1 The Algebra of Complex Numbers

1.1 Arithmetic Operations

Problem 1.1.1. Find the values of

$$(1+2i)^3, \quad \frac{5}{-3+4i}, \quad \left(\frac{2+i}{3-2i}\right)^2, \quad (1+i)^n+(1-i)^n.$$

Solution. These are all relatively simple calculations.

$$\begin{aligned}(1+2i)^3 &= (1+2i)(1+2i)(1+2i)\\ &= (-3+4i)(1+2i)\\ &= -11-2i\end{aligned}$$

$$\begin{aligned}\frac{5}{-3+4i} &= \frac{5(-3-4i)}{(-3+4i)(-3-4i)}\\ &= \frac{-15-20i}{25}\\ &= -\frac{3}{5}-\frac{4}{5}i\end{aligned}$$

$$\begin{aligned}\left(\frac{2+i}{3-2i}\right)^2 &= \left(\frac{(2+i)(3+2i)}{(3-2i)(3+2i)}\right)^2\\ &= \left(\frac{4+7i}{13}\right)^2\\ &= \frac{-33+56i}{169}\end{aligned}$$

For the final calculation, note that the number $a_n = (1+i)^n + (1-i)^n$ is twice the real part of $(1+i)^n$ due to using the conjugate. Furthermore, by Euler's identity we find $(1+i)^n = \sqrt{2}^n e^{in\pi/4}$ which has real part $\cos(n\pi/4)$. Thus

$$(1+i)^n+(1-i)^n = 2\sqrt{2}^n \cos\frac{n\pi}{4}$$

Note that this can be shown via modulo arguments but it is far more tedious. $\square$

Problem 1.1.2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}$$

Solution. Again, these are just computations.

$$\begin{aligned}(x+iy)^4 &= ((x+iy)(x+iy))^2\\ &= \left((x^2-y^2)+2xyi\right)\left((x^2-y^2)+2xyi\right)\\ &= (x^4-6x^2y^2+y^4)+(4x^3y-4xy^3)i\end{aligned}$$

$$\begin{aligned}\frac{1}{x+iy} &= \frac{x-iy}{x^2+y^2}\\ &= \frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i\end{aligned}$$

$$\begin{aligned}\frac{(x-1)+iy}{(x+1)+iy} &= \frac{((x-1)+iy)\,((x+1)-iy)}{((x+1)+iy)\,(((x+1)-iy)}\\ &= \frac{(x^2+y^2-1)+2iy}{x^2+2x+y^2+1}\end{aligned}$$

$$
\begin{aligned}
\frac{1}{(x+iy)^2} &= \frac{1}{(x^2-y^2)+2xyi} \\
&= \frac{(x^2-y^2)-2xyi}{((x^2-y^2)+2xyi)((x^2-y^2)-2xyi)} \\
&= \frac{(x^2-y^2)-2xyi}{x^4-2x^2y^2+y^4}
\end{aligned}
$$

□

Problem 1.1.3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

Solution. I'll only do one combination for each but it's easy to verify the others.

$$
\begin{aligned}
\left(\frac{-1+i\sqrt{3}}{2}\right)^3 &= \left(\frac{-2-2i\sqrt{3}}{4}\right)\left(\frac{-1+i\sqrt{3}}{2}\right) \\
&= \frac{2+6}{8} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{-1+i\sqrt{3}}{2}\right)^6 &= \left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right)^2 \\
&= 1^2 \\
&= 1
\end{aligned}
$$

□

1.2 Square Roots

Problem 1.2.1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$$

Solution. We find $x^2 = \frac{1}{2}(0+\sqrt{1}) = \frac{1}{2}$ so $x = \pm\frac{\sqrt{2}}{2}$. Similarly, $y^2 = \pm\frac{\sqrt{2}}{2}$. With the relation $2xy = 1$, we have a final solution of

$$\sqrt{i} = \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

We have a similar set of solutions for $\sqrt{-i}$ but now the product $2xy = -1$ so the signs of corresponding solutions are switched. Thus, we have

$$\sqrt{-i} = \pm\frac{\sqrt{2}}{2} \mp \frac{\sqrt{2}}{2}i$$

□