EE447: Assignment: Numerical Integration of Singular Integrals

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1 Reading Portion

Chapter on integration from both books. Section on Gaussian Quadratures.

2 Programming Portion - Singular Integrals

We wish to compute the following integral:

$$J = \int_1^3 \frac{e^{-x}}{J_1\left(\sqrt{-x^2+4x-3}\right)} dx \tag{1}$$

where $J_1(x)$ is the Bessel funciton of the first kind of order 1. It is available in Python as `scipy.special.jv`. Note that $J_1(x) \approx x$ for small arguments. So the denominator goes to zero when the argument of J_1 goes to zero.

1. Graph the integrand of Eq. 1 in Python from $x = 1$ to $x = 3$.
2. Use **quad** and see what it does.
3. Use the open Romberg method to integrate. Study its convergence.
4. What kind of singularities are present and where? Use the transformation to convert the integral to a non-singular one and then apply qromo. Study the convergence.

3 Programming Portion - Gaussian Quadratures

1. Consider the integral in Eq. 1. Transform the integral to -1 to 1 by a suitable transformation.
2. Evaluate the integral using Gauss-Chebyshev quadratures. Note that for this method, no program is needed to compute x_j and w_j. They are given by:

$$\begin{aligned} x_j &= \cos\left(\frac{\pi\left(j-\frac{1}{2}\right)}{N}\right) \\ w_j &= \frac{\pi}{N} \end{aligned}$$

   Write the program in python to compute the integral for different N and plot the accuracy vs. N. To find error, compute for $N = 20$ and assume that is exact.

We wish to compute the second integral **in the Romberg assignment:**

$$I_1 = \int_0^1 J_\nu^2(ku)u\,du \tag{2}$$

$$I_2 = \int_1^\infty K_\nu^2(gu)u\,du \tag{3}$$

where $k = \kappa a = 2.7$ and $g = \gamma a = 1.2$. There we assumed a large cutoff and did the integral. Here we will do Gaussian quadratures.

1. Define functions corresponding to both integrands, `f1` and `f2.`
2. Use **quad** and see the cost to evaluate both integrals to an accuracy of 10^{-12}.
3. Use Gauss-Legendre to evaluate I_1 and Gauss-Laguerre to evaluate I_2. Note that in the second integral you will have to transform the variables so that the integrand looks like $f(u)\exp(-u)$. The asymptotic behaviour of K_ν is given by

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \gg \nu \tag{4}$$

   Additionally note that the $1/\sqrt{x}$ behaviour does not need to be captured since it is meant for singular behaviour near $x = 0$. For large x, e^{-x} dominates.
4. Use Romberg to evaluate the first integral to the required accuracy.
5. Transform the infinite range to a finite range via $u = A\tan w$ and use Romberg to integrate the second integral. Do you need to use open or closed romberg? Why?
6. Compare the different methods and determine the best performing algorithm.