Probabilistic Method and Random Graphs

Lecture 6. The Method of (Conditional) Expectation

Xingwu Liu

Institute of Computing Technology, Chinese Academy of Sciences, Beijing, China

[1] The slides are mainly based on Chapter 5 of Probability and Computing.

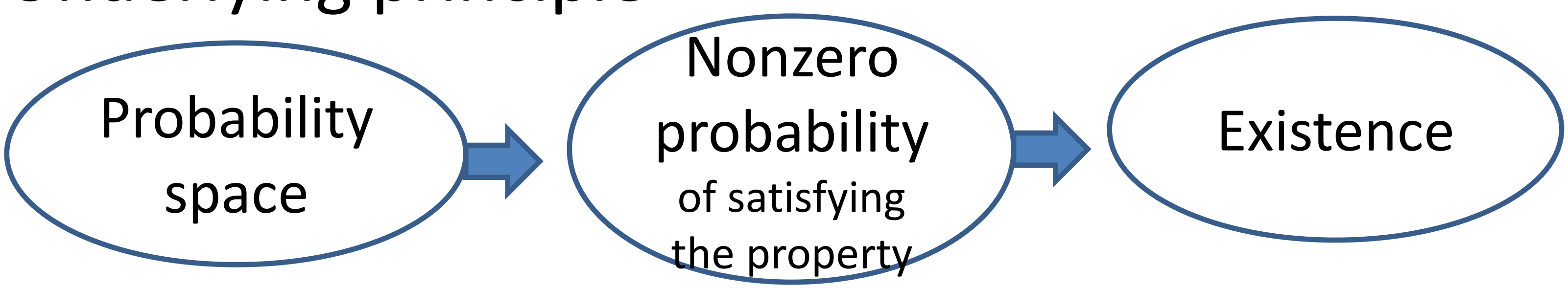
Comments, questions, or suggestions?

A Review of Lecture 5

- Bloom filter
  - Space/time efficient
  - Low false positivity
- Efficient Hamiltonian cycle algorithm
  - Valid for dense graphs
  - Reduced to coupon collection
  - Independent adjacency list model

A Review of Lecture 5

- **Probabilistic method**
  - **Proving the existence** of an object satisfying certain properties **without** constructing it
- Underlying principle



- Naturally lead to (randomized) algorithms
- Partially decide Ramsey number by counting

Is probabilistic method just counting?

- Expectation
- Variance
- Local lemma
- ……

First Moment method

- Use the expectation in probabilistic reasoning
- Two types of first moment method
  - Expectation argument

    $$\Pr(X \geq E[X]) > 0, \Pr(X \leq E[X]) > 0$$

  - Markov's inequality

    $$\mathrm{Prob}[X \geq a] \leq \mathrm{E}[X]/a$$

    $$\mathrm{Prob}[X \neq 0] = \mathrm{Prob}[X > 0] = \mathrm{Prob}[X \geq 1] \leq \mathrm{E}[X]$$

    (non-negative, integer-valued)

Expectation argument

- **MAX-3SAT**: Given a Boolean formula with 3 literals in each clause, find a truth assignment which satisfies the maximum number of clauses

$$(x_1 \vee x_2 \vee x_3) \wedge \ldots \wedge (\overline{x_1} \vee x_4 \vee \overline{x_3})$$

- Theorem: there is a truth assignment which satisfies $\frac{7}{8}$ fraction of the clauses

Proof

- Randomly assign truth values
- Define r.v X_i indicating if clause i is true
- $E[X_i] = \frac{7}{8}$
- $E[\sum X_i] = \frac{7}{8} n$

- Remark: probability of sampling a good truth assignment $\geq \frac{1}{n+8}$

Expectation argument

- Turan Theorem: Any graph G = (V, E) contains an independent set of size at least $\frac{|V|}{D+1}$, where $D = \frac{2|E|}{|V|}$
- Proof: Consider the following random process for constructing an independent set *S*:
  - Initialize *S* to be the empty set.
  - For each vertex *u* in *V* in random order:

    If no neighbors of *u* are in *S*, add *u* to *S*
  - Return *S*.

Clearly, S is an independent set, and each vertex u is selected with probability at least $\frac{1}{d(u)+1}$.

So, $E[|S|] = \sum \frac{1}{d(u)+1} \geq \frac{|V|}{D+1}$ due to convexity.

Remark: probability of sampling a good independent set is $\geq \frac{1}{2D|V|^2}$

Markov's Inequality

- Theorem: For any n-uniform hypergraph H with m edges, if $m < 2^{n-1}$ then the graph is 2-colorable.
- Proof: Color vertices of H with red/blue randomly with probability $1/2$.

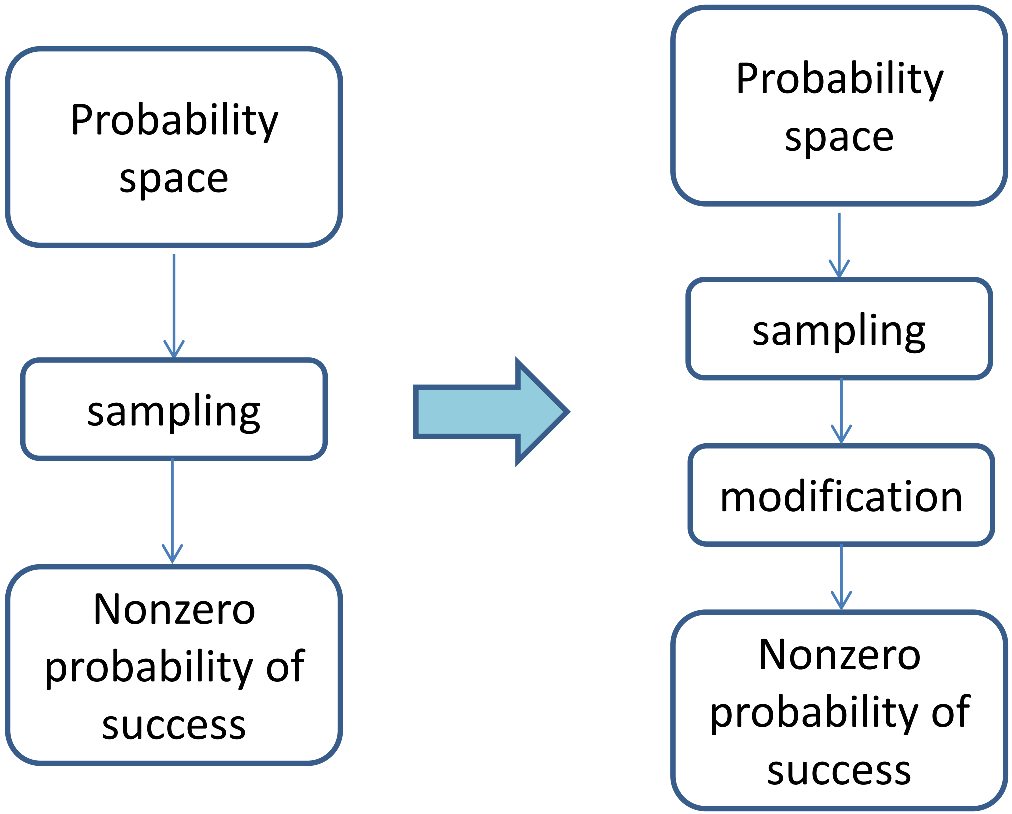
  X: the number of monochromatic edges.

  Then $\Pr(H$ is not properly colored)

  $$= \Pr(X>0) \leq E[X] = \frac{m}{2^{n-1}} < 1.$$
- Remark: probability of sampling a good coloring is $\geq 1 - \frac{m}{2^{n-1}}$

Sample and Modify

- Reflection on the paradigm



Big Chromatic Number and Big Girth

- Chromatic number vs local structure
  - Loose local structure $\Rightarrow$ small chro. number?
  - No!
- One of the first applications of prob. Method
- Theorem: for any integers $g, k > 0$, there is a graph with girth $\geq g$ and chromatic number $\geq k$
- We just prove the special case $g = 4$, i.e. triangle-free

Basic Idea of the Proof

- Pick a random graph *G* from $G_{n,p}$
- With high probability the max independent set size $\alpha(G)$ is small
  - $\alpha(G)\chi(G) \geq n$ implies that $\chi(G)$ is big
- With high probability *G* has few triangles
- Destroy the triangles

Proof: $\alpha(G)$ is small

- X: the number of independent sets of size $\frac{n}{2k}$
- $\Pr\left(\alpha(G) \geq \frac{n}{2k}\right) = \Pr(X \neq 0) \leq E[X]$

$$=\binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}}$$

$$< 2^n e^{-\frac{pn(n-2k)}{8k^2}}$$

- Small if n is large and p is not too small

Proof: triangles are few

- Y: the number of triangles of G
- $E[Y] = \binom{n}{3}p^3 < \frac{(np)^3}{6} < \frac{n}{6}$ if $p = n^{-2/3}$
- By Markov ineq., $\Pr\left(Y > \frac{n}{2}\right) \leq 1/3$
- Go back to $\Pr\left(\alpha(G) \geq \frac{n}{2k}\right) < 2^n e^{-\frac{pn(n-2k)}{8k^2}}$

$$< e^n \, e^{-\frac{pn^2}{16k^2}} = e^{n - n^{\frac{4}{3}}/16k^2} \quad \text{if } n > 4k$$

- If $n^{1/3} \geq 32k^2$, $\Pr\left(\alpha(G) \geq \frac{n}{2k}\right) < e^{-n}$<2/3

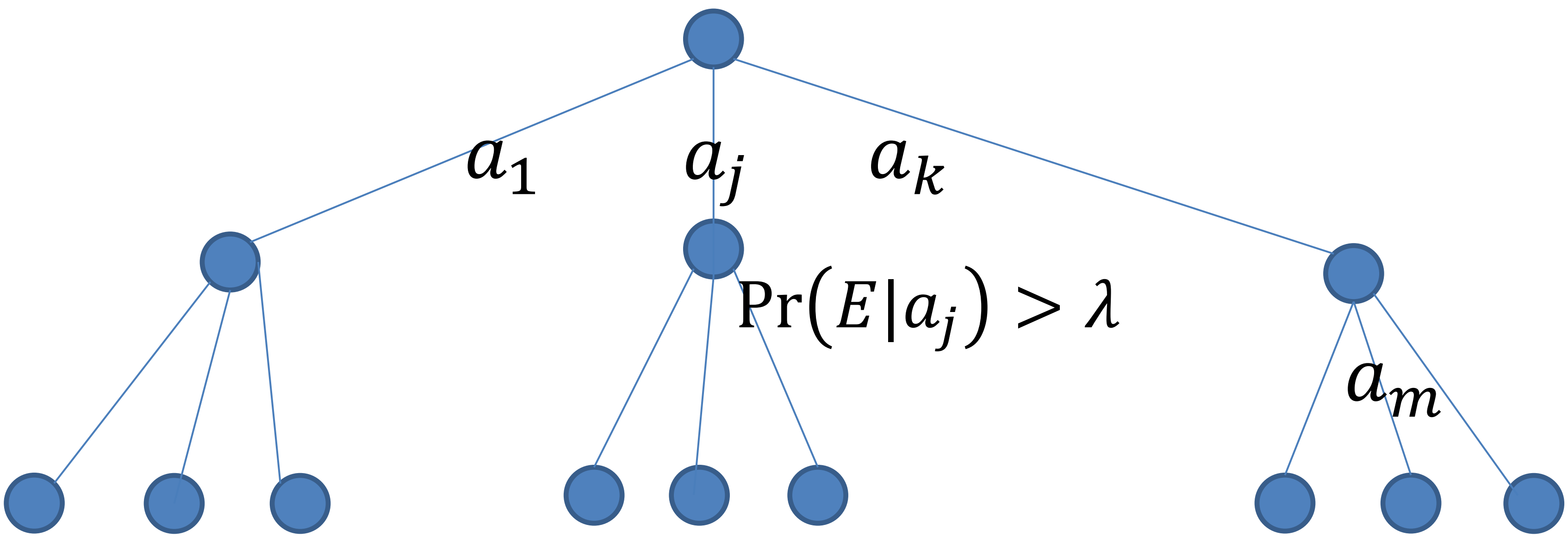
Proof: modification

- $\Pr\left(\alpha(G) < \frac{n}{2k}, Y \le \frac{n}{2}\right) < 1$
- Remove one vertex from each triangle of G, ending up with a graph $G'$with $n' = n - Y$
- $\alpha(G') \le \alpha(G) < \frac{n}{2k}$
- $\chi(G') \ge \frac{n'}{\alpha(G')} \ge \frac{n'}{\alpha(G)} \ge \frac{n-Y}{\frac{n}{2k}} \ge k$ with nonzero probability

De-randomization using conditional probability

- Convert a probabilistic proof into a deterministic algorithms
- Basic principle
  - A probabilistic proof shows existence by random sampling. The point is that one must be lucky
  - Replace random sampling by lucky choices which is guaranteed to exist
  - **sequential *small* random choices** $\Rightarrow$lucky choices
    - The best choice works, by total probability formula

$$\lambda < \Pr(E) = \sum \Pr(E|a_i)\Pr(a_i)$$



Application: MAX-3SAT

- **MAX-3SAT**: Given a Boolean formula with 3 literals in each clause, find a truth assignment which satisfies the maximum number of clauses
- We have proven that $\frac{7}{8}$ fraction can be satisfied

Deterministic $\frac{7}{8}$-algorithm

- The proof randomly samples all the variables
- randomly sampling $\cong$ sequentially, uniformly sampling
- Algo.:
  - Assume that we have set $x_i = 0/1, i \leq k$.
  - Let $x_i = 0$ or 1 such that given the partial assignment for $x_i, i \leq k + 1$, the expected number of satisfied clauses is maximized

Application: Turan Theorem

- What if the conditional probability is hard to compute?
- Any graph $G = (V, E)$ contains an independent set of size at least $\frac{|V|}{D+1}$, where $D = \frac{2|E|}{|V|}$
- In the proof, choose randomly one-by-one

Idea of the algorithm (1)

- Q: the size of the found independent set
- Initially $E[Q] \geq \frac{|V|}{D+1}$
- If each choice does not decrease $E[Q]$, the final $Q \geq \frac{|V|}{D+1}$. But how to compute $E[Q]$?
- $S^{(t)}$: the present S
- $R^{(t)}$: vertex set away from S by distance>1
- $X^{(t)} = \left|S^{(t)}\right| + \sum_{w \in R^{(t)}} \frac{1}{d(w)+1}$
  - **Pessimistic estimator**: $E\left[Q \middle| S^{(t)}\right] \geq X^{(t)}$

Idea of the algorithm (2)

- $X^{(0)} \geq \frac{|V|}{D+1}$
- OK if $X^{(t)}$ never decreases. Can we?
- Choose a vertex u in the $(t$+1)th step,
  $X^{(t+1)} = X^{(t)} + 1 - \sum_{w \in N(u) \cup u} \frac{1}{d(w)+1}$
- If u is random,
  $E\left[X^{(t+1)} - X^{(t)}\right] = 1 - \sum_{w \in R^{(t)}} \frac{1}{d(w)+1} \frac{d(w)+1}{|R^{(t)}|}$=0
- So, we can choose u not to decrease $X^{(t)}$

Deterministic algorithm

- Initialize S to be the empty set
- While
  - there exists a not-yet-considered vertex u with no neighbor in S
  - Add such a vertex u to S where u minimizes $\sum_{w \in N(u) \cup u} \frac{1}{d(w)+1}$
- Return S

References

- http://www.cse.buffalo.edu/~hungngo/classes/2011/Spring-694/lectures/sm.pdf
- http://en.wikipedia.org/wiki/Method_of_conditional_probabilities
- Erdos. Graph theory and probability I
- Erdos. Graph theory and probability II