Math 2552 Midterm 3

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1. (15 points) Consider the non-linear system below.

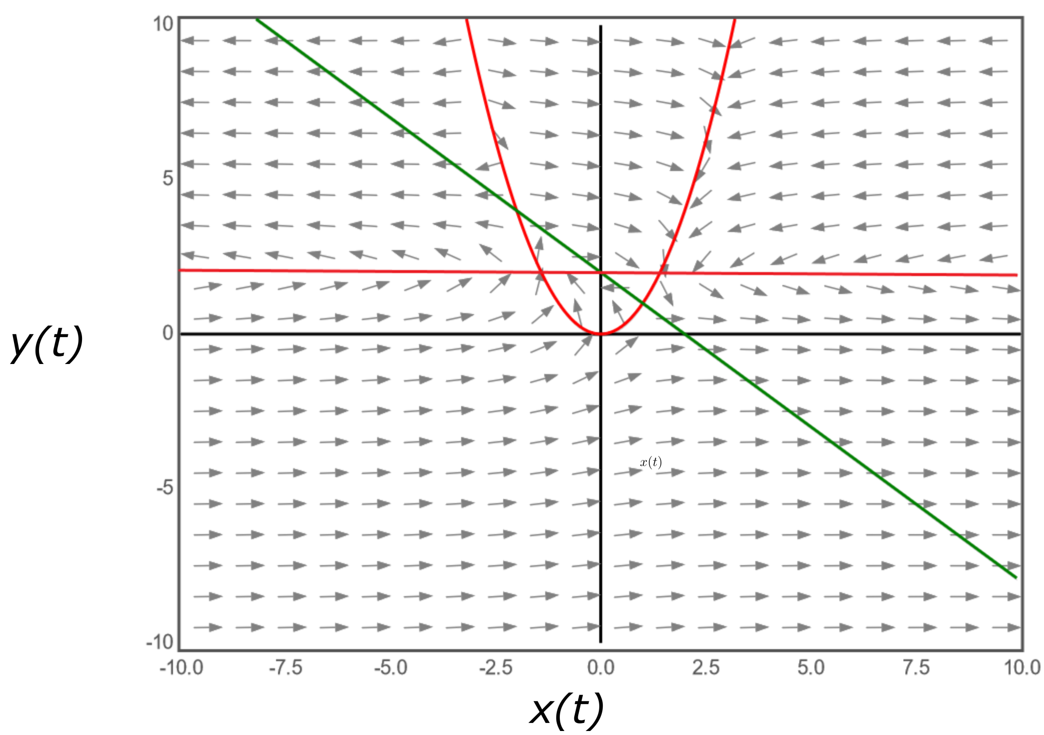
$$\frac{dx}{dt} = (y-2)(y-x^2), \quad \frac{dy}{dt} = 2-x-y$$

 (a) Plot and label the nullclines of the system. Please label your axes.
 (b) Identify all critical points of the system. Show your work.
 (c) Compute the Jacobian matrix. For each critical point you identified in part (b), use eigenvalues to classify the critical points according to stability (stable, unstable, asymptotically stable) and type (saddle, proper node, etc).

Solution. The nullclines are found by setting each equation in the system equal to zero, yielding the following.

x-nullclines	y-nullclines
$y-2=0$	$2-x-y=0$
$y-x^2=0$	

We plot these over the slope field of the graph, letting red lines be x-nullclines and green lines be y-nullclines.



To identify the critical points, we find the intersections of different nullclines. If $y=2$, then $2-x-2=0 \implies x=0$ so one critical point is $(0,2)$. If $y=x^2$, then the equation for the y-nullcline becomes $2-x-x^2=0 \implies (2+x)(1-x)=0 \implies x=-2, x=1$.

Letting $x = -2$ yields the critical point $(-2, 4)$ and letting $x = 1$ gives the critical point $(1, 1)$.

The Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 4x - 2xy & 2y - x^2 - 2 \\ -1 & -1 \end{pmatrix}$$

We can classify each critical point by evaluating the Jacobian at said points.

$$J(0, 2) = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix} \implies (-\lambda)(-1 - \lambda) + 2 = \lambda^2 + \lambda + 2 = 0$$
$$\implies \lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-7}$$

Since the eigenvalues are complex with real part less than zero, the critical point $(0, 2)$ is a stable spiral.

$$J(-2, 4) = \begin{pmatrix} 8 & 10 \\ -1 & -1 \end{pmatrix} \implies (8 - \lambda)(-1 - \lambda) + 10 = \lambda^2 - 7\lambda + 2 = 0$$
$$\implies \lambda = \frac{7}{2} \pm \frac{1}{2}\sqrt{47}$$

Since both eigenvalues are real and greater than zero, the critical point $(-2, 4)$ is an unstable node.

$$J(1, 1) = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix} \implies (2 - \lambda)(-1 - \lambda) - 1 = \lambda^2 - \lambda - 3 = 0$$
$$\implies \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{13}$$

Since both eigenvalues are real with one being negative and the other being positive, the critical point $(1, 1)$ is an unstable saddle. $\square$

2. (15 points) Use the Laplace transform to solve the following IVP. Please show your work.

$$y'' + 3y' + 2y = u_4(t), \quad y(0) = 0, \quad y'(0) = \frac{1}{2}$$

Solution. Taking the Laplace transform of both sides yields

$$s^2Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = \frac{e^{-4s}}{s}$$

Plugging and rearranging, we obtain

$$Y(s^2 + 3s + 2) = \frac{e^{-4s}}{s} + \frac{1}{2}$$

or

$$Y = \frac{e^{-4s}}{s^3 + 3s^2 + 2s} + \frac{1}{2s^2 + 6s + 4}.$$

We apply a shifting theorem to the left term, finding

$$L^{-1}\left\{\frac{e^{-4s}}{s^3 + 3s^2 + 2s}\right\} = u_4(t)L^{-1}\left\{\frac{1}{s^3 + 3s^2 + 2s}\right\}(t-4)$$

We use partial fractions to find the above inverse Laplace transform, which gives

$$\frac{1}{s^3 + 3s^2 + 2s} = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s-2)}$$

Therefore,

$$L^{-1}\left\{\frac{1}{s^3 + 3s^2 + 2s}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Similarly, we use partial fractions to find the inverse Laplace transform of the term on the right.

$$\frac{1}{2s^2 + 6s + 4} = \frac{1}{2(s+1)(s+2)} = \frac{1}{2(s+1)} - \frac{1}{2(s+2)}$$

so

$$L^{-1}\left\{\frac{1}{2s^2 + 6s + 4}\right\} = \frac{1}{2}(e^{-t} - e^{-2t})$$

Thus, we obtain a final answer of

$$y(t) = u_4(t)\left(\frac{1}{2} - e^{4-t} + \frac{1}{2}e^{8-2t}\right) + \frac{1}{2}\left(e^{-t} - e^{-2t}\right)$$

□

3. (8 points) Use the Laplace transform to solve the following IVP. Please show your work.

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$$

Solution. Taking the Laplace transform of both sides yields

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + 2Y = e^{-\pi s}$$

Rearranging, we get

$$Y(s^2 + 2s + 2) = e^{-\pi s} + 1$$

or

$$Y = \frac{e^{-\pi s}}{s^2 + 2s + 2} + \frac{1}{s^2 + 2s + 2}.$$

We apply a shifting theorem to simplify the inverse Laplace transform of the left term.

$$L^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 2s + 2}\right\} = u_\pi(t) L^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\}(t - \pi)$$

Thus, we merely need to calculate the inverse Laplace transform above, which we do by completing the square.

$$L^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin(t)$$

We have the solution

$$y(t) = u_\pi(t) e^{\pi - t}\sin(t - \pi) + e^{-t}\sin(t)$$

$\square$

4. (10 points) Consider the following IVP.

$$4y'' + 4y' + 17y = g(t), \quad y(0) = y'(0) = 0$$

(a) Use the Laplace transform to determine an explicit expression for $Y(s)$, which is the Laplace transform of $y(t)$. You can leave your answer in terms of $G(s)$, which is the Laplace transform of $g(t)$. Please show your work.

(b) Use your result from part (a), the convolution theorem, and the inverse Laplace transform to obtain an explicit expression for $y(t)$. Leave your answer in terms of a convolution integral involving $g(t)$. Please show your work.

Solution. We start by taking the Laplace transform of both sides, which gives

$$4(s^2Y - sy(0) - y'(0)) + 4(sY - y(0)) + 17Y = G(s)$$

Plugging in values and factoring yields

$$Y(s)(4s^2 + 4s + 17) = G(s)$$

or

$$Y = \frac{1}{4s^2 + 4s + 17} G(s).$$

By the convolution theorem, the right side is equal to the Laplace transform of the convolution of f and g where $L(f)$ is the left factor and $L(g) = G(s)$. We compute f by taking the inverse Laplace transform of the left factor, obtaining

$$\begin{aligned} f = L^{-1}\left\{\frac{1}{4s^2 + 4s + 17}\right\} &= \frac{1}{4} L^{-1}\left\{\frac{1}{(s + \frac{1}{2})^2 + 4}\right\} \\ &= \frac{1}{8} e^{-\frac{1}{2}t} \sin(2t) \end{aligned}$$

Then the convolution theorem implies that

$$y(t) = f * g = \int_0^t \frac{1}{8} e^{-\frac{1}{2}\tau} \sin(2\tau) g(t - \tau)\, d\tau$$

□