# Chapter 1

# Sets

## 1.1 Introduction to Sets

No exercises.

## 1.2 Properties

No exercises

## 1.3 The Axioms

**Problem 1.3.1.** Show that the set of all $x$ such that $x \in A$ and $x \notin B$ exists.

*Solution.* Consider the property $\mathbf{P}(x, B)$: "$x \notin B$." Then, by the Comprehension Schema, for every $A$ and $B$ there exists a set $C$ such that $x \in C$ if and only if $x \in A$ and $\mathbf{P}(x, B)$, or if and only if $x \in A$ and $x \notin B$. $\square$

**Problem 1.3.2.** Replace the Axiom of Existence by the following weaker postulate:

**Weak Axiom of Existence.** *Some set exists.*

Prove the Axiom of Existence using the Weak Axiom of Existence and the Comprehension Schema. [*Hint*: Let $A$ be a set known to exist; consider $\{x \in A \mid x \neq x\}$.]

*Solution.* Recall that the Axiom of Existence states that there exists a set which has no elements. Let $A$ be a set known to exist, which is guaranteed by the Weak Axiom of Existence. Consider the property $\mathbf{P}(x)$: "$x \neq x$." Then, by the Comprehension Schema, there is a set $B$ such that $x \in B$ if and only if $x \in A$

and $\mathbf{P}(x)$, or $x \in A$ and $x \neq x$. However, we have that $\forall x : x = x$. That is, there is no $x$ such that $x \neq x$. In particular, the set $B$ has no elements so there exists a set which has no elements. $\square$

**Problem 1.3.3.**

(a) Prove that a "set of all sets" does not exist. [*Hint*: if $V$ is a set of all sets, consider $\{x \in V \mid x \notin x\}$.]

(b) Prove that for any set $A$ there is some $x \notin A$.

*Solution.*

(a) Suppose that there exists a set $V$ containing all sets. Consider the property $\mathbf{P}(x)$: "$x \notin x$." By the Comprehension Schema, there exists a set $X = \{x \in V \mid \mathbf{P}(x)\} = \{x \in V \mid x \notin x\}$. That is, $X \in V$ because it's a set.

We either have that $X \in X$ or $X \notin X$. If $X \in X$, then $X \in V$ and $X \notin X$, a contradiction. If $X \notin X$, then $X \in V$ and $X \notin X$ imply that $X \in X$, another contradiction. Since every step is true, it must be the case that our original assumption of the existence of $V$ is false.

(b) Suppose there is a set $A$ such that $\forall x : x \in A$. Since every object $x$ we have constructed so far is a set, $A$ is a "set of all sets" which we have shown cannot exist.

$\square$

**Problem 1.3.4.** Let $A$ and $B$ be sets. Show that there exists a unique set $C$ such that $x \in C$ if and only if either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.

*Solution.* By Problem 1.3.1, the following two sets exist:

$$C_1 = \{x \mid x \in A \text{ and } x \notin B\}$$
$$C_2 = \{x \mid x \in B \text{ and } x \notin A\}$$

By the Axiom of Union, there is a set $C = C_1 \cup C_2$ such that $x \in C$ if and only if $x \in C_1$ or $x \in C_2$. That is,

$$C = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

Thus, $C$ exists. If $C'$ is another set satisfying the hypothesis, then $x \in C$ if and only if $x \in C'$ so by the Axiom of Extensionality, $C = C'$. That is, $C$ is unique. In general, the union of sets is unique. $\square$

**Problem 1.3.5.**

(a) Given $A, B,$ and $C$, there is a set $P$ such that $x \in P$ if and only if $x = A$ or $x = B$ or $x = C$.

(b) Generalize to four elements.

*Solution.*

(a) By the Axiom of Pair, the following three sets exist:

$$\begin{aligned} P_A &= \{x \mid x = A\} = \{A\} \\ P_B &= \{x \mid x = B\} = \{B\} \\ P_C &= \{x \mid x = C\} = \{C\} \end{aligned}$$

Then by the Axiom of Union, there is a set $P = \bigcup\{P_A, P_B, P_C\}$ such that $x \in P$ if and only if $x \in S$ for some $S \in P$. That is, $x \in S$ if and only if $x \in P_A$ or $x \in P_B$ or $x \in P_C$ if and only if $x = A$ or $x = B$ or $x = C$.

(b) Again by the Axiom of Pair, there exists a set $P_D = \{x \mid x = D\} = \{D\}$. Again by the Axiom of Union, there is a set $P = \bigcup\{P_A, P_B, P_C, P_D\}$ such that $x \in P$ if and only if $x = A$ or $x = B$ or $x = C$ or $x = D$.

□

**Problem 1.3.6.** Show that $\mathcal{P}(X) \subseteq X$ is false for any $X$. In particular, $\mathcal{P}(X) \neq X$ for any $X$. This proves again that a "set of all sets" does not exist. [*Hint*: Let $Y = \{u \in X \mid u \notin u\}$; $Y \in \mathcal{P}(X)$ but $Y \notin X$.]

*Solution.* Let $X$ be any set and define $Y = \{u \in X \mid u \notin u\}$. Certainly $Y \subseteq X$ because $x \in Y$ implies $x \in X$. Thus, $Y \in \mathcal{P}(X)$ by the Axiom of Powerset. Suppose $Y \in X$. We either have $Y \in Y$ or $Y \notin Y$. If $Y \in Y$ then we have $Y \notin Y$, a contradiction. On the other hand, if $Y \notin Y$, then it is implied that $Y \in Y$, another contradiction. Thus, our assumption that $Y \in X$ must be false. Therefore, there is an element in $\mathcal{P}(X)$ which does not belong to $X$ so $\mathcal{P}(X) \not\subseteq X$. □

**Problem 1.3.7.** The Axiom of Pair, the Axiom of Union, and the Axiom of Power Set can be replaced by the following weaker versions.

**Weak Axiom of Pair.** *For any $A$ and $B$, there is a set $C$ such that $A \in C$ and $B \in C$.*

**Weak Axiom of Union.** *For any set $S$, there exists $U$ such that if $X \in A$ and $A \in S$, then $X \in U$.*

**Weak Axiom of Power Set.** *For any set $S$, there exists $P$ such that $X \subseteq S$ implies $X \in P$.*

Prove the Axiom of Pair, the Axiom of Union, and the Axiom of Power Set using these weaker versions. [*Hint*: Use also the Comprehension Schema.]

*Solution.* By the Weak Axiom of Pair, there exists a set $C'$ such that $A \in C'$ and $B \in C'$. Consider the property $\mathbf{P}(x, A, B)$: "$x = A$ or $x = B$." By the Comprehension Schema, there exists a set $C$ such that $x \in C$ if and only if $x \in C'$ and $\mathbf{P}(x, A, B)$. That is, for any sets $A$ and $B$, there exists a set $C$ such that $x \in C$ if and only if $x = A$ or $x = B$, which is the Axiom of Pair.

By the Weak Axiom of Union, there exists a set $U'$ for all $S$ such that if $X \in A$ and $A \in S$ then $X \in U'$. Consider the property $\mathbf{P}(X, A, S)$: "$\exists A \in S$ such that $X \in A$." By the Comprehension Schema, there exists a set $U$ such that $X \in U$ if and only if $X \in U'$ and $\mathbf{P}(X, A, S)$. That is, for any set $S$, there exists a set $U$ such that $X \in U$ if and only if there exists $A \in S$ such that $X \in A$, which is the Axiom of Union.

By the Axiom of Power Set, there exists a set $P'$ such that $X \in P'$ if and only if $X \subseteq S$. Consider the property $\mathbf{P}(X, S)$: "$X \subseteq S$." By the Comprehension Schema, there exists a set $P$ such that $X \in C$ if and only if $X \in P'$ and $\mathbf{P}(X, S)$. That is, for any set $S$, there exists a set $P$ such that $X \in P$ if and only if $X \subseteq S$. $\square$