Probability and Computing - HW1

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1 Problem1

Deduce the expected run time of Quick-Sort, using conditional expectation.

Note that $C(N)$ be an compare times of length N list. Then $C(N) = C(Y-1) + C(N-Y) + N, p = i$. Then we apply expectation of both side.

$$\begin{aligned} E(C(N)) &= \sum_Y E(C(Y-1)|Y=i) + \sum_Y E(C(N-Y)|Y=i) + N - 1 \\ &= \frac{1}{N} \sum_Y E(C(i-1)) + E(C(N-i)) + N - 1 \\ &= \frac{2}{N} \sum_Y E(C(i-1)) + N - 1 \end{aligned} \tag{1}$$

We are guessing that $E(C(N)) \leq ci \ln i$, so

$$\begin{aligned} E(C(N)) &\leq (N-1) + \frac{2}{N} \sum_{i=1}^{N} (ci \ln i) \\ &\leq (N-1) + \frac{2}{N} \int_1^N (cx \ln x) dx \\ &\leq (N-1) + \frac{2}{N}((c/2)N^2 \ln N - cN^2/4 + c/4) \\ &\leq cN \ln N, for c = 2 \end{aligned} \tag{2}$$

$E(C(N)) \leq 2N \ln N$.

2 Problem2

Why does the probability that the second door hides a car changes after Monty Hall opens the third door? An intuitive interpretation is preferred.

Because the Monty Hall knew the result of the door. In term of the information theory, Monty Hall gave some information to the player. So if he change his choose will increase the probability of choosing a car.

Firstly, random choose a door has 1/3 probability. After open a door, the door remained has probability of 1/2 is bigger then the previous 1/3.

3 Problem3

The proof of Jensen's Inequality in the text book assumes that the convex function f is twice differentiable. Present a convex function that is not differentiable, and show that Jensen's Inequality holds even if the convex function is not differentiable. Only discrete probability is considered.

Consider the function of $f(X) = |X|$, which is a convex function but is not differentiable (at $X = 0$ is undefined).

$$\begin{aligned} E[f(X)] &= \sum_{X \in \Omega} f(X)Pr(X) \\ &= \sum_{X>=0} XPr(X) + \sum_{X<0}(-X)Pr(X) \end{aligned} \tag{3}$$

$$\begin{aligned} f[E(X)] &= f(\sum_{X>=0} XPr(X) + \sum_{X<0} XPr(X)) \\ &= |\sum_{X>=0} XPr(X) + \sum_{X<0} XPr(X)| \\ &\leq |\sum_{X>=0} XPr(X)| + |\sum_{X<0} XPr(X)| \\ &= \sum_{X>=0} XPr(X) + \sum_{X<0}(-X)Pr(X) \\ &= E[f(X)] \end{aligned} \tag{4}$$

So $E[f(X)] \geq f(E[X])$.

4 Problem4

Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Send the result to advanced_algorithm@163.com in a string $s_1 s_2 \cdots s_i \cdots s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.

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