Math 2552 Written HW Set 4

Akash Narayanan

February 16, 2021

Trench 10.3.9. Let

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix},\ y_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix},\ y_2 = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix},\ k = \begin{bmatrix} -19 \\ 11 \end{bmatrix}$$

(a) Verify that $\{y_1, y_2\}$ is a fundamental set of solutions for $y' = Ay$.

(b) Solve the initial value problem

$$y' = Ay,\ y(0) = k. \tag{A}$$

(c) Use the result of Exercise 6(b) to find a formula for the solution of (A) for an arbitrary initial vector k.

Solution. It is easy to see that the two vectors satisfy the system of differential equations. To show the two are linearly independent, it suffices to show that the Wronskian is non-zero.

$$W(y_1, y_2) = \begin{vmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{vmatrix} = -e^{3t} \neq 0$$

Thus, $\{y_1, y_2\}$ does indeed form a fundamental set of solutions. That is, the general solution has the form

$$y = c_1 y_1 + c_2 y_2$$

For the particular case $y(0) = k$, we have

$$\begin{aligned} -5c_1 + 2c_2 &= -19 \\ 3c_1 - c_2 &= 11 \end{aligned}$$

which yields $c_1 = 3$ and $c_2 = -2$ for a particular solution of

$$y(t) = 3y_1 - 2y_2 = \begin{bmatrix} -15e^{2t} - 4e^{t} \\ 9e^{2t} + 2e^{t} \end{bmatrix}$$

Given the fundamental set $\{y_1, y_2\}$, we form the matrices

$$Y(t) = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix},\ Y^{-1}(t) = \begin{bmatrix} e^{-2t} & 2e^{-2t} \\ 3e^{-t} & 5e^{-2t} \end{bmatrix}$$

Then for an arbitrary vector k, we have the following solution to (A):

$$y(t) = Y(t)Y^{-1}(0)k = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} k = \begin{bmatrix} -5e^{2t} + 6e^{t} & -10^{2t} + 10e^{t} \\ 3e^{2t} - 3e^{t} & 6e^{2t} - 5e^{t} \end{bmatrix} k$$

$\square$

Trench 10.5.11. Find the general solution.

$$y' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} y$$

Solution. Denote the matrix by A. Then we compute the eigenvalues of A by finding the roots of the characteristic polynomial

$$\det(A - \lambda I) = -\lambda^3 + 10\lambda^2 - 32\lambda + 32 = -(\lambda - 4)^2(\lambda - 2) = 0$$

Then the eigenvalues are $\lambda_1 = 2$ with multiplicity 1 and $\lambda_2 = 4$ with multiplicity 2. The eigenvector associated with $\lambda_1 = 2$ is

$$\vec{v}_1 = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

yielding the solution

$$y_1 = c_1 e^{2t} \vec{v}_1$$

The eigenvector associated with $\lambda_2 = 4$ is

$$\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

yielding the solution

$$y_2 = c_2 e^{4t} \vec{v}_2$$

A second eigenvector associated with λ_2 satisfies the equation $(A-4I)\vec{u} = \vec{v}_2$. Solving the equation yields

$$\vec{u} = \begin{pmatrix} \frac{1}{2} \\ -a \\ a \end{pmatrix}$$

where a is free. This forms the solution

$$y_3 = c_3 e^{4t}(\vec{u} + t\vec{v}_2)$$

The three solutions are independent and form the general solution

$$y = c_1 e^{2t} \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} \frac{1}{2} \\ -a - t \\ a + t \end{pmatrix}$$

$\square$

Trench 10.6.13. Find the general solution.

$$y' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} y$$

Solution. We find the eigenvalues by calculating the roots of the character-stic polynomial.

$$\det(A - \lambda I) = \lambda^3 + 2\lambda - 4 = -(\lambda + 2)(\lambda - (1 - i))(\lambda - (1 + i)) = 0$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1 - i$, and $\lambda_3 = 1 + i$. The eigenvector associated with $\lambda_1 = -2$ is

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

which yields the solution

$$y_1 = c_1 e^{-2t} \vec{v}_1$$

The eigenvector associated with $\lambda_2 = 1 - i$ is

$$v_2 = \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$$

This eigenvalue-vector pair yields two solution vectors to the original system, namely

$$w_1 = c_2 e^t \begin{pmatrix} \sin(t) \\ -\cos(t) \\ \cos(t) \end{pmatrix}, \quad w_2 = c_3 e^t \begin{pmatrix} -\cos(t) \\ -\sin(t) \\ \sin(t) \end{pmatrix}$$

Thus, the general solution to the system is

$$y = c_1 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(t) \\ -\cos(t) \\ \cos(t) \end{pmatrix} + c_3 e^t \begin{pmatrix} -\cos(t) \\ -\sin(t) \\ \sin(t) \end{pmatrix}$$

□