Unit-5

① Halting problem → Undecidable

PCP: Post Correspondence Problem (PCP)

- Undecidable Problem by Emil Leon.

- This is a Puzzle Problem
- PCP mean that no algorithm set of domino's can solve all instances of the problem.

$$\left[\frac{ab}{aba}\right] \left[\frac{ba}{abb}\right] \left[\frac{b}{ab}\right] \left[\frac{abb}{b}\right] \left[\frac{a}{bab}\right]$$

- we want to rearrange this.
- we can do this within the dominos or we can include some repetitions.

$$\left[\frac{ab}{aba}\right] \left[\frac{ba}{abb}\right] \left[\frac{a}{bab}\right]$$

combine numerators, denominators

abbaa ⟺ aba abb bab

strings formed by combining numerators = strings formed by combining denominators.

numerator	denominator
x	y
ab	aba
ba	abb
b	ab
abb	b
a	bab

Take xy pair

strings formed by X = strings formed by Y

Domino Problem ⇒ Key in understanding undecidability & complexity theory

Take domino which is having match in numerator & denominator

$\left[\frac{ab}{aba}\right]$

choose matching domino

$\left[\frac{\cancel{ab}}{\cancel{ab}a}\right]$

now choose domino that has a in the numerator.

$\left[\frac{\cancel{ab}}{\cancel{ab}\cancel{a}}\right]$ $\left[\frac{\cancel{a}}{bab}\right]$

$\left[\frac{}{ba\cancel{b}}\right]$ $\left[\frac{\cancel{ba}}{\cancel{ab}b}\right]$ $\left[\frac{\cancel{b}}{ab}\right]$

$\left[\frac{\cancel{abb}}{b}\right]$ $\left[\frac{\cancel{abb}}{\cancel{b}}\right]$ $\left[\frac{\cancel{b}}{\cancel{ab}}\right]$

$\left[\frac{ab\cancel{b}}{\cancel{b}}\right]$

- we can have n.no of solutions
- choosing domino is user choice.
- dominos can be repeated

Combine strings in numerator & denominator

ababab abbabb babb

⇕

ababab abbabbb abb

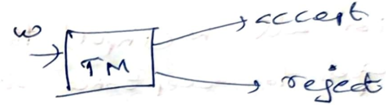
} both are equal.

Sometimes, we may get into undefined loop recursive.

That type of PCP does not have solution for it.

Recursive language → always halt.

Both acc & rej are properly defined

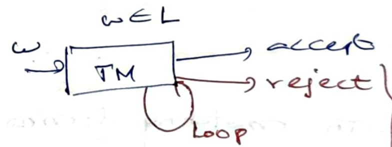


These languages are said to be Recursive lang.

→ decidable languages

→ there exists a TM that will always halt and decide whether a given string belongs to the lang (or) not

Recursive Enumerable lang



— acceptance alone will be properly defined.

→ semi decidable lang.

① TM will Accept a string if it belongs to the lang.

② May never halt if the string does not belong to the lang.

PCP is Undecidable (Proof)

Acceptance problem of TM:

- For all 'w' (input) the TM goes to acceptance state or not.

Modified PCP:

- Convert this acceptance problem of TM to PCP

MPCP-

Starts with matching domino "#/#"

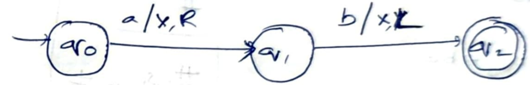
$\left[\frac{\#}{\#}\right]$ - one domino in the given PCP is always matched.

- This is the one modification we do in PCP.

7 steps

consider eg. prob of TM & convert to PCP.

TM: [check starting two symbols are ab]



$TM = (Q, \Sigma, \delta, \Gamma, q_0, B, F)$

$\Sigma = \{a, b\}, \quad \Gamma = \{a, b, B, x\}$

i/p $w = aba$

To do:

Convert this TM design to PCP.

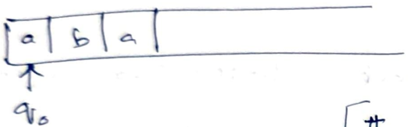
consider w=aba:

PCP ⇒ dominos.

If PCP have solution ⇒ TM also have a solution.

Step 1:

Create domino out of this transition.



$q_0 aba$

General form.

$$\left[\frac{\#}{\# q_0 w \#}\right]$$

Matching domino.

Starting domino:

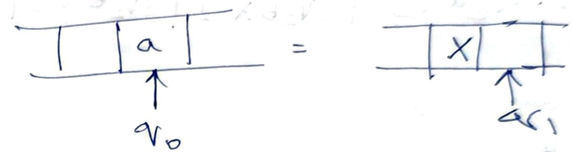
$$\left[\frac{\#}{\# q_0 aba \#}\right]$$

⇒ first domino as per modified version of PCP.

Step 2. Take all transitions with right move.

$$\delta(q_0, a) = (q_1, X, R)$$

domino here describes ID.



$q_0 a = X q_1$

$$\left[\frac{q_0 a}{X q_1}\right]$$

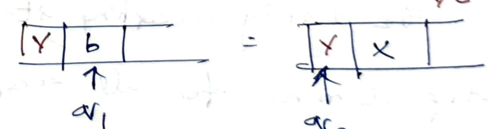
numerator contains → before transition

denominator contains → after transition

Step 3: leftside Transition

$$\delta(q_1, b) = (q_2, X, L).$$

$Y \in \Gamma$



$\Gamma = \{a, b, B, X\}$

$Y q_1 b = q_2 Y X.$

For all possible Tape symbols with domino

$$\left[\frac{a\,q_1\,b}{q_2\,a\,X}\right] \left[\frac{b\,q_1\,b}{q_2\,b\,X}\right] \left[\frac{B\,q_1\,b}{q_2\,B\,X}\right] \left[\frac{X\,q_1\,b}{q_2\,X\,X}\right]$$

Step 4:-

For all possible tape symbols, create domino.

$\Gamma = \{a, b, B, X\}$

$$\left[\frac{a}{a}\right] \left[\frac{b}{a}\right] \left[\frac{B}{B}\right] \left[\frac{X}{X}\right]$$

Step 5:-

For all possible $\Gamma = \{a, b, B, X\}$

$q_2 \rightarrow$ only accepting state.

creating domino for all accepting state.

$$\left[\frac{a\,q_2}{q_2}\right] \left[\frac{q_2\,a}{q_2}\right] \left[\frac{b\,q_2}{q_2}\right] \left[\frac{q_2\,b}{q_2}\right]$$

$$\left[\frac{B\,q_2}{q_2}\right] \left[\frac{q_2\,B}{q_2}\right] \left[\frac{X\,q_2}{q_2}\right] \left[\frac{q_2\,X}{q_2}\right]$$

↑

Domino's created after accepting state

Step 6:

Create 2 diff domino's

$$\left[\frac{\#}{\#}\right] \left[\frac{\#}{B\#}\right]$$

← whenever you have Blank state create domino like this.

Step 7.

$$\left[\frac{q_2\,\#\,\#}{\#}\right]$$

- Final domino that we will create.

There are the 7 steps that are used to convert given TM problem to PCP problem.

How to solve the domino? & find solution for the PCP problem.

Solution.

$$\left[\frac{\#}{\# q_0 a b b \#}\right] \left[\frac{q_0 a}{x q_1}\right]$$

$$\left[\frac{b}{b}\right] \left[\frac{a}{a}\right] \left[\frac{\#}{\#}\right]$$

round 1

round 2:

$$\left[\frac{\#}{\# x q_1 b a \#}\right]$$

use (step 3)

$$\left[\frac{\#}{\# x q_1 b a \#}\right] \left[\frac{x q_1 b}{q_2 x x}\right] \left[\frac{a}{a}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\#}{\# q_2 x x a \#}\right]$$

(use step 5 here, because we reached q_2 state final state)

$$\left[\frac{}{q_2 x x a \#}\right] \left[\frac{q_2 x}{q_2}\right] \left[\frac{x}{x}\right] \left[\frac{a}{a}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{}{q_2 x a \#}\right] \left[\frac{q_2 x}{q_2}\right] \left[\frac{a}{a}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\#}{\# q_2 a \#}\right] \left[\frac{q_2 a}{q_2}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\#}{\# q_2 \#}\right]$$

(use step 7)

$$\left[\frac{\cancel{\#}}{\#\cancel{q_2}\#}\right] \cdot \left[\frac{\cancel{q_2}\#\,\cancel{\#}}{\cancel{\#}}\right]$$

∴ No remaining element left.

If acceptance of TM is decidable the PCP is also decidable.

~~PCP~~.

$\boxed{\text{TM problem is undecidable} \Rightarrow \text{so problem of PCP is undecidable.}}$

PCP

① $\left[\frac{T}{TH}\right] \left[\frac{H}{TH}\right] \left[\frac{HTH}{HT}\right] \left[\frac{TT}{T}\right]$

This may end in recessive infinite loop

$\left[\frac{1}{111}\right]$ ① $\left[\frac{10111}{10}\right]$ ② $\left[\frac{10}{0}\right]$ ③ → cannot be choosen because there is no matching element.

$\left[\frac{10111}{10}\right]$

② $\left[\frac{\cancel{1}\cancel{0}\cancel{111}}{\cancel{1}\cancel{0}}\right]$ ① $\left[\frac{\cancel{1}}{111}\right]$ $\left[\frac{①0111}{\cancel{1}0}\right]$ → no match for this, so go for ①.

① $\left[\frac{\cancel{1}}{\cancel{111}}\right]$ $\left[\frac{1}{\cancel{111}}\right]$ 3 $\left[\frac{\cancel{10}}{\cancel{0}}\right]$

Solution: when domino's arranged in ~~Rain~~ sequence

2 1 1 3

$\left[\frac{\cancel{1011}}{\cancel{10}}\right]$ $\left[\frac{\cancel{1}}{\cancel{11}}\right]$ $\left[\frac{\cancel{1}}{\cancel{111}}\right]$ $\left[\frac{\cancel{10}}{\cancel{0}}\right]$

(2) (1) (1) (3)

The string formed, on numerator

$$\boxed{\begin{aligned} X &= 101111110 \\ Y &= 101111110 \end{aligned}}$$

- A domino's can be repeated any no. of time
- sometimes domino's can be excluded to find solution.

$\left[\frac{T}{TH}\right]$ $\left[\frac{H}{TH}\right]$ $\left[\frac{HTH}{HT}\right]$ $\left[\frac{TT}{T}\right]$

Solution:-

1 2 1 3 3 2 (4 domino is not used)

4 2

$\left[\frac{1}{111}\right]$ $\left[\frac{10111}{10}\right]$ $\left[\frac{10}{0}\right]$

	X	Y	
x_1	1	111	y_1
x_2	10111	10	y_2
x_3	10	0	y_3

$\left[\frac{ab}{abb}\right]$ $\left[\frac{ba}{aa}\right]$ $\left[\frac{ab}{bb}\right]$

(1) (2) (3)

(1) => start - only one domino having matching element

(1) $\left[\frac{\cancel{ab}}{\cancel{abb}}\right]$ (2) $\left[\frac{\cancel{ba}}{\cancel{aa}}\right]$ (3) $\left[\frac{\cancel{ab}}{\cancel{bb}}\right]$ (2) $\left[\frac{\cancel{ba}}{\cancel{aa}}\right]$ (3) $\left[\frac{\cancel{ab}}{\cancel{bb}}\right]$ (2) $\left[\frac{\cancel{b}a}{\cancel{a}a}\right]$

1 2 3 2 3 (2) → entering into loop, when you give such type of i/ps to Automata systems it may not halt.

Hence we call this PCP as Undecidable.

Undecidable

TM $\xrightarrow{w \in L}$ accept

$\xrightarrow{w \notin L}$ may halt
may not halt
enter into loop
reject it
} no prediction.

Ex: Eg. of MPCP

A = [aa, bb, abb]

B = [aab, ba, b]

Sol: 1213

aabbaaabb
aabbaaabb

MPCP: same as PCP, but the solution must start with the first pair in the list.

Construct a TM to identify the employee ID of an Institution. The ID'S are formed in such a way that it starts with a digit followed by a letter and if is then followed by zero or more letter or digit. 1, 2, and 3 are the only allowed digits that can be included, and 'R' is the only letter allowed. (ie $\Sigma = \{1, 2, 3, R\}$)

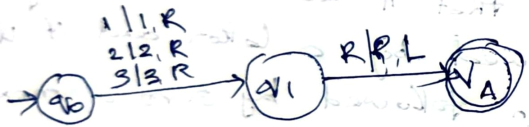
(1) Construct a TM to identify the valid ID'S

② construct the MPCP instance for the above TM

③ check whether the string '2R13' has solution or accepted using MPCP instance.

$\Gamma = \{1, 2, 3, R, B\}$.

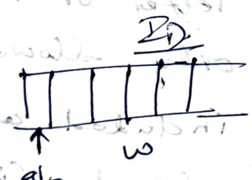
Step ①:



MPCP

Step ②.

Step 1:



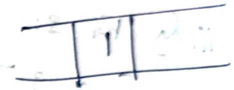
$$\left[\frac{\#}{\# q_0 w \#}\right]$$

ID ⇒ $q_0 w$

$\boxed{w = 2R13}$ — i/p

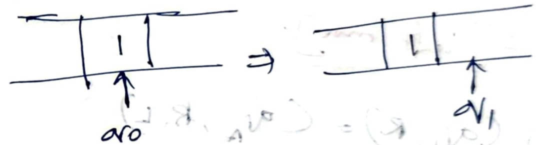
$$\left[\frac{\#}{\# q_0 2R13 \#}\right]$$

Step 2: (right move)



$\delta(q_0, 1) \Rightarrow (\delta(q_1, 1, R)$

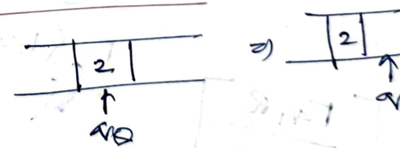
For transition 1:



$\boxed{q_0 1 \Rightarrow 1 q_1}$ ID

$$\left[\frac{q_0 1}{1 q_1}\right]$$ — ①

For transition 2:



$q_0 2 \rightarrow 2 q_1$ (ID)

$\delta(q_0, 2) \Rightarrow (q_1, 2, R)$

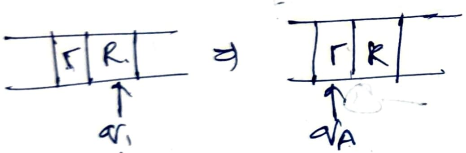
$\left[\frac{q_0 2}{2 q_1}\right] \rightarrow$ 2

IIrly For 3:-

$\left[\frac{q_0 3}{3 q_1}\right]$ — 3

Step 3 [left move].

$\delta(q_1, R) = (q_A, R, L)$



$q_A \Gamma R$.

$\Gamma q_1 R \longrightarrow q_A \Gamma R$

Before Transfer

After transfer

$\Gamma = \{1, 2, 3, R, B\}$

$\left[\frac{\Gamma q_1 R}{q_A \Gamma R}\right]$

$\left[\frac{1 q_1 R}{q_A 1 R}\right] \left[\frac{2 q_1 R}{q_A 2 R}\right] \left[\frac{3 q_1 R}{q_A 3 R}\right]$

$\left[\frac{R q_1 R}{q_A R R}\right] \left[\frac{B q_1 R}{q_A B R}\right]$

Step 4: $\Gamma = \{1, 2, 3, R, B\}$

For all tape symbols.

$\left[\frac{1}{1}\right] \left[\frac{2}{2}\right] \left[\frac{3}{3}\right] \left[\frac{R}{R}\right] \left[\frac{B}{B}\right]$

Step 5:- For accepting state.

$\left[\frac{\Gamma q_A}{q_A}\right] \left[\frac{q_A \Gamma}{q_A}\right]$

$\left[\frac{1 q_A}{q_A}\right] \left[\frac{q_A 1}{q_A}\right] \left[\frac{2 q_A}{q_A}\right] \left[\frac{q_A 2}{q_A}\right]$

$\left[\frac{3 q_A}{q_A}\right] \left[\frac{q_A 3}{q_A}\right] \left[\frac{R q_A}{q_A}\right] \left[\frac{q_A R}{q_A}\right]$

$\left[\frac{B q_A}{q_A}\right] \left[\frac{q_A B}{q_A}\right]$

Step 6:

$$\left[\frac{\#}{\#}\right] \left[\frac{\#}{B\#}\right]$$

For all problems step 6 & 7 are same.

Step 7:

$$\left[\frac{q_A\#\#}{\#}\right]$$

Eg: w = 2R13

$$\left[\frac{\#}{\#q_0 2R13\#}\right] \left[\frac{q_0 2}{2q_1}\right] \left[\frac{R}{R}\right] \left[\frac{1}{1}\right] \left[\frac{3}{3}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{2q_1R13\#}\right] \left[\frac{2q_1R}{q_A2R}\right] \text{step 3} \left[\frac{1}{1}\right] \left[\frac{3}{3}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{q_A2R13\#}\right] \left[\frac{q_A2}{q_A}\right] \left[\frac{R}{R}\right] \left[\frac{1}{1}\right] \left[\frac{3}{3}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{q_AR13\#}\right] \left[\frac{q_AR}{q_A}\right] \left[\frac{1}{1}\right] \left[\frac{3}{3}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{q_A13\#}\right] \left[\frac{q_A1}{q_A}\right] \left[\frac{3}{3}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{q_A3\#}\right] \left[\frac{q_A3}{q_A}\right] \left[\frac{\#}{\#}\right]$$

$$\left[\frac{\quad}{q_A\#}\right] \left[\frac{q_A\#\#}{\#}\right]$$

TM → accepting criteria

PCP → also enters into accepting state

PCP

$A = w_1, w_2 \dots w_n$
$B = x_1, x_2 \dots x_n$ — consider 2 list of equal length.

$$\boxed{w_1 w_2 \dots w_n = x_1 x_2 \dots x_n}$$

PCP has solution if $w_1 w_2 \dots w_n = x_1 x_2 \dots x_n$

Eg:

	A	B
1	1	111
2	10111	10
3	10	0

match

$\boxed{2 \ 1 \ 1 \ 3}$

10111 1 1 1 0
10111 1 1 1 0 — solution obtained.

Eg 2:

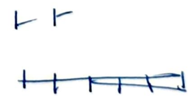
1	10	101
2	011	11
3	101	011

1 3 3 ..

1 0 1 0 1 1 0 1
1 0 1 0 1 1 0 1 1 1 ....

No solution.

Eg



MPCP:

Given list A, & B, of K strings

A = w1, w2 .. wK

B = x1, x2 .. xK does there exist a sequence of integers i1, i2 ... ir such that

w1 wi1 wi2 ... wir = x1 xi1 xi2 .. xir

A sol is required to start with the first string on each list.

Binary Encoding - Description of TM.

Universal TM:

- capable of simulating other turing machine.

$TM = \{ Q, \Sigma, \Gamma, \delta, q_0, B, F \}$

has to encode.

moves

L - 1

R - 11

0 → separator for each element.

$$TM = \{ \{ \overset{1}{q_0}, \overset{11}{q_{accept}} \}, \overset{00\ 10\ 11}{\{0, 1\}}, \{ \overset{1}{0}, \overset{11}{1}, \overset{111}{B} \},$$

$$\delta(q_0, 0) = (q_0, 0, R), \ \delta(q_0, 1) = (q_0, 1, R)$$

$$\delta(q_0, B) = (q_{accept}, 1, L), \ q_0, B, q_{accept} \}.$$

TM ⟶ enters Accept state

UTM ⟶ also "

$$\cancel{TM = \{ \underbrace{101100}_{Q} \underbrace{1011}_{\Sigma} 00 \underbrace{1011}_{} \underbrace{0111}_{\Gamma} 0010 }$$

Each and every element is represented in terms of no. of bytes.

$TM = \langle \underbrace{101100}_{Q}\ \underbrace{1011}_{\Sigma}\ 00\underbrace{1011011100}_{\Gamma}\ \underbrace{1010101011\,0}_{\delta}$

$1011010110110\cancel{1}0101110110110\,1$

$001001110011\ , 10 \rangle$

$\underbrace{1}_{q_0}\ \underbrace{111}_{B}\ \underbrace{11}_{q_{acc}}$

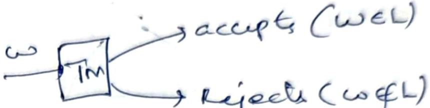
$\langle M, w \rangle$ – i/p, – machine – UTM representation

This Binary encoding is used in UTM.

UTM – which can make other TM to work inside.

UTM – can simulate any other machine.

Properties of Recursive language



Accept & reject state are properly defined. we call those lang as recursive lang TM accept or reject

Union,
Intersection,
Complement

① Union of 2 RL;

(L_1 & L_2) Both Languages are recursive union of both also recursive

L_1 – palindrom
L_2 – start with a