Solutions to Introduction to Set Theory by Hrbáček and Jech

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Chapter 1

Sets

1.1 Introduction to Sets

No exercises.

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1.3 The Axioms

Problem 1.3.1. Show that the set of all x such that $x \in A$ and $x \notin B$ exists.

Solution. Consider the property $\mathbf{P}(x, B)$: "$x \notin B$." Then, by the Comprehension Schema, for every A and B there exists a set C such that $x \in C$ if and only if $x \in A$ and $\mathbf{P}(x, B)$, or if and only if $x \in A$ and $x \notin B$. □

Problem 1.3.2. Replace the Axiom of Existence by the following weaker postulate:

Postulate (Weak Axiom of Existence)**.** Some set exists.

Prove the Axiom of Existence using the Weak Axiom of Existence and the Comprehension Schema. [*Hint*: Let A be a set known to exist; consider $\{x \in A \mid x \neq x\}$.]

Solution. Recall that the Axiom of Existence states that there exists a set which has no elements. Let A be a set known to exist, which is guaranteed by the Weak Axiom of Existence. Consider the property $\mathbf{P}(x)$: "$x \neq x$." Then, by the Comprehension Schema, there is a set B such that $x \in B$ if and only if $x \in A$

and $\mathbf{P}(x)$, or $x \in A$ and $x \neq x$. However, we have that $\forall x : x = x$. That is, there is no x such that $x \neq x$. In particular, the set B has no elements so there exists a set which has no elements. $\square$

Problem 1.3.3.

(a) Prove that a "set of all sets" does not exist. [*Hint*: if V is a set of all sets, consider $\{x \in V \mid x \notin x\}$.]

(b) Prove that for any set A there is some $x \notin A$.

Solution.

(a) Suppose that there exists a set V containing all sets. Consider the property $\mathbf{P}(x)$: "$x \notin x$." By the Comprehension Schema, there exists a set $X = \{x \in V \mid \mathbf{P}(x)\} = \{x \in V \mid x \notin x\}$. That is, $X \in V$ because it's a set.

We either have that $X \in X$ or $X \notin X$. If $X \in X$, then $X \in V$ and $X \notin X$, a contradiction. If $X \notin X$, then $X \in V$ and $X \notin X$ imply that $X \in X$, another contradiction. Since every step is true, it must be the case that our original assumption of the existence of V is false.

(b) Suppose there is a set A such that $\forall x : x \in A$. Since every object x we have constructed so far is a set, A is a "set of all sets" which we have shown cannot exist.

$\square$

Problem 1.3.4. Let A and B be sets. Show that there exists a unique set C such that $x \in C$ if and only if either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.

Solution. By Problem 1.3.1, the following two sets exist:

$$C_1 = \{x \mid x \in A \text{ and } x \notin B\}$$
$$C_2 = \{x \mid x \in B \text{ and } x \notin A\}$$

By the Axiom of Union, there is a set $C = C_1 \cup C_2$ such that $x \in C$ if and only if $x \in C_1$ or $x \in C_2$. That is,

$$C = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

Thus, C exists. If C' is another set satisfying the hypothesis, then $x \in C$ if and only if $x \in C'$ so by the Axiom of Extensionality, $C = C'$. That is, C is unique. In general, the union of sets is unique. $\square$

Problem 1.3.5.

(a) Given $A, B,$ and C, there is a set P such that $x \in P$ if and only if $x = A$ or $x = B$ or $x = C$.

(b) Generalize to four elements.