

Probability and Computing - HW3

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November 6, 2014

1 Problem1

Suppose that balls are thrown randomly into n bins. Show, for some constant c_1 , that if there are $c_1\sqrt{n}$ balls then the probability that no two land in the same bin is at most $1/e$. Similarly, show for some constant c_2 (and sufficiently large n) that, if there are $c_2\sqrt{n}$ balls, then the probability that no two land in the same bin is at least $1/2$. Make these constants as close to optimum as possible. Hint: you may need the fact that $e^{-x} \geq 1 - x$ and $e^{-x-x^2} \leq 1 - x$ for $x \leq 1/2$.

- Find c_1 The probability that max load is 1 is

$$\begin{aligned} Pr(MaxLoad = 1) &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{c_1\sqrt{n}}{n}\right) \\ &\leq \prod_{i=1}^{c_1\sqrt{n}-1} e^{-\frac{i}{n}} \\ &= e^{\frac{c_1\sqrt{n}-c_1^2n}{2n}} \\ &\leq e^{\frac{c_1-c_1^2}{2}} \end{aligned} \tag{1}$$

Then we set $\frac{c_1-c_1^2}{2} \leq -1$, and get $c_1 \geq 2$.

- The probability that max load is 1 is

$$\begin{aligned} Pr(MaxLoad = 1) &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{c_2\sqrt{n}}{n}\right) \\ &\geq \prod_{i=1}^{c_2\sqrt{n}-1} e^{-\frac{i}{n} - \frac{i^2}{n^2}} \\ &= e^{-\frac{c_2\sqrt{n}(c_2\sqrt{n}-1)}{2n} - \frac{(c_2\sqrt{n}-1)c_2\sqrt{n}(2c_2\sqrt{n}-1)}{6n^2}} \\ &\geq e^{-\frac{c_2^2}{2} - \frac{2c_2^3n\sqrt{n}}{6n^2}} \\ &\geq e^{-\frac{c_2^2}{2} - \frac{c_2^3}{3}} \\ &\geq 1 - \frac{c_2^2}{2} - \frac{c_2^3}{3} \end{aligned} \tag{2}$$

Then we set $1 - \frac{c_2^2}{2} - \frac{c_2^3}{3} \geq \frac{1}{2}$, and get $0 \leq c_2 \leq 0.8$.

2 Problem2

Let X be a Poisson random variable with mean μ , representing the number of errors on a page of this book. Each error is independently a grammatical error with probability p and a spelling error with probability $1 - p$. If Y and Z are random variables representing the numbers of grammatical and spelling errors (respectively) on a page of this book, Prove that Y and Z are Poisson random variables with means $p\mu$ and $(1 - p)\mu$, respectively. Also, prove that Y and Z are independent.

We know $X \text{ Poisson}(\mu)$. So

$$Pr(X = k) = e^{-\mu} \frac{\mu^k}{k!} \quad (3)$$

$$\begin{aligned} Pr(Y = k) &= \sum_{i=k}^{\infty} Pr(X = i) \binom{k}{i} p^k (1 - p)^{i-k} \\ &= \frac{e^{-\mu} p^k \mu^k}{k!} \sum_{i=k}^{\infty} \frac{\mu^{i-k} (1 - p)^{i-k}}{(i - k)!} \\ &= \frac{e^{-\mu} p^k \mu^k e^{\mu(1-p)}}{k!} \\ &= e^{-\mu p} \frac{(\mu p)^k}{k!} \end{aligned} \quad (4)$$

So $Y \text{ Poisson}(p\mu)$, and for the similar reason $Z \text{ Poisson}((1 - p)\mu)$.

Because

$$\begin{aligned} Pr(Y = k_1, Z = k_2) &= Pr(X = k_1 + k_2) \binom{k_1}{k_1 + k_2} p^{k_1} (1 - p)^{k_2} \\ &= e^{-\mu} \frac{\mu^{k_1 + k_2}}{(k_1 + k_2)!} \frac{(k_1 + k_2)!}{k_1! k_2!} p^{k_1} (1 - p)^{k_2} \\ &= e^{-p\mu} \frac{(p\mu)^{k_1}}{k_1!} * e^{-(1-p)\mu} \frac{((1 - p)\mu)^{k_2}}{k_2!} \\ &= Pr(Y = k_1) * Pr(Z = k_2) \end{aligned} \quad (5)$$

Y and Z are independent.

3 Problem3

Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to $n - 1$. There are $\log_2 n$ players. Each player randomly chooses a starting location l uniformly from $[0, n - 1]$ and then places one ball in each of the bins numbered l mode n , $l + 1$ mode n , \dots , $l + n / \log_2 n - 1$ mode n . Argue that the maximum load in this case is only $O(\log \log n / \log \log \log n)$ with probability that approaches 1 as $n \rightarrow \infty$.

4 Problem4

Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1s_2\cdots s_i\cdots s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.