Math 2552 Midterm 2

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March 11, 2021

1. (2 points) Determine all values of α, if any, for which all solutions tend to zero as $t \to \infty$.

$$y'' + (\alpha + 2)y' + 4y = 0, \quad \alpha \in \mathbb{R}$$

Show your work.

Solution. Consider the characteristic equation $r^2 + (\alpha + 2) + 4 = 0$. Using the quadratic formula, it has the following roots:

$$\begin{aligned} r &= \frac{-(\alpha+2) \pm \sqrt{(\alpha+2)^2 - 16}}{2} \\ &= -(1 + \frac{\alpha}{2}) \pm \frac{1}{2}\sqrt{\alpha^2 + 4\alpha - 12} \end{aligned}$$

We now treat separate cases based on the value of the sign of the discriminant $d = \alpha^2 + 4\alpha - 12$.

First consider the case $d < 0$. This occurs when $(\alpha + 2)^2 - 16 < 0$, or

$$\begin{aligned} (\alpha+2)^2 &< 16 \\ -4 < \alpha + 2 &< 4 \\ -6 < \alpha &< 2 \end{aligned}$$

For solutions to converge to zero under this condition, the real part of the roots must be negative. That is,

$$\begin{aligned} -(1 + \frac{\alpha}{2}) &< 0 \\ 2 + \alpha &> 0 \\ \alpha &> -2 \end{aligned}$$

Therefore, solutions converge to zero for $\alpha \in (-2, 2)$.

Now consider the case $d = 0$, which occurs when $\alpha = 2$. Then the root $r = -2$. Since this is the only root, the solution to the differential equation is

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

which clearly converges to zero as t goes to infinity.

Finally, consider the case $d > 0$. This occurs when $(\alpha + 2)^2 - 16 > 0$, or

$$\begin{aligned} (\alpha+2)^2 &> 16 \\ |\alpha + 2| &> 4 \\ \alpha + 2 > 4 \quad &\text{or} \quad -\alpha - 2 > 4 \\ \alpha > 2 \quad &\text{or} \quad \alpha < -6 \end{aligned}$$

Note that if $\alpha < -2$ then $r > 0$ for at least one of the roots (because the vertex is positive and there is one root greater than the vertex) so at least one of the solutions will diverge as t goes to infinity. Thus, we can restrict our attention to $\alpha > 2$. We require that the root closest to zero still be negative. As an inequality, we obtain

$$\begin{aligned}
-\frac{\alpha}{2} - 1 + \frac{1}{2}\sqrt{\alpha^2 + 4\alpha - 12} &< 0 \\
\sqrt{\alpha^2 + 4\alpha - 12} &< \alpha + 2 \\
\alpha^2 + 4\alpha - 12 &< \alpha^2 + 4\alpha + 4 \\
0 &< 16
\end{aligned}$$

Since the inequality holds trivially, we have that both roots of the characteristic equation are negative if $\alpha > 2$. Thus, all solutions to the differential equation converge to zero as t goes to infinity if and only if $\alpha > -2$. $\square$

2. (2 points) Determine whether the functions $y_1 = e^{2t}$ and $y_2 = e^{3t}$ are a fundamental set of solutions to the differential equation $y'' + 5y' + 6y = 0$ with $y = y(t)$. Show your work.

Solution. We can check by first seeing if the functions satisfy the differential equation. We find

$$y_1'' + 5y_1' + 6y_1 = 4e^{2t} + 10e^{2t} + 6e^{2t} \neq 0$$

Since y_1 does not satisfy the differential equation, the set $\{y_1, y_2\}$ is not a fundamental set of solutions. $\square$

3. (1 point) State whether the following differential equation is linear or non-linear, and whether it is homogeneous or non-homogeneous.

$$y'' + (1 - y)y' + 2y = 1$$

Solution. The differential equation is non-linear and non-homogeneous. It is non-linear because there is a term yy' which is not a linear combination of y and its derivatives. It is non-homogeneous because there is a non-zero constant term, namely 1. □

4. (3 points) Construct an initial value problem for the following situation. Show your work.

   A 0.05 Newtons (N) force stretches a spring 0.01 m. A mass weighing 2 kg is attached to the spring, and the spring is also attached to a viscous damper that applies a force of 0.4 N when the velocity of the mass is 0.1 m/s. The mass is pulled down 0.1 m below its equilibrium position and given an initial upward velocity of 0.6 m/s.

   Solution. By Newton's Laws, we have $F = kL$. Substituting, we find $0.05 = 0.01k$ so $k = 5$. Similarly, the damping force is given by the equation $F_d = \gamma v$. Substituting the corresponding values yields $0.4 = 0.1\gamma$ so $\gamma = 4$. Given the initial conditions, the corresponding initial value problem is

$$2y'' + 4y' + 5 = 0, \quad y(0) = 0.1, \quad y'(0) = -4$$

□

5. (3 points) If $Y = W[f, g]$ is the Wronskian of f and g, and $u = 3f + g$, $v = f - 3g$, express the Wronskian $W[u, v]$ of u and v in terms of Y. Show your work.

Solution. First, note that $Y = fg' - f'g$. We find that

$$\begin{aligned} W[u,v] &= uv' - u'v \\ &= (3f+g)(f'-3g') - (3f'+g')(f-3g) \\ &= (3ff' - 9fg' + gf' - 3gg' - 3ff' + 9f'g + fg' + 3gg' \\ &= -8(fg' - f'g) \\ &= -8Y \end{aligned}$$

$\square$

6. (3 points) Determine a suitable form for the particular solution $Y(t)$ if the method of undetermined coefficients is to be used. Please show your work.

$$y'' + 11y' + 24y = 4\sin(4t) + 2e^{-8t}$$

Solution. We start by considering the general solution to the homogeneous equation:

$$y_h'' + 11y_h' + 24y_h = 0$$

The characteristic equation is $r^2 + 11r + 24 = 0$ which has solutions $r_1 = -8$ and $r_2 = -3$. Then the solution to the homogeneous equation is

$$y_h = c_1e^{-8t} + c_2e^{-3t}$$

Through inspection, one might guess that a particular solution would have the form

$$Y(t) = a_1\sin(4t) + a_2\cos(4t) + a_3e^{-8t}$$

However, note that e^{-8t} is a solution to the homogeneous equation. We remedy this by including a factor of t. Therefore, a suitable form for a particular solution using the method of undetermined coefficients is

$$Y(t) = a_1\sin(4t) + a_2\cos(4t) + a_3te^{-8t}$$

□

7. (4 points) The position of a moving object, $y(t)$, for time $t \geq 0$ satisfies the IVP

$$y'' + 2y' + 4y = 0, \quad y(0) = 1,\ y'(0) = 4, \quad y = y(t)$$

(a) Express the differential equation in the IVP as a first-order system in the form $\vec{x}' = A\vec{x}$.

(b) Solve the DE using any method you like. You do not need to solve the IVP, but show your work.

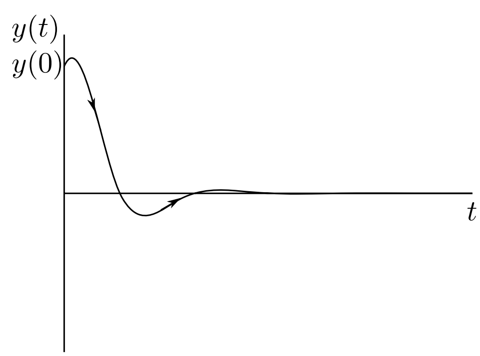
(c) Sketch the trajectory of the object for $t \geq 0$ in the phase plane. Indicate the location of the object at time $t = 0$, the direction of motion, and label your axes.

Solution. Let $x_1 = y$ and $x_2 = y'$. Then we find $x_1' = y' = x_2$ and $x_2' = -4y - 2y' = -4x_1 - 2x_2$. In matrix form, this can be expressed as

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

To solve the homogeneous differential equation, consider the characteristic equation $r^2 + 2r + 4 = 0$ which has solutions $r = -1 \pm i\sqrt{3}$. Then the general solution is

$$y = e^{-t}\left(c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\right)$$



$\square$

8. (10 points) Use the variation of parameters method to identify the general solution to

$$\vec{x}' = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

Solution. Let A be the coefficient matrix of $\vec{x}$. Then

$$\det(A - \lambda I) = \lambda^2 - 4\lambda - 12 = (\lambda + 2)(\lambda - 6) = 0$$

and the eigenvalues of the system are $\lambda_1 = -2$ and $\lambda_2 = 6$. The corresponding eigenvectors can be found by solving the equation $(A - \lambda I)\vec{v} = 0$. We have

$$(A + 2I)\vec{v}_1 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \vec{v}_1 = 0 \Longrightarrow 4x_1 + 4x_2 = 0 \Longrightarrow x_1 = -x_2$$

Letting $x_2 = -1$, we find the eigenvector corresponding to $\lambda_1 = -2$ to be

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solving for $\vec{v}_2$, we find

$$(A - 6I)\vec{v}_2 = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \vec{v}_2 = 0 \Longrightarrow 4x_1 - 4x_2 = 0 \Longrightarrow x_1 = x_2$$

Letting $x_2 = 1$, we find the eigenvector corresponding to $\lambda_2 = 6$ to be

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the solution to the homogeneous system is

$$y_h = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The corresponding fundamental matrix is

$$X = \begin{pmatrix} e^{-2t} & e^{6t} \\ -e^{-2t} & e^{6t} \end{pmatrix}$$

We calculate

$$\det(X) = e^{4t} + e^{4t} = 2e^{4t}$$

$$X^{-1} = \frac{1}{\det(X)} \begin{pmatrix} e^{6t} & -e^{6t} \\ e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{2t} & -e^{2t} \\ e^{-6t} & e^{-6t} \end{pmatrix}$$

A particular solution is

$$y_p = X \int X^{-1} g \, dt$$

where g is the non-homogeneous part of the system of differential equations. Calculating, we find

$$\begin{aligned}
y_p &= \frac{1}{2}X\int \begin{pmatrix} e^{2t} & -e^{2t} \\ e^{-6t} & e^{-6t} \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix} dt \\
&= \frac{1}{2}X\int \begin{pmatrix} 12e^{2t} \\ 12e^{-6t} \end{pmatrix} dt \\
&= \frac{1}{2} \begin{pmatrix} e^{-2t} & e^{6t} \\ -e^{-2t} & e^{6t} \end{pmatrix} \begin{pmatrix} 6e^{2t} \\ -2e^{-6t} \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -4 \end{pmatrix}
\end{aligned}$$

Then the general solution to the system of differential equations is

$$\vec{x} = c_1e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

□

9. (10 points) Solve the DE. Show your work.

$$y'' + 25y = 60\cos(5t), \quad y = y(t)$$

Solution. We first consider the homogeneous equation

$$y_h'' + 25y_h = 0$$

The characteristic equation is $r^2 + 25 = 0$ which has solutions $r = \pm 5i$. Then the general form for the homogeneous solution is

$$y_h = c_1\cos(5t) + c_2\sin(5t)$$

We find a particular solution using the method of undetermined coefficients. Assume that a particular solution has the form

$$y_p = At\cos(5t) + Bt\sin(5t)$$

Then we have

$$\begin{aligned} y_p' &= (5Bt + A)\cos(5t) + (B - 5At)\sin(5t) \\ y_p'' &= (10B - 25At)\cos(5t) - (25Bt + 10A)\sin(5t) \end{aligned}$$

Substituting this into the differential equation, we obtain

$$\begin{aligned} y_p'' + 25y_p &= (10B - 25At)\cos(5t) - (25Bt + 10A)\sin(5t) + 25(At\cos(5t) + Bt\sin(5t)) \\ &= 10B\cos(5t) - 10A\sin(5t) = 60\cos(5t) \end{aligned}$$

That is, $A = 0$ and $B = 6$ so a particular solution is

$$y_p = 6t\sin(5t)$$

Thus, the general solution to the differential equation is

$$y(t) = c_1\cos(5t) + c_2\sin(5t) + 6t\sin(5t)$$

□

10. (10 points) Solve the DE using variation of parameters. Solutions to the homogeneous problem are $y_1 = t^2$ and $y_2 = t^{-2}$. Please show your work.

$$t^2 y'' + ty' - 4y = t, \quad y = y(t), \quad t > 0$$

Solution. We start by rewriting the differential equation in standard form:

$$y'' + \frac{1}{t}y' - \frac{4}{t^2}y = \frac{1}{t}$$

Given the fundamental set of solutions, we can construct a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2$$

for functions u_1 and u_2. We first calculate the Wronskian

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\ &= t^2(-2t^{-3}) - (2t)(t^{-2}) \\ &= -4t^{-1} \end{aligned}$$

Then we have

$$\begin{aligned} u_1'(t) &= \frac{-y_2(t)g(t)}{W[y_1, y_2]} = \frac{-t^{-2} \cdot t^{-1}}{-4t^{-1}} = \frac{1}{4t^2} \\ u_2'(t) &= \frac{y_1(t)g(t)}{W[y_1, y_2]} = \frac{t^2 \cdot t^{-1}}{-4t^{-1}} = -\frac{t^2}{4} \end{aligned}$$

where $g(t)$ is the non-homogeneous part of the standard form differential equation. Integrating with respect to t yields

$$\begin{aligned} u_1 &= \int \frac{1}{4t^2}\,dt = -\frac{1}{4t} \\ u_2 &= \int -\frac{t^2}{4}\,dt = -\frac{t^3}{12} \end{aligned}$$

Then a particular solution to the differential equation is

$$\begin{aligned} y_p &= -\frac{1}{4t}t^2 - \frac{t^3}{12}t^{-2} \\ &= -\frac{t}{3} \end{aligned}$$

Finally, the general solution to the differential equation is

$$y(t) = c_1 t^2 + c_2 t^{-2} - \frac{t}{3}$$

□