Math 2552 Written HW Set 8

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March 30, 2021

Judson 5.1.3. For the below system,

a. Plot and label the nullclines for equation in the system.

b. Find all of the equilibrium solutions for the system.

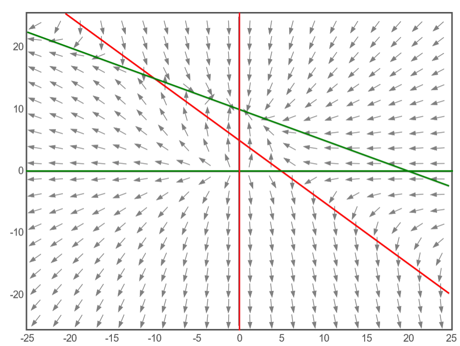
c. Use the Jacobian to classify each equilibrium solution (spiral source, nodal sink, etc.).

$$\begin{aligned}\frac{dx}{dt} &= x(5 - x - y)\\ \frac{dy}{dt} &= y(20 - x - 2y)\end{aligned}$$

Solution. We plot the slope field and the nullclines for the system of differential equations. The nullclines are found by setting each equation in the system equal to zero, yielding the four equations

$$\begin{aligned} x &= 0 & y &= 0\\ 5 - x - y &= 0 & 20 - x - 2y &= 0\end{aligned}$$

and the following graph. Red lines are x-nullclines and green lines are y-nullclines.



Equilibrium solutions are where both derivatives are zero, or where different nullclines intersect. Easily seen are the points $(0, 0), (0, 10)$, and $(5, 0)$. The last point is found by setting $5 - x - y = 20 - x - 2y$, or $y = 15$, yielding the point $(-10, 15)$.

To classify each solution, we start by calculating the Jacobian,

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x - y + 5 & -x \\ -y & -4y - x + 20 \end{pmatrix}$$

Evaluating the Jacobian at each critical point allows us to classify them. We begin with $(0, 0)$:

$$J(0, 0) = \begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix}$$

By inspection, the matrix has eigenvalues 5 and 20. Since both eigenvalues are positive and real, the point is a nodal source. Next, we have

$$J(0,10) = \begin{pmatrix} -5 & 0 \\ -10 & -20 \end{pmatrix}$$

Again by inspection, the matrix has eigenvalues -5 and -20. Both eigenvalues are negative and real so the point is a nodal sink. Evaluating the third point, we find

$$J(5,0) = \begin{pmatrix} -5 & -5 \\ 0 & 15 \end{pmatrix}$$

By inspection, the matrix has eigenvalues -5 and 15. Both eigenvalues are real but differ in sign so the point is a saddle. Finally, we find

$$J(-10,15) = \begin{pmatrix} 10 & 10 \\ -15 & -30 \end{pmatrix}$$

The characteristic polynomial of this matrix is $\lambda^2 + 20\lambda - 150$ and the eigenvalues are $-10 \pm 5\sqrt{10}$. Both eigenvalues are real but differ in sign so the point is a saddle. $\square$

Judson 5.1.5. For the below system,

a. Plot and label the nullclines for equation in the system.

b. Find all of the equilibrium solutions for the system.

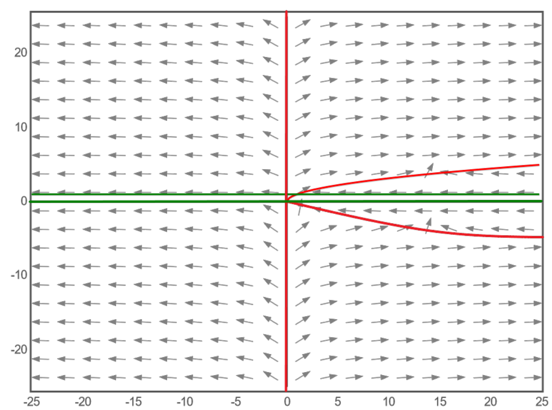
c. Use the Jacobian to classify each equilibrium solution (spiral source, nodal sink, etc.).

$$\frac{dx}{dt} = x(y^2 - x)$$
$$\frac{dy}{dt} = y(y-1)$$

Solution. As above, we plot the slope field and nullclines for the system with the same colorscheme. Setting the derivatives in the system equal to zero yields the equations

$$x = 0 \qquad\qquad y = 0$$
$$x = y^2 \qquad\qquad y = 1$$

and the following graph where red lines are x-nullclines and green lines are y-nullclines.



We find the equilibrium points by calculating where equilibrium points intersect. Letting $x = 0$ yields the points $(0, 0)$ and $(0, 1)$. The third point is found by letting $y = 1$ and seeing that $x = 1$, meaning the third point is $(1, 1)$.

Next, we calculate the Jacobian.

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} y^2 - 2x & 2xy \\ 0 & 2y - 1 \end{pmatrix}$$

Evaluating at the equilibrium points, we find

$$J(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

which, by inspection, has eigenvalues 0 and -1. Since one eigenvalue is negative and the other is zero, the critical point is a semi-stable node. The second equilibrium point yields

$$J(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has eigenvalue 1. Thus, the critical point is a nodal source. Finally, the last equilibrium point gives

$$J(1,1) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

By inspection, this matrix has eigenvalues -1 and 1. Since they are both real and differ in sign, the critical point is a saddle. $\square$

Trench 6.2.5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c. Find all values of c such that the free vibration of the weight has infinitely many oscillations.

Solution. Converting 16 lb to mass yields $16/32 = 0.5$ slugs. Furthermore, we have $k = 16/0.5 = 32$ lb/ft. so the equation of motion for the problem is

$$0.5y'' + cy' + 32y = 0.$$

The free vibration has infinitely many oscillations if and only if the motion is underdamped, or $c < \sqrt{4mk}$. The characteristic equation of the differential equation is

$$0.5r^2 + cr + 32 = 0$$

and it has roots

$$r = -c \pm \sqrt{c^2 - 64}$$

The motion is underdamped if $c < \sqrt{64} = 8$. Thus, the set of values of c such that the free vibration of the weight has infinitely many oscillations is the interval $0 \leq c < 8$. $\square$