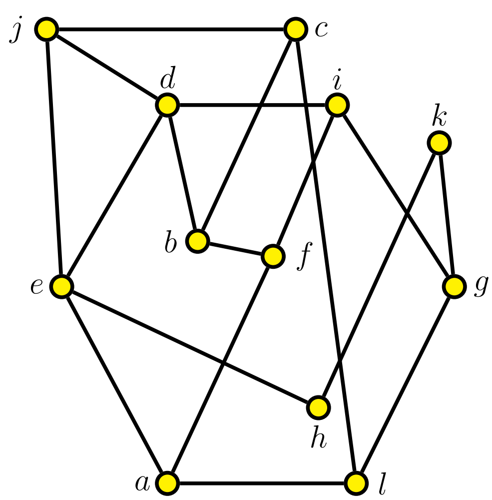
Applied Combinatorics Worksheet 3

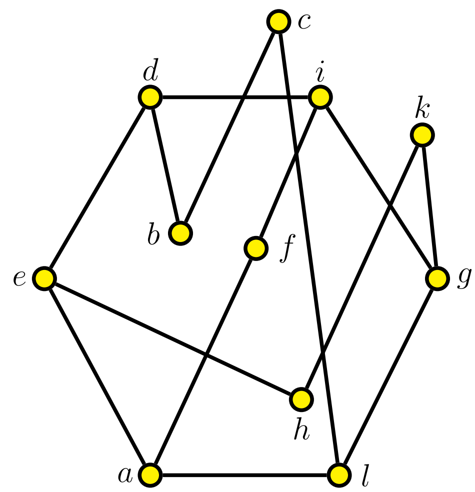
Akash Narayanan

September 29, 2020

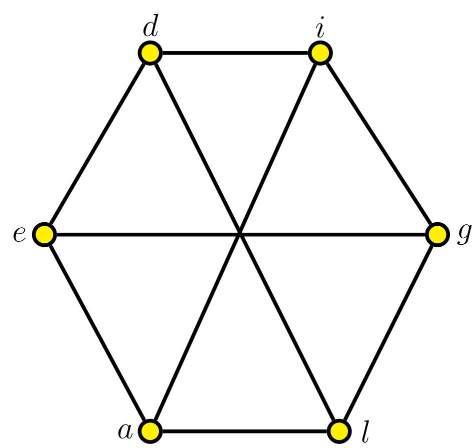
Problem 1. Is the graph below planar? If it is, find a drawing without edge crossings. If it is not, give a reason why it is not.



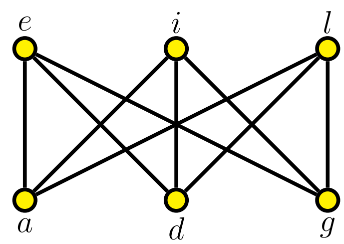
Solution. The graph is not planar because it contains a subgraph homeomorphic to $K_{3,3}$. Consider the following subgraph:



Certainly, this subgraph is homeomorphic to the following.



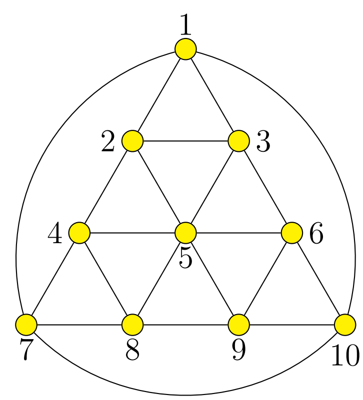
This is isomorphic to $K_{3,3}$ as shown below.



By Kuratowski's Theorem, the graph is not planar. $\square$

Problem 2. Exhibit a planar drawing of an Eulerian planar graph with 10 vertices and 21 edges.

Solution. The graph shown below is planar, has the specified number of edges and vertices, and has an Eulerian circuit as follows: (1, 3, 6, 10, 9, 8, 7, 4, 8, 5, 9, 6, 5, 4, 2, 5, 3, 2, 1, 10, 7) $\square$



Problem 3. We say that a relation R on a set X is symmetric if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$. If $X = \{a, b, c, d, e, f\}$, how many symmetric relations are there on X? How many of these are reflexive?

Solution. A relation R is merely a subset of $X \times X$. If $|X| = n$, R is a subset of a set with n^2 elements. If R is symmetric, then the elements of R can be treated as sets of size 2 since the order does not matter. These sets can be formed by choosing 2 elements out of n. That is, elements of R can be selected from a set of size $\binom{n}{2}$. However, this excludes elements of the form

(a, a). To account for these, we add n more ordered pairs to our set of size $\binom{n}{2}$. Thus, R is a subset of a set of size $\binom{n}{2} + n = \frac{n(n+1)}{2}$. Therefore, there are $2^{\frac{n(n+1)}{2}}$ symmetric relations on X.

A relation R is reflexive if for every $a \in X$, $(a, a) \in R$. Thus, R must contain the n ordered pairs of that form. Then R is a subset of a set containing the remaining $n^2 - n = n\,(n-1)$ relations. Thus, there are $2^{n(n-1)}$ reflexive relations on X. $\square$