CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 3 (Introduction to Probability)

Gowtham Atluri

9/4/2018

Questions

1. Seven friends decide to order pizzas by telephone from Pizza4U based on a flyer pushed through their letterbox. Pizza4U has only 4 kinds of pizza, and each person chooses a pizza independently. Bob phones Pizza4U and places the combined pizza order. How many different combined orders are possible? **[2 points]**

   Solution:

   Here the objective is to compute number of combined orders not sequence of pizza selection by the 7 friends. For example, *2 pizzas of type A, 2 pizzas of type B, 2 pizzas of type C, and 2 pizzas of type D* is a combined order. Whereas, ABABCCD and AABBCCD are two sequences of pizza selection by the 7 friends that result in the same above combined order.

   The problem of computing the number of combined orders can be treated as dividing the 7 pizzas into four groups. We need three separators to find four groups as shown here: $oo|oo|oo|o$

   So there are 7 pizzas and 3 separators. The number of combined orders is the number of ways we can place the 3 separators. That is, $\binom{10}{3} = 120$

2. For each of the probability tables (i) list the random variable, (ii) domain of the random variable, and (iii) determine if it is a probability distribution: **[3 points]**

   (a)

$$\begin{array}{lcl} p(a=1) & & 0.6 \\ p(a=2) & = & 0.04 \\ p(a=3) & & 0.34 \end{array}$$

   Solution:

   (i) a is the random variable
   (ii) domain is {1,2,3}
   (iii) As the probability values do not addup to 1, this is not a probability distribution.

   (b)

$$\begin{array}{lcl} p(x=a) & & 0.46 \\ p(x=b) & = & 0.16 \\ p(x=c) & & 0.38 \end{array}$$

   Solution:

   (i) x is the random variable

   (ii) domain is {a,b,c}

   (iii) As the probability values are all between 0 and 1, and addup to 1, this is a probability distribution.

   (iv)

$$\begin{array}{lcl} p(z=aa) & & 0.4 \\ p(z=bc) & = & 0.1 \\ p(z=de) & & 0.5 \end{array}$$

Solution:

(i) z is the random variable
(ii) domain is {aa,bc,de}
(iii) As the probability values are all between 0 and 1, and addup to 1, this is a probability distribution.

3. There are two boxes. *Box 1* contains three red and five white balls and *box 2* contains two red and five white balls. A box is chosen at random $p(box = 1) = p(box = 2) = 0.5$ and a ball chosen at random from this box turns out to be red. What is the posterior probability that the red ball came from box 1? **[3 points]**

   Solution: Given probabilities: $p(box = 1) = p(box = 2) = 0.5$

   Conditional probabilities: $p(ball = red|box = 1) = 3/8$ $p(ball = white|box = 1) = 5/8$ $p(ball = red|box = 2) = 2/7$ $p(ball = white|box = 2) = 5/7$

   We need to determine $p(box = 1|ball = red)$

   Using conditional probability

$$p(box = 1|ball = red) = \frac{p(box = 1, ball = red)}{p(ball = red)} = \frac{p(ball = red|box = 1)p(box = 1)}{p(ball = red)}$$

   $p(ball = red|box = 1)$ and $p(box = 1)$ are already available. We need to determine $p(ball = red)$

   Using marginalization and conditional probability

$$p(ball = red) = \sum_{box=1,2} p(box, ball = red) = \sum_{box=1,2} p(ball = red|box)p(box)$$

$$= p(ball = red|box = 1)p(box = 1) + p(ball = red|box = 2)p(box = 2) = (\frac{3}{8})(\frac{1}{2}) + (\frac{2}{7})(\frac{1}{2})$$

$$p(box = 1|ball = red) = \frac{p(ball = red|box = 1)p(box = 1)}{p(ball = red)} = \frac{(\frac{3}{8})(\frac{1}{2})}{(\frac{3}{8})(\frac{1}{2}) + (\frac{2}{7})(\frac{1}{2})} = \frac{\frac{3}{8}}{\frac{3}{8} + \frac{2}{7}} = 0.5675$$

4. Answer the following questions: **[5+3+3 points]**

   (i) Prove that

$$p(x|z) = \sum_y p(x|y, z)p(y|z) = \sum_{y,w} p(x|w, y, z)p(w|y, z)p(y|z)$$

   Solution

$$\begin{aligned}
p(x|z) &= \sum_y p(x, y|z) && \text{(using marginalization)} \quad (1)\\
&= \sum_y \frac{p(x, y, z)}{p(z)} && \text{(using conditional prob.)} \quad (2)\\
&= \sum_y \frac{p(x|y, z)p(y, z)}{p(z)} && \text{(using conditional prob.)} \quad (3)\\
&= \sum_y \frac{p(x|y, z)p(y|z)p(z)}{p(z)} && \text{(using conditional prob.)} \quad (4)\\
&= \sum_y p(x|y, z)p(y|z) && (5)
\end{aligned}$$

$$
\begin{aligned}
p(x|z) &= \sum_y p(x,y|z) && \text{(using marginalization)} && (6)\\
&= \sum_y \frac{p(x,y,z)}{p(z)} && \text{(using conditional prob.)} && (7)\\
&= \sum_{y,w} \frac{p(x,y,z,w)}{p(z)} && \text{(using marginalization)} && (8)\\
&= \sum_{y,w} \frac{p(x|y,z,w)p(y,z,w)}{p(z)} && \text{(using conditional prob.)} && (9)\\
&= \sum_{y,w} \frac{p(x|y,z,w)p(w|y,z)p(y,z)}{p(z)} && \text{(using conditional prob.)} && (10)\\
&= \sum_{y,w} \frac{p(x|y,z,w)p(w|y,z)p(y|z)p(z)}{p(z)} && \text{(using conditional prob.)} && (11)\\
&= \sum_{y,w} p(x|y,z,w)p(w|y,z)p(y|z)p(z) && && (12)
\end{aligned}
$$

(ii) When y is unconditionally idependent of z, prove that

$$p(x|z) = \sum_y p(x|y,z)p(y)$$

Solution

$$
\begin{aligned}
p(x|z) &= \sum_y p(x|y,z)p(y|z) && && (13)\\
&= \sum_y p(x|y,z)p(y) && \text{(as } p(y|z) = p(z) \text{ since } y,z \text{ are independent)} && (14)
\end{aligned}
$$

(iii) When x is conditionally idependent of y, given z, prove that

$$p(x|z) = \sum_y p(x|z)p(y|z)$$

Solution

$$
\begin{aligned}
p(x|z) &= \sum_y p(x|y,z)p(y|z) && && (15)\\
&= \sum_y p(x|z)p(y) && \text{(as } p(x|y,z) = p(x|z) \text{ since } x,y \text{ are independent, given } z) && (16)
\end{aligned}
$$

5. The joint distribution $p(A, B)$ is as follows: (rows: A, columns: B) **[4 points]**

	$b1$	$b2$	$b3$
$a1$	0.42	0.05	0.02
$a2$	0.02	0.02	0.01
$a3$	0.02	0.02	0.42

Compute the marginal probabilities $p(A)$ and $p(B)$.

Solution:

$$p(A) = \begin{array}{c} 0.49 \\ 0.05 \\ 0.46 \end{array} \qquad p(B) = \begin{array}{c} 0.46 \\ 0.09 \\ 0.45 \end{array}$$

6. Based on the above joint distribution, determine if A and B are independent. **[2 points]**

   Solution: We need to compute $p(A)p(B)$ and compare with $p(AB)$. Even if one of the joint probabilities do not match with those in $p(A)p(B)$, A and B are not independent.

$$p(A)p(B) = \begin{array}{cccc} & b1 & b2 & b3 \\ a1 & 0.23 & 0.023 & 0.21 \\ a2 & 0.023 & 0.005 & 0.049 \\ a3 & 0.21 & 0.04 & 0.21 \end{array}$$

   As none of these probabilities match with the joint probability, A and B are not independent.

7. **[Bonus question]** 1. Seven friends decide to order pizzas by telephone from Pizza4U based on a flyer pushed through their letterbox. Pizza4U has only 4 kinds of pizza, and each person chooses a pizza independently. Bob phones Pizza4U and places the combined pizza order, simply stating how many pizzas of each kind are required. Unfortunately, the precise order is lost, so the chef makes seven randomly chosen pizzas and then passes them to the delivery boy. What is the probability that the delivery boy has the right order?