Date / /
Page

Exercise 15

Q.1. $\text{Gamma}(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}$

Standard form: $P(y \mid \theta) = h(y)\exp\left[\eta(\theta)^{T} T(y) - \Psi(\theta)\right]$

$\text{Gamma}(\alpha, \beta) = \frac{1}{\lambda} \cdot \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot \lambda^{\alpha} e^{-\lambda/\beta}$

Taking log and exponents throughout...

$$\frac{1}{\lambda} \exp\left[-\log\Gamma(\alpha) - \alpha\log\beta + \alpha\log\lambda - \frac{\lambda}{\beta}\right]$$

comparing above eqn. with standard form:

$h(y) = 1/\lambda$

$\eta(\theta)^{T} = [\alpha, 1/\beta]$

$T(y) = [\log\lambda, -\lambda]$

$\Psi(\theta) = [-\log\Gamma(\alpha) - \alpha\log\beta]$

Scanned with CamScanner

2]

a] Mixture of priors :- $[p_1(\theta), p_2(\theta), P_k(\theta)]$

$$[\text{Gamma}(8, 0.1), \text{Gamma}(16, 0.1), \text{Gamma}(24, 0.1)]$$

Mixing probability :- $[\pi_1, \pi_2, \dots \pi_k]$

$$[0.33, 0.33, 0.34]$$

$$\Theta \sim \sum_{i=1}^{K} \pi_i \, p_i(\theta)$$

$$\sim \left[(0.33) \times \text{Gamma}(8, 0.1) + (0.33) \times \text{Gamma}(16, 0.1) + (0.34) \times \text{Gamma}(24, 0.1) \right]$$

b] $p(y) = \int p(y|\theta)\, p(\theta)$

Now $p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta)$

$$= \prod_{i=1}^{10} \frac{\lambda^{y_i}}{y_i!} e^{-\lambda}$$

$$\boxed{p(y_i|\theta) = \frac{\lambda^{\sum_i^n y_i} e^{-n\lambda}}{\prod_i^n y_i!}}$$

$$p_1(y) = \int_\theta p(y|\theta)\, p_1(\theta)$$

$$= \int_\lambda \frac{\lambda^{\sum_i^n y_i} e^{-n\lambda}}{\prod_i^n y_i!} \cdot \text{Gamma}(\alpha_0, \beta_0)$$

$$= \int_\lambda \frac{\lambda^{\sum_i^n y_i} e^{-n\lambda}}{\prod_i y_i!} \cdot \left[\frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha_0 - 1} \cdot e^{-\lambda/\beta_0}\right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int \frac{\lambda^{(\sum_i^n y_i + \alpha_0 - 1)} \cdot e^{-n\lambda - \lambda/\beta_0}}{\prod_i^n y_i!}$$

$$p_i(y) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}\prod_i^n y_i!}\int_\lambda \lambda^{\left(\sum_i^n y_i+\alpha_0-1\right)}\cdot e^{\left(-n\lambda-\lambda/\beta_0\right)}$$

Comparing the integral part with functional form of Gamma distribution.

$$\alpha - 1 = \sum_i^n y_i + \alpha_0 - 1$$

$$\frac{-\lambda}{\beta} = -n\lambda - \frac{\lambda}{\beta_0}$$

$$\frac{1}{\beta} = n + \frac{1}{\beta_0}$$

$$\beta = \frac{1}{n + 1/\beta_0} = \frac{\beta_0}{n\beta_0 + 1}$$

$$= \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}\prod_i^n y_i!}\left[\frac{1}{\Gamma\left(\sum_i^n y_i+\alpha_0\right) * \left(\frac{\beta_0}{n\beta_0+1}\right)^{\left(\sum y_i+\alpha_0\right)}}\right]$$

$$= \frac{\Gamma\left(\sum_i^n y_i + \alpha_0\right)\left(\frac{\beta_0}{n\beta_0+1}\right)^{\sum y_i+\alpha}}{\Gamma(\alpha)\,\beta^{\alpha}\prod_i^n y_i!}$$

Scanned with CamScanner

Posterior

c) $p(\theta/y) = \pi_1' \,\text{Gamma}\left(\alpha_1 + \Sigma y_i, \frac{\beta_1}{\beta_1 n + 1}\right)$

$+ \pi_2' \,\text{Gamma}\left(\alpha_2 + \Sigma y_i, \frac{\beta_2}{\beta_2 n + r}\right)$

$+ \pi_3' \,\text{Gamma}\left(\alpha_3 + \Sigma y_i, \frac{\beta_3}{\beta_3 n + 1}\right)$

d) $p(\theta/y) = \pi_1 \,\text{Gamma}\left(8 + \Sigma y_i, \frac{0.1}{0.1n + 1}\right)$

$+ \pi_2 \,\text{Gamma}\left(16 + \Sigma y_i, \frac{0.1}{0.1n + 1}\right)$

$+ \pi_3 \,\text{Gamma}\left(24 + \Sigma y_i, \frac{0.1}{0.1n + 1}\right)$

$$\pi_1' = \frac{0.33 p_1(y)}{0.33 p_1(y) + 0.33 p_2(y) + 0.34 p_3(y)}$$

$$\pi_2' = \frac{0.33 p_2(y)}{0.33 p_1(y) + 0.33 p_2(y) + 0.34 p_3(y)}$$

$$\pi_3' = \frac{0.34 p_3(y)}{0.33 p_1(y) + 0.33 p_2(y) + 0.34 p_3(y)}$$

Scanned with CamScanner

$$p_i(y) = \frac{\Gamma(\Sigma y_i + \alpha)\left(\frac{\beta}{\beta n+1}\right)^{\Sigma y_i + \alpha}}{\Pi y_i \left(\Gamma(\alpha)\ \beta^{\alpha}\right)} \times$$

$$\text{Gamma}\left(\Sigma y_i + \alpha; \frac{\beta}{\beta n+1}\right)$$

$$p_1(y) = \frac{\Gamma(\Sigma y_i + \alpha)\left(\frac{\beta_1}{\beta_1 n+1}\right)^{\Sigma y_i + \alpha} \text{Gamma}\left(\Sigma y_i + \alpha_1, \frac{\beta_1}{\beta_1 n+1}\right)}{\Pi y_i! \left(\Gamma(\alpha_1)\ \beta_1^{\alpha}\right)}$$

$$p_2(y) = \frac{\Gamma(\Sigma y_n + \alpha_2)\left(\frac{\beta_2}{\beta_2 n+1}\right)^{\Sigma y_2 + \alpha} \text{Gamma}\left(\Sigma y_n + \alpha, \frac{\beta_2}{\beta_2 n+1}\right)}{\Pi y_n! \left(\Gamma(\alpha_2)\ \beta_2^{\alpha}\right)}$$

$$p_3(y) = \frac{\Gamma(\Sigma y_n + \alpha_3)\left(\frac{\beta_3}{\beta_3 n+1}\right)^{\Sigma y_3 + \alpha} \text{Gamma}\left(\Sigma y_n + \alpha_3, \frac{\beta_3}{\beta_3 n+1}\right)}{\Pi y_n! \left(\Gamma(\alpha_3)\ \beta_3^{\alpha}\right)}$$

Scanned with CamScanner