Math 2552 Written HW Set 2

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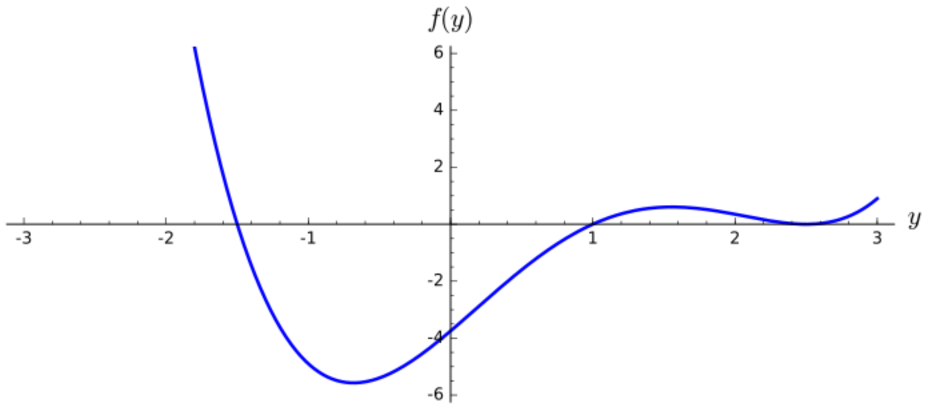
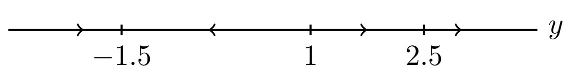
Judson 1.3.21. Consider the differential equation $y' = f(y)$, where the graph of $f(y)$ is given below. Draw the phase line for the equation and classify each equilibrium solution as a sink, a source, or a node.



Solution. The equilibrium solutions are those points where $f(y) = 0$. This occurs at $y = -1.5$, $y = 1$, and $y = 2.5$. We can classify the equilibrium solutions by evaluating the derivative at various values of y as shown in the table below.

y	y
$y < -1.5$	$y' > 0$
$-1.5 < y < 1$	$y' < 0$
$1 < y < 2.5$	$y' > 0$
$2.5 < y$	$y' > 0$

Using the table, we can draw the corresponding phase diagram.



From here, it becomes clear that $y = -1.5$ is a **sink**, $y = 1$ is a **source**, and $y = 2.5$ is **stable**. $\square$

Trench 2.3.7. Find all (x_0, y_0) for which the initial value problem

$$y' = \ln(1 + x^2 + y^2),\ y(x_0) = y_0$$

has **(a)** a solution **(b)** a unique solution on some open interval that contains x_0.

Solution. The Existence and Uniqueness Theorem states that the initial value problem has at least one solution if $f(x, y) = \ln(1 + x^2 + y^2)$ is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

containing (x_0, y_0). Furthermore, if both f and f_y are continuous on R, then the initial value problem has a unique solution on an open subinterval of (a, b) containing x_0.

$f(x, y)$ is continuous for $1 + x^2 + y^2 > 0$. Since $x^2 + y^2 > 0$ for all (x, y), **(a)** the initial value problem has at least one solution for all (x_0, y_0).

For **(b)**, we calculate

$$f_y(x, y) = \frac{2y}{1 + x^2 + y^2}$$

Rational functions are only discontinuous at points where the denominator is 0, but $1 + x^2 + y^2$ is greater than 0 for all x and y. Since $f_y(x, y)$ is continuous at all (x, y), the Existence and Uniqueness Theorem guarantees that the initial value problem has a unique solution for all (x_0, y_0). $\square$

Trench 4.2.5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperature at 12:15 and 12:20 are 120°C and 90°C, respectively.

1. At what time was the object placed outside?
2. When will its temperature be 40°C?

Solution. By Newton's Law of Cooling, the temperature of the object $T(t)$ may be modeled with the following differential equation and initial value:

$$\frac{dT}{dt} = k(T - 35),\ T(0) = 150$$

This is a separable equation which we can solve as follows:

$$\int \frac{dT}{T-35} dT = \int k dt$$
$$\ln(T - 35) = kt + C.$$

Using our initial value $T(0) = 150$ yields $C = \ln(115)$ and further rearranging yields

$$T - 35 = e^{kt+\ln(115)}$$
$$T = 115e^{kt} + 35$$

To solve for k, we use the two given values of T and the time difference of 5 minutes. We find

$$120 = 115e^{kt_0} + 35 \Longrightarrow 85 = 115e^{kt_0}$$
$$90 = 115e^{k(t_0+5)} + 35 \Longrightarrow 55 = 115e^{kt_0}e^{5k}$$

Dividing the first equation by the second gives

$$\frac{11}{17} = e^{5k}$$
$$k = \frac{1}{5}\ln\left(\frac{11}{17}\right) \approx -0.087$$

Thus, our equation for the temperature of the object is $T = 115e^{-0.087t} + 35$. We can solve (1) by setting $T = 120$ and solving for t, which yields $t = 3.47$ minutes or about 208 seconds. Subtracting this from the time when the object is 120°C (12:15) shows that it was approximately 12:11:32 when the object was placed outside.

We can solve (2) by letting $T = 40$ and finding $t = 36$ minutes. Adding this to the initial time it was placed outside shows that the object will be 40°C at about 12:47:32. □