Math 2552 Written HW Set 3

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Judson 3.2.11. Consider the system

$$
\begin{aligned}
x' &= ax + y \\
y' &= 2ax + 2y,
\end{aligned}
$$

where $a \in \mathbb{R}$. For what values of a do you find a bifurcation (a change in the type of phase portrait)? Sketch typical phase portraits for a value of a above and below the bifurcation point.

Solution. We start by rewriting the system in matrix form $\vec{x}\,' = A\vec{x}$ where

$$
A = \begin{pmatrix} a & 1 \\ 2a & 2 \end{pmatrix}
$$

The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - (a+2)\lambda$. Setting this equal to 0 and solving for λ yields the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a + 2$. From here it is clear that there is a bifurcation at $a = -2$, where λ_2 changes sign.

By solving the system $(A - \lambda I)\vec{v} = \vec{0}$, we obtain the following eigenvectors:

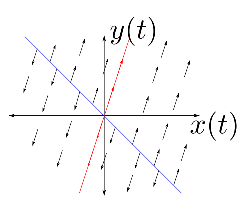
$$
\vec{v}_1 = \begin{pmatrix} 1 \\ -a \end{pmatrix} \qquad\qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

Note that the eigenvalue associated with $\vec{v}_1$ is 0. That is, the points along this line are equilibrium solutions to the system of equations.

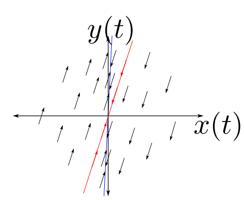
We have that the general solution to the system is

$$
\vec{x}\,' = c_1 \begin{pmatrix} 1 \\ -a \end{pmatrix} + c_2 e^{(a+2)t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

Now we draw two phase portraits for values of a above and below -2.



$a = 1$



$a = -4$

□

Judson 3.4.3. Find the general solution of the linear system shown below and give a sketch of the phase portrait with a few solution curves.

$$x' = -x - 4y$$
$$y' = 3x - 2y$$

Solution. We rewrite the system in matrix form $\vec{x}\,' = A\vec{x}$ where

$$A = \begin{pmatrix} -1 & -4 \\ 3 & -2 \end{pmatrix}$$

This matrix has the eigenvalues $\lambda_{1,2} = -3/2 \pm i\sqrt{47}/2$. The corresponding eigenvectors are

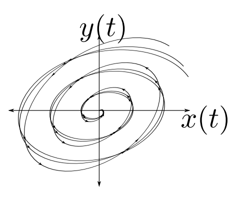
$$\vec{v}_1 = \begin{pmatrix} 1 + i\sqrt{47} \\ 6 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 - i\sqrt{47} \\ 6 \end{pmatrix}$$

Now we can define $\vec{w}_1 = e^{\lambda_1 t}\vec{v}_1$. Expanding this and using Euler's formula yields

$$\begin{aligned} \vec{w}_1 &= e^{-\frac{3}{2}t}\left(\cos\left(\frac{\sqrt{47}}{2}t\right) + i\sin\left(\frac{\sqrt{47}}{2}t\right)\right)\begin{pmatrix} 1 + i\sqrt{47} \\ 6 \end{pmatrix} \\ &= e^{-\frac{3}{2}t}\begin{pmatrix} \cos(\frac{\sqrt{47}}{2}t) - \sqrt{47}\sin(\frac{\sqrt{47}}{2}t) \\ 6\cos(\frac{\sqrt{47}}{2}t) \end{pmatrix} + ie^{-\frac{3}{2}t}\begin{pmatrix} \sqrt{47}\cos(\frac{\sqrt{47}}{2}t) + \sin(\frac{\sqrt{47}}{2}t) \\ 6\sin(\frac{\sqrt{47}}{2}t) \end{pmatrix} \end{aligned}$$

Then the general solution is

$$\vec{x}(t) = c_1 e^{-\frac{3}{2}t}\begin{pmatrix} \cos(\frac{\sqrt{47}}{2}t) - \sqrt{47}\sin(\frac{\sqrt{47}}{2}t) \\ 6\cos(\frac{\sqrt{47}}{2}t) \end{pmatrix} + c_2 e^{-\frac{3}{2}t}\begin{pmatrix} \sqrt{47}\cos(\frac{\sqrt{47}}{2}t) + \sin(\frac{\sqrt{47}}{2}t) \\ 6\sin(\frac{\sqrt{47}}{2}t) \end{pmatrix}$$



A phase portrait for the linear system

□

Judson 3.5.7. Solve the following linear system for the given initial values and give a sketch of the phase portrait with a few solution curves.

$$\begin{aligned} x' &= 9x + 4y \\ y' &= -9x - 3y \\ x(0) &= 2 \\ y(0) &= -3 \end{aligned}$$

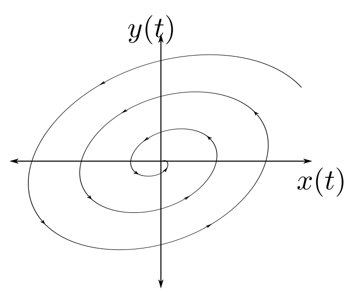
Solution. We start by rewriting the system in matrix form $\vec{x}\,' = A\vec{x}$ where

$$A = \begin{pmatrix} 9 & 4 \\ -9 & -3 \end{pmatrix}$$

The eigenvalue is $\lambda = 3$ with multiplicity 2. The corresponding eigenvector is

$$\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$\square$



A sketch of the solution curve