Chapter 1

Sets

1.1 Introduction to Sets

No exercises.

1.2 Properties

No exercises

1.3 The Axioms

Problem 1.3.1. Show that the set of all x such that $x \in A$ and $x \notin B$ exists.

Solution. Consider the property $\mathbf{P}(x, B)$: "$x \notin B$." Then, by the Comprehension Schema, for every A and B there exists a set C such that $x \in C$ if and only if $x \in A$ and $\mathbf{P}(x, B)$, or if and only if $x \in A$ and $x \notin B$. $\square$

Problem 1.3.2. Replace the Axiom of Existence by the following weaker postulate:

Weak Axiom of Existence. *Some set exists.*

Prove the Axiom of Existence using the Weak Axiom of Existence and the Comprehension Schema. [*Hint*: Let A be a set known to exist; consider $\{x \in A \mid x \neq x\}$.]

Solution. Recall that the Axiom of Existence states that there exists a set which has no elements. Let A be a set known to exist, which is guaranteed by the Weak Axiom of Existence. Consider the property $\mathbf{P}(x)$: "$x \neq x$." Then, by the Comprehension Schema, there is a set B such that $x \in B$ if and only if $x \in A$

and $\mathbf{P}(x)$, or $x \in A$ and $x \neq x$. However, we have that $\forall x : x = x$. That is, there is no x such that $x \neq x$. In particular, the set B has no elements so there exists a set which has no elements. $\square$

Problem 1.3.3.

(a) Prove that a "set of all sets" does not exist. [*Hint*: if V is a set of all sets, consider $\{x \in V \mid x \notin x\}$.]

(b) Prove that for any set A there is some $x \notin A$.

Solution.

(a) Suppose that there exists a set V containing all sets. Consider the property $\mathbf{P}(x)$: "$x \notin x$." By the Comprehension Schema, there exists a set $X = \{x \in V \mid \mathbf{P}(x)\} = \{x \in V \mid x \notin x\}$. That is, $X \in V$ because it's a set.

We either have that $X \in X$ or $X \notin X$. If $X \in X$, then $X \in V$ and $X \notin X$, a contradiction. If $X \notin X$, then $X \in V$ and $X \notin X$ imply that $X \in X$, another contradiction. Since every step is true, it must be the case that our original assumption of the existence of V is false.

(b) Suppose there is a set A such that $\forall x : x \in A$. Since every object x we have constructed so far is a set, A is a "set of all sets" which we have shown cannot exist.

$\square$

Problem 1.3.4. Let A and B be sets. Show that there exists a unique set C such that $x \in C$ if and only if either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.

Solution. By Problem 1.3.1, the following two sets exist:

$$C_1 = \{x \mid x \in A \text{ and } x \notin B\}$$
$$C_2 = \{x \mid x \in B \text{ and } x \notin A\}$$

By the Axiom of Union, there is a set $C = C_1 \cup C_2$ such that $x \in C$ if and only if $x \in C_1$ or $x \in C_2$. That is,

$$C = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

Thus, C exists. If C' is another set satisfying the hypothesis, then $x \in C$ if and only if $x \in C'$ so by the Axiom of Extensionality, $C = C'$. That is, C is unique. In general, the union of sets is unique. $\square$

Problem 1.3.5.

(a) Given $A, B,$ and C, there is a set P such that $x \in P$ if and only if $x = A$ or $x = B$ or $x = C$.

(b) Generalize to four elements.

Solution.

(a) By the Axiom of Pair, the following three sets exist:

$$P_A = \{x \mid x = A\} = \{A\}$$
$$P_B = \{x \mid x = B\} = \{B\}$$
$$P_C = \{x \mid x = C\} = \{C\}$$

Then by the Axiom of Union, there is a set $P = \bigcup\{P_A, P_B, P_C\}$ such that $x \in P$ if and only if $x \in S$ for some $S \in P$. That is, $x \in S$ if and only if $x \in P_A$ or $x \in P_B$ or $x \in P_C$ if and only if $x = A$ or $x = B$ or $x = C$.

(b) Again by the Axiom of Pair, there exists a set $P_D = \{x \mid x = D\} = \{D\}$. Again by the Axiom of Union, there is a set $P = \bigcup\{P_A, P_B, P_C, P_D\}$ such that $x \in P$ if and only if $x = A$ or $x = B$ or $x = C$ or $x = D$.

□

Problem 1.3.6. Show that $\mathcal{P}(X) \subseteq X$ is false for any X. In particular, $\mathcal{P}(X) \neq X$ for any X. This proves again that a "set of all sets" does not exist. [*Hint*: Let $Y = \{u \in X \mid u \notin u\}$; $Y \in \mathcal{P}(X)$ but $Y \notin X$.]

Solution. Let X be any set and define $Y = \{u \in X \mid u \notin u\}$. Certainly $Y \subseteq X$ because $x \in Y$ implies $x \in X$. Thus, $Y \in \mathcal{P}(X)$ by the Axiom of Powerset. Suppose $Y \in X$. We either have $Y \in Y$ or $Y \notin Y$. If $Y \in Y$ then we have $Y \notin Y$, a contradiction. On the other hand, if $Y \notin Y$, then it is implied that $Y \in Y$, another contradiction. Thus, our assumption that $Y \in X$ must be false. Therefore, there is an element in $\mathcal{P}(X)$ which does not belong to X so $\mathcal{P}(X) \not\subseteq X$. □

Problem 1.3.7. The Axiom of Pair, the Axiom of Union, and the Axiom of Power Set can be replaced by the following weaker versions.

Weak Axiom of Pair. *For any A and B, there is a set C such that $A \in C$ and $B \in C$.*

Weak Axiom of Union. *For any set S, there exists U such that if $X \in A$ and $A \in S$, then $X \in U$.*

Weak Axiom of Power Set. *For any set S, there exists P such that $X \subseteq S$ implies $X \in P$.*

Prove the Axiom of Pair, the Axiom of Union, and the Axiom of Power Set using these weaker versions. [*Hint*: Use also the Comprehension Schema.]

Solution. By the Weak Axiom of Pair, there exists a set C' such that $A \in C'$ and $B \in C'$. Consider the property $\mathbf{P}(x, A, B)$: "$x = A$ or $x = B$." By the Comprehension Schema, there exists a set C such that $x \in C$ if and only if $x \in C'$ and $\mathbf{P}(x, A, B)$. That is, for any sets A and B, there exists a set C such that $x \in C$ if and only if $x = A$ or $x = B$, which is the Axiom of Pair.

By the Weak Axiom of Union, there exists a set U' for all S such that if $X \in A$ and $A \in S$ then $X \in U'$. Consider the property $\mathbf{P}(X, A, S)$: "$\exists A \in S$ such that $X \in A$." By the Comprehension Schema, there exists a set U such that $X \in U$ if and only if $X \in U'$ and $\mathbf{P}(X, A, S)$. That is, for any set S, there exists a set U such that $X \in U$ if and only if there exists $A \in S$ such that $X \in A$, which is the Axiom of Union.

By the Axiom of Power Set, there exists a set P' such that $X \in P'$ if and only if $X \subseteq S$. Consider the property $\mathbf{P}(X, S)$: "$X \subseteq S$." By the Comprehension Schema, there exists a set P such that $X \in C$ if and only if $X \in P'$ and $\mathbf{P}(X, S)$. That is, for any set S, there exists a set P such that $X \in P$ if and only if $X \subseteq S$. □

1.4 Elementary Operations on Sets

Problem 1.4.1. Prove all the displayed formulas in this section and visualize them using Venn diagrams.

Solution. I'll omit the Venn diagrams but I may return and add them later.

(a) Commutativity:

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. That is, $x \in B$ and $x \in A$ so $x \in B \cap A$. The reverse argument is identical so $A \cap B = B \cap A$. A similar argument proves that $A \cup B = B \cup A$. □

(b) Associativity:

$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof. Let $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. That is, $x \in A$, $x \in B$, and $x \in C$. So $x \in B \cap C$. Therefore, $x \in A \cap (B \cap C)$. The reverse argument is identical and the proof is similar for set union. □

(c) Distributivity:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. That is, $x \in B$ or $x \in C$. Therefore, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. Thus, $x \in (A \cap B) \cup (A \cap C)$.

On the other hand, let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$. In the first case, $x \in A$ and $x \in B$, so $x \in A$ and $x \in (B \cup C)$. In the second case, $x \in A$ and $x \in C$, so $x \in A$ and $x \in (B \cup C)$. In either case, it follows that $x \in A \cap (B \cup C)$. A highly similar argument proves the second statement. □

(d) De Morgan's Laws:

$$C - (A \cap B) = (C - A) \cup (C - B)$$
$$C - (A \cup B) = (C - A) \cap (C - B)$$

Proof. Let $x \in C - (A \cap B)$. Then $x \in C$ and $x \notin A \cap B$. That is, $x \in C$, but $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in C - A$ so $x \in (C - A) \cup (C - B)$. Similarly, if $x \notin B$, then $x \in C - B$ so $x \in (C - A) \cup (C - B)$.

For the reverse direction, let $x \in (C - A) \cup (C - B)$. Then $x \in C - A$ or $x \in C - B$. In the first case, we have $x \in C$ and $x \notin A$, so $x \notin (A \cap B)$. Thus, $x \in C - (A \cap B)$. In the second case, we have $x \in C$ and $x \notin B$, so $x \notin (A \cap B)$. Again, we have $x \in C - (A \cap B)$. A similar argument proves the second statement.

□

Problem 1.4.2. Prove the following statements.

Solution. There are 7 :(

(a) $A \subseteq B$ if and only if $A \cap B = A$ if and only if $A \cup B = B$ if and only if $A - B = \emptyset$

Proof. Suppose $A \subseteq B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular, $x \in A$ so $A \cap B \subseteq A$. If $x \in A$, then $x \in B$ since $A \subseteq B$. Thus, $x \in A \cap B$ so $A \subseteq A \cap B$. This implies $A \cap B = A$.

Now suppose $A \cap B = A$. If $x \in B$, then $x \in A \cup B$ by definition. Thus, $B \subseteq A \cup B$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cap B$ so $x \in B$. In either case, $x \in B$ so $A \cup B \subseteq B$. This implies $A \cup B = B$.

Assume that $A \cup B = B$. Suppose $A - B$ is nonempty and let $x \in A - B$. Then $x \in A$ and $x \notin B$. If $x \in A$ then $x \in A \cup B$. However, if $x \notin B$ and $A \cup B = B$, we have $x \notin A \cup B$. A contradiction arises so we must have $A - B = \emptyset$.

Finally, assume $A - B = \emptyset$. Let $x \in A$. Then either $x \in B$ or $x \notin B$. If $x \notin B$, then we have $x \in A$ and $x \notin B$, so $x \in A - B$, a contradiction. Thus, $x \in B$ and $A \subseteq B$. □

(b) $A \subseteq B \cap C$ if and only if $A \subseteq B$ and $A \subseteq C$.

Proof. First suppose $A \subseteq B \cap C$. Let $x \in A$. Then $x \in B \cap C$, so $x \in B$ and $x \in C$. Thus, $A \subseteq B$ and $A \subseteq C$.

For the reverse direction, suppose $A \subseteq B$ and $A \subseteq C$. Let $x \in A$. Then $x \in B$ and $x \in C$, so $x \in B \cap C$. Hence, $A \subseteq B \cap C$. □

(c) $B \cup C \subseteq A$ if and only if $B \subseteq A$ and $C \subseteq A$.

Proof. Suppse $B \cup C \subseteq A$. If $x \in B$ then $x \in B \cup C$, so $x \in A$. Thus, $B \subseteq A$. Similarly, $C \subseteq C$.

Now suppose $B \subseteq A$ and $C \subseteq A$. Let $x \in B \cup C$. Then $x \in B$ or $x \in C$. In either case, $x \in A$, so $B \cup C \subseteq A$. □

(d) $A - B = (A \cup B) - B = A - (A \cap B)$

Proof. Let $x \in A - B$. That is, $x \in A$ and $x \notin B$. Then $x \in A \cup B$ and $x \notin B$ so $x \in (A \cup B) - B$. Thus, $A - B \subseteq (A \cup B) - B$. Now let $x \in (A \cup B) - B$ so that $x \in (A \cup B)$ and $x \notin B$. It follows that $x \in A$ and $x \notin B$ so $x \in A - B$ and $(A \cup B) - B \subseteq A - B$. This implies that $A - B = (A \cup B) - B$.

Suppose $x \in A - B$. It follows that $x \in A$ and $x \notin B$. Then $x \notin A \cap B$ so $x \in A - (A \cap B)$. Hence, $A - B \subseteq A - (A \cap B)$. For the reverse, suppose $x \in A - (A \cap B)$. That is, $x \in A$ and $x \notin A \cap B$. Then $x \notin B$ so $x \in A - B$, showing that $A - (A \cap B) \subseteq A - B$. Together, it is implied that $A - B = A - (A \cap B)$. □

(e) $A \cap B = A - (A - B)$.

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. It follows that $x \notin A - B$. Thus, $x \in A - (A - B)$ so $A \cap B \subseteq A - (A - B)$.

Now let $x \in A - (A - B)$. That is, $x \in A$ and $x \notin A - B$. The latter implies that $x \in B$ (since otherwise it would be in the difference). Therefore, $x \in A \cap B$ so $A - (A - B) \subseteq A \cap B$ and the two sets are equal. □

(f) $A - (B - C) = (A - B) \cup (A \cap C)$.

Proof. Let $x \in A - (B - C)$. That is, $x \in A$ and $x \notin B - C$. Suppose that $x \notin (A - B) \cup (A \cap C)$. Then it must be the case that $x \in B$ and $x \notin C$. If this were the case, $x \in B - C$. By contraposition, if $x \notin B - C$ it follows that $x \in (A - B) \cup (A \cap C)$. Therefore, $A - (B - C) \subseteq (A - B) \cup (A \cap C)$.

Now let $x \in (A - B) \cup (A \cap C)$ so $x \in A - B$ or $x \in A \cap C$. In the first case, $x \in A$ and $x \notin B$. If $x \notin B$ then $x \notin B - C$, so $x \in A - (B - C)$. In the second case, $x \in A$ and $x \in C$. If $x \in C$ then $x \notin B - C$, so $x \in A - (B - C)$. Thus, either case shows that $(A - B) \cup (A \cap C) \subseteq A - (B - C)$. Together, these statements imply $A - (B - C) = (A - B) \cup (A \cap C)$. □

(g) $A = B$ if and only if $A \triangle B = \emptyset$.

Proof. Assume $A = B$. Recall that $A \triangle B = (A - B) \cup (B - A)$. $A = B$ implies $A \subseteq B$ and $B \subseteq A$, and in (a) we proved that $A \subseteq B$ if and only if $A - B = \emptyset$. It follows that $A - B = \emptyset$ and $B - A = \emptyset$, so $A \triangle B = \emptyset$.

Now assume $A \triangle B = \emptyset$. That is, $A - B = \emptyset$ and $B - A = \emptyset$. Again by (a), this is the case if and only if $A \subseteq B$ and $B \subseteq A$, so we have $A = B$.

□

Problem 1.4.3. For each of the following (false) statements draw a Venn diagram in which it fails:

Solution. I'll add this when I'm more familiar with making Venn diagrams in LaTeX(AKA I'm procrastinating) □

Problem 1.4.4. Let A be a set; show that a "complement" of A does not exist. (The "complement" of A is the set of all $x \notin A$.)

Solution. Suppose such a set exists, say B. Then consider $A \cup B$ to get the universal set, which does not exist. Thus B does not exist. □

Problem 1.4.5. Let $S \neq \emptyset$ and A be sets.

(a) Set $T_1 = \{Y \in \mathcal{P}(A) \mid Y = A \cap X \text{ for some } X \in S\}$, and prove $A \cap \bigcup S = \bigcup T_1$ (generalized distributive law).

(b) Set $T_2 = \{Y \in \mathcal{P}(A) \mid Y = A - X \text{ for some } X \in S\}$, and prove

$$A - \bigcup S = \bigcap T_2$$
$$A - \bigcap S = \bigcup T_2$$

(generalized De Morgan laws).

Solution. Recall that $S \neq \emptyset$.

(a) $x \in A \cap \bigcup S$ if and only if $x \in A$ and $\exists X \in S$ such that $x \in X$. That is, $x \in A \cap \bigcup S$ if and only if $x \in T_1$.

(b) Let $x \in A - \bigcup S$. That is, $x \in A$ and $x \notin \bigcup S$. Thus, for all $X \in S$, we have $x \notin X$. Therefore, for all $X \in S$, we have $x \in A - X$. Hence, $x \in A - \bigcup S$ if and only if $x \in \bigcap T_2$.

Similarly, let $x \in A - \bigcap S$. Then $x \in A$ and $\exists X \in S$ such that $x \notin X$. That is, $\exists Y \in \mathcal{P}(A)$ such that $x \in Y$. Thus, $x \in A - \bigcap S$ if and only if $x \in \bigcup T_2$.

$\square$

Problem 1.4.6. Prove that $\bigcap S$ exists for all $S \neq \emptyset$. Where is the assumption $S \neq \emptyset$ used in the proof?

Solution. If $S \neq \emptyset$, then $\exists A \in S$. Consider the property $\mathbf{P}(x)$: "$x \in X$ for all $X \in S$." Then $\bigcap S = \{x \in A \mid \mathbf{P}(x)\}$ exists by the Axiom Schema of Comprehension.

If $S = \emptyset$, then $\bigcap S$ is a universal set. Indeed, if it were not then there would exist a set $A \in \emptyset$ such that $x \notin A$, but no such set exists. $\square$