Math 2552 Written HW Set 6

Akash Narayanan

March 9, 2021

Trench 5.4.7. Find a particular solution.

$$y'' - 4y' - 5y = -6xe^{-x}$$

Solution. We find a particular solution using the method of undetermined coefficients. First, we will find the general solution to the homogeneous equation. Note that its characteristic equation is $r^2 - 4r - 5 = (r-5)(r+1)$ so the general solution to the homogeneous equation is

$$y_h = c_1 e^{5x} + c_2 e^{-x}.$$

A standard first guess for the particular solution would be $y_p = (ax+b)e^{-x}$. However, note that this particular solution contains a term that is in the general solution to the homogeneous equation. This is problematic because it effectively means that attempting to solve for a and b by plugging the solution into the differential equation would only yield one equation despite having 2 variables. We remedy this by including a factor of x. This yields

$$\begin{aligned} y_p &= (ax+b)xe^{-x} \\ y_p' &= -(ax^2 + (b-2a)x - b)e^{-x} \\ y_p'' &= (ax^2 + (b-4a)x - 2b + 2a)e^{-x}. \end{aligned}$$

Plugging this into the differential equation yields (after a fair bit of simplification)

$$\begin{gathered} (ax^2 + (b-4a)x - 2b + 2a)e^{-x} + 4(ax^2 + (b-2a)x - b)e^{-x} - 5(ax+b)xe^{-x} = -6xe^{-x} \\ \Longrightarrow (-12ax + 2a - 6b)e^{-x} = -6xe^{-x} \end{gathered}$$

Equating terms of equal degree in the polynomial factor, we obtain two equations:

$$\begin{gathered} -12a = -6 \\ 2a - 6b = 0 \end{gathered}$$

This has the solution $a = 1/2$ and $b = 1/6$. Thus, a particular solution to the differential equation (after distributing the x to the linear equation) is

$$y_p = (\frac{1}{2}x^2 + \frac{1}{6}x)e^{-x}.$$

$\square$

Trench 5.7.7. Use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

$$x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \frac{1}{x}$$

Solution. Before starting, we will rewrite the differential equation so that y'' has coefficient 1:

$$y'' + \frac{1}{x}y - \frac{1}{x^2}y = 2 + \frac{2}{x^2} \tag{1}$$

Given the fundamental set of solutions to the homogeneous equation, we construct a particular solution of the form

$$y_p = u_1y_1 + u_2y_2$$

for function u_1 and u_2. We start by calculating the Wronskian

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -\frac{2}{x}.$$

Then we have

$$\begin{aligned} u_1'(x) &= \frac{-y_2(x)f(x)}{W(y_1, y_2)(x)} = \frac{-x^{-1}(2 + 2x^{-2})}{-2x^{-1}} = 1 + \frac{1}{x^2} \\ u_2'(x) &= \frac{y_1(x)f(x)}{W(y_1, y_2)(x)} = \frac{x(2 + 2x^{-2})}{-2x^{-1}} = -x^2 - 1 \end{aligned}$$

where $f(x)$ is the right side of (1). Integrating both functions with respect to x yields

$$\begin{aligned} u_1 &= \int 1 + \frac{1}{x^2}\,dx = x - \frac{1}{x} \\ u_2 &= \int -x^2 - 1\,dx = -\frac{x^3}{3} - x \end{aligned}$$

Thus, a particular solution to the differential equation is

$$\begin{aligned} y_p(x) &= \left(x - \frac{1}{x}\right)x + \left(-\frac{x^3}{3} - x\right)\frac{1}{x} \\ &= \frac{2}{3}x^2 - 2 \end{aligned}$$

□

Trench 10.7.1. Find a particular solution.

$$y' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} y + \begin{bmatrix} 21e^{4t} \\ 8e^{-3t} \end{bmatrix}$$

Solution. Let A denote the coefficient matrix of y. Then

$$\det(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$$

so the eigenvalues of the system are $\lambda_1 = 1$ and $\lambda_2 = -3$. The corresponding eigenvectors are found by solving $(A - \lambda I)\vec{v} = 0$. We have

$$(A - I)\vec{v} = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix} \vec{v} = 0 \Longrightarrow -x_1 - 2x_2 = 0 \Longrightarrow -x_1 = 2x_2$$

Letting $x_2 = 1$, we find the eigenvector corresponding to $\lambda_1 = 1$ to be

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Similarly, we find

$$(A + 3I)\vec{v} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \vec{v} = 0 \Longrightarrow -x_1 + 2x_2 = 0 \Longrightarrow 2x_2 = x_1$$

Letting $x_2 = 1$, we find the eigenvector corresponding to $\lambda_2 = 3$ to be

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then the general solution to the homogeneous system is

$$y_h = c_1 \begin{pmatrix} -2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \end{pmatrix}$$

The corresponding fundamental matrix is

$$Y = \begin{pmatrix} -2e^t & 2e^{-3t} \\ e^t & e^{-3t} \end{pmatrix}$$

We find

$$\det(Y) = -2e^{-2t} - 2e^{-2t} = -4e^{-2t}$$
$$Y^{-1} = \frac{1}{\det(Y)} \begin{pmatrix} e^{-3t} & -2e^{-3t} \\ -e^t & -2e^t \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix}$$

Then a particular solution is

$$y_p = Y \int Y^{-1} g \, dt$$

where g is nonhomogeneous part of the system of differential equations. Calculating, we have

$$\begin{aligned}
y_p &= \frac{1}{4}Y\int \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \begin{pmatrix} 21e^{4t} \\ 8e^{-3t} \end{pmatrix} \, dt \\
&= \frac{1}{4}Y\int \begin{pmatrix} -21e^{3t} + 16e^{-4t} \\ 21e^{7t} + 16 \end{pmatrix} \, dt \\
&= \frac{1}{4}\begin{pmatrix} -2e^{t} & 2e^{-3t} \\ e^{t} & e^{-3t} \end{pmatrix} \begin{pmatrix} -7e^{3t} - 4e^{-4t} \\ 3e^{7t} + 16t \end{pmatrix} \\
&= \begin{pmatrix} 8te^{-3t} + 5e^{4t} + 2e^{-3t} \\ 4te^{-3t} - e^{4t} - e^{-3t} \end{pmatrix}
\end{aligned}$$

□