Math 2552 Written HW Set 1

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Judson 1.2.17. Solve the initial value problem

$$xy' = \sqrt{1-y^2},\ y(1) = 0.$$

Solution. We start by rewriting the equation as

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{x}$$

or alternatively

$$\frac{1}{\sqrt{1-y^2}}dy = \frac{1}{x}dx.$$

Integrating both sides of the equation yields

$$\arcsin(y) = \ln|x| + C,$$

where C is an arbitrary constant. We can use the initial condition $y(1) = 0$ to find that

$$\begin{aligned} \arcsin(0) &= \ln|1| + C \\ \implies C &= 0. \end{aligned}$$

Therefore we have

$$\arcsin(y) = \ln|x|$$

or

$$y = \sin(\ln|x|).$$

If we drop the absolute value signs, we obtain a final solution of

$$y = \sin(\ln(x))$$

which is continuous on the interval $(0, \infty)$. □

Judson 1.3.10. Find the equilibrium solutions for the differential equation

$$\frac{dx}{dt} = (x^2 - 1)(x - 2).$$

Draw the phase line and classify each equilibrium solution as a sink, a source, or a node.

Solution. First we factor the differential equation to find

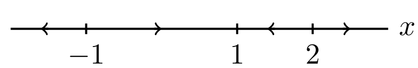
$$\frac{dx}{dt} = (x - 1)(x + 1)(x - 2)$$

and set each factor equal to zero to find where the function is constant. Then the solutions $x(t) = 1$, $x(t) = -1$, and $x(t) = 2$ are the equilibrium solutions.

Now we classify the equilibrium points by evaluating the derivative at various values of x as shown in the table below.

x	dx/dt
$x < -1$	$dx/dt < 0$
$-1 < x < 1$	$dx/dt > 0$
$1 < x < 2$	$dx/dt < 0$
$2 < x$	$dx/dt > 0$

From this, we draw the phase line diagram associated with the differential equation.



This makes it clear that $x = -1$ is a **source**, $x = 1$ is a **sink**, and $x = 2$ is a **source**. $\square$

Judson 1.5.21. A 600-liter tank initially contains 200 liters of water containing 10 kilograms of salt. Suppose that water containing 0.1 kilograms of salt flows into the top of the tank at a rate of 10 liters per minute. The water in the tank is kept well mixed, and 5 liters per minute are removed from the bottom of the tank. How much salt is in the tank when the tank is full?

Solution. Let $x(t)$ be the amount of salt in the tank at time t. Then the initial condition tells us that $x(0) = 10$. We also let $V(t)$ be the amount of water in the tank at time t. Then $V(t) = 200 + 5t$.

Furthermore, we know the rate at which salt flows in and out of the tank. Then we have

$$\frac{dx}{dt} = 10(0.1) - \frac{5x}{V(t)} = 1 - \frac{x}{40+t}.$$

Rearranging, we get the first order linear differential equation

$$\frac{dx}{dt} + \frac{1}{40+t}x = 1.$$

We solve for the integrating factor

$$\mu(t) = \exp\left(\int \frac{1}{40+t}dt\right) = 40 + t.$$

Multiplying both sides of the differential equation by the integrating factor yields

$$(40+t)\frac{dx}{dt} + x = 40 + t$$

or

$$\frac{d}{dt}\left[(40+t)x\right] = 40 + t.$$

We can integrate both sides with respect to t to get

$$(40+t)x = 40t + \frac{t^2}{2} + C.$$

If we use our initial condition of $x(0) = 10$, then we find $C = 400$. Rearranging our above equation, we obtain a solution of

$$x(t) = \frac{t^2 + 80t + 800}{2t + 80}.$$

Since the maximum capacity of the tank is 600 liters, the tank is full at time $t = 80$. Finally, we find that $x(80) = \frac{170}{3} \approx 56.667$. That is, the tank contains about 56.667 kilograms of salt when it is full. $\square$