Math 2552 Written HW Set 5

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Trench 5.1.11. Find a second solution y_2 that isn't a constant multiple of the solution y_1. Choose K to conveniently simply y_2.

$$y'' - 6y' + 9y = 0; \quad y_1 = e^{3x}$$

Solution. Given one nontrivial solution of the differential equation, we can find another using Abel's formula. Exercise 9 shows that a second solution has the form $y_2 = uy_1$ where

$$u' = K\frac{e^{-P(x)}}{y_1^2(x)}$$

for arbitrary nonzero constant K. For the given differential equation, $p(x) = -6$ so we have $P(x) = \int p(x)\,dx = -6x$. Then we find

$$\begin{aligned} u' &= K\frac{e^{-6x}}{(e^{3x})^2} = K \\ u &= \int K\,dx = Kx \end{aligned}$$

which shows that $y_2 = Kxe^{3x}$. We can verify that y_2 satisfies the differential equation. Indeed,

$$y_2'' - 6y_2' + 9y_2 = (9kx + 6k)e^{3x} - (18kx + 6k)e^{3x} + 9kxe^{3x} = 0$$

Certainly y_2 is not a constant multiply of y_1 since it has a factor of x. Setting $K = 1$ yields a final solution of

$$y_2 = xe^{3x}$$

$\square$

Trench 5.2.13. Solve the initial value problem.

$$y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$$

Solution. The characterstic polynomial of the differential equation is

$$p(r) = r^2 + 14r + 50 = (r+7)^2 + 1$$

The roots of the characterstic polynomial are $r_1 = -7 + i$ and $r_2 = -7 - i$. Then the general solution to the differential equation is

$$y = c_1 e^{-7x} \cos(x) + c_2 e^{-7x} \sin(x)$$

Differentiating yields

$$y' = -7c_1 e^{-7x} \cos(x) - c_1 e^{-7x} \sin(x) - 7c_2 e^{-7x} \sin(x) + c_2 e^{-7x} \cos(x)$$

From here, we can use the initial conditions to form the system of equations

$$\begin{aligned} 2 &= c_1 \\ -17 &= -7c_1 + c_2 \end{aligned}$$

which has the solution $c_1 = 2$ and $c_2 = -3$. Thus, the solution to the initial value problem is

$$y(x) = 2e^{-7x} \cos(x) - 3e^{-7x} \sin(x)$$

$\square$

Trench 6.1.7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

Solution. The equation of motion is

$$my'' + ky = 0,$$

or

$$y'' + \frac{k}{m}y = 0.$$

Given that $mg = k\Delta l$, we have $k/m = g/\Delta l$. Using imperial units as in the problem, we have $g = 32\text{ft/s}^2$. Furthermore, we have $\Delta l = 1/8$ ft. Thus, we find

$$\frac{k}{m} = \frac{32}{1/8} = 256$$

and the differential equation becomes

$$y'' + 256y = 0, \quad y(0) = \frac{2}{3}, \quad y'(0) = -4.$$

The differential equation has the characteristic equation

$$r^2 + 256 = 0$$

which has the solutions $r = \pm 16i$. Then the general solution to the differential equation is

$$y = c_1 \cos 16t + c_2 \sin 8t.$$

Differentiating yields

$$y' = -16c_1 \sin 16t + 16c_2 \cos 16t.$$

Using the initial conditions yields the system of equations

$$\begin{aligned} \frac{2}{3} &= c_1 \\ -4 &= 16c_2 \end{aligned}$$

which has the solution $c_1 = 2/3$ and $c_2 = -1/4$. Thus, the displacement is given by

$$y = \frac{2}{3}\cos 16t - \frac{1}{4}\sin 16t.$$

$\square$