Applied Combinatorics Homework 4

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Problem 1. How many positive integers less than or equal to 1000 are divisible by none of 3, 8, and 25?

Solution. We can apply the principle of Inclusion-Exclusion to positive integers less than or equal to 1000. First note that the number of positive integers less than or equal to 1000 divisible by k is $\lfloor \frac{1000}{k} \rfloor$. In particular, this counts the number of multiples of k less than or equal to 1000. Then we have

$$1000 - \lfloor \frac{1000}{3} \rfloor - \lfloor \frac{1000}{8} \rfloor - \lfloor \frac{1000}{25} \rfloor + \lfloor \frac{1000}{3 \cdot 8} \rfloor + \lfloor \frac{1000}{3 \cdot 25} \rfloor + \lfloor \frac{1000}{8 \cdot 25} \rfloor - \lfloor \frac{1000}{3 \cdot 8 \cdot 25} \rfloor = 560$$

Thus, there are 560 positive integers less than or equal to 1000 that are divisible by none of 3, 8, and 25. $\square$

Problem 2. Find the coefficient on x^{10} in the generating function $(1+x)^{12}$.

Solution. We can find the coefficient of x^{10} by using the binomial theorem, which states that the coefficient of x^{10} is $\binom{12}{10} = 66$. $\square$

Problem 3. Solve the recurrence equation $r_{n+2} = r_{n+1} + 2r_n$ if $r_0 = 1, r_2 = 3$.

Solution. We can rewrite the recurrence as a polynomial in the advancement operator.

$$(A^2 - A - 2)f(n) = 0$$

The polynomial in A factors into $(A-2)(A+1)$. The corresponding bases for the solution space are 2^n and $(-1)^n$. Thus, the general solution to the recurrence relation is $r_n = c_0 2^n + c_1(-1)^n$.

We're given two conditions. Using the first, we have $r_0 = c_0 2^0 + c_1(-1)^0$. That is, $c_0 + c_1 = 1$. Similarly, the second condition yields $8c_0 + c_1 = 3$. Using the first equation, we solve for $c_1 = 1 - c_0$, which we can substitute into the second equation. This shows that $7c_0 = 2$ which implies that $c_0 = \frac{2}{7}$. Finally, we can substitute this back into one of the previous equations to solve for $c_1 = \frac{5}{7}$. Thus our recurrence is

$$r_n = \frac{2}{7} \cdot 2^n + \frac{5}{7} \cdot (-1)^n$$

$\square$