0.5
0.1, 0.5, 0.8

Lec - 13 Ex

1]

a]
$$\left.\begin{aligned} P(\theta = 0.2) &= 0.1 \\ P(\theta = 0.4) &= 0.1 \\ P(\theta = 0.6) &= 0.3 \\ P(\theta = 0.8) &= 0.5 \end{aligned}\right\} \leftarrow P(\theta) \Rightarrow \text{Prior.}$$

θ = Probability that coin flip will result in head.

$N_H = 10 \qquad N_T = 40$

Equation for Posterior is :-

$$P(\theta | y_1, y_2 \ldots y_{50}) \propto P(\theta) . \theta^{N_H} (1-\theta)^{N_T}$$

For $\theta = 0.2$

$$\begin{aligned} P(\theta = 0.2 | y_1, \ldots y_{50}) &\propto P(\theta = 0.2)(0.2)^{10} . (0.8)^{40} \\ &\propto (0.1)(0.2)^{10} . (0.8)^{40} \\ &\propto 1.36 \times 10^{-12} \end{aligned}$$

For $\theta = 0.4$

$$\begin{aligned} P(\theta = 0.4 | y_1 \ldots y_{50}) &\propto P(\theta = 0.4)(0.4)^{10} . (0.6)^{40} \\ &\propto (0.1)(0.4)^{10} . (0.6)^{40} \\ &\propto 1.4 \times 10^{-14} \end{aligned}$$

For $\theta = 0.6$

$$P(\theta = 0.6 | y_1 \ldots y_{50}) \propto P(\theta = 0.6) \cdot (0.6)^{10} \cdot (0.4)^{40}$$
$$\propto (0.3)(0.6)^{10} \cdot (0.4)^{40}$$
$$\propto 2.19 \times 10^{-19}$$

For $\theta = 0.8$

$$P(\theta = 0.8 | y_1 \ldots y_{50}) \propto P(\theta = 0.8)(0.8)^{10} \cdot (0.2)^{40}$$
$$\propto (0.5)(0.8)^{10} \cdot (0.2)^{40}$$
$$\propto 5.9 \times 10^{-30}$$

Now,

$$P(\theta = 0.2 | y_1 \ldots y_{50}) + P(\theta = 0.4 | y_1 \ldots y_{50}) + P(\theta = 0.6 | y_1 \ldots y_{50}) + P(\theta = 0.8 | y_{\ldots})$$

$$= 1.374 \times 10^{-12}$$

$$\Rightarrow P(\theta = 0.2 | y_1 \ldots y_{50}) = 0.9898 \approx 0.99$$
$$P(\theta = 0.4 | y_1 \ldots y_{50}) = 0.0101 \approx 0.01$$
$$P(\theta = 0.6 | y_1 \ldots y_{50}) \approx 0$$
$$P(\theta = 0.8 | y_1 \ldots y_{50}) \approx 0$$

The resultant Posterior is not in agreement with the Prior belief because of the Data. 80% of the data was suggesting that a flip will result in Tail.
Hence the posterior is highly influenced by the data.

b] Since θ is continuous; for a flat prior $\Rightarrow P(\theta) = k$

$$\int_0^1 P(\theta)\, d\theta = 1$$

$$[k\theta]_0^1 = 1$$

$$k = 1$$

$$\therefore \; P(\theta) = 1$$

$$P(\theta \mid y_1 \dots y_{50}) \propto P(\theta)\, \theta^{N_H} (1-\theta)^{N_T}$$
$$\propto \theta^{N_H} (1-\theta)^{N_T}$$

$$P(\theta \mid y_1 \dots y_{50}) = \frac{1}{c}\, \theta^{N_H} (1-\theta)^{N_T}$$

$$= \frac{1}{c} (\theta)^{10} \cdot (1-\theta)^{40}$$

This can also be seen as a beta Distribution with 'c' as normalizing constant.

$\therefore \quad \alpha = N_H + 1 \quad , \beta = N_T + 1$

$\alpha = 11 \quad , \beta = 41$

The above posterior equation can now be written as $Beta(N_H+1, N_T+1) \Rightarrow Beta(11, 41)$

~~[illegible]~~

$$p(\theta \mid y_1 \ldots y_{50}) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\Rightarrow \underbrace{\frac{\Gamma(52)}{\Gamma(11)\Gamma(41)}}_{\text{constant}} \left[\theta^{10} \cdot (1-\theta)^{40}\right]$$

c] Scene 2:- $N_H = 40 \quad N_T = 10$

$$p(\theta \mid y_1 \ldots y_{50} \ldots y_{100}) \propto p(\theta)\, p(y_1 \ldots y_{50} \ldots y_{100} \mid \theta)$$

$$\propto p(\theta)\, p(y_1 \ldots y_{50} \mid \theta)\, p(y_{51} \ldots y_{100} \mid \theta)$$

$$\propto p(\theta \mid y_1 \ldots y_{50})\, p(y_{51} \ldots y_{100} \mid \theta)$$

$$\Rightarrow \frac{\Gamma(52)}{\Gamma(11)\Gamma(41)} \left[\theta^{10} \cdot (1-\theta)^{40}\right] \left[\theta^{40} \cdot (1-\theta)^{10}\right]$$

$$\Rightarrow \text{Constant} \times \left[\theta^{50} \cdot (1-\theta)^{50}\right]$$

$\therefore P(\theta|y_1 \ldots y_{50} \ldots y_{100}) = \text{Constant} \times \left[\theta^{50}(1-\theta)^{50}\right]$

d] Taking Prior $P(\theta) = 1$ ($y|\alpha d$): $N_H = 50$, $N_T = 50$

$$P(\theta|y_1 \ldots y_{100}) \propto P(\theta)\, \theta^{50} \cdot (1-\theta)^{50}$$

$$= \frac{1}{C}\, \theta^{50} \cdot (1-\theta)^{50}$$

Taking this as Beta Distribution;

$\alpha = 51$

$\beta = 51$

$$= \frac{\Gamma(102)}{\Gamma(51)\Gamma(51)}\, \theta^{50} \cdot (1-\theta)^{50}$$

$$P(\theta|y_1 \ldots y_{100}) = \text{Constant} \cdot \left[\theta^{50} \cdot (1-\theta)^{50}\right]$$

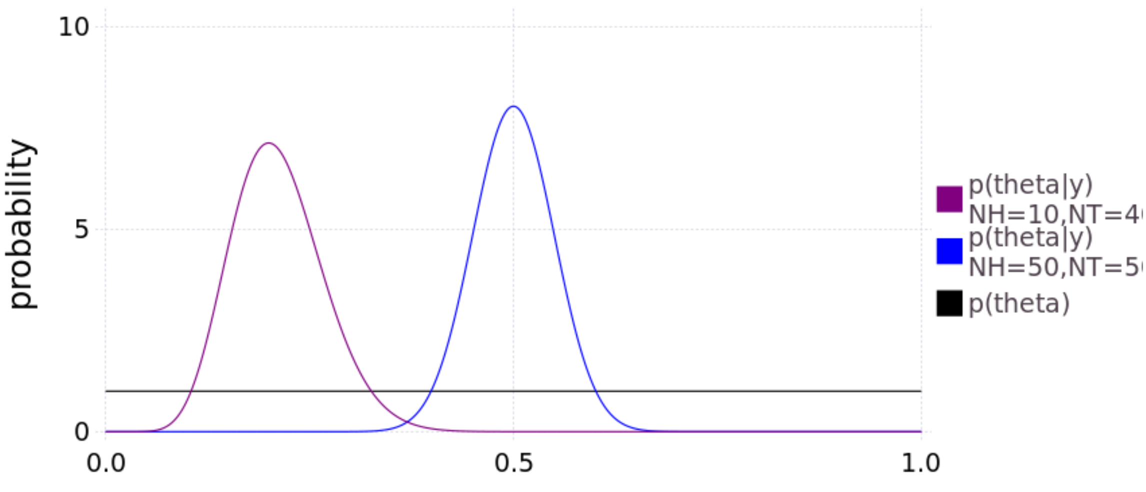
$\therefore$ The Functional form of posterior Distribution of (c) & (d) are same with different Constants.

In [13]:

```
using Distributions, Gadfly, Cairo;
```

In [21]:

```
white_panel = Theme(
panel_fill=colorant"white",
default_color=colorant"purple",bar_spacing=3mm,
major_label_font_size=18pt,
minor_label_font_size=14pt,
key_title_font_size = 18pt,
key_label_font_size = 14pt,
major_label_color=colorant"black",
minor_label_color=colorant"black"
);
x = collect(0:0.001:1);
prior = ones(length(x));
d1 = Beta(11,41);
d3 = Beta(51,51);
posterior1 = pdf.(d1,x);
posterior3 = pdf.(d3,x);
myplot = plot(
layer(x=x,y=posterior1,Geom.line,Theme(default_color=colorant"purple")),
layer(x=x,y=posterior3,Geom.line,Theme(default_color=colorant"blue")),
layer(x=x,y=prior,Geom.line,Theme(default_color=colorant"black")),
Coord.Cartesian(xmin=0, xmax=1,ymax=10.2), Guide.ylabel("probability"),
Guide.xlabel(""),
Guide.manual_color_key("", ["p(theta|y)
NH=10,NT=40",
"p(theta|y)
NH=50,NT=50", "p(theta)"],
["purple","blue","black"]), white_panel);
draw(PNG(9inch, 4inch),myplot)
```



Out[21]:

```
false
```

The functional form of posterior distribution of part c and d are equivalent with different normalizing constant. So we calculate beta distribution as Beta(51,51)

In [ ]:

```

```