Home Work of Week 7

Deadline: 9:00am, December 16 (Tuesday), 2014

1. Consider a graph in $G_{n,p}$ with $p = c\frac{\ln n}{n}$. Use the second moment method to prove that if $c < 1$ then, for any constant $\epsilon > 0$ and for n sufficiently large, the graph has isolated vertices with probability at least $1 - \epsilon$.

2. Prove the Asymmetric Lovasz Local Lemma: Let $\mathbb{A} = \{A_1, ...A_n\}$ be a set of finite events over a probability space, and for each $1 \leq i \leq n$, $\Gamma(A_i) \subseteq \mathbb{A}$ is such that A_i is mutually independent of all events not in $\Gamma(A_i)$. If $\sum_{A_j \in \Gamma(A_i)} Pr(A_j) \leq 1/4$ for all i, then $Pr(\bigwedge_{i=1}^{n} \overline{A_i}) \geq \prod_{i=1}^{n}(1 - 2Pr(A_i)) > 0$. [Hint: let $x(A_i) = 2Pr(A_i)$ and use the general Lovasz Local Lemma.]

3. Given $\beta > 0$, a vertex-coloring of a graph G is said to be β-frugal if (i) each pair of adjacent vertices has different colors, and (ii) no vertex has β neighbors that have the same color. Prove that if G has maximum degree $\Delta \geq \beta^{\beta}$ with $\beta \geq 2$, then G has a β-frugal coloring with $16\Delta^{1+1/\beta}$ colors. [Hint: you may want to define two types of events corresponding to the two conditions of being β-frugal. Then the result in question 1 can be used.]

4. Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different. You may want to let $A_{u,v,c}$ be the event that u and v are both colored with color c and then consider the family of such events.

5. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.