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ON THE OSCILLATORY INTEGRATION OF SOME ORDINARY  
DIFFERENTIAL EQUATIONS

OCTAVIAN G. MUSTAFA

ABSTRACT. Conditions are given for a class of nonlinear ordinary differential equations  $x'' + a(t)w(x) = 0$ ,  $t \geq t_0 \geq 1$ , which includes the linear equation to possess solutions  $x(t)$  with prescribed oblique asymptote that have an oscillatory pseudo-wronskian  $x'(t) - \frac{x(t)}{t}$ .

## 1. INTRODUCTION

A certain interest has been shown recently in studying the existence of bounded and positive solutions to a large class of elliptic partial differential equations which can be displayed as

$$(1) \quad \Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0, \quad x \in G_R,$$

where  $G_R = \{x \in \mathbb{R}^n : |x| > R\}$  for any  $R \geq 0$  and  $n \geq 2$ . We would like to mention the contributions [3], [1], [8] – [11], [13, 14], [18] and their references in this respect.

It has been established, see [8, 9], that it is sufficient for the functions  $f, g$  to be Hölder continuous, respectively continuously differentiable in order to analyze the asymptotic behavior of the solutions to (1) by the comparison method [15]. In fact, given  $\zeta > 0$ , let us assume that there exist a continuous function  $A: [R, +\infty) \rightarrow [0, +\infty)$  and a nondecreasing, continuously differentiable function  $W: [0, \zeta] \rightarrow [0, +\infty)$  such that

$$0 \leq f(x, u) \leq A(|x|)W(u) \quad \text{for all } x \in G_R, u \in [0, \zeta]$$

and  $W(u) > 0$  when  $u > 0$ . Then we are interested in the positive solutions  $U = U(|x|)$  of the elliptic partial differential equation

$$\Delta U + A(|x|)W(U) = 0, \quad x \in G_R,$$

for the role of super-solutions to (1).

M. Ehrnström [13] noticed that, by imposing the restriction

$$x \cdot \nabla U(x) \leq 0, \quad x \in G_R,$$

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upon the super-solutions  $U$ , an improvement of the conclusions from the literature is achieved for the special subclass of equations (1) where  $g$  takes only nonnegative values. Further developments of Ehrnström's idea are given in [3, 1, 11, 14].

Translated into the language of ordinary differential equations, the research about  $U$  reads as follows: given  $c_1, c_2 \geq 0$ , find (if any) a positive solution  $x(t)$  of the nonlinear differential equation

$$(2) \quad x'' + a(t)w(x) = 0, \quad t \geq t_0 \geq 1,$$

where the coefficient  $a: [t_0, +\infty) \rightarrow \mathbb{R}$  and the nonlinearity  $w: \mathbb{R} \rightarrow \mathbb{R}$  are continuous and given by means of  $A, W$ , such that

$$(3) \quad x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \rightarrow +\infty$$

and

$$(4) \quad \mathcal{W}(x, t) = \frac{1}{t} \left| \frac{x'(t)}{x(t)} - \frac{1}{t} \right| = x'(t) - \frac{x(t)}{t} < 0, \quad t > t_0.$$

The symbol  $o(f)$  for a given functional quantity  $f$  has here its standard meaning. In particular, by  $o(1)$  we refer to a function of  $t$  that decreases to 0 as  $t$  increases to  $+\infty$ .

The papers [2, 1, 22, 21, 20] present various properties of the functional quantity  $\mathcal{W}$ , which shall be called *pseudo-wronskian* in the sequel. Our aim in this note is to complete their conclusions by giving some sufficient conditions upon  $a$  and  $w$  which lead to the existence of a solution  $x$  to (2) that verifies (3) while having an *oscillatory pseudo-wronskian* (this means that there exist the unbounded from above sequences  $(t_n^\pm)_{n \geq 1}$  and  $(t_n^0)_{n \geq 1}$  such that  $t_{2n-1}^0 < t_n^+ < t_{2n}^0 < t_n^- < t_{2n+1}^0$  and  $\mathcal{W}(t_n^+) > \mathcal{W}(t_n^0) = 0 > \mathcal{W}(t_n^-)$  for all  $n \geq 1$ ). We answer thus to a question raised in [1, p. 371], see also the comment in [2, pp. 46–47].

## 2. THE SIGN OF $\mathcal{W}$

Let us start the discussion with a simple condition to settle the sign issue of the pseudo-wronskian.

**Lemma 1.** *Given  $x \in C^2([t_0, +\infty), \mathbb{R})$ , suppose that  $x''(t) \leq 0$  for all  $t \geq t_0$ . Then  $\mathcal{W}(x, \cdot)$  can change from being nonnegative-valued to being negative-valued at most once in  $[t_0, +\infty)$ . In fact, its set of zeros is an interval (possibly degenerate).*

**Proof.** Notice that

$$\frac{d^2}{dt^2} [x(t)] = \frac{1}{t} \cdot \frac{d}{dt} [t\mathcal{W}(x, t)], \quad t \geq t_0.$$

The function  $t \mapsto t\mathcal{W}(x, t)$  being nonincreasing, it is clear that, if it has zeros, it has either a unique zero or an interval of zeros.  $\square$

The result has an obvious counterpart.

**Lemma 2.** *Given  $x \in C^2([t_0, +\infty), \mathbb{R})$ , suppose that  $x''(t) \geq 0$  for all  $t \geq t_0$ . Then,  $\mathcal{W}(x, \cdot)$  can change from being nonpositive-valued to being positive-valued at most once in  $[t_0, +\infty)$ . Again, its set of zeros is an interval (possibly reduced to one point).*

Consider that  $x$  is a positive solution of equation (2) in the case where  $a(t) \geq 0$  in  $[t_0, +\infty)$  and  $w(u) > 0$  for all  $u > 0$ . Then, we have

$$\frac{d\mathcal{W}}{dt} = -\frac{\mathcal{W}}{t} - a(t)w(x(t)), \quad t \geq t_0,$$

which leads to

$$(5) \quad \mathcal{W}(x, t) = \frac{1}{t} \left[ t_0 \mathcal{W}_0 - \int_{t_0}^t sa(s)w(x(s)) ds \right], \quad \mathcal{W}_0 = \mathcal{W}(x, t_0),$$

throughout  $[t_0, +\infty)$  by means of Lagrange's variation of constants formula.

The integrand in (5) being nonnegative-valued, we regain the conclusion of Lemma 1. In fact, if  $T \in [t_0, +\infty)$  is a zero of  $\mathcal{W}(x, \cdot)$  then it is a solution of the equation

$$(6) \quad t_0 \mathcal{W}_0 = \int_{t_0}^T sa(s)w(x(s)) ds.$$

On the other hand, if the pseudo-wronskian of  $x$  is positive-valued throughout  $[t_0, +\infty)$  then it is necessary to have

$$(7) \quad (t_0 \mathcal{W}_0 \geq) \quad \int_{t_0}^{+\infty} sa(s)w(x(s)) ds < +\infty.$$

It has become clear at this point that whenever the equation (2) has a positive solution  $x$  such that  $\mathcal{W}_0 \leq 0$ , the functional coefficient  $a$  is nonnegative-valued and has at most isolated zeros and  $w(u) > 0$  for all  $u > 0$ , the pseudo-wronskian  $\mathcal{W}$  satisfies the restriction (4). Now, returning to the problem stated in the Introduction, we can evaluate the main difficulty of the investigation: *if the positive solution  $x$  has prescribed asymptotic behavior, see formula (3) or a similar development, then we cannot decide upfront whether or not  $\mathcal{W}_0 \leq 0$* . The formula (6) shows that there are also certain difficulties to estimate the zeros of the pseudo-wronskian.

### 3. THE BEHAVIOR OF $\mathcal{W}$

Let us survey in this section some of the recent results regarding the pseudo-wronskian.

It has been established that its presence in the structure of a nonlinear differential equation

$$(8) \quad x'' + f(t, x, x') = 0, \quad t \geq t_0 \geq 1,$$

where the nonlinearity  $f: [t_0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous, allows for a remarkable flexibility of the hypotheses when searching for solutions with the asymptotic development (3) (or similar).

**Theorem 1** ([22, p. 177]). *Assume that there exist the nonnegative-valued, continuous functions  $a(t)$  and  $g(s)$  such that  $g(s) > 0$  for all  $s > 0$  and  $xg(s) \leq g(x^{1-\alpha}s)$ , where  $x \geq t_0$  and  $s \geq 0$ , for a certain  $\alpha \in (0, 1)$ . Suppose further that*

$$|f(t, x, x')| \leq a(t)g\left(\left|x' - \frac{x}{t}\right|\right) \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{a(s)}{s^\alpha} ds < \int_{c+|\mathcal{W}_0|t_0^{1-\alpha}}^{+\infty} \frac{du}{g(u)}.$$

Then the solution of equation (8) given by (5) exists throughout  $[t_0, +\infty)$  and has the asymptotic behavior

$$(9) \quad x(t) = c \cdot t + o(t), \quad x'(t) = c + o(1) \quad \text{when } t \rightarrow +\infty$$

for some  $c = c(x) \in \mathbb{R}$ .

To compare this result with the standard conditions in asymptotic integration theory regarding the development (9), see the papers [2, 1, 24] and the monograph [19].

Another result is concerned with the presence of the pseudo-wronskian in the function space  $L^1((t_0, +\infty), \mathbb{R})$ .

**Theorem 2** ([1, p. 371]). *Assume that  $f$  does not depend explicitly of  $x'$  and there exists the continuous function  $F : [t_0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ , which is nondecreasing with respect to the second variable, such that*

$$|f(t, x)| \leq F\left(t, \frac{|x|}{t}\right) \quad \text{and} \quad \int_{t_0}^{+\infty} t \left[1 + \ln\left(\frac{t}{t_0}\right)\right] F\left(t, |c| + \frac{\varepsilon}{t_0}\right) dt < \varepsilon$$

for certain numbers  $c \neq 0$  and  $\varepsilon > 0$ . Then there exists a solution  $x(t)$  of equation (8) defined in  $[t_0, +\infty)$  such that

$$x(t) = c \cdot t + o(1) \quad \text{when } t \rightarrow +\infty \quad \text{and} \quad \mathcal{W}(x, \cdot) \in L^1.$$

The effect of perturbations upon the pseudo-wronskian is investigated in the papers [2, 22, 21].

**Theorem 3** ([22, p. 183]). *Consider the nonlinear differential equation*

$$(10) \quad x'' + f(t, x, x') = p(t), \quad t \geq t_0 \geq 1,$$

where the functions  $f : [t_0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $p : [t_0, +\infty) \rightarrow \mathbb{R}$  are continuous and verify the hypotheses

$$|f(t, x, x')| \leq a(t) \left| x' - \frac{x}{t} \right|, \quad \int_{t_0}^{+\infty} ta(t) dt < +\infty$$

and

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_{t_0}^t sp(s) ds = C \in \mathbb{R} - \{0\}.$$

Then, given  $x_0 \in \mathbb{R}$ , there exists a solution  $x(t)$  of equation (10) defined in  $[t_0, +\infty)$  such that

$$x(t_0) = x_0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathcal{W}(x, t) = C.$$

In particular,

$$\lim_{t \rightarrow +\infty} \frac{x(t)}{t \ln t} = C.$$

A slight modification of the discussion in [21, Remark 3], see [2, p. 47], leads to the next result.

**Theorem 4.** Assume that  $f$  in (10) does not depend explicitly of  $x'$  and there exists the continuous function  $F: [t_0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ , which is nondecreasing with respect to the second variable, such that

$$|f(t, x)| \leq F(t, |x|) \quad \text{and} \quad \int_t^{+\infty} sF\left(s, |P(s)| + \sup_{\tau \geq s} \{q(\tau)\}\right) ds \leq q(t), \quad t \geq t_0,$$

for a certain positive-valued, continuous function  $q(t)$  possibly decaying to 0 as  $t \rightarrow +\infty$ . Here,  $P$  is the twice continuously differentiable antiderivative of  $p$ , that is  $P''(t) = p(t)$  for all  $t \geq t_0$ . Suppose further that

$$\limsup_{t \rightarrow +\infty} \left[ t \frac{\mathcal{W}(P, t)}{q(t)} \right] > 1 \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \left[ t \frac{\mathcal{W}(P, t)}{q(t)} \right] < -1.$$

Then equation (10) has a solution  $x(t)$  throughout  $[t_0, +\infty)$  such that

$$x(t) = P(t) + o(1) \quad \text{when} \quad t \rightarrow +\infty$$

and  $\mathcal{W}(x, \cdot)$  oscillates.

Finally, the presence of the pseudo-wronskian in the structure of a nonlinear differential equation can lead to multiplicity when searching for solutions with the asymptotic development (3).

**Theorem 5** ([20, Theorem 1]). Given the numbers  $x_0, x_1, c \in \mathbb{R}$ , with  $c \neq 0$ , and  $t_0 \geq 1$  such that  $t_0 x_1 - x_0 = c$ , consider the Cauchy problem

$$(11) \quad \begin{cases} x'' = \frac{1}{t} g(tx' - x), & t \geq t_0 \geq 1, \\ x(t_0) = x_0, & x'(t_0) = x_1, \end{cases}$$

where the function  $g: \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $g(c) = g(3c) = 0$  and  $g(u) > 0$  for all  $u \neq c$ . Assume further that

$$\int_{c+}^{2c} \frac{du}{g(u)} < +\infty \quad \text{and} \quad \int_{2c}^{(3c)-} \frac{du}{g(u)} = +\infty.$$

Then problem (11) has an infinity of solutions  $x(t)$  defined in  $[t_0, +\infty)$  and developable as

$$x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \rightarrow +\infty$$

for some  $c_1 = c_1(x)$  and  $c_2 = c_2(x) \in \mathbb{R}$ .

The asymptotic analysis of certain functional quantities attached to the solutions of equations (2), (8) and (10), as in our case the pseudo-wronskian, might lead to some surprising consequences. Among the functional quantities that gave the impetus to spectacular developments in the qualitative theory of linear/nonlinear ordinary differential equations we would like to refer to

$$\mathcal{K}(x)(t) = x(t)x'(t), \quad t \geq t_0,$$

employed in the theory of *Kneser-solutions*, see the papers [6, 7] for the linear and respectively the nonlinear case and the monograph [19], and

$$\mathcal{H}\mathcal{W}(x) = \int_{t_0}^{+\infty} x(s)w(x(s)) ds.$$

The latter quantity is the core of the nonlinear version of *Hermann Weyl's limit-point/limit-circle classification* designed for equation (2), see the well-documented monograph [5] and the paper [23].

#### 4. THE NEGATIVE VALUES OF $\mathcal{W}$

We shall assume in the sequel that the nonlinearity  $w$  of equation (2) verifies some of the hypotheses listed below:

$$(12) \quad |w(x) - w(y)| \leq k|x - y|, \quad \text{where } k > 0,$$

and

$$(13) \quad w(0) = 0, \quad w(x) > 0 \quad \text{when } x > 0, \quad |w(xy)| \leq w(|x|)w(|y|)$$

for all  $x, y \in \mathbb{R}$ . We notice that restriction (13) implies the existence of a majorizing function  $F$ , as in Theorem 2, given by the estimates

$$|f(t, x)| = |a(t)w(x)| \leq |a(t)| \cdot w(t)w\left(\frac{|x|}{t}\right) = F\left(t, \frac{|x|}{t}\right).$$

We can now use the paper [24] to recall the main conclusions of an asymptotic integration of equation (2). It has been established that whenever  $\int_{t_0}^{+\infty} tw(t)|a(t)| dt < +\infty$ , all the solutions of (2) have asymptotes (3) and their first derivatives are developable as

$$(14) \quad x'(t) = c_1 + o(t^{-1}) \quad \text{when } t \rightarrow +\infty.$$

Consequently,  $\mathcal{W}(x, t) = -c_2 t^{-1} + o(t^{-1})$  for all large  $t$ 's. In this case (the functional coefficient  $a$  has varying sign), when dealing with the sign of the pseudo-wronskian, of interest would be the subcase where  $c_2 = 0$ . Here, the asymptotic development does not even ensure that  $\mathcal{W}$  is eventually negative. Enlarging the family of coefficients to the ones subjected to the restriction  $\int_{t_0}^{+\infty} t^\varepsilon w(t)|a(t)| dt < +\infty$ , where  $\varepsilon \in [0, 1)$ , the developments (3), (14) become

$$(15) \quad x(t) = ct + o(t^{1-\varepsilon}), \quad x'(t) = c + o(t^{-\varepsilon}), \quad c \in \mathbb{R},$$

yielding the less precise estimate  $\mathcal{W}(x, t) = o(t^{-\varepsilon})$  when  $t \rightarrow +\infty$ . We have again a lack of precision in the asymptotic development of  $\mathcal{W}(x, \cdot)$  with respect to the sign issue. We also deduce on the basis of (3), (15) that some of the coefficients  $a$  in these classes verify (7), a fact that complicates the discussion.

The next result establishes the existence of a positive solution to (2) subjected to (4), (15) for the largest class of functional coefficients:  $\varepsilon = 0$ . By taking into account Lemmas 1, 2 and the non-oscillatory character of equation (2) when the nonlinearity  $w$  verifies (13), we conclude that *for an investigation within this class of coefficients  $a$  of the solutions with oscillatory pseudo-wronskian it is necessary that  $a$  itself oscillates*. Also, when  $a$  is non-negative valued we recall that the condition

$$\int_{t_0}^{+\infty} a(t) dt < +\infty$$

is necessary for the linear case of equation (2) to be non-oscillatory, see [16], while in the case given by  $w(x) = x^\lambda$ ,  $x \in \mathbb{R}$ , with  $\lambda > 1$  (such an equation is usually called an *Emden-Fowler equation*, see the monograph [19]) the condition

$$(16) \quad \int_{t_0}^{+\infty} ta(t) dt = +\infty$$

is necessary and sufficient for oscillation, see [4]. In the case of Emden-Fowler equations with  $\lambda \in (0, 1)$  and a continuously differentiable coefficient  $a$  such that  $a(t) \geq 0$  and  $a'(t) \leq 0$  throughout  $[t_0, +\infty)$ , another result establishes that equation (2) has no oscillatory solutions provided that condition (16) fails, see [17].

Regardless of the oscillation of  $a$ , it is known [1, p. 360] that the linear case of equation (2) has bounded and positive solutions with eventually negative pseudo-wronskian.

**Theorem 6.** *Assume that the nonlinearity  $w$  verifies hypothesis (13) and is non-decreasing. Given  $c, d > 0$ , suppose that the functional coefficient  $a$  is nonnegative-valued, with eventual isolated zeros, and*

$$\int_{t_0}^{+\infty} w(t) a(t) dt \leq \frac{d}{w(c+d)}.$$

*Then, the equation (2) has a solution  $x$  such that  $\mathcal{W}_0 = 0$ ,*

$$(17) \quad c - d \leq x'(t) < \frac{x(t)}{t} \leq c + d \quad \text{for all } t > t_0$$

*and*

$$(18) \quad \lim_{t \rightarrow +\infty} x'(t) = \lim_{t \rightarrow +\infty} \frac{x(t)}{t} = c.$$

**Proof.** We introduce the set  $D$  given by

$$D = \{u \in C([t_0, +\infty), \mathbb{R}) : ct \leq u(t) \leq (c+d)t \text{ for every } t \geq t_0\}.$$

A partial order on  $D$  is provided by the usual pointwise order “ $\leq$ ”, that is, we say that  $v_1 \leq v_2$  if and only if  $v_1(t) \leq v_2(t)$  for all  $t \geq t_1$ , where  $v_1, v_2 \in D$ . It is not hard to see that  $(D, \leq)$  is a complete lattice.

For the operator  $V: D \rightarrow C([t_0, +\infty), \mathbb{R})$  with the formula

$$V(u)(t) = t \left\{ c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^s \tau a(\tau) w(u(\tau)) d\tau ds \right\}, \quad u \in D, \quad t \geq t_0,$$



the next estimates hold

$$\begin{aligned}
c &\leq \frac{V(u)(t)}{t} = c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^s \tau a(\tau) \cdot w(\tau) w\left(\frac{u(\tau)}{\tau}\right) d\tau ds \\
&\leq c + \sup_{\xi \in [0, c+d]} \{w(\xi)\} \cdot \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^s \tau w(\tau) a(\tau) d\tau ds \\
&= c + w(c+d) \left[ \frac{1}{t} \int_{t_0}^t \tau w(\tau) a(\tau) d\tau + \int_t^{+\infty} w(\tau) a(\tau) d\tau \right] \\
&\leq c + w(c+d) \int_{t_0}^{+\infty} w(\tau) a(\tau) d\tau \leq c + d
\end{aligned}$$

by means of (13). These imply that  $V(D) \subseteq D$ .

Since  $c \cdot t \leq V(c \cdot t)$  for all  $t \geq t_0$ , by applying the Knaster-Tarski fixed point theorem [12, p. 14], we deduce that the operator  $V$  has a fixed point  $u_0$  in  $D$ . This is the pointwise limit of the sequence of functions  $(V^n(c \cdot \text{Id}_I))_{n \geq 1}$ , where  $V^1 = V$ ,  $V^{n+1} = V^n \circ V$  and  $I = [t_0, +\infty)$ .

We deduce that

$$u'_0(t) = [V(u_0)]'(t) = \frac{u_0(t)}{t} - \frac{1}{t} \int_{t_0}^t \tau a(\tau) w(u_0(\tau)) d\tau < \frac{u_0(t)}{t},$$

when  $t > t_0$ , and thus (17), (18) hold true.

The proof is complete.  $\square$

## 5. THE OSCILLATORY INTEGRATION OF EQUATION (2)

Let the continuous functional coefficient  $a$  with varying sign satisfy the restriction

$$\int_{t_0}^{+\infty} t^2 |a(t)| dt < +\infty.$$

We call the problem studied in the sequel *an oscillatory (asymptotic) integration* of equation (2).

**Theorem 7.** *Assume that  $w$  verifies (12),  $w(0) = 0$  and there exists  $c > 0$  such that*

$$(19) \quad L_+^c > 0 > L_-^c,$$

where

$$L_+^c = \limsup_{t \rightarrow +\infty} \frac{t \int_t^{+\infty} s w(cs) a(s) ds}{\int_t^{+\infty} s^2 |a(s)| ds}, \quad L_-^c = \liminf_{t \rightarrow +\infty} \frac{t \int_t^{+\infty} s w(cs) a(s) ds}{\int_t^{+\infty} s^2 |a(s)| ds}.$$

*Then the equation (2) has a solution  $x(t)$  with oscillatory pseudo-wronskian such that*

$$(20) \quad x(t) = c \cdot t + o(1) \quad \text{when} \quad t \rightarrow +\infty.$$

**Proof.** There exist  $\eta > 0$  such that  $L_+^c > \eta$ ,  $L_-^c < -\eta$  and two increasing, unbounded from above sequences  $(t_n)_{n \geq 1}$ ,  $(t^n)_{n \geq 1}$  of numbers from  $(t_0, +\infty)$  such that  $t^n \in (t_n, t_{n+1})$  and

$$(21) \quad t_n \int_{t_n}^{+\infty} s w(cs) a(s) ds + k\eta \int_{t_n}^{+\infty} s^2 |a(s)| ds < 0$$

and

$$(22) \quad t^n \int_{t^n}^{+\infty} s w(cs) a(s) ds - k\eta \int_{t^n}^{+\infty} s^2 |a(s)| ds > 0$$

for all  $n \geq 1$ .

Assume further that

$$\int_{t_0}^{+\infty} \tau^2 |a(\tau)| d\tau \leq \frac{\eta}{k(c + \eta)}$$

and introduce the complete metric space  $S = (D, \delta)$  given by

$$D = \{y \in C([t_0, +\infty), \mathbb{R}) : t|y(t)| \leq \eta \text{ for every } t \geq t_0\}$$

and

$$\delta(y_1, y_2) = \sup_{t \geq t_0} \{t|y_1(t) - y_2(t)|\}, \quad y_1, y_2 \in D.$$

For the operator  $V: D \rightarrow C([t_0, +\infty), \mathbb{R})$  with the formula

$$V(y)(t) = \frac{1}{t} \int_t^{+\infty} s a(s) w\left(s \left[c - \int_s^{+\infty} \frac{y(\tau)}{\tau} d\tau\right]\right) ds, \quad y \in D, t \geq t_0,$$

the next estimates hold (notice that  $|w(x)| \leq k|x|$  for all  $x \in \mathbb{R}$ )

$$(23) \quad t|V(y)(t)| \leq k \int_t^{+\infty} s^2 |a(s)| \left[c + \eta \int_s^{+\infty} \frac{d\tau}{\tau^2}\right] ds \leq \eta$$

and

$$\begin{aligned} t|V(y_2)(t) - V(y_1)(t)| &\leq k \int_t^{+\infty} s^2 |a(s)| \left(\int_s^{+\infty} \frac{d\tau}{\tau^2}\right) ds \cdot \delta(y_1, y_2) \\ &\leq \frac{k}{t_0} \int_t^{+\infty} s^2 |a(s)| ds \leq \frac{\eta}{c + \eta} \cdot \delta(y_1, y_2). \end{aligned}$$

These imply that  $V(D) \subseteq D$  and thus  $V: S \rightarrow S$  is a contraction.

From the formula of operator  $V$  we notice also that

$$(24) \quad \lim_{t \rightarrow +\infty} tV(y)(t) = 0 \quad \text{for all } y \in D.$$

Given  $y_0 \in D$  the unique fixed point of  $V$ , one of the solutions to (2) has the formula  $x_0(t) = t \left[c - \int_t^{+\infty} \frac{y_0(s)}{s} ds\right]$  for all  $t \geq t_0$ . Via (24) and L'Hospital's rule, we

provide also an asymptotic development for this solution, namely

$$\begin{aligned} \lim_{t \rightarrow +\infty} [x_0(t) - c \cdot t] &= - \lim_{t \rightarrow +\infty} t \int_t^{+\infty} \frac{y_0(s)}{s} ds = - \lim_{t \rightarrow +\infty} t y_0(t) \\ &= - \lim_{t \rightarrow +\infty} t V(y_0)(t) = 0. \end{aligned}$$

The estimate

$$\begin{aligned} \left| t y_0(t) - \int_t^{+\infty} s w(cs) a(s) ds \right| &\leq k \int_t^{+\infty} s^2 |a(s)| \left[ \int_s^{+\infty} \frac{|y_0(\tau)|}{\tau} d\tau \right] ds \\ &\leq k\eta \cdot \frac{1}{t} \int_t^{+\infty} s^2 |a(s)| ds, \quad t \geq t_0, \end{aligned}$$

accompanied by (21), (22), leads to

$$(25) \quad y_0(t_n) = \mathcal{W}(x_0, t_n) < 0 \quad \text{and} \quad y_0(t^n) = \mathcal{W}(x_0, t^n) > 0.$$

The proof is complete.  $\square$

**Remark 1.** When Equation (2) is linear, that is  $w(x) = x$  for all  $x \in \mathbb{R}$ , the formula (19) can be recast as

$$L_+ = \limsup_{t \rightarrow +\infty} \frac{t \int_t^{+\infty} s^2 a(s) ds}{\int_t^{+\infty} s^2 |a(s)| ds} > 0 > \liminf_{t \rightarrow +\infty} \frac{t \int_t^{+\infty} s^2 a(s) ds}{\int_t^{+\infty} s^2 |a(s)| ds} = L_-.$$

We claim that *for all  $c \neq 0$  there exists a solution  $x(t)$  with oscillatory pseudo-wronskian which verifies (20)*. In fact, replace  $c$  with  $c_0$  in the formulas (21), (22) for a certain  $c_0$  subjected to the inequality  $\min\{L_+, -L_-\} > \frac{\eta}{c_0}$ . It is obvious that, when  $L_+ = -L_- = +\infty$ , formulas (21), (22) hold for all  $c_0, \eta > 0$ . Given  $c \in \mathbb{R} - \{0\}$ , there exists  $\lambda \neq 0$  such that  $c = \lambda c_0$ . The solution of Equation (2) that we are looking for has the formula  $x = \lambda \cdot x_0$ , where  $x_0(t) = t \left[ c_0 - \int_t^{+\infty} \frac{y_0(s)}{s} ds \right]$  for all  $t \geq t_0$  and  $y_0$  is the fixed point of operator  $V$  in  $D$ . Its pseudo-wronskian oscillates as a consequence of the obvious identity

$$\lambda \cdot \mathcal{W}(x_0, t) = \mathcal{W}(x, t), \quad t \geq t_0.$$

**Example 1.** An immediate example of functional coefficient  $a$  for the problem of linear oscillatory integration is given by  $a(t) = t^{-2} e^{-t} \cos t$ , where  $t \geq 1$ .

We have

$$\int_t^{+\infty} s^2 a(s) ds = \frac{1}{\sqrt{2}} \cos\left(t + \frac{\pi}{4}\right) e^{-t} \quad \text{and} \quad \int_t^{+\infty} s^2 |a(s)| ds \leq e^{-t}$$

throughout  $[1, +\infty)$  which yields  $L_+ = +\infty, L_- = -\infty$ .

Sufficient conditions are provided now for an oscillatory pseudo-wronskian to be in  $L^p((t_0, +\infty), \mathbb{R})$ , where  $p > 0$ . Since  $\lim_{t \rightarrow +\infty} \mathcal{W}(x, t) = 0$  for any solution  $x(t)$  of equation (2) with the asymptotic development (20), (14), we are interested in the case  $p \in (0, 1)$ .

**Theorem 8.** Assume that, in the hypotheses of Theorem 7, the coefficient  $a$  verifies the condition

$$(26) \quad \int_{t_0}^{+\infty} \left[ \frac{t}{\int_t^{+\infty} s^2 |a(s)| ds} \right]^{1-p} t^2 |a(t)| dt < +\infty \quad \text{for some } p \in (0, 1).$$

Then the equation (2) has a solution  $x(t)$  with an oscillatory pseudo-wronskian in  $L^p$  and the asymptotic expansion (20).

**Proof.** Recall that  $y_0$  is the fixed point of operator  $V$ . Then, formula (23) implies that

$$|y_0(t)| \leq k(c + \eta) \cdot \frac{1}{t} \int_t^{+\infty} s^2 |a(s)| ds, \quad t \geq t_0.$$

Via an integration by parts, we have

$$\begin{aligned} \frac{1}{[k(c + \eta)]^p} \int_t^T |y_0(s)|^p ds &\leq \frac{T^{1-p}}{1-p} \left[ \int_T^{+\infty} s^2 |a(s)| ds \right]^p \\ &\quad + \frac{p}{1-p} \int_t^T \left[ \frac{s}{\int_s^{+\infty} \tau^2 |a(\tau)| d\tau} \right]^{1-p} s^2 |a(s)| ds \end{aligned}$$

for all  $T \geq t \geq t_0$ .

The estimates

$$\begin{aligned} \frac{T^{1-p}}{1-p} \left[ \int_T^{+\infty} s^2 |a(s)| ds \right]^p &= \frac{T^{1-p}}{1-p} \int_T^{+\infty} \left[ \frac{1}{\int_T^{+\infty} \tau^2 |a(\tau)| d\tau} \right]^{1-p} s^2 |a(s)| ds \\ &\leq \frac{1}{1-p} \int_T^{+\infty} \left[ \frac{s}{\int_s^{+\infty} \tau^2 |a(\tau)| d\tau} \right]^{1-p} s^2 |a(s)| ds \end{aligned}$$

allow us to establish that

$$\frac{1}{[k(c + \eta)]^p} \int_t^T |y_0(s)|^p ds \leq \frac{1+p}{1-p} \int_t^{+\infty} \left[ \frac{s}{\int_s^{+\infty} \tau^2 |a(\tau)| d\tau} \right]^{1-p} s^2 |a(s)| ds.$$

The conclusion follows by letting  $T \rightarrow +\infty$ .

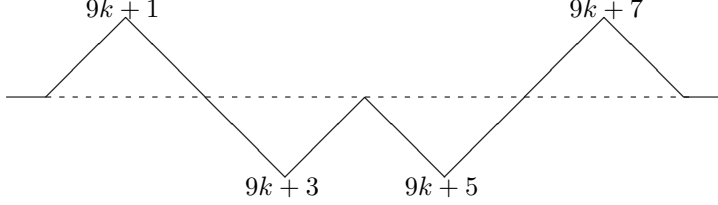
The proof is complete.  $\square$

**Example 2.** An example of functional coefficient  $a$  in the linear case that verifies the hypotheses of Theorem 8 is given by the formula

$$t^2 a(t) = b(t) = \begin{cases} a_k(t - 9k), & t \in [9k, 9k + 1], \\ a_k(9k + 2 - t), & t \in [9k + 1, 9k + 3], \\ a_k(t - 9k - 4), & t \in [9k + 3, 9k + 4], \\ a_k(9k + 4 - t), & t \in [9k + 4, 9k + 5], \\ a_k(t - 9k - 6), & t \in [9k + 5, 9k + 7], \\ a_k(9k + 8 - t), & t \in [9k + 7, 9k + 8], \\ 0, & t \in [9k + 8, 9(k + 1)], \end{cases} \quad k \geq 1.$$

Here, we take  $a_k = k^{-\alpha} - (k+1)^{-\alpha}$  for a certain integer  $\alpha > \frac{2-p}{p}$ .

To help the computations, the  $k$ -th “cell” of the function  $b$  can be visualized next.



It is easy to observe that

$$\int_{9k}^{9k+4} b(t) dt = \int_{9k+4}^{9k+8} b(t) dt = 0 \quad \text{for all } k \geq 1.$$

We have

$$\int_{9k+2}^{+\infty} b(t) dt = \int_{9k+2}^{9k+4} b(t) dt = -a_k, \quad \int_{9k+6}^{+\infty} b(t) dt = \int_{9k+6}^{9k+8} b(t) dt = a_k$$

and respectively

$$\int_{9k+2}^{+\infty} |b(t)| dt = 3a_k + 4 \sum_{m=k+1}^{+\infty} a_m, \quad \int_{9k+6}^{+\infty} |b(t)| dt = a_k + 4 \sum_{m=k+1}^{+\infty} a_m.$$

By noticing that

$$L_+ = \lim_{k \rightarrow +\infty} \frac{(9k+6) \int_{9k+6}^{+\infty} b(t) dt}{\int_{9k+6}^{+\infty} |b(t)| dt}, \quad L_- = \lim_{k \rightarrow +\infty} \frac{(9k+2) \int_{9k+2}^{+\infty} b(t) dt}{\int_{9k+2}^{+\infty} |b(t)| dt},$$

we obtain  $L_+ = \frac{9\alpha}{4}$  and  $L_- = -\frac{9\alpha}{4}$ .

To verify the condition (26), notice first that

$$\begin{aligned} I_k &= \int_{9k}^{9(k+1)} \left[ \frac{t}{\int_t^{+\infty} |b(s)| ds} \right]^{1-p} t^2 |a(t)| dt \\ &\leq \int_{9k}^{9(k+1)} \left[ \frac{9(k+1)}{\int_{9(k+1)}^{+\infty} |b(s)| ds} \right]^{1-p} a_k dt, \quad k \geq 1. \end{aligned}$$

The elementary inequality  $a_k \leq (2^\alpha - 1)(k+1)^{-\alpha}$  implies that

$$I_k \leq \frac{c_\alpha}{(k+1)^{(1+\alpha)p-1}}, \quad \text{where } c_\alpha = 9 \left( \frac{9}{4} \right)^{1-p} (2^\alpha - 1),$$

and the conclusion follows from the convergence of the series  $\sum_{k \geq 1} (k+1)^{1-(1+\alpha)p}$ .

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