Probability and Computing - HW3

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1 Problem1

Suppose that balls are thrown randomly into n bins. Show, for some constant c_1, that if there are $c_1\sqrt{n}$ balls then the probability that no two land in the same bin is at most $1/e$. Similarly, show for some constant c_2 (and sufficiently large n) that, if there are $c_2\sqrt{n}$ balls, then the probability that no two land in the same bin is at least $1/2$. Make these constants as close to optimum as possible. Hint: you may need the fact that $e^{-x} \geq 1 - x$ and $e^{-x-x^2} \leq 1 - x$ for $x \leq 1/2$.

- Find c_1 The probability that max load is 1 is

$$\begin{aligned} Pr(MaxLoad = 1) &= (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{c_1\sqrt{n}}{n}) \\ &\leq \prod_{i=1}^{c_1\sqrt{n}-1} e^{-\frac{i}{n}} \\ &= e^{\frac{c_1\sqrt{n} - c_1^2 n}{2n}} \\ &\leq e^{\frac{c_1 - c_1^2}{2}} \end{aligned} \tag{1}$$

  Then we set $\frac{c_1 - c_1^2}{2} \leq -1$, and get $c_1 \geq 2$.

- The probability that max load is 1 is

$$\begin{aligned} Pr(MaxLoad = 1) &= (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{c_2\sqrt{n}}{n}) \\ &\geq \prod_{i=1}^{c_2\sqrt{n}-1} e^{-\frac{i}{n} - \frac{i^2}{n^2}} \\ &= e^{-\frac{c_2\sqrt{n}(c_2\sqrt{n})}{2n} - \frac{(c_2\sqrt{n}-1)c_2\sqrt{n}(2c_2\sqrt{n}-1)}{6n^2}} \\ &\geq e^{-\frac{c_2^2}{2} - \frac{2c_2^3 n\sqrt{n}}{6n^2}} \\ &\geq e^{-\frac{c_2^2}{2} - \frac{c_2^3}{3}} \\ &\geq 1 - \frac{c_2^2}{2} - \frac{c_2^3}{3} \end{aligned} \tag{2}$$

Then we set $1-\frac{c_2^2}{2}-\frac{c_2^3}{3}\geq\frac{1}{2}$, and get $0\leq c_2\leq 0.8$.

2 Problem2

Let X be a Poisson random variable with mean μ, representing the number of errors on a page of this book. Each error is independently a grammatical error with probability p and a spelling error with probability $1-p$. If Y and Z are random variables representing the numbers of grammatical and spelling errors (respectively) on a page of this book, Prove that Y and Z are Poisson random variables with means $p\mu$ and $(1-p)\mu$, respectively. Also, prove that Y and Z are independent.

We know $X\ Poisson(\mu)$. So

$$Pr(X=k)=e^{-\mu}\frac{\mu^k}{k!} \tag{3}$$

$$\begin{aligned}Pr(Y=k)&=\sum_{i=k}^{\infty}Pr(X=i)\left(\begin{array}{c}k\\i\end{array}\right)p^k(1-p)^{i-k}\\&=\frac{e^{-\mu}p^k\mu^k}{k!}\sum_{i=k}^{\infty}\frac{\mu^{i-k}(1-p)^{i-k}}{(i-k)!}\\&=\frac{e^{-\mu}p^k\mu^k e^{\mu(1-p)}}{k!}\\&=e^{-\mu p}\frac{(\mu p)^k}{k!}\end{aligned} \tag{4}$$

So $Y\ Poisson(p\mu)$, and for the similar reason $Z\ Poisson((1-p)\mu)$.

Because

$$\begin{aligned}Pr(Y=k_1,Z=k_2)&=Pr(X=k_1+k_2)\left(\begin{array}{c}k_1\\(k_1+k_2)\end{array}\right)p^{k_1}(1-p)^{k_2}\\&=e^{-\mu}\frac{\mu^{k_1+k_2}}{(k_1+k_2)!}\frac{(k_1+k_2)!}{k_1!k_2!}p^{k_1}(1-p)^{k_2}\\&=e^{-p\mu}\frac{(p\mu)^(k_1)}{k_1!}*e^{-(1-p)\mu}\frac{((1-p)\mu)^{k_2}}{k_2!}\\&=Pr(Y=k_1)*Pr(Z=k_2)\end{aligned} \tag{5}$$

Y and Z are independent.

3 Problem3

Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to $n-1$. There are log_2n players. Each player randomly chooses a starting location l uniformly from $[0,n-1]$ and then places one ball in each of the bins numbered l

mode n, $l+1$ mode $n, \ldots l + n/\log_2 n - 1$ mode n. Argue that the maximum load in this case is only $O(\log\log n/\log\log\log n)$ with probability that approaches 1 as $n \to \infty$.

Have no idea.

4 Problem4

Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 \cdots s_i \cdots s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.

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