- Binomial Distribution
  - Sum of n Bernoulli w.p. p
  - $S \sim \text{Bin}(n,p) = \sum_{i=1}^{n} X_i$
  - $E[S] = n \cdot p$
  - $\text{Var}(S) = np(1-p)$
  - $P(S{=}k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Geometric Distribution
  - $Z \sim \text{Geo}(p)$
  - $E[Z] = \frac{1}{p}$
  - $\text{Var}(Z) = \frac{1-p}{p^2}$
  - $P(Z=k) = p(1-p)^{k-1}$
- Poisson Distribution
  - $X \sim \text{Pois}(\lambda)$
  - $E[X] = \lambda$
  - $\text{Var}(X) = \lambda$
  - $P(X=k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$
  - $X \sim \text{Bin}(n,p) \approx \text{Pois}(np)$ $n \gg p$
- Negative Binomial Distribution
  - Sum of Geometric
  - $S_r \sim \text{NB}(r,p)$
  - $E[S_r] = \frac{r}{p}$
  - $\text{Var}(S_r) = \frac{r(1-p)}{p^2}$
  - $P(S_r{=}n) = p^r \binom{n-1}{r-1} (1-p)^{n-r}$
- Markov Inequality
  - $P(Y \geq c) \leq \frac{E[Y]}{c} = \frac{\mu_Y}{c}, \quad Y \geq 0, c > 0$
- Chebychev Inequality
  - $P(|Y - E[Y]| \geq c) \leq \frac{\sigma_Y^2}{c^2}$
  - Confidence Interval $\quad X \sim Bin(n,p)$
    - $P\left(|X - n \cdot p| \geq \frac{a \cdot \sqrt{n}}{2}\right) \leq \frac{1}{a^2}$
    - $P\left(\left|\frac{X}{n} - p\right| < \frac{a}{2\sqrt{n}}\right) \geq 1 - \frac{1}{a^2}$
    - $P\left(p \in \left[\widehat{p} - \frac{a}{2\sqrt{n}}, \widehat{p} + \frac{a}{2\sqrt{n}}\right]\right) \geq 1 - \frac{1}{a^2}, \quad \widehat{p} = \frac{X}{n}$
- $N_s \perp\!\!\!\perp N_t - N_s \;(t > s) \qquad N_t - N_s \sim \text{Pois}(\lambda(t-s))$
- $P\{N_2 = 2, N_3 = 5\} = P\{N_2 = 2, N_3 - N_2 = 3\}$ $= P\{N_2 = 2\} \cdot P\{N_3 - N_2 = 3\}$
- $U_1,\ U_2,\ U_3 \perp\!\!\!\perp \sim \exp(\lambda) \rightarrow$ inter-count
- Exactly 1 bus before time 3 $\quad X \sim \text{unif}[0,3]$
- Now 5 busses before 3
  - # busses before time 2
  - $P(bus\ b4\ 2) = F_X(2) = \frac{2}{3}$
  - $S \sim \text{Bin}\left(5, \frac{2}{3}\right) \quad E[S] = 5 \cdot \frac{2}{3} = \frac{10}{3}$

- Continuous Random Variables
  - $f_X(u) = F'_X(u)$
  - $P(X=u) = 0$
  - $F_x(u)$ always continuous, $\Delta F_X(u) = 0$
  - $\int_{-\infty}^{\infty} f_X(u)\,du = F_X(+\infty) + F_X(-\infty) = 1$
  - $P\{a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$
  - $P\left\{u_0 - \frac{\epsilon}{2} < X < u_0 + \frac{\epsilon}{2}\right\} = \int_{u_0-\epsilon/2}^{u_0+\epsilon/2} f_X(u)\,du \approx f_X(u_0) \cdot \epsilon$
  - $E[X] = \int_{-\infty}^{\infty} u \cdot f_X(u)\,du$
  - $E[X^2] = \int_{-\infty}^{\infty} u^2 \cdot f_X(u)\,du$
  - LOTUS $\rightarrow E[g(X)] = \int_{-\infty}^{\infty} g(u) \cdot f_X(u)\,du$

Uniform Distribution

- $X \sim \text{unif}[a,b]$
- $E[X] = \frac{b+a}{2}$
- $\text{Var}(X) = \frac{(a-b)^2}{12}$
- $f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & else \end{cases}$
- $F_X(c) = \begin{cases} 0 & c < a \\ \frac{c-a}{b-a} & a \leq c \leq b \\ 1 & c > b \end{cases}$

Exponential Distribution

- $X \sim \exp(\lambda)$
- Memoryless $P(X > s+t | X > t) = P(X > s)$
- $E[X] = \frac{1}{\lambda}$
- $\text{Var}(X) = \frac{1}{\lambda^2}$
- $f_X(u) = \begin{cases} 0 & u < 0 \\ \lambda \cdot e^{-\lambda \cdot u} & u \geq 0 \end{cases}$
- $F_X(c) = \begin{cases} 0 & c < 0 \\ 1 - e^{-\lambda \cdot u} & c \geq 0 \end{cases}$

Erlang Distribution

- $T_r \sim \text{erlang}(r, \lambda)$
- $f_{T_r}(r) = -\frac{dP\{T_r > t\}}{dt} = \frac{e^{-\lambda t}\lambda^r t^{r-1}}{(r-1)!}$
- $P\{T_r > t\} = \sum_{k=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$
- $E[T_r] = E\left[\sum_{i=1}^{r} U_i\right] = \sum_{i=1}^{r} E[U_i] = \frac{r}{\lambda}$
- $\text{Var}(X) = \frac{r}{\lambda^2}$

$$E[T^n] = \frac{n!}{\lambda^n}.$$

expo

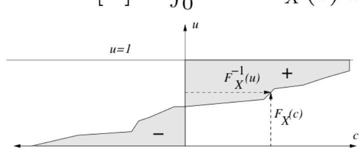
- Gaussian Approximation
  - $X \sim \text{BIN}(n,p) \quad \tilde{X} \sim N(np, np(1-p))$
  - Continuity Correction
    - $P\{X \leq k\} \approx P\left\{\tilde{X} \leq k + 0.5\right\}$
    - $P\{X \geq k\} \approx P\left\{\tilde{X} \geq k - 0.5\right\}$

- Scaling pdfs
  - $Y = aX + b$
  - $f_Y(u) = \frac{1}{a} f_X\left(\frac{u-b}{a}\right)$
  - $F_Y(c) = F_X\left(\frac{u-b}{a}\right)$
  - $E[Y] = a \cdot E[X] + b$
  - $\text{Var}(Y) = a^2 \cdot \text{Var}(X)$
  - stretch out in X direction by factor of a
  - shrink in Y direction by factor of a
  - shift b units to the right
- Gaussian (Normal) Distribution
  - $X \sim N(\mu, \sigma^2)$
  - $\mu \pm \sigma : 68.3\%, \qquad \mu \pm 2\sigma : 95.5\%$
  - $f_X(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}}$
  - $E[X] = \mu$
  - $\text{Var}(X) = \sigma^2$
  - for $N(0,1)$, $F_X(c) = \Phi(c) = Q(-c) = 1 - Q(c)$

- Central Limit Theorem
  - $X \sim \text{Bin}(n,p)$
  - $P\{X \leq c\} = P\left\{\frac{X-\mu}{\sigma} \leq \frac{c-\mu}{\sigma}\right\} \xrightarrow{n \rightarrow \infty} \Phi\left(\frac{c-\mu}{\sigma}\right)$
- Functions of a Random Variable
  - $g(X) = Y$
  - Option 1
    - Find support of Y
    - Find CDF of Y given definition $F_Y(c) = P\{Y \leq c\} = P\{g(X) \leq c\}$
    - find pdf of y
  - Option 2
    - X with CDF F_X given using $U \sim \text{unif}[0,1]$
    - choose $g(u) = F_X^{-1} = min\{F(u) \geq 1\}$ set $u = F_Y(c)$, solve for c, this is $g(u)$
  - $g(U) = Y$, CDF of Y is F_X
- Failure Rate Function
  - $h(t) = \lim_{\epsilon \rightarrow 0} \frac{P\{t < T \leq t + \epsilon | T > t\}}{\epsilon}$
  - For $T > 0$, $h(t) = \frac{pdf}{CCDF} = \frac{f_X(t)}{1 - F_X(t)}$
  - For $T \sim \exp(\lambda) \rightarrow h(t) = f_T(0) = \lambda$
  - $F(t) = 1 - e^{-\int_0^t h(s)\,ds}$

$$E[Y|X=u] \;=\; \int_{-\infty}^{\infty} v f_{Y|X}(v|u) dv$$

- Area Rule for Expectation
  - $E[X] = A^+ - A^-$
  - $E[X] = \int_0^1 F_X^{-1}(u)\,du$
  - $E[X] = \int_0^{\infty} 1 - F_X(c)\,dc - \int_{-\infty}^{0} F_X(c)\,dc$



- Joint CDF
  - $F_{X,Y}(u_0, v_0) = P\{X \leq u_0, Y \leq v_0\}$
  - $P\{(X,Y) \in (a,b] \times (c,d]\} =$ $F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c)$
  - $F(u,v) \in [0,1] \,\forall (u,v) \in \mathbb{R}^2$
  - non-decreasing and right-continuous in u and v
- Joint pdf
  - $F_{X,Y} = \int_{-\infty}^{u_0} \int_{-\infty}^{v_0} f_{X,Y}(u,v)\,dv\,du$
  - $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,dv\,du$

arginal

- $F_X(u_0) = \int_{-\infty}^{u_0} \left[\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\right] du$
- $F_Y(v_0) = \int_{-\infty}^{v_0} \left[\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\right] dv$
- $f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv$
- $f_Y(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,du$

- Conditional
  - $f_{X|Y}(u|v) = \frac{f_{XY}(u,v)}{f_Y(v)}$
  - $f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(u)}$
- Independence of Random Variables
  - Discrete: $p_{XY}(u,v) = p_X(u) \cdot p_Y(v) \Leftrightarrow X \perp\!\!\!\perp Y$
  - Continuous: $f_{XY}(u,v) = f_X(u) \cdot f_Y(v) \Leftrightarrow X \perp\!\!\!\perp Y$
    - $X,\ Y \sim$ Unif over S, $\ X \perp\!\!\!\perp Y \Leftrightarrow S$ is a product set
    - $X,\ Y$ any r.v., $X \perp\!\!\!\perp Y \Rightarrow S$ is a product set only prove $X \not\perp\!\!\!\perp Y$

$$E[X^k] = \int_0^1 u^k du = \frac{u^{k+1}}{k+1}\bigg|_0^1 = \frac{1}{k+1} \quad \text{(if } U \text{ is uniformly distributed over } [0,1]\text{)}, \qquad f_Y(v) = \int_{-\infty}^{\infty} f_{Y|X}(v|u) f_X(u) du. \qquad Q(u) = \int_u^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv = 1 - \Phi(u) = \Phi(-u)$$

Solution The mean, μ_X, is the midpoint of the interval $[a,b]$, and the standard deviation is $\sigma_X=\frac{(b-a)}{2\sqrt{3}}$ (see Section 3.3). The pdf for $X-\mu_X$ is obtained by shifting the pdf of X to be centered at zero. Thus, $X-\mu_X$ is uniformly distributed over the interval $[-\frac{b-a}{2},\frac{b-a}{2}]$. When this random variable is divided by σ_X, the resulting pdf is shrunk horizontally by the factor σ_X. This results in a uniform distribution over the interval $[-\frac{b-a}{2\sigma_X},\frac{b-a}{2\sigma_X}]=[-\sqrt{3},\sqrt{3}]$. This makes sense, because the uniform distribution over the interval $[-\sqrt{3},\sqrt{3}]$ is the unique uniform distribution with mean zero and variance one.

The *standard normal distribution* is the normal distribution with $\mu=0$ and $\sigma^2=1$. It is also called the $N(0,1)$ distribution. The CDF of the $N(0,1)$ distribution is traditionally denoted by the letter Φ (Phi):

$$\Phi(u)=\int_{-\infty}^{u}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv,$$

Solution: (a) Let $Y=-X$, or equivalently, $Y=g(X)$ where $g(u)=-u$. We shall find the pdf of Y after first finding the CDF. For any constant c, $F_Y(c)=P\{Y\le c\}=P\{-X\le c\}=P\{X\ge -c\}=1-F_X(-c)$. Differentiating with respect to c yields $f_Y(c)=f_X(-c)$. Geometrically, the graph of f_Y is obtained by reflecting the graph of f_X about the vertical axis.
(b) Suppose now that $Y=aX+b$. The pdf of Y in case $a>0$ is given in Section 3.6.1. So suppose $a<0$. Then $F_Y(c)=P\{aX+b\le c\}=P\{aX\le c-b\}=P\{X\ge\frac{c-b}{a}\}=1-F_X\left(\frac{c-b}{a}\right)$. Differentiating with respect to c yields

$$f_Y(c)=f_X\left(\frac{c-b}{a}\right)\frac{1}{|a|}, \qquad (3.7)$$

where we use the fact that $a=-|a|$ for $a<0$. Actually, (3.7) is also true if $a>0$, because in that case it is the same as (3.5). So (3.7) gives the pdf of $Y=aX+b$ for any $a\neq 0$.

$$\begin{aligned}F_X(v) &= P\{0\le X\le v\}=P\{0\le Z\le v\}K=P\left\{-1\le\frac{Z-2}{2}\le\frac{v-2}{2}\right\}K\\ &=\left(\Phi\left(\frac{v-2}{2}\right)-\Phi(-1)\right)K=\left(\Phi\left(\frac{v-2}{2}\right)-0.1587\right)K.\end{aligned}$$

$$F_X(v)=\begin{cases}0 & \text{if } v\le 0\\ \left(\Phi\left(\frac{v-2}{2}\right)-0.1587\right)K & \text{if } 0<v\le 4\\ 1 & \text{if } v\ge 4.\end{cases}$$

Solution: (a) The random variable X has the binomial distribution with parameters $n=1000$ and $p=0.5$. It thus has mean $\mu_X=np=500$ and standard deviation $\sigma=\sqrt{np(1-p)}=\sqrt{250}\approx 15.8$. By the Gaussian approximation with the continuity correction,

$$P\{X\ge K\}=P\{X\ge K-0.5\}=P\left\{\frac{X-\mu}{\sigma}\ge\frac{K-0.5-\mu}{\sigma}\right\}\approx Q\left(\frac{K-0.5-\mu}{\sigma}\right).$$

Since $Q(2.325)\approx 0.01$ we thus want to select K so $\frac{K-0.5-\mu}{\sigma}\approx 2.325$ or $K=\mu+2.325\sigma+0.5=537.26$. Thus, $K=537$ or $K=538$ should do. So, if the coin is flipped a thousand times, there is about a one percent chance that heads shows for more than 53.7% of the flips.

$$F_Y(c)=P\{Y\le c\}=P\{X\le g^{-1}(c)\}=F_X(g^{-1}(c)).$$

The derivative of the inverse of a function is one over the derivative of the function itself[7]: $g^{-1}(c)'=\frac{1}{g'(g^{-1}(c))}$, where $g'(g^{-1}(c))$ denotes the derivative, $g'(u)$, evaluated at $u=g^{-1}(c)$. Thus, differentiating F_Y yields:

$$f_Y(c)=\begin{cases}f_X(g^{-1}(c))\frac{1}{g'(g^{-1}(c))} & A<c<B\\ 0 & \text{else.}\end{cases} \qquad (3.8)$$

Example 3.8.2 Suppose $Y=X^2$, where X has the $N(\mu,\sigma^2)$ distribution with $\mu=2$ and $\sigma^2=3$. Find the pdf of Y.

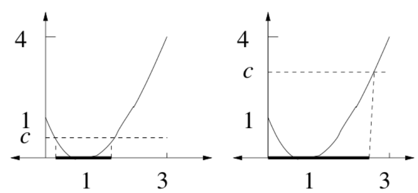
Solution: Note that $Y=g(X)$ where $g(u)=u^2$. The support of the distribution of X is the whole real line, and the range of g over this support is $\mathbb{R}_+$. Next we find the CDF, F_Y. Since $P\{Y\ge 0\}=1$, $F_Y(c)=0$ for $c<0$. For $c\ge 0$,

$$\begin{aligned}F_Y(c) = P\{X^2\le c\} &= P\{-\sqrt{c}\le X\le\sqrt{c}\}\\ &= P\left\{\frac{-\sqrt{c}-2}{\sqrt{3}}\le\frac{X-2}{\sqrt{3}}\le\frac{\sqrt{c}-2}{\sqrt{3}}\right\}\\ &= \Phi\left(\frac{\sqrt{c}-2}{\sqrt{3}}\right)-\Phi\left(\frac{-\sqrt{c}-2}{\sqrt{3}}\right).\end{aligned}$$

Differentiate with respect to c, using the chain rule and the fact: $\Phi'(s)=\frac{1}{\sqrt{2\pi}}\exp(-\frac{s^2}{2})$, to obtain

$$f_Y(c)=\begin{cases}\frac{1}{\sqrt{24\pi c}}\left\{\exp\left(-\frac{(\sqrt{c}-2)^2}{6}\right)+\exp\left(-\frac{(\sqrt{c}+2)^2}{6}\right)\right\} & \text{if } c\ge 0\\ 0 & \text{if } c<0.\end{cases} \qquad (3.6)$$

Solution. Since X ranges over the interval $[0,3]$, Y ranges over the interval $[0,4]$. The expression for $F_Y(c)$ is qualitatively different for $0\le c\le 1$ and $1\le c\le 4$, as seen in the following sketch:



In each case, $F_Y(c)$ is equal to one third the length of the shaded interval. For $0\le c\le 1$,

$$F_Y(c)=P\{(X-1)^2\le c\}=P\{1-\sqrt{c}\le X\le 1+\sqrt{c}\}=\frac{2\sqrt{c}}{3}.$$

For $1\le c\le 4$,

$$F_Y(c)=P\{(X-1)^2\le c\}=P\{0\le X\le 1+\sqrt{c}\}=\frac{1+\sqrt{c}}{3}.$$

Combining these observations yields:

$$F_Y(c)=\begin{cases}0 & c<0\\ \frac{2\sqrt{c}}{3} & 0\le c<1\\ \frac{1+\sqrt{c}}{3} & 1\le c<4\\ 1 & c\ge 4.\end{cases}$$

Differentiating yields

$$f_Y(c)=\frac{dF_Y(c)}{dc}=\begin{cases}\frac{1}{3\sqrt{c}} & 0\le c<1\\ \frac{1}{6\sqrt{c}} & 1\le c<4\\ 0 & \textit{else.}\end{cases}$$

By LOTUS,

$$E[Y]=E[(X-1)^2]=\int_0^3(u-1)^2\frac{1}{3}du=1$$