Eigenvalues and eigenvectors

LATEST SUBMISSION GRADE

90%

1. This assessment will test your ability to apply your knowledge of eigenvalues and eigenvectors to some special cases.

1/1 po

Use the following code blocks to assist you in this quiz. They calculate eigenvectors and eigenvalues respectively:

```
# Eigenvalues
M = np.array([[0.1, 0.1, 0.1, 0.7],
              [0.7,0.1, 0.1, 0.1],
              [0.1, 0.7, 0.1, 0.1],
              [0.1, 0.1, 0.7, 0.1]])
vals, vecs = np.linalg.eig(M)
vals
```

Run

Reset

```
# Eigenvectors - Note, the eigenvectors are the columns of the output.
M = np.array([[0.1, 0.1, 0.1, 0.7],
              [0.7,0.1, 0.1, 0.1],
              [0.1, 0.7, 0.1, 0.1],
              [0.1, 0.1, 0.7, 0.1]])
vals, vecs = np.linalg.eig(M)
vecs

```

Run

Reset

To practice, select all eigenvectors of the matrix, $A = \begin{bmatrix} 4 & -5 & 6 \\ 7 & -8 & 6 \\ 3/2 & -1/2 & -2 \end{bmatrix}$.

☐ $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

☐ $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$

☐ $\begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$

☑ $\begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$

✓ **Correct**

This is one of the eigenvectors.

☐ None of the other options.

☑ $\begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}$

✓ **Correct**

This is one of the eigenvectors. Note eigenvectors are only defined upto a scale factor.

☑ $\begin{bmatrix} -2/\sqrt{9} \\ -2/\sqrt{9} \\ 1/\sqrt{9} \end{bmatrix}$

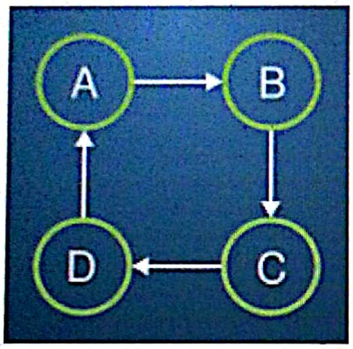
✓ **Correct**

This is one of the eigenvectors. Note eigenvectors are only defined upto a scale factor.

2. Recall from the *PageRank* notebook, that in PageRank, we care about the eigenvector of the link matrix, L, that has eigenvalue 1, and that we can find this using *power iteration method* as this will be the largest eigenvalue.

1 / 1 poi

PageRank can sometimes get into trouble if closed-loop structures appear. A simplified example might look like this,



With link matrix, $L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Use the calculator in Q1 to check the eigenvalues and vectors for this system.

What might be going wrong? Select all that apply.

- [ ] Some of the eigenvectors are complex.
- [ ] The system is too small.
- [x] Because of the loop, *Procrastinating Pat*s that are browsing will go around in a cycle rather than settling on a webpage.

✓ **Correct**

If all sites started out populated equally, then the incoming pats would equal the outgoing, but in general the system will not converge to this result by applying power iteration.

- [x] Other eigenvalues are not small compared to 1, and so do not decay away with each power iteration.

☑ Other eigenvalues are not small compared to 1, and so do not decay away with each power iteration.

✓ **Correct**

The other eigenvectors have the same size as 1 (they are -1, i, $-i$)

☐ None of the other options.

3. The loop in the previous question is a situation that can be remedied by damping.

If we replace the link matrix with the damped, $L' = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.7 \\ 0.7 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.7 & 0.1 \end{bmatrix}$, how does this help?

☐ None of the other options.

☐ It makes the eigenvalue we want bigger.

☑ The other eigenvalues get smaller.

✓ **Correct**

So their eigenvectors will decay away on power iteration.

☐ The complex number disappear.

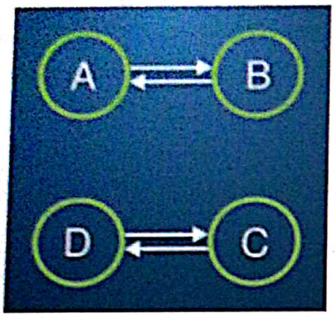
☑ There is now a probability to move to any website.

✓ **Correct**

This helps the power iteration settle down as it will spread out the distribution of Pats

4. Another issue that may come up, is if there are disconnected parts to the Internet. Take this example,

1 / 1 po



with link matrix, $L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

This form is known as block diagonal, as it can be split into square blocks along the main diagonal, i.e., $L = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, with $A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in this case.

What is happening in this system?

☑ There isn't a unique PageRank.

✓ **Correct**

The power iteration algorithm could settle to multiple values, depending on its starting conditions.

☐ The system has zero determinant.

☑ There are loops in the system.

✓ **Correct**

There are two loops of size 2. (A ⇄ B) and (C ⇄ D)

- [x] There are two eigenvalues of 1.

Correct
The eigensystem is degenerate. Any linear combination of eigenvectors with the same eigenvalue is also an eigenvector.

- [ ] None of the other options.

5. By similarly applying damping to the link matrix from the previous question. What happens now?

- [ ] There becomes two eigenvalues of 1.
- [ ] The negative eigenvalues disappear.
- [x] Damping does not help this system.

! This should not be selected
Adding damping will settle the system to a single value.

- [ ] The system settles into a single loop.
- [ ] None of the other options.

6. Given the matrix $A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$, calculate its characteristic polynomial.

- (●) $\lambda^2 - 2\lambda + \frac{1}{4}$
- ( ) $\lambda^2 - 2\lambda - \frac{1}{4}$
- ( ) $\lambda^2 + 2\lambda + \frac{1}{4}$
- ( ) $\lambda^2 + 2\lambda - \frac{1}{4}$

✓ **Correct**

Well done - this is indeed the characteristic polynomial of A.

7. By solving the characteristic polynomial above or otherwise, calculate the eigenvalues of the matrix $A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$.

- ( ) $\lambda_1 = 1 - \frac{\sqrt{5}}{2}, \lambda_2 = 1 + \frac{\sqrt{5}}{2}$
- (●) $\lambda_1 = 1 - \frac{\sqrt{3}}{2}, \lambda_2 = 1 + \frac{\sqrt{3}}{2}$
- ( ) $\lambda_1 = -1 - \frac{\sqrt{3}}{2}, \lambda_2 = -1 + \frac{\sqrt{3}}{2}$
- ( ) $\lambda_1 = -1 - \frac{\sqrt{5}}{2}, \lambda_2 = -1 + \frac{\sqrt{5}}{2}$

✓ **Correct**

Well done! These are the roots of the above characteristic polynomial, and hence these are the eigenvalues of A.

8. Select the two eigenvectors of the matrix $A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$.

8. Select the two eigenvectors of the matrix $A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$.

- ○ $\mathbf{v_1} = \begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}$
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- ◉ $\mathbf{v_1} = \begin{bmatrix} -1-\sqrt{3} \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1+\sqrt{3} \\ 1 \end{bmatrix}$

✓ **Correct**

These are the eigenvectors for the matrix A. They have the eigenvalues λ_1 and λ_2 respectively.

9. Form the matrix C whose left column is the vector $\mathbf{v_1}$ and whose right column is $\mathbf{v_2}$ from immediately above.

By calculating $D = C^{-1}AC$ or by using another method, find the diagonal matrix D.

- ◉ $\begin{bmatrix} 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 1-\frac{\sqrt{3}}{2} \end{bmatrix}$
- ○ $\begin{bmatrix} -1-\frac{\sqrt{3}}{2} & 0 \\ 0 & -1+\frac{\sqrt{3}}{2} \end{bmatrix}$
- ○ $\begin{bmatrix} 1-\frac{\sqrt{5}}{2} & 0 \\ 0 & 1+\frac{\sqrt{5}}{2} \end{bmatrix}$
- ○ $\begin{bmatrix} -1-\frac{\sqrt{5}}{2} & 0 \\ 0 & -1+\frac{\sqrt{5}}{2} \end{bmatrix}$

- (selected) $\begin{bmatrix} 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 1-\frac{\sqrt{3}}{2} \end{bmatrix}$
- $\begin{bmatrix} -1-\frac{\sqrt{3}}{2} & 0 \\ 0 & -1+\frac{\sqrt{3}}{2} \end{bmatrix}$
- $\begin{bmatrix} 1-\frac{\sqrt{5}}{2} & 0 \\ 0 & 1+\frac{\sqrt{5}}{2} \end{bmatrix}$
- $\begin{bmatrix} -1-\frac{\sqrt{5}}{2} & 0 \\ 0 & -1+\frac{\sqrt{5}}{2} \end{bmatrix}$

✓ **Correct**

Well done! Recall that when a matrix is transformed into its diagonal form, the entries along the diagonal are the eigenvalues of the matrix - this can save lots of calculation!

10. By using the diagonalisation above or otherwise, calculate A^2.

- (selected) $\begin{bmatrix} 11/4 & -2 \\ -1 & 3/4 \end{bmatrix}$
- $\begin{bmatrix} 11/4 & -1 \\ -2 & 3/4 \end{bmatrix}$
- $\begin{bmatrix} -11/4 & 1 \\ 2 & -3/4 \end{bmatrix}$
- $\begin{bmatrix} -11/4 & 2 \\ 1 & -3/4 \end{bmatrix}$

✓ **Correct**

Well done! In this particular case, calculating A^2 directly is probably easier - so always try to look for the method which solves the question with the least amount of pain possible!