1 Basis for a Topology

Exercise 1.1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution. Consider the set $B = \bigcup U$. Since B is the union of open sets, it is open. We claim that $A = B$. Indeed, let $a \in A$. Then $a \in U$ for some open set U. But then $a \in \bigcup U = B$ so $A \subseteq B$. Now let $b \in B$. That is, $b \in U$ for some $U \subset A$. Then $b \in A$ so $B \subseteq A$, proving that $A = B$. Therefore, A is open. $\square$

Exercise 1.2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.