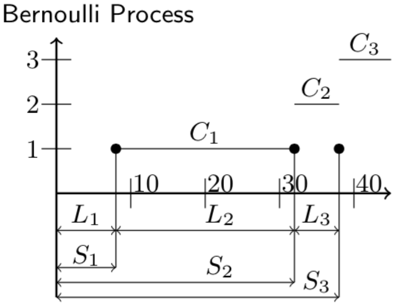
Chapter 1

- Axioms of Probability
  - $P(A) \le 1$
  - $P(A) \ge 0$
  - $P(\Omega) = 1$
  - $P(A^C) = 1 - P(A)$
  - $(AB)^C = A^C \cup B^C$
  - $(A \cup B)^C = A^C B^C$
  - $P(A) \le P(B)$ for $A \subset B$
  - $P(\emptyset) = 0$
  - $P(A \cup B) = P(A) + P(B) - P(AB)$
- Counting and Over Counting
  - balls and bins
  - poker hand
  - rearranging letters in words
  - $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
- Variance (spread of distribution)
  - $\text{Var}(X) = E\left[(X - E[X])^2\right] = E[X^2] - \mu_X^2 = \sigma_X^2$
  - Scaling
    - $\text{Var}(aX) = a^2 \cdot \text{Var}(X)$
  - Addition
    - $\text{Var}(X + b) = \text{Var}(X)$
- Conditional Probability
  - $P(A|B) = \begin{cases} \frac{P(AB)}{P(B)} & P(B) \neq 0 \\ \text{undefined} & P(B) = 0 \end{cases}$

Chapter 2

- Random Variables
  - Function on Ω, generalizes an event
- Probability Mass Function
  - $P_X(i) = P(X = i)$
  - Sum of PMF = 1
- Expected Value (average)
  - $E[X] = \mu_X = \Sigma u_i \cdot P_x(u_i)$
  - LOTUS
    - $E[Y] = E[g(X)] = \Sigma g(u_i) \cdot P_X(u_i)$
  - Scaling
    - $E[aX] = a \cdot E[X]$
  - Addition
    - $E[X + b] = b + E[X]$
- Binomial Distribution
  - Sum of n Bernoulli w.p. p
  - $S \sim \text{Bin}(n,p) = \sum_{i=1}^{n} X_i$
  - $E[S] = n \cdot p$
  - $\text{Var}(S) = n \cdot p \cdot (1-p)$
  - $P(S = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$
- Geometric Distribution
  - $Z \sim \text{Geo}(p)$
  - $E[Z] = \frac{1}{p}$
  - $\text{Var}(Z) = \frac{1-p}{p^2}$
  - $P(Z = k) = p(1-p)^{k-1}$
- Poisson Distribution
  - $X \sim \text{Pois}(\lambda)$
  - $E[X] = \lambda$
  - $\text{Var}(X) = \lambda$
  - $P(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$
  - $X \sim \text{Bin}(n,p) \approx \text{Pois}(np),\ n \gg p$
- Negative Binomial Distribution
  - Sum of Geometric
  - $S_r \sim \text{NB}(r,p)$
  - $P(S_r = n) = p^r \binom{n-1}{r-1} \cdot (1-p)^{n-r}$
  - $E[S_r] = \frac{r}{p}$
  - $\text{Var}(S_r) = \frac{r(1-p)}{p^2}$
- Bernoulli Process



$L \to Geo$
$S \to NB$
$C \to Bin$

- Independence
  - $A \perp\!\!\!\perp B \iff P(AB) = P(A) \cdot P(B)$
  - $A, B, C \perp\!\!\!\perp \iff A, B, C$ pairwise independent and $P(ABC) = P(A) \cdot P(B) \cdot P(C)$
  - Given $X \perp\!\!\!\perp Y$
    - $f(X) \perp\!\!\!\perp g(Y)$
    - $E[XY] = E[X] \cdot E[Y]$
    - $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
- Bernoulli Distribution
  - $X \sim \text{Ber}(p) = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$
  - $E[X] = p$
  - $\text{Var}(X) = p \cdot (1-p)$
- Union Bound
  - $P(A \cup B) = P(A) + P(B) - P(AB) \le P(A) + P(B)$
  - Good when $A \perp\!\!\!\perp B$, $P(A)$, $P(B)$ small
  - $P(A_1 \cup A_2 \cup \ldots \cup A_n) \le \sum_{i=1}^{n} P(A_i)$
- Network Reliability and Flow Rate
  - Use Law of Total Probability to Simplify
  - Use Union Bound for Accurate and Easy Answer
- Array Code
  - Error Detection
- Maximum Likelihood Estimation
  - Take Derivative, Set = 0 (can use ln)
  - observe $X = k$, p unknown
    - $Bin(n,p)$: $\hat{p}_{ML} = \frac{k}{n}$
    - $Geo(n)$: $\hat{p}_{ML} = \frac{1}{k}$
    - $Pois(\lambda)$: $\hat{\lambda}_{ML}(k) = k$
- Markov Inequality
  - $P(Y \ge c) \le \frac{E[Y]}{c} = \frac{\mu_Y}{c}, \quad Y \ge 0, c > 0$
- Chebychev Inequality
  - $P(|Y - E[Y]| \ge c) \le \frac{\sigma_Y^2}{c^2}$
  - Confidence Interval $\quad X \sim Bin(n,p)$
    - $P\left(|X - n \cdot p| \ge \frac{a \cdot \sqrt{n}}{2}\right) \le \frac{1}{a^2}$
    - $P\left(\left|\frac{X}{n} - p\right| < \frac{a}{2\sqrt{n}}\right) \ge 1 - \frac{1}{a^2}$
    - $P\left(p \in \left[\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}}\right]\right) \ge 1 - \frac{1}{a^2}, \quad \hat{p} = \frac{X}{n}$
- Law of Total Probability
  - $P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A|E_i), \quad \bigcup_{i=1}^{n} E_i = \Omega$
- Bayes Formula
  - $P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B) \cdot P(A|B)}{P(A)}$
- Hypothesis testing
  - $P_{\text{false alarm}} = P(\text{declare } H_1 | H_0)$
  - $P_{\text{miss}} = P(\text{declare } H_0 | H_1)$
  - $P_e = P(H_0) \cdot P_{\text{false alarm}} + P(H_1) \cdot P_{\text{miss}}$
  - $\Lambda(j) = \frac{P(X = j | H_1)}{P(X = j | H_0)}$
  - Maximum Likelihood
    - $\Lambda(X) \underset{H_0}{\overset{H_1}{\gtrless}} 1$
  - Maximum a Posteriori Probability
    - $\Lambda(X) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)} = \frac{\pi_0}{\pi_1}$

deb2 ECE313 Notes with HKN@UIUC ©