TEICHMÜLLER SPACE AND THE NIELSEN REALIZATION PROBLEM

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Abstract. Teichmüller theory has played an essential role in the study of mapping class groups of surfaces. In this paper, we introduce the complex structure and the Weil-Petersson metric on Teichmüller space. Additionally, we present a positive solution to the Nielsen realization problem.

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1. Introduction

Let S_g denote a closed oriented surface of genus g. The mapping class group of S_g is the group of orientation-preserving self-homeomorphisms of S_g modulo those that are isotopic to the identity. That is,

$$\operatorname{Mod}(S_g) = \operatorname{Homeo}^+(S_g)/\operatorname{Homeo}_0(S_g).$$

The Nielsen realization problem was first posed by Jakob Nielsen in 1932. One formulation is as follows.

Question 1.1 (Nielsen Realization Problem)**.** Let G be a finite subgroup of $\operatorname{Mod}(S_g)$. Does there exist a Riemann surface X whose underlying topological space is homeomorphic to S_g such that G may be realized as a finite subgroup of $\operatorname{Aut}(X)$?

Rephrasing, the question asks whether there exist a complex structure on S_g such that each element of G admits a representative which acts as a conformal automorphism. Recall that $\operatorname{Mod}(S_g)$ acts on the Teichmüller space T_g of marked Riemann surfaces of genus g. An equivalent formulation is whether there exists a point of T_g which is fixed by every element of G.

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The positive solution for $g = 1$ is a classical result, in which case $\mathrm{Mod}(S_g) = SL(2, \mathbb{Z})$ and $T_g = \mathbb{H}^2$. Partial results for the case where $g \geq 2$ were obtained in the years after Nielsen posed the question, but the general problem remained open until Steven Kerckhoff provided an affirmative solution in 1983. Since then, several other solutions have been put forward.

In this paper, we develop the relevant complex analytic theory of Teichmüller space, showing that it is a complex manifold. We construct the Weil-Petersson metric and show that it is Kähler. That is, the metric osculates to order 2 to the Euclidean metric. This is essential for performing computations along Weil-Petersson geodesics in local coordinates. We then sketch a solution to the Nielsen realization problem due to Scott Wolpert [7], which proceeds via a comparison of the Beltrami equation with the classical theory of Eichler integrals.

Throughout the paper, we will consider the Teichmüller space T_g of marked Riemann surfaces with genus g for some fixed $g \geq 2$. We denote by Γ a discrete subgroup of $PSL(2, \mathbb{R})$ such that $\mathbb{H}/\Gamma$ is a Riemann surface of genus g.

2. The analytic structure on Teichmüller space

We start by recalling the measurable Riemann mapping theorem, which describes solutions to the Beltrami equation and their dependence on parameters. Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $\|\mu\|_\infty < 1$. A mapping $f : \mathbb{C} \to \mathbb{C}$ is conformal with respect to the metric $ds = |dz + \mu d\overline{z}|$ if and only if f satisfies the Beltrami equation

$$f_{\overline{z}} = \mu f_z. \tag{2.1}$$

A homeomorphism of $\mathbb{C}$ satisfying (2.1) extends uniquely to a homeomorphism of the Riemann sphere of $\hat{\mathbb{C}}$.

Theorem 2.2 ([3]). *There exists a unique quasiconformal homeomorphism f^μ of $\hat{\mathbb{C}}$ fixing 0, 1, and ∞ which satisfies the Beltrami equation (2.1). Furthermore, if $\mu = \mu(t)$ depends analytically on a real parameter t, then $f^\mu(z)$ depends analytically on t for every fixed z.*

Now suppose μ is only defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Then μ admits a natural extension to $\mathbb{C}$ by setting

$$\mu(z) = \begin{cases} 0 & \mathrm{Im}\, z = 0, \\ \overline{\mu(\overline{z})} & \mathrm{Im}\, z < 0. \end{cases}$$

We will identify μ with its extension. The unique solution f^μ to (2.1) then satisfies $f^\mu(\overline{z}) = \overline{f^\mu(z)}$. In particular, f^μ preserves the real line and takes $\mathbb{H}$ to itself.

Remark 2.3. One may instead extend μ by setting $\mu(z) = 0$ for $\mathrm{Im}\, z \leq 0$. The unique solution to the associated Beltrami equation is denoted $w^\mu(z)$. It will at times be useful to work with $w^\mu(z)$ as well, so throughout this section we record the relevant properties.

Using Theorem 2.2, one defines an exponential map from the space $L^\infty(\Gamma)$ of Γ-equivariant Beltrami coefficients to the Teichmüller space T_g of a Riemann surface uniformized by Γ. One then defines a complex structure on Teichmüller space such that this map is a holomorphic submersion. The rest of the section will carry out this procedure in detail.

We briefly recall one construction of Teichmüller space as a representation space. For details of this construction, see Section 2.1 of [1]. Consider the fundamental group of a surface of genus g

$$\Gamma_g = \pi_1(S_g) = \left\langle a_1, b_1, \ldots, a_g, b_g \middle| \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

Then for any Riemann surface of genus g uniformized by a group $\Gamma < PSL(2,\mathbb{R})$, we have $\Gamma \cong \Gamma_g$. We denote the set of such groups Γ by Σ. We consider the space of discrete faithful representations $\Gamma_g \to PSL(2,\mathbb{R})$, where two representations θ and θ' are considered equivalent if there exists $A \in PSL(2,\mathbb{R})$ such that $\theta(\gamma)A = A\theta'(\gamma)$ for all $\gamma \in \Gamma_g$. The space of equivalence classes of such representations is the Teichmüller space T_g.

Let θ be a representative for a point of T_g. After conjugating, we may assume that the fixed points of $\theta(a_1)$ are 0 and ∞, and that the attractive fixed point of $\theta(b_1)$ is 1. We call such representations *normalized* and we see that each point of T_g has a unique normalized representative, hence we identify normalized representations with their images. A dimension count reveals that a normalized representation is determined by $6g-6$ real parameters. As a result, we may realize T_g as a subspace of $\mathbb{R}^{6g-6}$.

We say a Beltrami differential $\nu \in L^\infty(\mathbb{H})$ is Γ-equivariant if

$$\nu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \nu(z)$$

for all $\gamma \in \Gamma$. That is, ν descends to a $(-1,1)$-form on $\mathbb{H}/\Gamma$. We denote the set of Γ-equivariant Beltrami differentials by $B(\Gamma)$. Suppose $\mu \in B_1(\Gamma) = \{\nu \in B(\Gamma) : \|\nu\|_\infty < 1\}$. Let f^μ be the normalized solution to the Beltrami equation as in Theorem 2.2. Then for any $\gamma \in \Gamma$, a computation reveals that the mapping $f^\mu \circ \gamma$ also satisfies (2.1). It follows from the uniqueness part of Theorem 2.2 that there exists a unique element $\gamma^\mu \in PSL(2,\mathbb{R})$ such that

$$f^\mu \circ \gamma = \gamma^\mu \circ f^\mu. \tag{2.4}$$

Now fix a normalized representation θ with image Γ. The group Γ serves as a basepoint for T_g. We define a map $\theta^\mu : \Gamma_g \to PSL(2,\mathbb{R})$ by

$$\theta^\mu(\gamma) = \theta(\gamma)^\mu$$

where the right hand side is defined as in (2.4). Then θ^μ is a normalized representation of Γ_g whose image we denote by Γ^μ. Thus, we have defined a map $\Phi : B_1(\Gamma) \to T_g,\ \mu \mapsto \Gamma^\mu$. Analysis of Φ and its derivative will yield an explicit description of the tangent space to Teichmüller space, which is necessary to define the complex structure.

2.1. **Variations of solutions.** The starting point of this development is the analytic dependence of solutions to the Beltrami equation on the differential. To analyze infinitesimal variations of solutions, we define for $\mu, \nu \in L^\infty(\mathbb{H})$ with $\|\mu\|_\infty < 1$ the variation

$$\dot{f}^\mu[\nu](z) \coloneqq \lim_{t\to 0} \frac{f^{\mu+t\nu}(z) - f^\mu(z)}{t} = \left.\frac{d}{dt}\right|_{t=0} f^{\mu+t\nu}(z).$$

For $\mu = 0$, we simplify the notation to $\dot{f}[\nu]$. The variation may be thought of as a quasiconformal vector field on $\mathbb{H}$.

More generally, for any function depending on $\mu \in L^\infty(\mathbb{H})$, we define the variation

$$\dot{F}(\mu)[\nu](z) \coloneqq \lim_{t\to 0} \frac{F(\mu+t\nu)(z) - F(\mu)(z)}{t} \tag{2.5}$$

and write $\dot{F}(0)[\nu] = \dot{F}[\nu]$.

A direct computation shows that $\dot{f}^\mu[\nu]$ is $\mathbb{R}$-linear in ν and

$$(\dot{f}^\mu[\nu])_{\overline{z}} = \mu(\dot{f}^\mu[\nu])_z + \nu f^\mu_z.$$

When $\mu = 0$, we obtain the simpler

$$\dot{f}[\nu]_{\overline{z}} = \nu. \tag{2.6}$$

The normalization of solutions to the Beltrami equation implies that $\dot{f}[\nu](z) = 0$ for $z = 0, 1, \infty$. Furthermore, through an analysis of the boundary behavior of solutions, one can show that $\dot{f}[\nu](z)$ is $o(|z|^2)$ as $z \to \infty$. In fact, this is enough to uniquely specify $\dot{f}[\nu]$. Ahlfors and Bers obtained the following explicit description of $\dot{f}$ in their solution to the Beltrami equation.

Proposition 2.7 ([2])**.**

$$\dot{f}[\nu](\zeta) = -\frac{2}{\pi}\mathrm{Re}\iint_{\mathbb{H}} \nu(z)R(z,\zeta)dxdy \tag{2.8}$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

The proof is essentially an application of the generalized Cauchy integral formula but involves some careful estimates. Note that the above observations are immediate from the proposition.

Remark 2.9. If one instead considers the variation $\dot{w}[\nu]$, then one has

$$\dot{f}[\nu](z) = \dot{w}[\nu](z) + \overline{\dot{w}[\nu](\overline{z})}. \tag{2.10}$$

Indeed, one finds that $\dot{w}[\nu](z) = 0$ for $z = 0, 1, \infty$, $\dot{w}_{\overline{z}} = \nu$ for $z \in \overline{\mathbb{H}}$, and $\dot{w}_{\overline{z}} = 0$ for $z \in \mathbb{L}$. These properties completely characterize $\dot{w}[\nu]$ ([5], Theorem 4.37), and the statement follows from the uniqueness of $\dot{f}[\nu]$.

We now study variations of translations in the following sense. Given $\mu, \lambda \in B_1(\Gamma)$, consider the corresponding points Γ^μ and Γ^λ of T_g. Suppose there exists a Beltrami differential $\rho \in B_1(\Gamma^\lambda)$ such that

$$f^\rho \circ \gamma^\lambda = \gamma^\mu \circ f^\rho.$$

That is, f^ρ takes Γ^λ to Γ^μ in T_g. By Theorem 2.2, this occurs if and only if $f^\mu = f^\rho \circ f^\lambda$. A direct computation shows that this holds if and only if

$$\rho \circ f^\lambda = \frac{\mu - \lambda}{1 - \overline{\lambda}\mu}\left(\frac{f^\lambda_z}{|f^\lambda_z|}\right)^2, \tag{2.11}$$

in which case we write $\rho = \mu \mid \lambda$.

To understand the variation of this translation, we define $\rho : B_1(\Gamma) \to B_1(\Gamma^\lambda)$ by $\rho(\mu) = \mu \mid \lambda$. The function ρ will play the role of the change of coordinates map when defining the manifold structure on T_g. By (2.11), we have

$$\rho(\lambda + t\nu) = \left[\frac{\nu t}{1 - \overline{\lambda}(\lambda + \nu t)} \left(\frac{f_z^\lambda}{|f_z^\lambda|} \right)^2 \right] \circ (f^\lambda)^{-1}.$$

Differentiating both sides with respect to t yields

$$\dot{\rho}(\lambda)[\nu] = L^\lambda \nu \tag{2.12}$$

where

$$L^\lambda \nu = \left[\frac{\nu \cdot (f_z^\lambda)^2}{|f_z^\lambda|^2 - |f_{\overline{z}}^\lambda|^2} \right] \circ (f^\lambda)^{-1}.$$

The map $\nu \mapsto L^\lambda \nu$ provides a $\mathbb{C}$-linear isomorphism between $B(\Gamma)$ and $B(\Gamma^\lambda)$. For any function f defined on $B_1(\Gamma)$, we define a new function f_λ on $B_1(\Gamma^\lambda)$ by requiring

$$f(\mu) = f_\lambda(\mu \mid \lambda) = f_\lambda \circ \rho. \tag{2.13}$$

Then one sees

$$\dot{f}(\lambda)[\nu] = \dot{f}_\lambda[L_\nu^\lambda]. \tag{2.14}$$

In particular, for the function $\mu \mapsto f^\mu$ we find that

$$\dot{f}^\lambda[\nu] = \dot{f}[L^\lambda \nu] \circ f^\lambda.$$

As a result, it suffices to understand variations of solutions at the basepoint Γ of T_g.

2.2. **The differential of Φ.** In this subsection, we describe the differential of Φ at the basepoint Γ. We then compute its kernel.

Let $\nu \in B(\Gamma)$. Replacing μ in (2.4) by $t\nu$ and differentiating with respect to t yields

$$\dot{f}[\nu] \circ \gamma(z) = \dot{\gamma}[\nu](z) + \gamma'(z)\dot{f}[\nu](z) \tag{2.15}$$

where $\dot{\gamma}[\nu]$ is defined as in (2.5).

Lemma 2.16. *The function $\dot{\gamma}[\nu]/\gamma'$ is a quadratic polynomial, which we denote by $Q_\gamma[\nu](z)$.*

Proof. For this proof, we work in the unit disk model $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, translating from the upper half plane by the conformal automorphism taking $0, 1, \infty$ to $1, -1, i$. Differentiating (2.15) with respect to $\overline{z}$ gives

$$(\nu \circ \gamma)\overline{\gamma'} = \dot{\gamma}[\nu]_{\overline{z}} + \gamma' \dot{f}[\nu]_{\overline{z}} = \dot{\gamma}[\nu]_{\overline{z}} + \gamma' \nu.$$

where the second equality holds by (2.6). Since ν is Γ-equivariant, this implies $\dot{\gamma}[\nu]_{\overline{z}} = 0$. That is, $\dot{\gamma}[\nu]$ is holomorphic. In addition, Möbius transformations preserve the boundary of the disk, hence if $|z| = 1$, then $\operatorname{Re}(\dot{\gamma}[\nu]/\gamma) = 0$ and $\operatorname{Im}(z\gamma'/\gamma) = 0$. It follows that $\operatorname{Re}(\dot{\gamma}[\nu]/z\gamma') = 0$. By the Schwarz reflection principle, $\dot{\gamma}[\nu]/z\gamma'$ admits an extension to all of $\hat{\mathbb{C}}$ with a pole of order at most 1 at 0 and ∞. That is, $\dot{\gamma}[\nu]/\gamma'$ is holomorphic everywhere except for a pole of order at most 2 at ∞. We conclude that $\dot{\gamma}[\nu]/\gamma' = Q_\gamma$ is a quadratic polynomial. □

Remark 2.17. It follows from (2.10) that the analogously defined period of $\dot{w}[\nu]$ is also a quadratic polynomial.

Note that the variation $\dot{f}$ is real linear, but not complex linear. We may decompose it into its complex linear and complex antilinear parts. The latter is given by

$$\phi[\nu](z) = \dot{f}[\nu](z) + i\dot{f}[i\nu](z).$$

By (2.6),

$$\phi[\nu]_{\overline{z}} = \nu - \nu = 0$$

hence $\phi[\nu]$ is holomorphic. Viewing $\dot{f}$ as a quasiconformal vector field on $\mathbb{H}$, $\phi[\nu]$ is its holomorphic part.

If $\nu \in B(\Gamma)$, the vector field $\phi[\nu]$ is not necessarily Γ-equivariant. The obstruction to being equivariant is measured by the period

$$P_\gamma[\nu] \coloneqq \frac{\phi[\nu] \circ \gamma}{\gamma'} - \phi[\nu]. \tag{2.18}$$

We find that

$$P_\gamma[\nu] = Q_\gamma[\nu] + iQ_\gamma[i\nu],$$

so by Lemma 2.16, $P_\gamma[\nu]$ is a quadratic polynomial as well.

Remark 2.19. One can also recover $Q_\gamma[\nu]$ from $P_\gamma[\nu]$ as

$$Q_\gamma[\nu] = \frac{1}{2}\left(P_\gamma[\nu] - z^2\overline{P_\gamma}(1/\overline{z})\right).$$

See [1] for a proof.

Differentiating enough times, one recovers a Γ-equivariant object. Indeed, a direct computation shows that the third derivative $\varphi[\nu] \coloneqq \phi[\nu]'''$ is a holomorphic quadratic differential satisfying

$$(\varphi[\nu] \circ \gamma)(\gamma')^2 = \varphi[\nu].$$

That is, $\varphi[\nu]$ is a holomorphic quadratic differential for Γ.

Let $Q(\Gamma)$ denote the complex vector space of Γ-equivariant holomorphic quadratic differentials. By the Riemann-Roch theorem, $\dim Q(\Gamma) = 3g - 3$. Let $N(\Gamma)$ be the subspace of $B(\Gamma)$ consisting of Beltrami differentials ν for which $\varphi[\nu] = 0$.

Theorem 2.20. *The mapping $B(\Gamma)/N(\Gamma) \to Q(\Gamma)$ defined by $\nu \mapsto \varphi[\nu]$ is an antilinear isomorphism.*

Proof. The mapping is injective by the definition of $N(\Gamma)$. It remains to show that it is surjective. Let $\varphi \in Q(\Gamma)$. Explicit integration yields a unique function ϕ defined on the closure of $\mathbb{H}$ such that $\phi''' = \varphi$ and ϕ vanishes at 0, 1, and ∞. We define a Beltrami differential ν by

$$\nu(z) = (z - \overline{z})^2\overline{\varphi}(z). \tag{2.21}$$

Solving the differential equation (2.6), one obtains a unique solution for $\dot{f}[\nu](z)$. From here, a computation yields

$$\varphi[\nu] = \phi[\nu]''' = (\dot{f}[\nu] + i\dot{f}[i\nu])''' = \varphi,$$

showing surjectivity. For details of the computation, see Theorem 2 of [1]. □

The inverse map $Q(\Gamma) \to B(\Gamma)$ defined by (2.21) may be used to construct a basis of the quotient space, and such differentials are significant for computational purposes.

Definition 2.22. Beltrami differentials of the form described in (2.21) are called *harmonic*.

Harmonic Beltrami differentials span a $6g-6$ dimensional real subspace of $B(\Gamma)$, which we denote by $HB(\Gamma)$.

Remark 2.23. Let $\nu = (z-\overline{z})^2\overline{\varphi}(z) \in HB(\Gamma)$. One may compute an integral representation for $\dot{w}[\nu]$ as in Proposition 2.7 given by

$$\dot{w}[\nu](\zeta) = -\frac{1}{\pi}\iint_{\mathbb{H}} \nu(z)\frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}dxdy.$$

From a series expansion, one computes $\dot{w}[\nu]''' = 2\overline{\varphi}(\overline{z})$. A full proof may be found in [5], Theorem 6.10.

We now provide several useful characterizations of $N(\Gamma)$.

Proposition 2.24. *Let $\nu \in B(\Gamma)$. The following are equivalent:*

(i) $\nu \in N(\Gamma)$,
(ii) $\phi[\nu] = 0$,
(iii) $P_\gamma[\nu] = 0$ *for all* $\gamma \in \Gamma$,
(iv) $\dot{\gamma}[\nu] = 0$ *for all* $\gamma \in \Gamma$,
(v) $\dot{f}[\nu] = 0$ *on* $\mathbb{R}$,
(vi) $\iint_{\mathbb{H}/\Gamma} \nu\varphi dxdy = 0$ *for all* $\varphi \in Q(\Gamma)$.

For a proof, see Lemma 1 of Chapter 6 of [2].

2.3. **The complex structure.** We now use the mapping $\Phi : B(\Gamma) \to T_g$ taking $\mu \mapsto \Gamma^\mu$ to show that T_g admits the structure of a complex manifold. As a complex vector space, $B(\Gamma)/N(\Gamma)$ admits a natural complex structure. By Theorem 2.20, $B(\Gamma)/N(\Gamma)$ has complex dimension $3g-3$.

Fix a collection of elements $\mu_1, \ldots, \mu_{3g-3}$ whose residue classes modulo $N(\Gamma)$ form a basis for the quotient space. In particular, any element $\mu \in B(\Gamma)$ admits a unique representation

$$\mu = \zeta_1\mu_1 + \cdots + \zeta_{3g-3}\mu_{3g-3} + \nu \tag{2.25}$$

where $\nu \in N(\Gamma)$. We define a map $m : \mathbb{C}^{3g-3} \to B(\Gamma)/N(\Gamma)$ by

$$m(\zeta) = m(\zeta_1, \ldots, \zeta_{3g-3}) = \zeta_1\mu_1 + \cdots + \zeta_{3g-3}\mu_{3g-3}.$$

We identify $\mathbb{C}^{3g-3}$ with $\mathbb{R}^{6g-6}$ via $\zeta_j = \xi_j + i\eta_j \mapsto (\xi_j, \eta_j)$.

We first show that T_g is a manifold. Consider the map $\zeta \mapsto \Gamma^{m(\zeta)}$ from the unit ball of $\mathbb{R}^{6g-6}$ to T_g viewed as a subspace of $\mathbb{R}^{6g-6}$. Letting $u = (u_1, \ldots, u_{6g-6})$ denote the coordinates of the codomain, it is shown in [2] that the u_i are smooth functions of ζ. The columns of the Jacobian matrix at the point $m = m(\zeta_1, \ldots, \zeta_{3g-3})$ are given by

$$\frac{\partial u}{\partial \xi_k} = \dot{u}(m)[\mu_k], \qquad \frac{\partial u}{\partial \eta_k}(m) = \dot{u}[i\mu_k]. \tag{2.26}$$

Suppose the matrix is singular at the origin. Then there is some $\mathbb{R}$-linear combination of the μ_k and $i\mu_k$ which lies in $N(\Gamma)$, contradicting the assumption that the μ_k form a $\mathbb{C}$-linear basis of $B(\Gamma)/N(\Gamma)$. Thus, the Jacobian determinant is nonzero. By the inverse function theorem, some open neighborhood of the origin in $\mathbb{R}^{6g-6}$ is mapped homeomorphically to an open neighborhood of $\Gamma \in T_g \subseteq \mathbb{R}^{6g-6}$. Thus,

T_g is an open subset of $\mathbb{R}^{6g-6}$. This yields the manifold structure on Teichmüller space.

Now we define the complex structure. Note that $B(\Gamma)$ inherits a complex structure from its quotient $B(\Gamma)/N(\Gamma)$, where a function f is said to be holomorphic if $f(\zeta_1\mu_1 + \cdots + \zeta_{3g-3}\mu_{3g-3} + \nu)$ is holomorphic in μ_i for all i.

Definition 2.27. We say a function $f : T_g \to \mathbb{C}$ is holomorphic at Γ^μ if $f(\Gamma^\mu)$ is holomorphic in μ for μ in some neighborhood of 0 in $B(\Gamma)$.

This definition of being holomorphic may depend on the choice of basepoint $\Gamma = \theta(\Gamma_g)$. To verify that it does not, let $f_0(\mu) = f_\lambda(\mu \mid \lambda)$ as in (2.13). Then the explicit expression for $\rho = \mu \mid \lambda$ given by (2.11) shows that f_0 and f_λ are simultaneously holomorphic in μ.

Equipped with the notion of a holomorphic function on T_g, it remains to see that there exists a holomorphic coordinate system around each point.

Theorem 2.28. *The coordinates ζ_k are holomorphic.*

Proof. For $\mu \in B(\Gamma)$ with the representation as in (2.25), we write

$$\zeta(\mu) = (\zeta_1, \dots, \zeta_{3g-3}).$$

Then we have by definition that $\Gamma^{m(\zeta(\mu))} = \Gamma^\mu$, hence

$$u(\Gamma^{m(\zeta(\mu))}) = u(\Gamma^\mu).$$

Replacing μ by $\mu + t\nu$ and differentating with respect to t yields

$$\dot{u}(m)\left[\sum_{k=1}^{3g-3} \dot{\zeta}_k(\mu)[\nu]\mu_k\right] = \dot{u}(\mu)[\nu].$$

where $m = m(\zeta(\mu))$. By (2.14), this is equivalent to

$$\dot{u}_m\left[\sum_{k=1}^{3g-3} \dot{\zeta}_k(\mu)[\nu]L^m\mu_k\right] = \dot{u}_\mu[L^\mu\nu].$$

As $\dot{u}_m = \dot{u}_\mu$, it follows from (iv) of Proposition 2.24 that

$$\sum_{k=1}^{3g-3} \dot{\zeta}_k(\mu)[\nu]L^m\mu_k - L^\mu\nu \in N(\Gamma).$$

Replacing ν by $i\nu$, we obtain

$$\sum_{k=1}^{3g-3} \dot{\zeta}_k(\mu)[i\nu]L^m\mu_k - L^\mu i\nu \in N(\Gamma).$$

The $\mathbb{C}$-linearity of L^μ then implies that

$$\sum_{k=1}^{3g-3} \left(\dot{\zeta}_k(\mu)[i\nu] - i\dot{\zeta}_k(\mu)[\nu]\right) L^m\mu_k \in N(\Gamma).$$

or equivalently,

$$\dot{u}(m)\left[\sum_{k=1}^{3g-3} \left(\dot{\zeta}_k(\mu)[i\nu] - i\dot{\zeta}_k(\mu)[\nu]\right)\mu_k\right] = 0.$$

By (2.26) and the linear independence of the partial derivatives of u in a neighborhood of $\mu = 0$ established above, this in turn implies

$$\dot{\zeta}_k(\mu)[i\nu] = i\dot{\zeta}_k(\mu)[\nu]$$

for all k. In particular, the derivative of the coordinate ζ_k is $\mathbb{C}$-linear at μ. That is, ζ_k is holomorphic. □

This establishes the complex structure on T_g.

Remark 2.29. There are several constructions of the complex structure on Teichmüller space. One such method invokes the simultaneous uniformization theorem of Bers to embed T_g into $Q(\Gamma)$ and involves a more detailed study of the solutions w^μ. This approach has a number of advantages over the one presented in this paper, such as bounds on the image of the embedding. However, the author finds the construction presented here to be more geometrically motivated. The interested reader may find details of the Bers embedding in [2] or Chapter 6 of [5].

3. The Weil-Petersson metric

The Weil-Petersson metric on the Teichmüller space T_g was introduced by Weil, generalizing the Petersson inner product on the space of modular forms. Let $ds^2 = \lambda(z)|dz|^2$ be the Poincaré metric on $\mathbb{H}$, where $\lambda(z) = (z - \overline{z})^2$. Then $\lambda(z)dxdy$ is the corresponding area element, which yields an area element for the hyperbolic metric on Riemann surfaces $\mathbb{H}/\Gamma$.

The results of Section 2.3 show that the tangent space to T_g at $\mathbb{H}/\Gamma$ is identified with $B(\Gamma)/N(\Gamma)$. The Weil-Petersson metric on T_g is defined at the basepoint by

$$\langle \mu, \nu \rangle := \int_{\mathbb{H}/\Gamma} \varphi[\mu](z)\overline{\varphi[\nu](z)}\lambda(z)dxdy. \tag{3.1}$$

To see that it is independent of the choice of representatives for μ and ν, observe that the proof of surjectivity in Theorem 2.20 shows that

$$\overline{\varphi[\nu]} = \nu(z - \overline{z})^{-2}$$

hence we may equivalently define the metric by

$$\langle \mu, \nu \rangle = \iint_{\mathbb{H}/\Gamma} \varphi[\mu]\nu dxdy.$$

Then (vi) of Proposition 2.24 shows that $\langle -, \nu \rangle = 0$ for $\nu \in N(\Gamma)$.

Choosing a set $\mu_1, \dots, \mu_{3g-3}$ of harmonic Beltrami differentials which form a basis of the quotient, we obtain an identification of the tangent space with $HB(\Gamma)$. By Gram-Schmidt orthogonalization, one can take the Beltrami differentials to be orthonormal with respect to the metric at the basepoint Γ. At other points $m = m(\zeta)$ of T_g, recall the identification of $B(\Gamma)$ with $B(\Gamma^m)$ described by (2.12). We then obtain linearly independent Beltrami differentials at points $m(\zeta)$ in a neighborhood of $\zeta = 0$ by

$$\mu_k(\zeta) = L^{m(\zeta)}\mu_k. \tag{3.2}$$

Then the Weil-Petersson metric at Γ^m is defined by

$$\begin{aligned}\langle \mu_j(\zeta), \mu_k(\zeta)\rangle &= \int_{\mathbb{H}/\Gamma^m} \varphi[\mu_j(\zeta)](z)\overline{\varphi[\mu_k(\zeta)](z)}\lambda(z)dxdy \\ &= \iint_{\mathbb{H}/\Gamma^m} \varphi[\mu_j(\zeta)](z)\mu_k(\zeta)(z)dxdy\end{aligned}$$

where the second equality is derived as above. Let

$$\rho_\zeta(z) = (|f^m_z(z)|^2 - |f^m_{\overline{z}}(z)|^2)\lambda(f^m(z)).$$

The function ρ describes the Poincaré density in the image of $f^m(\mathbb{H})$. Then an expansion of $L^{m(\zeta)}\mu_k$ and a change of coordinates shows that

$$\langle \mu_j(\zeta), \mu_k(\zeta)\rangle = \iint_{\mathbb{H}/\Gamma} v_j(\zeta)(z)\overline{v_k}(\zeta)(z)\rho_\zeta(z)^{-1}(1-|m(\zeta)(z)|^2)^2 dxdy \tag{3.3}$$

where

$$v_k(\zeta)(z) = (\varphi[\nu_k(\zeta)] \circ f^m(z)) \cdot (f^m_z(z))^2.$$

As before, we may equivalently define it by

$$\langle \mu_j(\zeta), \mu_k(\zeta)\rangle = \iint_{\mathbb{H}/\Gamma} v_j(\zeta)\mu_k dxdy. \tag{3.4}$$

From the definition, one sees that the Weil-Petersson metric is Hermitian. The associated Riemannian metric is given by

$$(\mu, \nu) = 2\mathrm{Re}\,\langle \mu, \nu\rangle.$$

Theorem 3.5. *Let $\mu_1(\zeta), \ldots, \mu_{3g-3}(\zeta)$ be an orthonormal frame for the Weil-Petersson metric consisting of harmonic Beltrami differentials. Then the components of the Weil-Petersson metric*

$$h_{ij}(\zeta) = \langle \mu_i(\zeta), \mu_j(\zeta)\rangle$$

satisfy

$$\frac{\partial h_{ij}}{\partial \zeta_k} = \frac{\partial h_{ij}}{\partial \overline{\zeta}_k} = 0. \tag{3.6}$$

In particular, the Weil-Petersson metric is Kähler. That is, the Weil-Petersson metric in these coordinates takes the form

$$ds^2 = \sum_{i=1}^{3g-3} |dz_i|^2 + O(|z|^2).$$

The proof of the theorem relies on the following lemma.

Lemma 3.7. *If ν is a harmonic Beltrami differential, then $\dot{\rho}[\nu] = 0$.*

Proof. The proof is computational, using the explicit form for $\dot{f}[\nu]$ that may be derived in the proof of Theorem 2.20. For details, see Lemma 2 of [1]. □

An equivalent formulation is that the first derivatives of $\rho(\zeta)$ vanish at $\zeta = 0$.

Proof of Theorem 3.5. Differentiating both sides of (3.3) with respect to ζ_k at $\zeta = 0$ yields

$$\frac{\partial h_{ij}}{\partial \zeta_k}(0) = \iint_{\mathbb{H}/\Gamma} \left(\frac{\partial v_i}{\partial \zeta_k}(0)(z) \cdot \overline{v_j}(0)(z) + v_i(0)(z) \cdot \frac{\partial \overline{v_j}}{\partial \zeta_k}(0)(z) \right) \lambda(z) dxdy$$
$$+ \iint_{\mathbb{H}/\Gamma} v_i(0)(z) \overline{v_j(0)}(z) \frac{\partial (\rho_\zeta^{-1})}{\partial \zeta_k}(0)(z) dxdy.$$

The second term vanishes by Lemma 3.7. On the other hand, differentiating (3.4) yields

$$\frac{\partial h_{ij}}{\partial \zeta_k}(0) = \iint_{\mathbb{H}/\Gamma} \frac{\partial v_i}{\partial \zeta_k}(0)(z) \cdot \overline{\varphi}_j(z) \lambda(z) dxdy,$$

where we have explicitly used the definition of a harmonic Beltrami differential. Noting that $v_j(0) = \varphi_j$, it follows that

$$\iint_{\mathbb{H}/\Gamma} \varphi_i(z) \cdot \frac{\partial \overline{v_i}}{\partial \zeta_k}(0)(z) \lambda(z) dxdy = 0.$$

An entirely analogous computation shows that

$$\iint_{\mathbb{H}/\Gamma} \varphi_i(z) \cdot \frac{\partial \overline{v_j}}{\partial \overline{\zeta}_k}(0)(z) \lambda(z) dxdy = 0.$$

Interchanging i and j, we conclude that (3.6) holds. □

Remark 3.8. An equivalent formulation of the above is that the orthonormal frame of harmonic Beltrami differentials form normal coordinates. In these coordinates, the unique geodesic through the origin with tangent direction v is the line $\gamma(t) = tv$. Furthermore, the covariant Hessian of a function f in normal coordinates is simply

$$\text{Hess}_f(v, v) = \left. \frac{d^2 f(\gamma(t))}{dt^2} \right|_{t=0} \tag{3.9}$$

since the Christoffel symbols vanish. Thus, it suffices to show (3.9) is positive definite to conclude that f is strictly convex along geodesics for the Weil-Petersson metric. This will be the key to the computation in Section 5.

4. Eichler integrals and the Beltrami equation

Now that we have developed the analytic theory of Teichmüller space, we begin working towards a solution to the Nielsen realization problem. This will involve showing the convexity of a certain function. For the purposes of computing the Hessian of a function on T_g, it is necessary to have a solution to the Beltrami equation (2.1) which we may approximate to order two. To this end, we review the theory of Eichler integrals, introduced by Eichler in his study of automorphic forms.

Let $\psi(z)$ be a holomorphic quadratic differential in the lower half plane $\mathbb{L} = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. Fix a point $w_0 \in \mathbb{L}$. Define the *Eichler integral* $E_\psi : \mathbb{L} \to \mathbb{C}$ of ψ by

$$E_\psi(w) = \int_{w_0}^{w} (w - v)^2 \psi(v) dv. \tag{4.1}$$

A brief computation reveals that $E_\psi'''(w) = 2\psi(w)$ and $(E_\psi)_{\overline{w}} = 0$. One may think of E_ψ as a holomorphic vector field on $\mathbb{L}$.

If $\psi \in Q(\Gamma)$, the Eichler integral E_ψ need not be Γ-equivariant. As in (2.18), the obstruction to being such is measured by the Eichler period

$$P_\gamma(w) = \frac{E_\psi(\gamma(w))}{\gamma'(w)} - E_\psi(w), \qquad \gamma \in \Gamma. \tag{4.2}$$

In fact, the Eichler period is also quadratic in w.

Lemma 4.3.

$$P_\gamma(w) = \int_{\gamma^{-1}(w_0)}^{w_0} (w-v)^2 \psi(v) dv. \tag{4.4}$$

Proof. First note the identity

$$\frac{(\gamma w - \gamma v)^2}{\gamma'(w)\gamma'(v)} = (w-v)^2.$$

Then a direct calculation yields

$$\begin{aligned} P_\gamma(w) &= E_\psi(\gamma(w))/\gamma'(w) - E_\psi(w) \\ &= \int_{w_0}^{\gamma w} \frac{(\gamma w - v)^2}{\gamma'(w)} \psi(v) dv - \int_{w_0}^{w} (w-v)^2 \psi(v) dv \\ &= \int_{\gamma^{-1} w_0}^{w} \frac{(\gamma w - \gamma v)^2}{\gamma'(w)\gamma'(v)} \psi(v) dz - \int_{w_0}^{w} (w-v)^2 \psi(v) dv \\ &= \int_{\gamma^{-1} w_0}^{w_0} (w-v)^2 \psi(v) dv. \end{aligned}$$

□

Remark 4.5. It is no coincidence that the Eichler periods and the periods in (2.18) are quadratic polynomials. A more conceptual explanation for their appearance is as follows. By the Schwarz reflection principle, both E_ψ and $\varphi[\nu]$ may be extended to holomorphic vector fields on all of the Riemann sphere $\hat{\mathbb{C}}$. Note that $\Gamma < PSL(2, \mathbb{C})$, which acts on $\hat{\mathbb{C}}$ by automorphisms. The Lie algebra of $PSL(2, \mathbb{C})$, denoted $\mathfrak{sl}_2\mathbb{C}$, may be identified with the space of quadratic polynomials via the standard representation. For $\gamma \in \Gamma$, the periods P_γ as defined in (2.18) and (4.2) represent infinitesimal perturbations of γ, hence they are identified with elements of $\mathfrak{sl}_2\mathbb{C}$. This observation connects to the theory of Eichler integrals and Kodaira-Spencer deformation theory. The curious reader is encouraged to see Chapter 7.2.4 of [5].

Suppose now we are given a holomorphic quadratic differential φ defined on $\mathbb{H}$. In analogy with the Eichler integral (4.1), fix a point $z_0 \in \mathbb{H}$ and consider the function $F_\varphi : \mathbb{H} \to \mathbb{C}$ defined by

$$F_\varphi(z) \coloneqq \overline{\int_{z_0}^{z} (\overline{z} - t)^2 \varphi(t) dt}.$$

Then $(F_\varphi)_{\overline{z}} = (z - \overline{z})^2 \overline{\varphi(z)}$.

Remark 4.6. Observe that $(F_\varphi)_{\overline{z}}$ is the harmonic Beltrami differential obtained from φ, which indicates that F_φ may serve as a suitable replacement for the variation $\dot{f}[\nu]$ where $\nu = (z - \overline{z})^2 \overline{\varphi}$.

As before, if the quadratic differential φ is Γ-equivariant, it does not necessarily follow that F_φ is Γ-equivariant. The obstruction to being such is measured by the analogously defined period.

Lemma 4.7. *If $\psi(z) = \overline{\varphi(\overline{z})}$ and $w_0 = \overline{z_0}$, then the periods E_ψ and F_φ are equal.*

Proof. Let $\gamma \in \Gamma$. As in the proof of Lemma 4.3, we compute

$$\frac{F_\varphi(\gamma(z))}{\gamma'(z)} - F_\varphi(z) = \overline{\int_{\gamma^{-1}z_0}^{z_0} (\overline{z}-t)^2\varphi(t)dt} = \int_{\gamma^{-1}z_0}^{z_0} (z-\overline{t})^2\overline{\varphi(t)}d\overline{t}.$$

On the other hand, the Eichler period is

$$\int_{\gamma^{-1}\overline{z_0}}^{\overline{z_0}} (\overline{z}-v)^2\overline{\varphi(\overline{v})}dv.$$

Since $\Gamma < PSL(2,\mathbb{R})$, we have $\gamma^{-1}(\overline{z_0}) = \overline{\gamma^{-1}(z_0)}$, hence the change of variables $v = \overline{t}$ yields the result. $\square$

Now for $\varphi \in Q(\Gamma)$, consider the function

$$\mathscr{F}_\varphi(z) = \begin{cases} F(z), & z \in \mathbb{H}, \\ E(z), & z \in \mathbb{L} \text{ with } \psi(z) = \overline{\varphi(\overline{z})} \text{ and } w_0 = \overline{z_0}. \end{cases}$$

Heuristically, one may view $\mathscr{F}$ as a global potential for the harmonic Beltrami differential ν constructed from φ. The following proposition justifies this.

Proposition 4.8. *Let $\varphi \in Q(\Gamma)$. For $\nu = (z-\overline{z})^2\overline{\varphi}$,*

$$\dot{w}[\nu](z) = \mathscr{F}_\varphi(z) + q(z) \tag{4.9}$$

where $q(z)$ is a quadratic polynomial.

Proof. First observe that on $\mathbb{L}$ we have by definition that

$$\mathscr{F}_\varphi'''(z) = E'''(z) = 2\overline{\phi(\overline{z})} = \dot{w}[\nu]'''(z)$$

where the last equality holds by Remark 2.23. Therefore, $\mathscr{F}_\phi$ and $\dot{w}$ differ by a quadratic polynomial $q(z)$ on $\mathbb{L}$. Now consider

$$G(z) = \dot{w}[\nu] - \mathscr{F}_\phi - q(z).$$

By definition, $G(z)$ vanishes for $z \in \mathbb{L}$. For $z \in \mathbb{H}$, note that $\dot{w}[\nu]_{\overline{z}} = \nu = F_{\overline{z}}$, hence $G_{\overline{z}} = q(z)_{\overline{z}} = 0$. That is, G is a holomorphic vector field on $\mathbb{H} \cup \mathbb{L}$. For $\gamma \in \Gamma$, the period

$$\frac{G(\gamma(z))}{\gamma'(z)} - G(z)$$

is a polynomial since the periods of the summands are polynomials (see Remark 2.17). On the other hand, the period of G vanishes identically on $\mathbb{L}$. By the identity theorem for holomorphic functions, the period vanishes on all of $\mathbb{H} \cup \mathbb{L}$, hence G is a Γ-equivariant holomorphic vector field. That is, G descends to a global holomorphic vector field on the Riemann surface $\mathbb{H}/\Gamma$. By the Riemann-Roch theorem, the degree of the holomorphic tangent bundle of a compact Riemann surface is $2-2g$. Thus, for $g \geq 2$, the only global holomorphic vector field is zero everywhere. We conclude that $G = 0$. $\square$

By (2.10), we immediately deduce that

$$\dot{f}[\nu] = (\mathscr{F} + q)(z) + \overline{(\mathscr{F} + q)(\overline{z})}. \tag{4.10}$$

Writing $q(z) = az^2 + bz + c$, we are in fact able to derive the coefficients a and c.

Lemma 4.11. *With notation as above,*

$$a = \frac{1}{1-\lambda}\overline{\int_{\lambda^{-1}z_0}^{z_0} \phi(t)dt}, \qquad c = \frac{\lambda}{\lambda - 1}\overline{\int_{\lambda^{-1}z_0}^{z_0} t^2\phi(t)dt}.$$

Proof. Indeed, by normalization, the period $\dot{f}[\nu](\lambda z)/\lambda - \dot{f}[\nu]$ vanishes at 0 and is $o(|z|^2)$ as $z \to \infty$. On the other hand, the period is a quadratic polynomial, hence it is necessarily a multiple of z. Therefore, by Proposition 4.8 we have that

$$\frac{\mathscr{F}_\phi(\lambda z) + q(\lambda z)}{\lambda} - \mathscr{F}_\phi(z) - q(z)$$

is a multiple of z. Expanding the above and setting the constant term and coefficient of z^2 to be equal to zero yields the result. □

5. Convexity of geodesic length functions

Fix a simple closed curve γ on the surface S_g.

Definition 5.1. The geodesic length function $l_\gamma : T_g \to \mathbb{R}$ is the map taking Γ to the length of the unique geodesic representative in the free homotopy class of γ on the Riemann surface $\mathbb{H}/\Gamma$.

The goal of this section is to compute the second derivative of l_γ in the direction of a harmonic Beltrami differential μ and show that it is positive. By Remark 3.8, this will show that geodesic length functions are strictly convex along Weil-Petersson geodesics. We derive some general formulas and then sketch the computation.

We begin by deriving the formula for the first variation of the geodesic length function. Conjugating if necessary, we may assume that the geodesic representative of γ lifts to the imaginary axis and corresponds to the transformation $z \mapsto \lambda z$ where $\lambda = e^{l_\gamma(R)} > 1$. Thus, for the remainder of the section we assume that $z \mapsto \lambda z$ is contained in Γ. A Beltrami differential $\mu \in B(\Gamma)$ determines a curve $\Gamma^{t\mu}$ in T_g for $t \in (-\varepsilon, \varepsilon)$ for ε small.

Proposition 5.2 ([4], Theorem 2)**.**

$$\left.\frac{dl_\gamma}{dt}\right|_{t=0} \Gamma^{t\mu} = \frac{2}{\pi}\mathrm{Re}\left(\int_D \mu\zeta^{-2} d\xi d\eta\right) \tag{5.3}$$

where $D = \{\zeta \in \mathbb{H} : 1 < |\zeta| < \lambda\}$.

Proof. The transformation $z \mapsto \lambda(t)z$ corresponding to the geodesic in $\mathbb{H}/\Gamma^{t\mu}$ is defined by the equation

$$f^{t\mu}(\lambda z) = \lambda(t) f^{t\mu}(z) \tag{5.4}$$

(compare with (2.4)). Differentiating both sides of (5.4) at $t = 0$ yields

$$\dot{f}[\mu](\lambda z) = \lambda'(0)z + \lambda \dot{f}[\mu](z). \tag{5.5}$$

By the definition of hyperbolic length,

$$l_\gamma(\Gamma^{t\mu}) = \log \lambda(t).$$

Therefore,

$$\left.\frac{dl_\gamma}{dt}\right|_{t=0} \Gamma^{t\mu} = \frac{\lambda'(0)}{\lambda} = \frac{\dot{f}[\mu](\lambda z)}{\lambda z} - \frac{\dot{f}[\mu](z)}{z}. \tag{5.6}$$

Recall the integral representation for $\dot{f}[\mu]$ described in Proposition 2.7. Substituting into (5.6) yields

$$\left.\frac{dl_\gamma}{dt}\right|_{t=0} \Gamma^{t\mu} = \operatorname{Re}\left(-\frac{2}{\pi}\iint_{\mathbb{H}} \mu(\zeta)\left(\frac{R(\lambda z,\zeta)}{\lambda z} - \frac{R(z,\zeta)}{z}\right) d\xi d\eta\right).$$

Rewriting the above integral by summing over translations of the domain $D = \{1 < |\zeta| < \lambda\}$ and using that $\mu(\lambda\zeta) = \mu(\zeta)$ by Γ-invariance of μ, we obtain

$$\left.\frac{dl_\gamma}{dt}\right|_{t=0} \Gamma^{t\mu} = -\frac{2}{\pi}\operatorname{Re}\left(\iint_D \mu(\zeta)\sum_{n=-\infty}^{\infty}\lambda^{2n}\left(\frac{R(\lambda z,\lambda^n\zeta)}{\lambda z} - \frac{R(z,\lambda^n\zeta)}{z}\right) d\xi d\eta\right).$$

Observe the identity

$$\lambda^{2n}R(z,\zeta) = -\frac{\lambda^n z}{\zeta}\left(\frac{\lambda^n}{\zeta-1} - \frac{\lambda^n}{\zeta - z}\right),$$

from which we compute

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\lambda^{2n}\left(\frac{R(\lambda z,\lambda^n\zeta)}{\lambda z} - \frac{R(z,\lambda^n\zeta)}{z}\right) &= -\frac{1}{\zeta}\sum_{n=-\infty}^{\infty}\left(\frac{\lambda^n}{\lambda^n\zeta - z} - \frac{\lambda^{n-1}}{\lambda^{n-1}\zeta - z}\right)\\ &= -\frac{1}{\zeta}\lim_{n\to\infty}\frac{\lambda^n}{\lambda^n\zeta - z}\\ &= -\frac{1}{\zeta^2}\end{aligned}$$

where we use that the series telescopes and that $\lambda > 1$. We conclude that

$$\left.\frac{dl_\gamma}{dt}\right|_{t=0} \Gamma^{t\mu} = \frac{2}{\pi}\operatorname{Re}\left(\iint_D \mu\zeta^{-2} d\xi d\eta\right)$$

as claimed. □

We now proceed with a computation of the second derivative. We note that the tangent to the curve at the point $\Gamma^{t\mu}$ is represented by the Beltrami differential $L^{t\mu}\mu$ (see (2.12)).

Theorem 5.7.

$$\left.\frac{d^2 l_\gamma}{dt^2}\right|_{t=0} \Gamma^{t\mu} = \frac{4}{\pi}\operatorname{Re}\int_D \mu\left(\frac{z\dot{f}[\mu]_z - \dot{f}[\mu]}{z^3}\right) d\xi d\eta \tag{5.8}$$

where $D = \{\zeta \in \mathbb{H} : 1 < |\zeta| < \lambda\}$.

Proof. By Proposition 5.2, the first derivative of l_γ as a function of t is given by

$$\frac{dl_\gamma}{dt}\Gamma^{t\mu} = \frac{2}{\pi}\operatorname{Re}\int_{1<|\zeta|<\lambda(t)} \zeta^{-2}L^{t\mu}(\zeta) d\xi d\eta$$

where $\lambda(t)$ is defined as in (5.4). Changing variables to $\zeta = f^\varepsilon(z)$ pulls the integral back to the domain D, yielding

$$\frac{dl_\gamma}{dt}\Gamma^{t\mu} = \frac{2}{\pi}\operatorname{Re}\int_D (f^{t\mu}(z))^{-2}\cdot\left(\mu\frac{(f_z^{t\mu}(z))^2}{|f_z^{t\mu}(z)|^2 - |f_{\overline{z}(z)}^{t\mu}|^2}\right)|D(f^{t\mu})|dxdy$$

where $|D(f^{t\mu})|$ denotes the Jacobian determinant. Since

$$|D(f^{t\mu})| = |f_z^{t\mu}|^2 - |f_{\overline{z}}^{t\mu}|^2$$

we obtain

$$\frac{dl_\gamma}{dt}\Gamma^{t\mu} = \frac{2}{\pi}\mathrm{Re}\left(\int_D \mu(z)\left(\frac{f_z^{t\mu}(z)}{f^{t\mu}(z)}\right)^2 dxdy\right).$$

Finally, we use the first order approximation $f^{t\mu}(z) = z + t\dot{f}[\mu] + O(t^2)$ and differentiate under the integral sign with respect to t to obtain (5.8). □

A quadratic differential $\varphi \in Q(\Gamma)$ admits a local representation as $\varphi(z)dz^2$ where $\varphi(\lambda z) = \varphi(z)$. We want to obtain a series expansion for φ for computational purposes. To do so, observe that the function $f(z) := \varphi(\lambda^z)$ is 1-periodic, hence admits a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}.$$

It follows that

$$\varphi(z) = f\left(\frac{\log z}{\log \lambda}\right) = \sum_{n=0}^{\infty} a_n z^{\varepsilon n}, \quad \varepsilon = \frac{2\pi i}{\log\lambda}. \tag{5.9}$$

For the remainder of the section, we fix a holomorphic branch of log to be real at 1. The series converges uniformly on compact subsets of $\mathbb{H}$.

We proceed with the computation of (5.8). By combining (4.10) and Lemma 4.11, we establish

$$\dot{f}(z) = \dot{w}(z) + \overline{\dot{w}(\overline{z})} = z^2 2\mathrm{Re}\, A + z 2\mathrm{Re}\, B + 2\mathrm{Re}\, C + z 2\mathrm{Re}\, b. \tag{5.10}$$

where

$$A(z) = \int_{z_0}^{z} \varphi(t)dt + \frac{1}{1-\lambda}\int_{\lambda^{-1}z_0}^{z_0} \varphi(t)dt$$

$$B(z) = -2\int_{z_0}^{z} t\varphi(t)dt$$

$$C(z) = \int_{z_0}^{z} t^2\varphi(t)dt + \frac{\lambda}{\lambda-1}\int_{\lambda^{-1}z_0}^{z_0} t^2\varphi(t)dt$$

For h a holomorphic function, the Cauchy-Riemann equations imply that $(\mathrm{Re}\, h)_z = \frac{1}{2}h_z$. It follows that

$$z\dot{f}_z(z) - \dot{f}(z) = z^2 2\mathrm{Re}\, A - 2\mathrm{Re}\, C. \tag{5.11}$$

The series expansion for φ now yields a series expansion for the left hand side of (5.11).

Lemma 5.12.

$$z\dot{f}_z(z) - \dot{f}(z) = z^2 2\mathrm{Re}\sum_{n=-\infty}^{\infty}\frac{a_n z^{\varepsilon n-1}}{\varepsilon n - 1} - 2\mathrm{Re}\sum_{n=-\infty}^{\infty}\frac{a_n z^{\varepsilon n+1}}{\varepsilon n + 1}. \tag{5.13}$$

Proof. Noting the identity

$$(\lambda t)^{\varepsilon n} = e^{\varepsilon n \log\lambda} t^{\varepsilon n} = t^{\varepsilon n},$$

an explicit integration of the series expansion of φ yields the result. □

We now compute (5.8) using the series expansions derived above. The series converge uniformly on compact subsets, enabling term-by-term integration. We sketch the integration for each series separately. For details of both computations, see Section 4 of [7].

Lemma 5.14. *For $\alpha, \beta \in \mathbb{C}$,*

$$\operatorname{Re}\int_D (z-\overline{z})^2 \left(\frac{\overline{\alpha z^{\varepsilon m}}}{z^2}\right) \frac{1}{z}\operatorname{Re}\frac{\beta z^{\varepsilon n-1}}{\varepsilon n-1} dxdy = 0, \quad \text{if } m \neq \pm n,$$

$$\operatorname{Re}\int_D (z-\overline{z})^2 \left(\frac{\overline{\alpha z^{\varepsilon m} + \beta z^{-\varepsilon n}}}{z^2}\right) \frac{1}{z}\operatorname{Re}\left(\frac{\alpha z^{\varepsilon n-1}}{\varepsilon n-1} + \frac{\beta z^{-\varepsilon n-1}}{-\varepsilon n-1}\right) dxdy \geq 0,$$

where equality holds if and only if $\alpha = \beta = 0$.

Proof. Convert the integrals to polar coordinates. In the first case, one finds that the integral vanishes due to orthogonality of trigonometric functions. In the second case, the integral reduces to

$$\frac{2}{1+|\varepsilon n|^2}\int_0^\pi \left|\alpha e^{i\theta(\varepsilon n-1)} + \overline{\beta} e^{i\theta(1-\varepsilon n)}\right|^2 \sin^2\theta d\theta$$

and the result follows. □

An entirely similar method of proof yields

Lemma 5.15. *For $\gamma, \delta \in \mathbb{C}$,*

$$-\operatorname{Re}\int_D (z-\overline{z})^2 \left(\frac{\overline{\alpha z^{\varepsilon m}}}{z^2}\right) \frac{1}{z^3}\operatorname{Re}\frac{\beta z^{\varepsilon n+1}}{\varepsilon n+1} dxdy = 0, \quad \text{if } m \neq \pm n,$$

$$-\operatorname{Re}\int_D (z-\overline{z})^2 \left(\frac{\overline{\alpha z^{\varepsilon m} + \beta z^{-\varepsilon n}}}{z^2}\right) \frac{1}{z^3}\operatorname{Re}\left(\frac{\alpha z^{\varepsilon n+1}}{\varepsilon n+1} + \frac{\beta z^{-\varepsilon n+1}}{-\varepsilon n+1}\right) dxdy \geq 0,$$

where equality holds if and only if $\alpha = \beta = 0$.

Corollary 5.16. *With notation as in Theorem 5.7, if $\mu \in B(\Gamma)$ is a non-zero harmonic Beltrami differential, then*

$$\left.\frac{d^2 l_\gamma}{dt^2}\right|_{t=0} \Gamma^{t\mu} > 0.$$

Proof. If $\mu = (z-\overline{z})^2\overline{\varphi}$ is non-zero, then applying the results of Lemma 5.12, Lemma 5.14, and Lemma 5.15 to the right hand side of Theorem 5.7 yield the result. □

By the discussion in Remark 3.8, we have proven the following.

Theorem 5.17. *The geodesic length function l_γ is strictly convex along Weil-Petersson geodesics.*

6. The Nielsen realization problem

We are finally in a position to prove the positive answer to the Nielsen realization problem.

Theorem 6.1. *Let G be a finite subgroup of the mapping class group $Mod(S)$ of a closed surface of genus g. Then there exists a point $x \in T_g$ such that $g \cdot x = x$ for all $g \in G$.*

We establish a preliminary result.

Definition 6.2. A collection of simple closed curves $\{\alpha_i\}_{i=1}^n$ is said to fill up S if, when they are chosen to have minimal pairwise intersections, $S - \bigcup_{i=1}^n \alpha_i$ is a union of disks.

For a simple closed curve α on S, recall the geodesic length function l_α as in Definition 5.1.

Lemma 6.3 ([6], Lemma 3.1)**.** *If $A = \{\alpha_i\}_{i=1}^n$ fills up S, then the length function $l_A := \sum_{i=1}^n l_{\alpha_i}$ is a proper function.*

Proof. It suffices to show that $B_A(M) = \{x \in T_g : l_A(x) < M\}$ is bounded for any $M \in \mathbb{R}$. We recall Mumford's compactness theorem, which implies that given an unbounded sequence in T_g, there exists a simple closed curve γ on S such that $l_\gamma(S) \to 0$. Then by the collar lemma, any curve which intersects γ has length tending to infinity. Therefore, a subset of T_g is bounded if and only if the length of every simple closed geodesic is bounded.

Since A fills up S, any simple closed geodesic γ in S can be homotoped so that it lies in $\bigcup_{i=1}^n \alpha_i$ and intersects any point of S at most N times, where N depends only on the homotopy class of γ. Therefore,

$$l_\gamma(x) \leq N l_A(x) < MN,$$

so the lengths of simple closed geodesics are bounded on $B_A(M)$. □

An immediate corollary is that l_A realizes a minimum in T_g.

Proof of Theorem 6.1. Fix a family $A = \{\alpha_i\}_{i=1}^n$ of simple closed curves that fill up S. Since the mapping class group acts on Teichmüller space by isometries for the Weil-Petersson metric, the length function l_A is G-invariant. It follows that the set of minima for l_A is G-invariant.

As observed above, the function l_A admits a minimum in T_g, say x. Suppose there exists another minimum $y \neq x$. Connect x and y by a Weil-Petersson geodesic. By Theorem 5.17, l_A is strictly convex along the geodesic, but this contradicts the assumption that both points are minima. Thus, the minimum of l_A is unique, hence it is a fixed point of G. □

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