Math 2552 Written HW Set 10

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Trench 8.6.3a. Find a formula for the solution of the initial value problem.

$$y'' + 3y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

Solution. Taking Laplace transforms yields

$$(s^2 + 3s + 1)Y(s) = F(s)$$

where $Y(s)$ is the Laplace transform of $y(t)$. Then

$$Y(s) = \frac{1}{(s^2 + 3s + 1)} F(s)$$

To compute the inverse Laplace transform of the left factor on the right side, we start by completing the square, which gives

$$\frac{1}{(s^2 + 3s + 1)} = \frac{1}{(s + \frac{3}{2})^2 - \frac{5}{4}}$$

The shifting theorem and table of Laplace transforms show that

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s^2 + 3s + 1)}\right\} &= e^{-\frac{3}{2}t} L^{-1}\left\{\frac{1}{s^2 - \frac{5}{4}}\right\} \\ &= \frac{2}{\sqrt{5}} e^{-\frac{3}{2}t} \sinh\left(\frac{\sqrt{5}}{2} t\right) \end{aligned}$$

Then the convolution theorem implies

$$L^{-1}\left\{\frac{1}{(s^2 + 3s + 1)} F(s)\right\} = \frac{2}{\sqrt{5}} \int_0^t e^{-\frac{3}{2}\tau} \sinh\left(\frac{\sqrt{5}}{2}\tau\right) f(t - \tau)\, d\tau$$

Thus, a formula for the solution to the initial value problem is

$$y(t) = \frac{2}{\sqrt{5}} \int_0^t e^{-\frac{3}{2}\tau} \sinh\left(\frac{\sqrt{5}}{2}\tau\right) f(t - \tau)\, d\tau$$

□

Trench 8.6.5a. Use the convolution theorem to evaluate the integral.

$$\int_0^t (t-\tau)^7 \tau^8 \, d\tau$$

Solution. By definition, the above integral is $f * g$ where $f(t) = t^8$ and $g(t) = t^7$. If $F = L(f)$ and $G = L(g)$, we can calculate

$$F(s) = L(t^8) = \frac{8!}{s^9}$$

and

$$G(s) = L(t^7) = \frac{7!}{s^8}$$

By the convolution theorem, we have

$$f * g = L^{-1}\{FG\}$$

Using the table of Laplace transforms, we find

$$\begin{aligned} L^{-1}\{FG\} &= L^{-1}\left\{\frac{8!7!}{s^{17}}\right\} \\ &= \frac{8!7!}{16!} t^{16} \end{aligned}$$

Thus, we find

$$\int_0^t (t-\tau)^7 \tau^8 \, d\tau = \frac{8!7!}{16!} t^{16}$$

□

Trench 8.7.9. Solve the initial value problem and graph the solution.

$$y'' + 3y' + 2y = 1 + \delta(t-1), \quad y(0) = 1, \quad y'(0) = -1$$

Solution. First we define $\hat{y}$ to be the solution of

$$y'' + 3y' + 2y = 1, \quad y(0) = 1, \quad y'(0) = -1$$

The characteristic polynomial of the differential equation is $r^2 + 3r + 2$ which has zeroes at $\lambda_1 = -2$, $\lambda_2 = -1$. Then the solution to the homogeneous equation is

$$y_h = c_1 e^{-2t} + c_2 e^{-t}$$

By inspection, a particular solution to the differential equation is $y_p = \frac{1}{2}$. Therefore, we find

$$\hat{y} = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{2}.$$

Plugging in the initial conditions yields the system of equations

$$\begin{aligned} c_1 + c_2 + \frac{1}{2} &= 1 \\ -2c_1 - c_2 &= -1 \end{aligned}$$

which has the solution $c_1 = \frac{1}{2}, c_2 = 0$. Thus, we have

$$\hat{y} = \frac{1}{2} e^{-2t} + \frac{1}{2}.$$

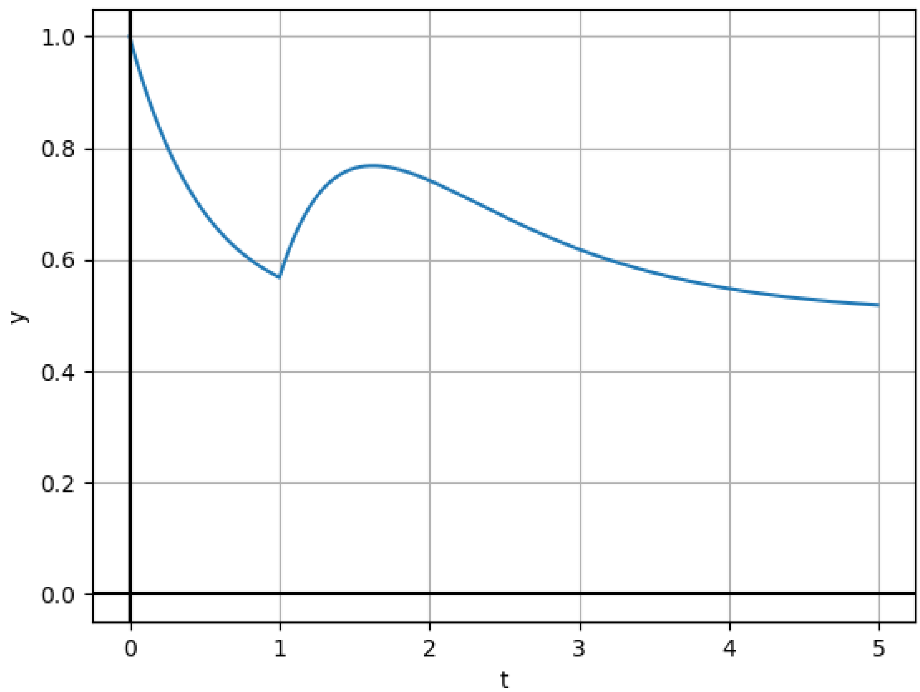
Furthermore, we find

$$\begin{aligned} w(t) &= L^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = L^{-1}\left\{\frac{1}{(s+2)(s+1)}\right\} \\ &= L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t} \end{aligned}$$

Then the solution to the initial value problem is

$$\begin{aligned} y &= \hat{y} + u(t-1)w(t-1) \\ &= \frac{1}{2}e^{-2t} + \frac{1}{2} + u(t-1)\left(e^{-(t-1)} - e^{-2(t-1)}\right). \end{aligned}$$

Shown below is a graph of the function.


1.0
0.8
0.6
0.4
0.2
0.0
y
0
1
2
3
4
5
t

□