Math 8803 HW 4

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Exercise 10.2.1. Let f satisfy the hypotheses of Lemma 10.11, and suppose that

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty.$$

(a) Show that there are numbers A and B and a non-negative integer K such that $f(z) = z^K e^{A+Bz} g(z)$ where $g(z) = \prod_{k=1}^{\infty}(1 - z/z_k)$.

(b) Observe that for any complex number w, $|1 - w| \leq e^{|w|}$ and show that there is a number C such that that $|g(z)| \leq e^{C|z|}$.

(c) Deduce that $\sum_{\rho} 1/|\rho| = \infty$ where the sum is over all non-trivial zeros of the zeta function.

Solution. Recall that the hypotheses of Lemma 10.11 state that f is an entire function with a zero of order K at 0, and that $f(z)$ vanishes at the non-zero numbers $z_1, z_2, z_3, \ldots$. Furthermore, there is a constant $\theta \in (1, 2)$ such that

$$\max_{|z| \leq R} |f(z)| \leq \exp(R^{\theta})$$

for all sufficiently large R.

(a) By Lemma 10.11, there exist numbers A, B' such that

$$f(z) = z^K e^{A+B'z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{z/z_k}.$$

Observe that

$$\prod_{k=1}^{\infty} e^{z/z_k} = \exp\left(z \sum_{k=1}^{\infty} \frac{1}{z_k}\right) = e^{Cz}$$

where $C = \sum_{k=1}^{\infty} 1/z_k$. The series converges by hypothesis. Letting $B = B' + C$ and rewriting the above expression for f, we have

$$f(z) = z^K e^{A+Bz} g(z)$$

as desired.

(b) Observe that

$$|g(z)| = \prod_{k=1}^{\infty} \left|1 - \frac{z}{z_k}\right| \leq \prod_{k=1}^{\infty} e^{|z/z_k|} = \exp\left(|z| \sum_{k=1}^{\infty} \frac{1}{|z_k|}\right) = e^{C|z|}$$

for some constant C.

(c) Suppose for the sake of contradiction that $\sum_\rho 1/|\rho| < \infty$. Applying (a), (b), and Theorem 10.12 to $\xi(s)$, we find that there exists a constant C such that

$$|\xi(s)| = \frac{1}{2}\left|e^{Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}\right| \ll e^{C|s|}$$

for all large s. However, by Stirling's formula,

$$\xi(\sigma) = \exp(\frac{1}{2}\sigma\log\sigma + O(\sigma))$$

as $\sigma\to\infty$, contradicting the above bound. We conclude that $\sum_\rho 1/|\rho| = \infty$.

□

Exercise 11.1.2. Suppose that δ is fixed, $0 < \delta < 1$. Show that

$$\sum_{p|q} \frac{\log p}{p^s - 1} \ll (\log q)^{1-\delta}$$

uniformly for $\sigma \geq \delta$.

Solution. First observe that for $\sigma \geq \delta$, we have

$$\left| \sum_{p|q} \frac{\log p}{p^s - 1} \right| \leq \sum_{p|q} \frac{\log p}{p^\delta - 1}.$$

Next note that since we are summing over the distinct prime factors of q, the sum achieves a local maximum whenever q is the product of the first k primes. It follows that if $n_k \leq q < n_{k+1}$, where n_k is the product of the first k primes, then

$$\sum_{p|q} \frac{\log p}{p^\delta - 1} \leq \sum_{p|n_k} \frac{\log p}{p^{\delta-1}}.$$

Thus, it suffices to consider the case where q is the product of the first k primes. In this case, the sum reduces to

$$\sum_{p \leq \log q} \frac{\log p}{p^\delta - 1}.$$

(I'm honestly not sure how to get this upper bound of the summation, but it's the bound that I need to make the rest of the work hold so I'm just going to use it). By Riemann-Stieltjes integration, we may rewrite this sum as

$$\int_1^{\log q} \frac{1}{u^\delta - 1} d\vartheta(u) = \left. \frac{\vartheta(u)}{u^\delta - 1} \right|_1^{\log q} + \delta \int_1^{\log q} \frac{\vartheta(u) u^{\delta-1}}{(u^\delta - 1)^2} du$$

By Chebyshev's estimate, $\vartheta(u) \asymp u$, hence the first term above is $\ll (\log q)^{1-\delta}$ as desired. For the integral, observe that the integrand is $\leq \vartheta(u) u^{\delta-3}$. We bound $\vartheta(u)$ above by $\vartheta(\log q)$, and the second term is then bounded by

$$\delta \vartheta(\log q) \int_1^{\log q} u^{\delta-3} du = \delta \vartheta(\log q) \left. \frac{u^{\delta-2}}{\delta - 2} \right|_1^{\log q} \ll (\log q)^{\delta-1},$$

where the asymptotic again arises from Chebyshev's estimate. This is certainly less than the bound for the first summand, hence we obtain the asymptotic bound $\square$

Exercise 11.1.1. Let $S(x;q)$ denote the number of integers n, $0 < n \le x$, such that $(n,q) = 1$, and put $R(x;q) = S(x;q) - (\varphi(q)/q)x$.

(a) Show that if $\sigma > 0$, $x > 0$, and $s \neq 1$, then

$$L(s,\chi_0) = \sum_{n\le x} \chi_0(n)n^{-s} + \frac{\varphi(q)}{q} \cdot \frac{x^{1-s}}{s-1} - \frac{R(x;q)}{x^s} + s\int_x^\infty R(u;q)u^{-s-1}du.$$

Show that this includes Theorem 1.12 as a special case.

(b) Let $\delta > 0$ be fixed. Show that if $\sigma \ge \delta$, then

$$L(s,\chi_0) = \frac{\varphi(q)}{q} \cdot \frac{x^{1-s}}{s-1} + \sum_{n\le x} \chi_0(n)n^{-s} + O(d(q)|s|x^{-\sigma}).$$

Solution.

(a) We write

$$L(s,\chi_0) = \sum_{n\le x} \chi_0(n)n^{-s} + \sum_{n>x} \chi_0(n)n^{-s}.$$

By Riemann-Stieltjes integration, we have

$$\sum_{n>x} \chi_0(n)n^{-s} = \int_x^\infty u^{-s}dS(u;q) = \frac{\varphi(q)}{q}\int_x^\infty u^{-s}du + \int_x^\infty u^{-s}dR(u;q).$$

Direct computation shows

$$\int_x^\infty u^{-s}du = \frac{x^{1-s}}{s-1}$$

while an application of integration by parts yields

$$\int_x^\infty u^{-s}dR(u;q) = [u^{-s}R(u;q)]_x^\infty + s\int_x^\infty R(u;q)u^{-s-1}du = -\frac{R(x;q)}{x^s} + s\int_x^\infty R(u;q)u^{-s-1}du.$$

We conclude that

$$L(s,\chi_0) = \sum_{n\le x} \chi_0(n)n^{-s} + \frac{\varphi(q)}{q} \cdot \frac{x^{1-s}}{s-1} - \frac{R(x;q)}{x^s} + s\int_x^\infty R(u;q)u^{-s-1}du.$$

In particular, for $q = 1$, we have $S(x;1) = [x]$ and $R(x;1) = [x] - x = -\{x\}$. We recover

$$\zeta(s) = \sum_{n\le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \{u\}u^{-s-1}du$$

which is the statement of Theorem 1.12.

(b) Observe that

$$S(x;q) = \sum_{n\le x}\sum_{d|(n,q)} \mu(d) = \sum_{d|q}\mu(d)\sum_{\substack{n\le x\\ d|n}} 1 = \sum_{d|q}\mu(d)[x/d].$$

Observe that $[x/d] = x/d - \{x/d\}$, hence

$$S(x;q) = x\sum_{d|q}\frac{\mu(d)}{d} + \sum_{d|q}\mu(d)\{x/d\} = \frac{\varphi(q)}{q}x + R(x;q).$$

In particular,

$$|R(x;q)| < \sum_{d|q} 1 = d(q).$$

Applying this bound to the expression in (a), we have

$$\left|\frac{R(x;q)}{x^s}\right| \leq \frac{d(q)}{x^s}$$

and

$$\left|s\int_x^\infty R(u;q)u^{-s-1}du\right| \leq |s|d(q)\int_x^\infty u^{-\sigma-1}du = d(q)|s|x^{-\sigma}.$$

Since this term dominates $d(q)$, we conclude that

$$L(s,\chi_0) = \frac{\varphi(q)}{q}\cdot\frac{x^{1-s}}{s-1} + \sum_{n\leq x}\chi_0(n)n^{-s} + O(d(q)|s|x^{-\sigma}).$$

□

11.3.7. Let c_1 be the constant of Theoem 11.16, suppose that $q \leq \exp(2c_1\sqrt{\log x})$ and that χ is a character modulo q. Show that

$$M(x,\chi) \ll x\exp(-c_1\sqrt{\log x})$$

when $L(s,\chi)$ has no exceptional zero, and that

$$M(x,\chi) = \frac{x^{\beta_1}}{L'(\beta_1,\chi)\beta_1} + O(x\exp(-c_1\sqrt{\log x}))$$

when $L(s,\chi)$ has an exceptional zero β_1.

Solution. The function $M(x,\chi)$ is defined by

$$M(x,\chi) = \sum_{n\leq x}\chi(n)\mu(n).$$

By the same analysis as in the case of the Riemann zeta function, we have

$$M(x,\chi) = \frac{1}{L(s,\chi)}.$$

Furthermore, by applying Perron's formula for $\sigma_0 > 1$, we have

$$M(x,\chi) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT}\frac{1}{L(s,\chi)}\frac{x^s}{s}ds + R$$

where

$$R \ll \sum_{x/2<n<2x}\mu(n)\min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{\sigma_0}}.$$

For $2 \leq T < x$ and $\sigma_0 = 1 + 1/x$, the error term is $\ll \frac{x}{T}(\log x)^2$ via the same analysis as in the proof of Theorem 11.16 and Theorem 6.9 (and in fact can probably be strengthened since $\mu(n)$ is generally smaller than $\Lambda(n)$). Let C denote the contour connecting $\sigma_0 - iT$, $\sigma_0 + iT$, $\sigma_1 + iT$, $\sigma_1 - iT$, where σ_1 is chosen appropriately in the cases below.

Suppose $L(s,\chi)$ has no exceptional zero. Take $\sigma_1 = 1 - c/(5\log qT)$, where c is chosen such that $\{\sigma > 1 - c/\log q\tau\}$ is a zero-free region. Since the region is zero-free, the integrand $1/L(s,\chi)$ is analytic inside the region bounded by the contour C. By Cauchy's theorem, the integral vanishes. Bounds for the integral along the sides of the contour apply as in Theorem 6.9 and Theorem 11.16, hence

$$M(x,\chi) \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(\frac{-c\log x}{5\log qT}\right)\right)$$

With the value of T as in the proof of Theorem 1.16, the desired estimate is obtained.

Now suppose $L(s,\chi)$ has an exceptional zero β_1. First suppose $\beta_1 \geq 1 - c/(4\log qT)$. Taking $\sigma_1 = 1 - c/(3\log qT)$, the integrand has a single pole in the region bounded by C, namely at β_1. The computation of the residue is given by

$$\lim_{s\to\beta_1}\frac{s-\beta_1}{L(s,\chi)}\frac{x^s}{s} = \frac{x^{\beta_1}}{L'(\beta_1,\chi)\beta_1}$$

hence

$$M(x,\chi) - \frac{x^{\beta_1}}{L'(\beta_1,\chi)\beta_1} \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(\frac{-c\log x}{5\log qT}\right)\right).$$

Finally, if $\beta_1 < 1 - c/(4\log qT)$, then we take σ_1 as in the first case so that β_1 is outside of the contour and we obtain the estimate as in the above case. Since we have in this case that

$$\frac{x^{\beta_1}}{L'(\beta_1,\chi)\beta_1} \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(\frac{-c\log x}{5\log qT}\right)\right)$$

we obtain the the same estimate as in the second case.

Finally, setting T as in the proof of Theorem 1.16, we obtain the desired estimate when there is an exceptional zero. □