Parametric Curve Parameter Estimation using L1 Optimization

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***Abstract*—This work estimates the unknown parameters (θ, M, X) of a given parametric curve using observed points sampled over $6 \le t \le 60$. The curve is modeled as a rotated and translated version of a latent oscillatory signal of the form $(t, e^{M|t|}\sin(0.3t))$. Initial estimates for θ, M, and X are obtained using PCA-based orientation inference, exponential envelope regression, and mean-alignment respectively. The estimates are refined by minimizing the L1 distance between observed and predicted points. Final results: $\theta = 28.119°$, $M = 0.02138$, $X = 54.9111$, with L1 error $= 37865.11$.**

I. Introduction

We are given a set of n points (x_i, y_i) that lie on a parametric curve defined by:

$$x(t) = t\cos\theta - e^{M|t|}\sin(0.3t)\sin\theta + X, \tag{1}$$

$$y(t) = 42 + t\sin\theta + e^{M|t|}\sin(0.3t)\cos\theta, \tag{2}$$

where θ is a rotation angle, M controls the exponential growth of the sinusoidal term, and X is a horizontal shift.

We are given that:

$$0° < \theta < 50°, \quad -0.05 < M < 0.05, \quad 0 < X < 100,$$

and the sampled parameter range is:

$$6 \le t \le 60.$$

The objective is to **estimate** (θ, M, X) such that the model curve is consistent with the given data while minimizing:

$$L_1 = \sum_{i=1}^{n} \left(|x_i - \hat{x}_i| + |y_i - \hat{y}_i|\right).$$

II. Methodology

A. Geometric Interpretation

The model can be viewed as a rotation and translation of a latent signal:

$$(t,\ s(t)), \quad s(t) = e^{M|t|}\sin(0.3t).$$

Thus, the primary direction of the curve encodes θ.

B. Estimating θ (Rotation) via PCA

Perform PCA on the observed (x, y) coordinates. The direction of maximum variance approximates the main axis of the curve. The angle of this axis yields an initial estimate θ_0 [1].

C. Estimating X (Translation)

Because $\sin(0.3t)$ oscillates around zero, the average horizontal alignment gives:

$$X_0 \approx \bar{x} - \bar{t}\cos(\theta_0).$$

D. Estimating M (Envelope Growth)

Dividing out the oscillation $\sin(0.3t)$ yields:

$$\log|s(t)| \approx M|t| + C,$$

so M is estimated via linear regression of $\log(|s|)$ vs $|t|$ [3].

E. Refinement via L1 Optimization

The final step refines (θ, M, X) by minimizing:

$$\min_{\theta, M, X} \sum_{i=1}^{n} \left(|x_i - \hat{x}_i| + |y_i - \hat{y}_i|\right),$$

using Powell's derivative-free optimization algorithm [2], [4].

III. Results

- $\theta = 28.119275°$
- $M = 0.0213837$
- $X = 54.911105$
- Final L_1 distance $= 37865.11$

These values accurately reconstruct the given data curve with minimal error.

IV. Conclusion

By treating the curve as a rotated, shifted, and amplitude-modulated sinusoid, we combine geometric reasoning (PCA), signal processing (envelope extraction), and numerical optimization (L1 minimization) to robustly estimate the parameters (θ, M, X).

References

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