# Probability and Computing - HW2

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## 1 Problem1

Prove the following extensions of the Chernoff bound. Let $X = \sum_{i=1}^{n} X_i$, where the $X_i$ are independent 0-1 random variables. Let $\mu = E[X]$. Choose any $\mu_L$ and $\mu_H$ such that $\mu_L \leq \mu \leq \mu_H$. Then, for any $\delta > 0$, $Pr(X \geq (1+\delta)\mu_H) \leq (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})\mu_H$. Similarly, for any $0 < \delta < 1$, $Pr(X \leq (1-\delta)\mu_L) \leq (\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}})\mu_L$.

Apply Markov's inequality, for any $t > 0$ we have

$$\begin{aligned} Pr(X \geq (1+\delta)\mu_H) &= Pr(e^{tX} \geq e^{t(1+\delta)\mu_H}) \\ &\leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu_H}} \\ &\leq \frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu_H}} \\ &\leq \frac{e^{(e^t-1)\mu_H}}{e^{t(1+\delta)\mu_H}} \end{aligned} \tag{1}$$

For any $\delta > 0$, we can set $t = ln(1+\delta)$ then

$$\begin{aligned} Pr(X \geq (1+\delta)\mu_H) &\leq \frac{e^{(e^t-1)\mu_H}}{e^{t(1+\delta)\mu_H}} \\ &= (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})\mu_H \end{aligned} \tag{2}$$

Apply Markov's inequality, for any $t < 0$ we have

$$\begin{aligned} Pr(X \leq (1-\delta)\mu_L) &= Pr(e^{tX} \geq e^{t(1-\delta)\mu_L}) \\ &\leq \frac{E[e^{tX}]}{e^{t(1-\delta)\mu_L}} \\ &\leq \frac{e^{(e^t-1)\mu}}{e^{t(1-\delta)\mu_L}} \\ &\leq \frac{e^{(e^t-1)\mu_L}}{e^{t(1-\delta)\mu_L}} \end{aligned} \tag{3}$$

For any $0 < \delta < 1$, we can set $t = ln(1-\delta) < 0$ then

$$\begin{aligned} Pr(X \le (1-\delta)\mu_L) &\le \frac{e^{(e^t-1)\mu_L}}{e^{t(1-\delta)\mu_L}} \\ &= (\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}})\mu_L \end{aligned} \tag{4}$$

## 2 Problem2

Consider a collection $X_1, \cdots X_n$ of n independent integers chosen uniformly from the set $\{0,1,2\}$. Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$. Derive a Chernoff bound for $Pr(X \le (1-\delta)n)$.

First we need to calculate the generating function of $X_i$, $M_{X_i}(t)$

$$\begin{aligned} M_{X_i}(t) &= E[e^{tX_i}] \\ &= \frac{1}{3}(1 + e^t + e^{2t}) \end{aligned} \tag{5}$$

Then we take the product of the n generating functions to obtain

$$\begin{aligned} M_X(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= \prod_{i=1} n\frac{1}{3}(1 + e^t + e^{2t}) \\ &= (\frac{1}{3}(1 + e^t + e^{2t}))^n \end{aligned} \tag{6}$$

Apply Markov's inequality, for any $t < 0$ we have

$$\begin{aligned} Pr(X \le (1-\delta)n) &= Pr(e^{tX} \ge e^{t(1-\delta)n}) \\ &\le \frac{E[e^{tX}]}{e^{t(1-\delta)n}} \\ &= \frac{[(1 + e^t + e^{2t})/3]^n}{e^{t(1-\delta)n}} \end{aligned} \tag{7}$$

let $t = ln\delta < 0$

$$\begin{aligned} Pr(X \le (1-\delta)n) &\le \frac{[(1 + e^t + e^{2t})/3]^n}{e^{t(1-\delta)n}} \\ &= [\frac{(1 + \delta + 2\delta)}{3\delta^{(1-\delta)}}]^n \end{aligned} \tag{8}$$

## 3 Problem3

Let $X_1, \cdots X_n$ be independent Poisson trials such that $Pr(X_i = 1) = p_i$ and let $a_1, \cdots a_n$ be real numbers in $[0,1]$. let $X = \sum_{i=1}^{n} a_i X_i$ and $\mu = E[X]$. Then the following Chernoff bound holds: for any $\delta > 0$, $Pr(X \geq (1+\delta)\mu) \leq (\frac{e^\delta}{(1+\delta)^{(1+\delta)}})^\mu$. Also prove a similar bound for the probability $Pr(X \leq (1-\delta)\mu)$ for any $0 < \delta < 1$.

First we can see that $\mu = E[X] = \sum_{i=1}^n a_i p_i$, and use this to calculate $M_X(t)$. We can see that $M_{X_i}(t) = 1 + p_i(e_t - 1)$, so

$$\begin{aligned} E(e^{tX}) = M_X(t) &= \prod_{i=1}^n (a_i + a_i p_i(e^t - 1)) \\ &\leq \prod_{i=1}^n (1 + a_i p_i(e^t - 1)) \\ &\leq \prod_{i=1}^n e^{a_i p_i (e^t - 1)} \\ &= \exp \sum_{i=1}^n a_i p_i (e^t - 1) \\ &= e^{(e^t - 1)\mu} \end{aligned} \tag{9}$$

Then we get the same the product of the n generating functions as the independent Poisson trials has. So we get the same Chernoff bound. That is to say $Pr(X \geq (1+\delta)\mu) \leq (\frac{e^\delta}{(1+\delta)^{(1+\delta)}})^\mu$ for any $\delta > 0$ and $Pr(X \leq (1-\delta)\mu) \leq (\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}})^\mu$ for any $0 < \delta < 1$.

# 4 Problem4

Recall that a function $f$ is said to be convex if, for any $x_1, x_2$ and for $0 \leq \lambda \leq 1$, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$.

- Let $Z$ be a random variable that takes on a finite set of values in $[0, 1]$, and let $p = E[Z]$. Define the Bernoulli random variable $X$ by $Pr(X = 1) = p$ and $Pr(X = 0) = 1 - p$. Show that $E[f(Z)] \leq E[f(X)]$ for any convex function $f$.
- Use the fact that $f(x) = e^{tx}$ is convex for any fixed $t \geq 0$ to obtain a Chernoff-like bound for $Z$.

- First we have

$$\begin{aligned} \int_0^1 p_z dz = 1 \\ \int_0^1 z p_z dz = p \end{aligned} \tag{10}$$

  Then use the convexity of $f$

$$\begin{aligned} f(z) = f(0 * (1-z) + 1 * z) &\leq f(0) * (1-z) + f(1) * z \\ &= (f(1) - f(0)) * z + f(0) \end{aligned} \tag{11}$$

Now we evaluate $E[f(X)]$

$$E[f(X)] = f(0) * (1-p) + f(1) * p \tag{12}$$

And evaluate $E[f(Z)]$

$$\begin{aligned} E[f(Z)] &= \int_0^1 f(z) p_z dz \\ &\leq \int_0^1 (f(1) - f(0)) * z p_z + f(0) p_z dz \\ &= f(0) * (1-p) + f(1) * p = E[f(X)] \end{aligned} \tag{13}$$

So we have $E[f(Z)] \leq E[f(X)]$.

- First we have $E[e^{tX}] = (1-p) + pe^t$. Apply Markov's inequality, for any $t > 0$ we have

$$\begin{aligned} Pr(Z \geq a) = Pr(e^{tZ} \geq e^{ta}) &\leq \frac{E(e^{tZ})}{e^{ta}} \\ &\leq \frac{E(e^{tX})}{e^{ta}} = \frac{pe^t - p + 1}{e^{ta}} \end{aligned} \tag{14}$$

So we have $Pr(Z \geq a) \leq \frac{pe^t - p + 1}{e^{ta}}$ for any $t > 0$.

# 5 Problem5

Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 \cdots s_i \cdots s_{20}$, where $s_i$ is 1 if the $i^{th}$ trial gets Head, and otherwise is 0.

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