Home Work of Week 4

Deadline: 9:00am, November 25 (Tuesday), 2014

1. Consider the probability that every bin receives exactly one ball when n balls are thrown randomly into n bins.

    - Give an upper bound on this probability using the Poisson approximation.
    - Determine the exact probability of this event.

2. The following problem models a simple distributed system wherein agents contend for resources but *back off* in the face of contention. Balls represent agents, and bins represent resources.

    The system evolves over rounds. Every round, balls are thrown independently and uniformly at random into n bins. Any ball that lands in a bin by itself is served and removed from consideration. The remaining balls are thrown again in the next round. We begin with n balls in the first round, and we will finish when every ball is served.

    - If there are b balls at the start of a round, what is the expected number of balls at the start of the next round?
    - Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\ln \ln n)$ rounds. (Hint: If x_j is the expected number of balls left after j rounds, show and use that $x_{j+1} \leq x_j^2/n$.)

3. Let $X_i^{(m)}$ be the number of balls in bin i when m balls are independently and uniformly thrown at random into n bins, and $Y_i^{(m)}, 1 \leq i \leq n$, are independent Poisson random variables each having expectation m/n. Assume that f is a nonnegative function.

    - Prove that if $E[f(X_1^{(m)}, ...X_n^{(m)})]$ is monotonically increasing in m, then $E[f(X_1^{(m)}, ...X_n^{(m)})] \leq 2E[f(Y_1^{(m)}, ...Y_n^{(m)})]$. (Hint: Show that $E[f(Y_1^{(m)}, ...Y_n^{(m)})] \geq E[f(X_1^{(m)}, ...X_n^{(m)})]Pr(\sum Y_i^{(m)} \geq m)$ and $Pr(\sum Y_i^{(m)} \geq m) \geq 1/2$.)
    - (**Bonus score 5 points**) If $E[f(X_1^{(m)}, ...X_n^{(m)})]$ is monotonically decreasing in m, then $E[f(X_1^{(m)}, ...X_n^{(m)})] \leq 2E[f(Y_1^{(m)}, ...Y_n^{(m)})]$.

4. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.