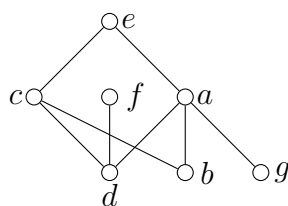


Math 3012 Midterm 2

Akash Narayanan

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1. (20 points) Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation by computing the down-sets and up-sets in the space provided. Then use the First Fit coloring algorithm to find the width ω and a partition of the poset into ω chains.



Solution:

$D(a) = \{b, d, g\}$	$U(a) = \{e\}$
$D(b) = \emptyset$	$U(b) = \{a, c, e\}$
$D(c) = \{b, d\}$	$U(c) = \{e\}$
$D(d) = \emptyset$	$U(d) = \{a, c, e, f\}$
$D(e) = \{a, b, c, d, g\}$	$U(e) = \emptyset$
$D(f) = \{d\}$	$U(f) = \emptyset$
$D(g) = \emptyset$	$U(g) = \{a, e\}$

Ranking the down-sets and up-sets, we find the following interval

representation:

$$I(a) = [4, 4]$$

$$I(b) = [1, 2]$$

$$I(c) = [3, 4]$$

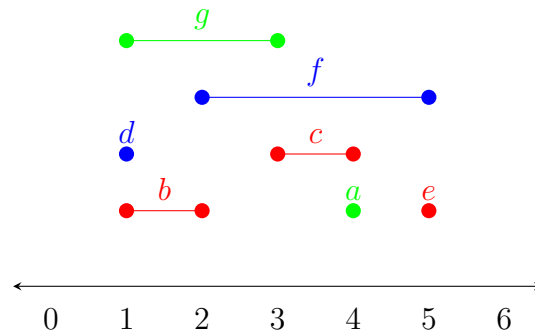
$$I(d) = [1, 1]$$

$$I(e) = [5, 5]$$

$$I(f) = [2, 5]$$

$$I(g) = [1, 3]$$

The corresponding interval graph is shown below, along with a coloring using the First Fit algorithm.

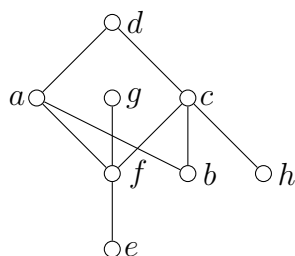


Since the First Fit algorithm is optimal for interval graphs, the chromatic number is 3. In particular, we've partitioned the graph into 3 chains: $\{a, g\}$, $\{b, c, e\}$, $\{d, f\}$. Furthermore, there are antichains of size 3, one being $\{a, c, f\}$. Since the width is bounded below by the size of an antichain and bounded above by a partitioning into n chains, the width of the graph must be 3.

2. (20 points) Consider the poset shown below. The ground set is

$$X = a, b, c, d, e, f, g, h.$$

Write the reflexive, antisymmetric, and transitive relation on X which defines this poset.



Solution:

$$P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g), (h, h), \\ (a, d), (b, a), (b, c), (b, d), (c, d), (e, a), (e, c), (e, d), \\ (e, f), (e, g), (f, a), (f, c), (f, d), (f, g), (h, c), (h, d)\}$$

3. (20 points) Let 2^{15} be the poset consisting of all subsets of $\{1, 2, 3, \dots, 15\}$ ordered by inclusion.
- (a) (5 points) What is the height of this poset?
 - (b) (5 points) What is the width of this poset?
 - (c) (5 points) How many maximal chains does the poset have?
 - (d) (5 points) How many maximal chains in this poset pass through the set $\{2, 3, 8, 13\}$?

Solution:

- (a) One maximal chain has size 16. Since every maximal chain is maximum, the height is 16.
- (b) By Sperner's theorem, the width of the poset is $C(15, 7) = 6435$.
- (c) There are $15!$ maximal chains formed by adding the elements of the set in any order.
- (d) The set $\{2, 3, 8, 13\}$ has 4 elements. There are $4!11!$ maximal chains that pass through the set.

4. (20 points)

(a) (20 points) Let $X = \{a, b, c, d, e\}$ and let

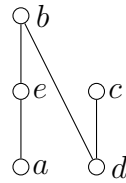
$$P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (d, b), (d, c), (a, e), (a, b), (e, b)\}.$$

Draw a diagram for the poset (X, P) .

(b) (10 points) How many symmetric binary relations are there on $\{1, 2, \dots, n\}$? Of these, how many are reflexive?

Solution:

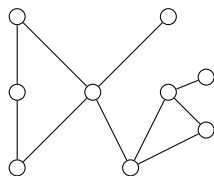
(a) A diagram for the poset (X, P) is shown below.



(b) A relation R is symmetric if whenever $(a, b) \in R$ then $(b, a) \in R$. This amounts to choosing 2 element subsets of the base set on n elements. There are precisely $C(n, 2)$ subsets of size 2. However, this disregards elements of the form (a, a) , of which there are n . Thus, there are $C(n, 2) + n = \frac{n(n+1)}{2}$ possible choices for elements of a symmetric relation. A symmetric relation is just a subset of these choices so there are $2^{\frac{n(n+1)}{2}}$ symmetric relations.

Of these symmetric relations, a reflexive relation contains (a, a) for all a in the underlying set. That is, we are restricted to choosing distinct pairs of elements to be in our relation. Then there are $2^{C(n, 2)}$ reflexive and symmetric relations.

5. (20 points) Verify Euler's formula for this planar graph.



Solution: Euler's formula is $V - E + F = 1 + T$ where V is the number of vertices, E is the number of edges, F is the number of faces (including the exterior face), and T is the number of components. This graph is 9 vertices, 10 edges, 3 faces, and 1 component. Indeed, $9 - 10 + 3 = 1 + 1$.