Solutions to Topology by Munkres

Akash Narayanan

Contents

Chapter I

Set Theory and Logic

1 Fundamental Concepts

Exercise 1.1. Check the distributive laws for $\cup$ and $\cap$ and DeMorgan's laws.

Solution. First, we prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B$ or $x \in C$. But this implies that $x \in A \cap B$ or $x \in A \cap C$. Therefore, we have $x \in (A \cap B) \cup (A \cap C)$. Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

For the opposite direction, let $x \in (A \cap B) \cup (A \cap C)$. This implies that $x \in A \cap B$ or $x \in A \cap C$. In either case, $x \in A$. Furthermore, we have $x \in B$ or $x \in C$. But then $x \in A \cap (B \cup C)$ so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, proving the equality. The proof for the second statement is entirely analogous.

Now we will prove the first of DeMorgan's laws, that $A - (B \cup C) = (A - B) \cap (A - C)$. Let $x \in A - (B \cup C)$. That is, $x \in A$ and $x \notin B \cup C$. But then $x \notin B$ and $x \notin C$ so $x \in A - B$ and $x \in A - C$. Thus, $x \in (A - B) \cap (A - C)$ so $A - (B \cup C) \subseteq (A - B) \cap (A - C)$.

For the other direction, let $x \in (A - B) \cap (A - C)$. Then $x \in A - B$ and $x \in A - C$ so $x \in A$, $x \notin B$, and $x \notin C$. But then $x \notin B \cup C$ so $x \in A - (B \cup C)$. Therefore, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$, proving equality of sets. The proof of the second statement is again analogous. $\square$

Exercise 1.2. Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subset$ or $\supset$.

(a) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cup C)$.

(b) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cup C)$.

(c) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$.

(d) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cap C)$.

(e) $A - (A - B) = B$.

(f) $A - (B - A) = A - B$.

(g) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

(h) $A \cup (B - C) = (A \cup B) - (A \cup C)$.

(i) $(A \cap B) \cup (A - B) = A$.

(j) $A \subset C$ and $B \subset D \Rightarrow (A \times B) \subset (C \times D)$.

(k) The converse of (j).

(l) The converse of (j), assuming that A and B are nonempty.

(m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

(n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(o) $A \times (B - C) = (A \times B) - (A \times C)$.

(p) $(A - B) \times (C - D) = (A \times C - B \times C) - (A \times D)$.

(q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

Solution. The forward direction of (a) is true but the reverse direction is false. To see this, let $a \in A$. Then $a \in B$ and $a \in C$ so $a \in B \cup C$. Therefore, $A \subset (B \cup C)$. However, consider $A \subset B$ and $A \not\subset C$. Certainly $A \subset B \cup C$ but the left side does not hold.

The forward direction of (b) is true but the reverse direction is false. To prove the forward direction, let $a \in A$. If $A \subset B$, then $a \in B$ so $a \in B \cup C$. If $A \subset C$, then $a \in B$ and still we have $a \in B \cup C$. In either case, $A \subset (B \cup C)$. For a counterexample to the reverse direction, consider $A = \{1, 2\}$, $B = \{1\}$, $C = \{2, 3\}$. Clearly $A \subset B \cup C$ but $A \not\subset B$ and $A \not\subset C$.

Both directions of (c) are true. Indeed, suppose $A \subset B$ and $A \subset C$ and let $a \in A$. Then $a \in B$ and $a \in C$ so $a \in B \cap C$, showing that $A \subset (B \cap C)$. Now suppose $A \subset (B \cap C)$ and let $a \in A$. Then $a \in B \cap C$ so $a \in B$ and $a \in C$. This implies that $A \subset B$ and $A \subset C$.

The forward direction of (d) is false but the reverse direction is true. To prove the reverse direction, suppose $A \subset (B \cap C)$ and let $a \in A$. Then $a \in B \cap C$ so $a \in B$ and $a \in C$, satisfying the left side of the statement. For a counterexample to the forward direction, consider $A = \{0, 1\}$, $B = \{0, 1, 2\}$, and $C = \{2\}$. Then $A \subset B$ so the left side is satisfied by $B \cap C = \{2\}$ which A is not a subset of.

Statement (e) holds if the equality is replaced with $\subset$. To see this, let $x \in A - (A - B)$. Then $x \in A$ but $x \notin A - B$. That is, $x \in B$ so $A - (A - B) \subset B$. For a counterexample to the other direction, consider the sets $A = \{0, 1\}$ and $B = \{1, 2\}$. Then $A - (A - B) = \{1\} \neq B$, showing equality does not hold.

Statement (f) holds if equality is replaced with $\supset$. Let $x \in A - B$. Then $x \in A$ and $x \notin B$. But if $x \notin B$ then $x \notin B - A$. Thus, $x \in A - (B - A)$ so $A - (B - A) \supset A - B$. For a counterexample to the other direction, consider $A = \{0, 1\}$ and $B = \{1, 2\}$. Then $A - (B - A) = \{0, 1\}$ while $A - B = \{0\}$, showing equality does not hold.

The equality in statement (g) holds. To prove this, let $x \in A \cap (B - C)$. Then $x \in A$ and $x \in B$, but $x \notin C$. That is, $x \in A \cap B$ and $x \notin A \cap C$. Thus, $x \in (A \cap B) - (A \cap C)$, showing that $\subset$ holds. Now let $x \in (A \cap B) - (A \cap C)$. This implies that $x \in A \cap B)$ but $x \notin A \cap C$. The first part shows that $x \in A$ and $x \in B$, so the second statement implies $x \notin C$. But then $x \in B - C$ so $x \in A \cap (B - C)$ and $\supset$ holds. Since both sides are subsets of each other, equality holds.

Statement (h) holds if equality is replaced with $\supset$. Indeed, let $x \in (A \cup B) - (A \cup C)$. Then $x \in A \cup B$ and $x \notin A \cup C$. This implies that $x \notin A$, $x \in B$, and $x \notin C$. But then $x \in B - C$, so $x \in A \cup (B - C)$, proving $\supset$ holds. For a counterexample to the other inclusion, consider $A = \{0\}, B = \{1, 2\}, C = \{2\}$. Then $A \cup (B - C) = \{0, 1\}$ while $(A \cup B) - (A \cup C) = \{1\}$.

The equality in statement (i) holds. First suppose $x \in (A \cap B) \cup (A - B)$. If $x \in A \cap B$ then $x \in A$ so $\subset$ holds. If $x \in A - B$ then $x \in A$ so $\subset$ holds. Now suppose $x \in A$. If $x \in B$ then $x \in A \cap B$ so $\supset$ holds. If $x \notin B$ then $x \in A - B$ so $\supset$ holds. Thus, the two sets are equal.

Statement (j) always holds. Indeed, suppose the left side is true and let $(a, b) \in A \times B$. Since $a \in A \subset C$, we have $a \in C$. Similarly, $b \in D$. Thus, $(a, b) \in C \times D$ so $(A \times B) \subset (C \times D)$.

Statement (k) does not always hold. The converse of (j) is that $(A \times B) \subset (C \times D) \Rightarrow A \subset C$ and $B \subset D$. For a counterexample, consider $A = \{0\}$, $B = \emptyset$, $C = \{1\}$, $D = \{2\}$. Then $A \times B = \emptyset \subset C \times D$ but clearly $A \not\subset C$.

Statement (l) always holds. Let $a \in A$. Since B is nonempty, choose some $b \in B$. Then $(a, b) \in A \times B$ so $(a, b) \in C \times D$. But then $a \in C$ and $b \in D$. That is, $A \subset C$ and $B \subset D$.

Statement (m) holds if equality is replaced with $\subset$. Let $(x, y) \in (A \times B) \cup (C \times D)$. If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$. But then $x \in A \cup C$ and $y \in B \cup D$ so $(x, y) \in (A \cup C) \times (B \cup D)$ showing $\subset$ holds. The same reasoning applies if $(x, y) \in C \times D$. To show the reverse inclusion does not always holds, consider $A = \{0\}$, $B = \{1\}$, $C = \{2\}$, $D = \{3\}$. Then $(A \times B) \cup (C \times D) = \{(0, 1), (2, 3)\}$ while $(A \cup C) \times (B \cup D) = \{(0, 1), (0, 3), (2, 1), (2, 3)\}$.

Statement (n) always holds. Suppose $(x, y) \in (A \times B) \cap (C \times D)$. Then $x \in A$ and $x \in C$ so $x \in A \cap C$. Similarly, $y \in B \cap D$. But then $(x, y) \in (A \cap C) \times (B \cap D)$, showing $\subset$ holds. Now let $(x, y) \in (A \cap C) \times (B \cap D)$. Then $x \in A \cap C$ and $y \in B \cap D$. This implies that $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Thus, it is in their intersection and $\supset$ holds, implying equality.

Statement (o) always holds. Suppose $(x, y) \in A \times (B - C)$. Then $x \in A$, $y \in B$, and $y \notin C$. But then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$ so it is in their difference and $\subset$ holds. If $(x, y) \in (A \times B) - (A \times C)$ then $x \in A$, $y \in B$, and $y \notin C$. Then $y \in B - C$ so $(x, y) \in A \times (B - C)$.

Statement (p) always holds. Suppose $(x, y) \in (A - B) \times (C - D)$. Then $x \in A$, $x \notin B$, $y \in C$, and $y \notin D$. This implies that $(x, y) \in A \times C - B \times C$ and $(x, y) \notin A \times D$. But then $(x, y) \in (A \times C - B \times C) - A \times D$ so $\subset$ holds. Now suppose $(x, y) \in (A \times C - B \times C) - A \times D$. Then $(x, y) \in A \times C - B \times C$ and $(x, y) \notin A \times D$. The first part implies that $x \in A$ and $y \in C$. We can deduce that $x \notin B$ and $\notin D$. Therefore $x \in A - B$ and $\in C - D$ so $(x, y) \in (A - B) \times (C - D)$ and $\supset$ holds, making equality true.

Statement (q) holds if equality is replaced with $\supset$. To see this, let $(x, y) \in (A-C)\times(B-D)$. Then $x \in A$, $x \notin C$, $y \in B$, and $y \notin D$. But then $(x, y) \in A \times B$ and $(x, y) \notin C \times D$ so it is in their difference and $\supset$ holds. To see that the reverse inclusion does not always holds, consider $A = \{0, 1\}$, $B = \{1\}$, $C = \{1, 2\}$, $D = \{3\}$. Then $(A \times B) - (C \times D) = \{(0, 1), (1, 1)\}$ while $(A - C) \times (B - D) = \{(0, 1)\}$. $\square$

Chapter II

Topological Spaces and Continuous Functions

2 Topological Spaces

This section has no exercises.

3 Basis for a Topology

Exercise 3.1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution. Consider the set $B = \bigcup U$. Since B is the union of open sets, it is open. We claim that $A = B$. Indeed, let $a \in A$. Then $a \in U$ for some open set U. But then $a \in \bigcup U = B$ so $A \subseteq B$. Now let $b \in B$. That is, $b \in U$ for some $U \subset A$. Then $b \in A$ so $B \subseteq A$, proving that $A = B$. Therefore, A is open. $\square$

Exercise 3.2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.