Physics notes for edX 8.01x "Classical Mechanics"

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Contents

Part I

Introduction and mathematics

Chapter 1: Introduction

Hello!

These are the notes I've taken, as a student, while taking this course.
As such, they may contain errors, and are most certainly incomplete (there's no way you could learn the content of this course from these notes!), and so on. Keep in mind that I'm just a student – if a reliable source contradicts something in here, that source is most likely correct!

I will use some citations in these notes, which aren't really intended as citations the way you'd use them in a proper scientific paper. Instead, they are used to show where I got the majority of the information for a small part of the notes, in a less strict and more "relaxed" way than you would see in a published paper.
Wherever there are no citations at all, the source is by default the week's lecture videos and/or other course materials. I will only add citations for external sources (such as other books, web pages etc.) as there would be way too many citations otherwise.

Any unsourced references to "the book" or "the textbook" is to (of course) the book used by the course: "Classical Mechanics" by Peter Dourmashkin.

I write down my thought processes and solutions to homework and exam problems while solving them. I look through them after having read the staff's official solutions (available by going back to the homework after deadline and clicking "Show answer(s)"); however, unless my answers are incorrect (such that I just got lucky with the green checkmark), I don't really revise them after the fact.
Therefore, the official solutions are often neater than mine! I still write these, mostly as a learning tool for myself, though.
Feel free to look through them, but be aware that they may be overly convoluted at times!

Chapter 2: Vector mathematics

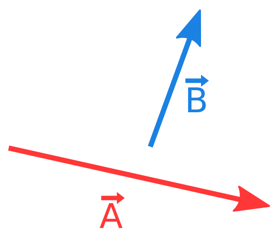
Many quantities in physics can be represented as a single number (a *scalar*): mass, temperature and distance are some examples. However, other quantities - such as force and velocity - have a *direction* as well as a magnitude. These quantities are represented by *vectors*, which are made up of two parts: magnitude (the "size" of the vector) and direction.
This distinction is what makes up the difference between velocity (a vector) and speed (the *magnitude* of velocity, i.e. a number/scalar that is always either zero or positive). The two words are often used interchangeably outside of physics.

To make it easy to differentiate between scalars (e.g. a) and vectors, we write an arrow above all vectors: $\vec{a}$

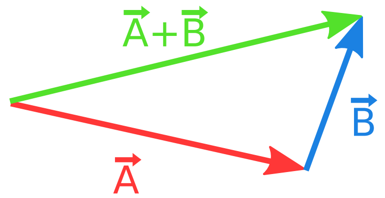
Vectors are generally drawn as arrows, where the length of the arrow is the *magnitude*, and the direction it's pointing is the direction. The direction can be represented by the angle formed between the positive x-axis and the vector (measured counter-clockwise), in much the same way as the trigonometric functions work in the unit circle, which makes it easy to apply those functions to vectors.

2.1 Vector addition and subtraction

Perhaps the simplest way to graphically add vectors is to move the *tail* of one of them to the *head* (i.e. the pointy end) of the other one - order doesn't matter - and then draw the sum as the vector from the tail of the combination to the head of the combination. Images say more than words for graphical problems, so let's try it. We start out with two separate vectors:



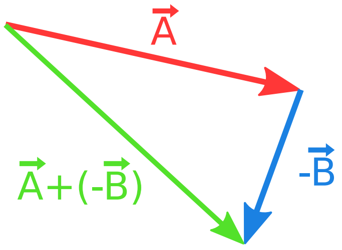
Move them together head-tail and draw the sum as the vector between the "total tail" and "total head":



This method works equally well with any number of vectors - just link them up tail-head as above; the sum vector will be the vector from the tail of the "group" to the head of the group, exactly as above.
The reason that we can do this is that vectors are completely characterized by their magnitude and direction - location in space is irrelevant. Two vectors with the same magnitude and direction are always equal, irrespective of their location. Therefore, we can move the vectors around to help us visualize vector addition.

So why does this method work? Well, imagine first traveling along $\vec{A}$, and then along $\vec{B}$. You would end up at $\vec{A} + \vec{B}$. It really is that simple!

As for subtraction, one way to think about it is to *add* the *negative* vector instead, i.e. $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. So what is $-\vec{b}$, exactly? It's simply the vector b with the direction reversed. The magnitude is the same; the only difference is that you draw the arrow on the opposite side of the line.



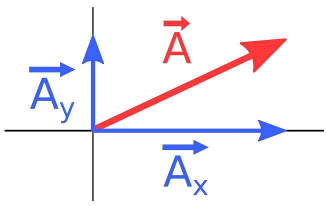
Note that the vectors in the above graphic are the same vectors as in the addition example; however, the blue vector graphic is now $-\vec{b}$, which is then added to $\vec{a}$ to produce the result $\vec{a}-\vec{b}$.

That covers the basics of the graphical way to add and subtract vectors; what about the mathematical way? Well, in order to cover that, we need to first introduce the concept of vector components, and *vector decomposition*.

What happens if we multiply a vector by a scalar (a dimensionless number)? Well, vectors work like numbers in that regard, in that $2 \cdot \vec{a} = \vec{a} + \vec{a}$. Draw that addition out on paper, and you'll see that the geometrical meaning is that the magnitude doubles, but the direction is unchanged.
As we saw above, when multiplying by negative numbers, the magnitude changes as appropriate, but the direction is flipped. $-3 \cdot \vec{a}$ would be a vector that is three times a big as $\vec{a}$, but points in the opposite direction. Multiplication by fractions (and even irrationals) work just as well, too.

2.2 Vector components

We can represent a vector as the sum of multiple vectors. In the most useful case, we can represent it as the sum of one vector "per dimension" the vector requires.
For a 2D vector in the Cartesian coordinate system, we can break a vector $\vec{A}$ into two vectors $\vec{A_x}$ and $\vec{A_y}$, such that the two vector components' directions are perpendicular; $\vec{A_x}$ points along the positive x axis, while $\vec{A_y}$ points along the positive y axis.



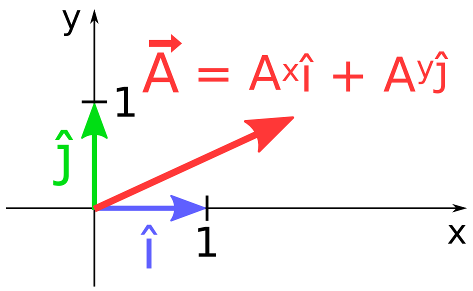
Imagine moving $\vec{A_y}$ to the head of $\vec{A_x}$; if we draw the sum vector, we would get exactly $\vec{A}$. That is, $\vec{A} = \vec{A_x} + \vec{A_y}$.

We can also think of the magnitude of the partial vectors, A_x and A_y. Those magnitudes represent how long the vector is in each dimension separately, and are very useful - more useful than the partial vectors themselves.

So *why*, or where, is this useful? For one, when this technique is combined with unit vectors (below), it makes mathematical manipulation of vectors much easier. Vector decomposition can also be a powerful tool in solving physics problems, as it can break down problems in multiple dimensions to multiple smaller, one-dimensional problems, which are often easier to solve.

2.2.1 Unit vectors

Unit vectors are simple, but the concept is still very powerful. We define unit vectors to be vectors of magnitude 1 that point in the direction of their respective axes. The unit vector $\hat{\imath}$ ("i-hat"), points in the positive x direction, while $\hat{\jmath}$ points in the positive y direction, and $\hat{k}$ in the positive z direction. These unit vectors are also sometimes known as $\hat{x}$, $\hat{y}$ and $\hat{z}$, respectively. In addition, the "hat" suffix is sometimes called "roof", as in "x roof".



Let's go back to the vector components above. The two components of $\vec{A}$ are both vectors, $\vec{A_x}$ and $\vec{A_y}$. We can write these components as the product of their magnitude (a scalar) and the unit vector in that direction (which is obviously a vector). Let's call the magnitude of vector $\vec{A_x}$ simply A_x.
Think of the vector $\vec{A_x}$ in the vector components figure as a longer version of the unit vector $\hat{i}$ - that is, it's the result of scalar multiplication of a magnitude, A_x, and the unit vector $\hat{i}$: $\vec{A_x} = A_x\hat{i}$
The same can be said for the component vector $\vec{A_y}$, which can be written as $A_y\hat{j}$.

Since we know that the sum of these two vectors equal $\vec{A}$, we now have

$$\vec{A} = \vec{A_x} + \vec{A_y}$$

$$\vec{A_x} = A_x\hat{i}$$

$$\vec{A_y} = A_y\hat{j}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

We have now written vector $\vec{A}$ as the sum of two separate, one-dimensional vectors. If we are working on a two-dimensional projectile motion problem, we can now calculate the motion along the x-axis as one problem, then calculate the motion along the y-axis as a separate problem, and then add the two together to get the combined motion. Doing so is generally much easier than solving the two-dimensional problem as-is.

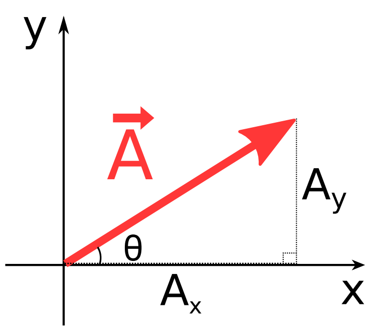
We can also write that last equation ($\vec{A} = A_x\hat{i} + A_y\hat{j}$) as

$$\vec{A} = \langle A_x, A_y \rangle$$

Here, we use a more compact notation, where the x and y components are listed, with the implicit meaning that me can construct the vector $\vec{A}$ by multiplying them by their respective unit vector, and adding the results.

2.2.2 Vector decomposition

Now that we know about vector components and unit vectors, let's apply these concepts, and also see how to actually do this decomposition mathematically. We've yet to see it from anything but a geometrical perspective. Sure, such perspectives are very useful for intuition, but in return, they are mostly useless for precise computation.
We can still use a picture to easier understand the mathematical decomposition, though:



Here, we see that A_x, A_y and $\vec{A}$ form a right triangle.

Via the Pythagorean theorem, we see that

$$|\vec{A}| = \sqrt{{A_x}^2 + {A_y}^2}$$

That is, the magnitude of the vector $\vec{A}$ is given by the sum of the squares of the component magnitudes.

If the vector were three-dimensional, we would simply add a "$+{A_z}^2$" under the square root; the same goes for even higher dimensions.

However, while that is extremely useful, it doesn't help us *find* A_x and A_y to begin with!
Let's stop stalling. Again, note how the three lines (if we consider the vector a line for now) form a right triangle. If we know the magnitude of the vector, i.e. the length of the hypotenuse, and the angle θ (theta) between the vector and the positive x axis, we can use trigonometry to find the components:

$$\cos\theta = \frac{A_x}{|\vec{A}|}$$

$$\sin\theta = \frac{A_y}{|\vec{A}|}$$

These come from the definitions of the sine and cosine functions - the cosine is the adjacent side (A_x) over the hypotenuse (the magnitude of $\vec{A}$), while sine is opposite (A_y) over hypotenuse.
We can now simply solve these equations for the components by multiplying both sides (of both equations) by the magnitude:

$$A_x = |\vec{A}|\cos\theta$$

$$A_y = |\vec{A}|\sin\theta$$

2.2.3 Example

Consider a vector with length/magnitude $|\vec{A}| = 5$ meters and angle $\theta = 30° = \frac{\pi}{6}$ radians from the x axis.
In other words, it's pointing "to the right" and slightly upwards, as seen from the origin.
In order to write this as a set of components, we can simply calculate the components as above:

$$A_x = 5\cos\left(\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2} \approx 4.33 \text{ meters}$$

$$A_y = 5\sin\left(\frac{\pi}{6}\right) = 2.5 \text{ meters}$$

We can therefore write the vector as either of these forms (keeping in mind that we rounded the x value):

$$\vec{A} = 4.33\hat{\imath} + 2.5\hat{\jmath}$$
$$\vec{A} = \langle 4.33, 2.5\rangle$$

2.3 Vector addition and subtraction, continued

We now know what we need in order to talk about the mathematical way of vector addition and subtraction. Thankfully, once we've separated a vector into its components, addition and subtraction becomes incredibly easy!

Let's take the example of adding two vectors:

$$\vec{a} = 5\hat{\imath} + 3\hat{\jmath}$$
$$\vec{b} = 2\hat{\imath} - 1\hat{\jmath}$$

$$\vec{a} + \vec{b} = (5\hat{\imath} + 3\hat{\jmath}) + (2\hat{\imath} - 1\hat{\jmath})$$

$$\vec{a} + \vec{b} = 7\hat{\imath} + 2\hat{\jmath}$$

Yes, it's that easy - just add the parts separately, and you have the answer. Subtraction works as you would expect at this point. Let's try the more compact notation; the vectors used are the same as the ones in the addition example above.

$$\vec{a} = \langle 5, 3 \rangle$$
$$\vec{b} = \langle 2, -1 \rangle$$
$$\vec{a} - \vec{b} = \langle 5 - 2, 3 - (-1) \rangle$$
$$\vec{a} - \vec{b} = \langle 2, 4 \rangle$$

2.4 The dot product / scalar product

Now that we have addition and subtraction down, let's have a look at vector multiplication. There are two ways to multiply two vectors: the dot product, and the cross product.
The dot product gives a scalar result (a single number), and is therefore sometimes called the *scalar product*, while the result of a cross product is a third vector. Note that the two aren't simply different ways of doing the same thing, but fundamentally different operations, with completely different meanings and results.
The dot product is usually considered easier, so let's tackle that one first.

First off, notation. The name dot product comes from the notation used for the operation:

$$\vec{a} \cdot \vec{b}$$

Before explaining the purpose of the dot product, let's go with the definitions and an example. It's rather easy to calculate, at least when you have the vector components.

Two definitions of the dot product, for two vectors with components $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, are:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

Or, generally, for a vector with n components/n dimensions:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Let me just add one last definition before we look at this from a geometrical point of view:

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

Keeping in mind that the angle between a vector and itself must be 0, and that $\cos(0) = 1$, this should make sense if you believe the formulas above.

Therefore, the magnitude of a vector can also be written as:

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

Like vector addition, the dot product operation is commutative; that is, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. As we'll see later, however, the same is *not* true for the cross product!

Now, if you've never seen the dot product before, I would assume you are a bit confused at this point. No worries - the above is meant as a "reference", not a tutorial. Let's get to the geometrical interpretation.

2.4.1 Geometrical interpretation of the dot product

Let's repeat the second definition of the dot product from above:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

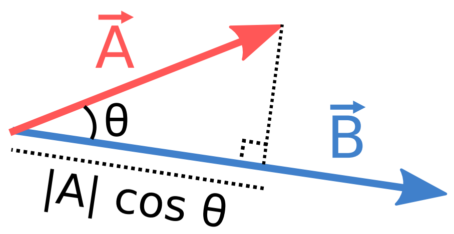
Note that it's clear that the sign of the dot product is determined by (the cosine of) the angle and by that alone; the other terms are both magnitudes, which are always positive, by definition.
If the dot product is positive, the two vectors are pointing "mostly" in the same direction, i.e. the angle between them is less than 90 degrees; the angle is acute. The dot product is zero if and only if the two vectors are perpendicular, as that's where the cosine term would be zero, and make the dot product zero as well.

And, if the angle is greater than 90 degrees, so that the vectors are pointing in different directions (with an obtuse angle between them), the dot product would be negative, as the cosine of the angle would be negative.

In a bit more concise form:

$$\vec{a} \cdot \vec{b} = \begin{cases} > 0 & \text{for } \theta < 90°\text{, acute angle} \\ 0 & \text{for } \theta = 90°\text{, right angle} \\ < 0 & \text{for } \theta > 90°\text{, obtuse angle} \end{cases}$$

Now, let's look at this from a geometrical perspective, as promised.



Here, we have two vectors, $\vec{A}$ and $\vec{B}$. We draw a line from $\vec{B}$, *perpendicular* to $\vec{B}$, that meets $\vec{A}$ at the head. By definition, the angle between the line and $\vec{B}$ is 90 degrees.

Let's now use the definition of the cosine ($\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$) to find the length of the segment denoted by the dotted line, which we'll denote $|A_B|$, or "the projection of $\vec{A}$ onto $\vec{B}$":

$$\frac{|A_B|}{|\vec{A}|} = \cos\theta$$

Solve for $|A_B|$ by multiplying both sides by the magnitude of $\vec{A}$:

$$|A_B| = |\vec{A}| \cos\theta$$

Now we know, as the picture suggests, that the projection of $\vec{A}$ onto $\vec{B}$ is given by the magnitude of $\vec{A}$ times the cosine of the angle that separates the two vectors.
Now, remember the definition of the dot product:

$$\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$$

We can rearrange the terms to give:

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cos\theta\ |\vec{B}|$$

Using the identity just above, this is the same thing as:

$$\vec{A} \cdot \vec{B} = |A_B|\ |\vec{B}|$$

So, we see that the geometrical interpretation of the dot product is, in one way to put it, the length that $\vec{A}$ goes in the direction of $\vec{B}$ (or the projection of $\vec{A}$ onto $\vec{B}$), times the magnitude of $\vec{B}$.

If this doesn't quite make sense, it will probably be easier to grasp when in actual use, such as when multiplying a force vector with a displacement vector to find work.

Another way (the same way, really) to think about it is this: imagine that the vector $\vec{B}$ is horizontal, i.e. parallel with the x axis, pointing to the right (the positive x axis).
Now, $|A_B|$ is just the x component of $\vec{A}$! Therefore, in general, we can think of $|A_B|$ as the "B direction component" of $\vec{A}$, so the dot product is the "B direction component" of $\vec{A}$ times the magnitude of $\vec{B}$.

2.5 The cross product / vector product

As the second naming suggests, this method of multiplying two vectors yields a third vector, namely one that is perpendicular to BOTH the vectors multiplied.
The notation used is, as the *first* naming suggests, a cross:

$$\vec{A} \times \vec{B} = \vec{C}$$

The cross product is only properly defined for 3- and 7-dimensional vectors. We will of course only work with the former in this course.

Okay, so we know that the cross product gives a third vector, that is perpendicular to both the vectors multiplied. It's also very important to know that the cross product operator is not commutative. That is,

$$\vec{B} \times \vec{A} \neq \vec{A} \times \vec{B}$$

It is *anti-commutative*:

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$$

It also works alongside scalar multiplication, so that

$$(r\vec{A}) \times \vec{B} = \vec{A} \times (r\vec{B}) = r(\vec{A} \times \vec{B})$$

Okay, okay, enough with the side-definitions. What is the definition of the cross product?
Well, as previously, there are two definitions we'll use: one that uses magnitude and angle, and one that uses vector components. The latter is rather complex, but here's the first one, to begin with:

2.5.1 Definition: Magnitude and angle

$$\vec{A} \times \vec{B} = |\vec{A}|\ |\vec{B}|\ \sin\theta\ \hat{n}$$

This is, you might note, very similar to the dot product, except it has a sine rather than a cosine, and also has a direction (we'll get to that soon), since it's a vector.
If we want just the magnitude of the cross product, it's eerily similar to the dot product:

$$|\vec{A} \times \vec{B}| = |\vec{A}|\ |\vec{B}|\ \sin\theta$$

The sine instead of the cosine is now the only difference.

One way to think about it is that the *dot* product measures "how parallel" two lines are. When completely parallel, the dot product is at its maximum. (Mathematically, the $\cos\theta$ term is 1, its maximum, when $\theta = 0$, i.e. the angle between the two is 0, i.e. they are completely parallel.)
The dot product is then zero when the vectors are perpendicular (not parallel at all), and negative when they point in different directions ($\theta > 90°$).

What about the *magnitude of* the cross product (not just the cross product itself!)? It's pretty much the opposite: you can think of it as measuring "how perpendicular" two vectors are. With two fully parallel vectors, the cross product equals zero (the angle $\theta = 0$, and $\sin(0) = 0$). When they are perpendicular, the cross product is at its maximum, since $\sin(90°) = 1$.

Okay, so that covers the magnitude, what about the direction, $\hat{n}$? As the hat/roof suggests, that is a unit vector... but in what direction? Hang on; we'll discuss that in the geometrical interpretation, after the component definition.

2.5.2 Definition: components

The second definition, using components - in its worst possible form (it'll get better soon) - is:

For $\vec{C} = \vec{A} \times \vec{B}$:

$$\vec{C} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$

Oh dear. Thankfully, there are ways to remember the above. First, what we do - for the mnemonic to work - is to rename the variables, and instead compute

$$\vec{A} = \vec{B} \times \vec{C}$$

Without this change, the mnemonic is probably *harder* to remember than the above mess, so go with me. After that change, we write the component equations separately, instead of all on one line. The sums-of-products are the same as above, though, if we account for the variable renames:

$$A_x = B_y C_z - B_z C_y$$
$$A_y = B_z C_x - B_x C_z$$
$$A_z = B_x C_y - B_y C_x$$

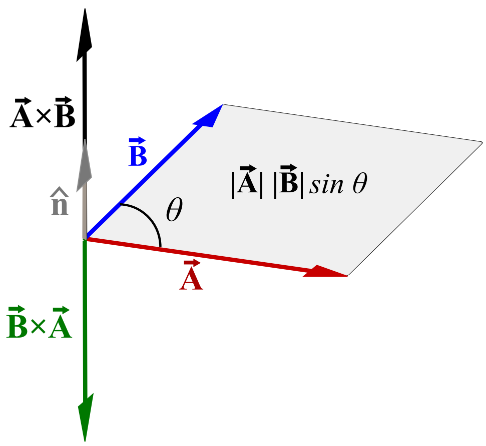
Still awful. Heck, it might just look harder like this! Don't panic - there's a pattern: **XYZZY**.
Note that the subscripts of the first equation spell XYZZY, and that the vector order is alphabetical for all equations ($A_* = B_* C_* - B_* C_*$).
That makes the first one relatively easy (compared to memorizing the entire thing), but what about the rest?
A-ha! Here's the pattern: to convert from the first equation to the second, "increase" the subscript by one letter; if at z, go back to x. That is, A_x becomes A_y (one letter ahead), A_y becomes A_z (one letter ahead), and A_z wraps around and becomes A_x; the same thing applies to the B and C components.
The same method is used to convert from the second to the third equation. Have a close look at them and make sure you realize this is true!

As an additional sanity check, note that the reverse of the first subscript pair is the one you then subtract: yz - zy, zx - xz and xy - yx. (Look at the subscripts in the three equations again if you don't get what I mean.)

Since $\vec{C}$ is supposed to be perpendicular to both $\vec{A}$ and $\vec{B}$, we can use the dot product to check whether our answer appears to be correct or not. Remember that the dot product is always zero for two perpendicular vectors - so we could check our work by testing that the two dot products $\vec{A} \cdot \vec{C}$ and $\vec{B} \cdot \vec{C}$ are both zero. If either or both is *not* zero, the cross product calculation was done incorrectly. If both *are* zero, that doesn't guarantee that the answer is correct, however. More on that later (there are two vectors perpendicular to both $\vec{A}$ and $\vec{B}$: $\vec{C}$ and -$\vec{C}$).

2.5.3 Geometrical interpretation of the cross product

Let's try to make sense of all the above.
We can imagine a parallelogram plane in 3D space, with two sides $\vec{A}$ and $\vec{B}$. This is certainly one of those times where an image is worth (more than) a thousand words:



(Image license is CC0. By Wikimedia user Svjo; modified by me.)

Here, we can see several key things:

- The area of the parallelogram is the magnitude of the cross product, $|\vec{A}|\,|\vec{B}|\ \sin\theta$ (this is one of the ways to calculate the area of a parallelogram).
- The unit vector $\hat{n}$ (and therefore the cross product) is indeed perpendicular to *both* $\vec{A}$ and $\vec{B}$; another way of saying this is that it's perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$.)
- $\vec{B}\times\vec{A}$ points in the opposite direction as $\vec{A}\times\vec{B}$ does, since $\vec{B}\times\vec{A} = -(\vec{A}\times\vec{B})$. (Remember that the negative of a vector is a vector pointing in the opposite direction, i.e. with the arrowhead on the other side of the line.)

However, we also see a problem: if both the upwards-pointing vector $\vec{A}\times\vec{B}$ and the downwards-pointing vector $\vec{B}\times\vec{A}$ are perpendicular to both $\vec{A}$ and $\vec{B}$, and they are (obviously!) not equal... how do we know which of the two to use? How do we identify which is which?

We'll have to use a rule known as the *right-hand rule* for this.
The right-hand rule is often taught in different ways, all with the same end result. The way I prefer is one using your whole right arm, simply because because I find it easier.
The rule is this: if your entire right arm points along the first vector ($\vec{A}$), angle your index through pinky (four fingers) in the direction of the second vector ($\vec{B}$); if this means you have to turn your arm, do so. Now, with your arm pointing along vector $\vec{A}$ and your fingers pointing along vector $\vec{B}$, extend the thumb straight out. Your thumb should now be perpendicular to both $\vec{A}$ and $\vec{B}$, and point along $\vec{A}\times\vec{B}$ (and not $\vec{B}\times\vec{A}$ - try swapping the arm and the finger vectors, and you'll find that the result is the thumb pointing in the opposite direction!).

That is:

- Right hand/entire arm points parallel to $\vec{A}$
- Fingers are curled to point parallel to $\vec{B}$
- Thumb now points parallel to $\vec{A}\times\vec{B}$ (perpendicular to both $\vec{A}$ and $\vec{B}$).

Note that this is only true for certain coordinate systems, namely *right-handed* ones. These are the ones used by all sane persons in physics, and the only ones used in this document.

One useful definition to test whether a system is right-handed or not, is that this SHOULD be true:

$$\hat{\imath}\times\hat{\jmath} = \hat{k}$$

If the above is false for your coordinate system (i.e. you get $-\hat{k}$ instead), your coordinate system is left-handed and simply won't work according to the definitions generally used in physics!

Part II

Lecture notes

Chapter 3: Week 1

3.1 Lecture 1: Units, dimensions and scaling arguments

The lecture begins with a quick intro to units, followed by a movie showing 40 orders of magnitude (from inside a proton, to a perspective 100 million lightyears from Earth).

After that, we begin talking about dimensional analysis and the metric system. The three SI base units, and their respective dimensions, are introduced: the meter (m) for measuring length [L], the second (s) for measuring time [T] and the kilogram [kg] for measuring mass [M].
We use the square brackets to notate that we are not talking about a *unit*, but a *dimension* - such as the three shown above (length, time, mass), or speed, acceleration, temperature, charge, etc.
One dimension can have many units (meters, yards, kilometers, miles and light-years all describe length), but one unit always describes exactly one dimension. (If it were not so, we could perhaps measure temperature in meters, or length in amperes!)

As an important side note, keep in mind that capitalization is extremely important in physics: 1 mm is 1/1000 of a meter, a very short distance, while 1 Mm is a million meters, or 1000 kilometers - a distance larger than many countries. The same goes for units: k means kilo (the prefix for 1000), while K means Kelvin, a unit of absolute temperature. Capital G is the symbol for the gravitational constant (about $6.67 \cdot 10^{-11}$ $\mathrm{N(m/kg)^2}$), while a lowercase g is the symbol for the gravitational acceleration near Earth, about 9.8 $\mathrm{m/s^2}$. The two are related, but still completely different, so they must not be confused for one another.

Many other units can be described as combinations of the three base units shown above, for example:

$$[\text{speed}] = \frac{[\text{L}]}{[\text{T}]} \tag{3.1}$$

All units of speed are in length per time - meters per second, kilometers per hour, inches per year, etc. Therefore, we say that the *dimension* of speed is the dimension of length per time, as shown above in a more mathematical notation.

Other examples are:

$$[\text{volume}] = [\text{L}]^3 \tag{3.2}$$

$$[\text{density}] = \frac{[\text{M}]}{[\text{L}]^3} \tag{3.3}$$

$$[\text{acceleration}] = \frac{[\text{L}]}{[\text{T}]^2} \tag{3.4}$$

The last one may seem strange if you have not studied physics before - an example of a unit of acceleration is meters per second squared, or meters per second per second (m/s^2 or $(m/s)/s$). It's quite simple though, once you get past the wording of it.
When measuring a change in something, we always add another "per second" (or another unit of time), so when the unit we are measuring the change in is already meters per second, we get meters per second per second.
For example, a car might start out at 0 m/s (standing still), and be moving at 5 m/s one second later. In that case, the car's average acceleration is 5 meters per second per second.

3.1.1 Uncertainty, and an experiment

Prof. Lewin stresses very strongly: "Any measurement you make without knowledge of its uncertainty is *meaningless*". He repeats this a few times throughout the lecture.

Using two rulers accurate to about ± 1 mm, he measures a student first standing up, and then lying down – after measuring an aluminum bar, to show that the two rulers agree. They do, within 1 mm - the uncertainty. The results of the experiment are a bit surprising: the student was about 2.5 cm ± 0.2 cm taller lying down! The reason being that gravity compresses our bodies slightly when standing up, while that effect would be gone lying down (since gravity then acts perpendicular to our length).

Because of the small uncertainty, compared to the relatively large height difference, we can be sure that the student indeed was taller lying down. Had the uncertainty of the measurement instead been ±3 cm, how could we know? The two values 185.7 cm and 183.2 cm are indistinguishable from each other if measured with a meter stick where the uncertainty is ±3 cm! The first could be anything between 182.7-188.7 cm, while the second could be anything from 180.2-186.2 cm. There is considerable overlap, which means the two could indeed be equal – we could only know by making a more accurate measurement.

Calculating uncertainty properly can be quite complex, and the correct methods will not be taught or used in this class, as it is simply out of the scope. Instead, we use simplified methods, "poor man's" as the professor called them.

Uncertainty in addition and subtraction

For addition and subtractions, it couldn't be much easier: the uncertainty of the sum or difference is simply the sum of the two uncertainties:

$$(A \pm a) + (B \pm b) = (A + B) \pm (a + b) \tag{3.5}$$

$$(A \pm a) - (B \pm b) = (A - B) \pm (a + b) \tag{3.6}$$

You can find this result by calculating with the extremes. For example, for adding 1.5 ± 0.003 m + 3 ± 0.005 m:

$$\text{min} = 1.497 \text{ m} + 2.995 \text{ m} = 4.492 \text{ m} \tag{3.7}$$

$$\text{max} = 1.503 \text{ m} + 3.005 \text{ m} = 4.508 \text{ m} \tag{3.8}$$

Both results are 0.008 m away from $3 + 1.5 = 4.5$, and so the uncertainty is ±0.008 m, the sum of the two uncertainties. If we use the same method where we subtract, we will find the same result: the uncertainties *add*, and the results will differ from the simple difference by $+0.008$ and -0.008, respectively.

Uncertainty in multiplication and division

First, keep in mind that some numbers are exact. If we multiply a length by 2 – a constant, not a measurement – then the length and the uncertainty are both multiplied by 2 exactly. No further work is necessary.
If the two are measurements, however, care needs to be taken.

One way to get a *rough* uncertainty value when dividing is to choose the largest and smallest values, respectively, for the numerator and denominator, and then subtract the nominal value from that.
As an example, let's say we want to calculate the approximate gravitational acceleration of the Earth based on measurements of the time for an object to fall from a certain height. The equation used is

$$g = \frac{2h}{t^2} \tag{3.9}$$

The 2 here is an exact value, so we don't need to worry about it.
If the height is $3.000 \pm 0.003\,\mathrm{m}$ and the time taken is $0.781 \pm 0.002\,\mathrm{s}$, we then find:

$$g = \frac{2 \cdot 3.000\,\mathrm{m}}{(0.781\,\mathrm{s})^2} \approx 9.8367\,\mathrm{m/s^2} \approx 9.84\,\mathrm{m/s^2} \tag{3.10}$$

We can then calculate the uncertainty as mentioned above. For the numerator, we add the +0.003 m, and in the denominator, we subtract the −0.002 s. Finally, we subtract the nominal value that we found above.

$$\text{error} = \frac{2 \cdot 3.003\,\mathrm{m}}{(0.779\,\mathrm{s})^2} - g = 9.8971\,\mathrm{m/s^2} - 9.8367\,\mathrm{m/s^2} = 0.0604 \approx 0.06\,\mathrm{m/s^2} \tag{3.11}$$

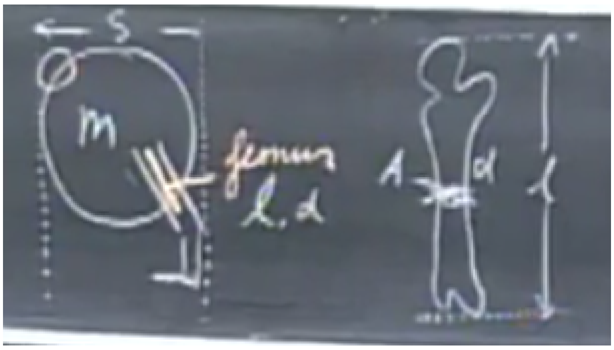
There has not yet been an example with multiplication used in the course, but I would assume that you still try to find the maximum possible value (by choosing the maximum for both terms) and then subtract the nominal value, just as above.

3.1.2 Scaling arguments and Galileo Galilei

Long ago, Galileo Galilei asked himself: why are the mammals the sizes they are, and not much bigger? The short version of a possible answer he came up with is that if they were more massive, their bones would break. Below is a more detailed analysis of what he might have thought about.

Say we have a mammal. It has a size S - very roughly defined, of course: there is no single length that defines the actual size of an animal properly. Let's just say that a mouse is perhaps 10 cm (plus the tail), and a horse is couple of meters – and let's not worry about the details.

The animal has a thigh bone, or *femur*, of length ℓ, and a thickness d (at the thinnest point). The cross-sectional area at that point is A. We can safely say that $A \propto d^2$ (A is proportional to d squared): doubling d will multiply the cross-sectional area by 4. We call the mass of the animal m.



Let's now have a look at a scaling argument.
We assume that the length of the femur scales linearly with the size of the animal. That is, if the animal is twice as large as another, its femur will be twice as long as the other animal's femur. A reasonable assumption, one would think.

We then assume that the animal's mass is proportional to the cube of the size – also very reasonable, as the size to the third is related to the animal's volume. Twice the volume, twice the mass, assuming the density is similar, of course.
Because of the previous relationship ($\ell \propto S$), this also implies that the mass is proportional to the length of the femur cubed. In mathematical notation, so far we have:

$$\ell \propto S \tag{3.12}$$
$$m \propto S^3 \propto \ell^3 \tag{3.13}$$

The *pressure* on the femur is proportional to the weight of the animal, divided by the femur's cross sectional area. The weight (which the course will talk about later) is proportional to the mass, and as stated earlier, A is proportional to d^2, so we have

$$\text{pressure} \propto \frac{m}{A} \propto \frac{m}{d^2} \tag{3.14}$$

Because the bones will break if the pressure on them is too great, m cannot increase without d^2 increasing by the same factor, if the animal is fairly close to the breaking limit already. This is key in this argument.

Because of this, we find

$$m \propto d^2 \tag{3.15}$$

... or the above cannot be true. Combining equation (3.13) with equation (3.15) just above, we find

$$d^2 \propto \ell^3 \tag{3.16}$$

or, taking the square root of both sides,

$$d \propto \ell^{3/2} \tag{3.17}$$

The above is the result we have been looking for. What this means is that if we have two animals, one being 10 times larger than the other (S being 10 times larger, which implies ℓ being 10 times larger via (3.12), via the above relation, the diameter of the femur d must be $10^{3/2} \approx 32$ times greater!
If we compare e.g. a mouse and an elephant, the difference in size being perhaps 100 times, via the same relationships, d must be $100^{3/2} = 1000$ times greater for the elephant!

Galileo Galilei may have thought this to be a good explanation as to why mammals are the size they are, and not much bigger: much larger animals would have bones so large, that they barely consist of anything else than bones to hold their weight up. Let's see if this appears to be correct by making some calculations on actual measurements of animal femurs.

If we take equation (3.17) and divide both sides by ℓ, we find

$$\frac{d}{\ell} \propto \sqrt{\ell} \tag{3.18}$$

This is then plotted from the professor's measurements of the bones. If the above is correct, we would expect that if ℓ is 4 times greater (such as a horse vs a raccoon), d/ℓ should be $\sqrt{4} = 2$ times greater.

The professor then showed the result of the experiment, by measuring these values (d and ℓ) for bones from various animals: a mouse, an opossum, a raccoon, an antelope, a horse, and an elephant. There was no evidence that the ratio of d/ℓ was different as we would have been expected. Even for the case of a mouse vs an elephant, where the difference in size (and thus ℓ) would be about a factor of 100, so that we expect d/ℓ for the elephant to be about 10 times greater than for the mouse, we find less than a factor of two!
Similar relationships were shown between all animal sizes: in no case was d/ℓ significantly different, as the hypothesis predicted. It looks like we, and Galilei, must admit defeat. The hypothesis simply doesn't hold up to experiment!

3.1.3 Dimensional analysis

Let's now look at some basic kinematics (the physics of motion) and dimensional analysis in closer detail. We drop an object, such as an apple, from a height h, and use a stopwatch to measure the time t before it hits the ground. How does the time t relate to the height h?

We can assume that the time is proportional to the height, to some unknown power, which we will call α:

$$t \propto h^\alpha$$

The mass of the apple might matter, so we might expect to find it to be proportional to the mass to some unknown power β:

$$t \propto h^\alpha m^\beta$$

Finally, it might be related to the Earth's gravitational acceleration g (not to be confused with the gravitational constant G; both of these will be introduced properly later in the course):

$$t \propto h^\alpha m^\beta g^\gamma \tag{3.19}$$

We can now start trying to figure this out. We know that the left-hand side has the dimension of time, [T]. This means that the product on the right-hand side must also have the dimension of time. Using the dimensional analysis notation, we must have

$$[\mathrm{T}]^1 = [\mathrm{L}]^\alpha [\mathrm{M}]^\beta \left(\frac{[\mathrm{L}]}{[\mathrm{T}]^2}\right)^\gamma \tag{3.20}$$

... where we have simply replaced the variable names with their respective dimensions, the dimension of h being length, m being mass, and g being acceleration (length per time^2).

We can now start working. There is only one [M] in this equation, and it's on the right-hand side. There is no possible way to get it to cancel out with anything else, so β must be 0 so that it disappears "by itself", so to speak.

We have two [L] on the right hand side, but there is no [L] on the left-hand side. That means that the two must cancel each other out, in one way another. That is,

$$\alpha + \gamma = 0$$

must be true.

Finally, we have [T] to the power one on the left-hand side, and to the power -2γ (since it is in the denominator, it is negative) on the right-hand side, and the two must be equal. All in all, we find

$$\begin{aligned} \beta &= 0 \\ \alpha + \gamma &= 0 \\ -2\gamma &= 1 \end{aligned}$$

We can solve the last equation for γ, and stick that value into the second equation, to find the final answers:

$$\begin{aligned} -2\gamma &= 1 \\ \gamma &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \alpha - 1/2 &= 0 \\ \alpha &= \frac{1}{2} \end{aligned}$$

And, so, we find these values, and these relationships with the variable names we had chosen earlier:

$$t \propto h^{1/2} g^{-1/2} \tag{3.21}$$

$$t \propto \sqrt{\frac{h}{g}} \tag{3.22}$$

Since the meaning of a proportionality is that some (still unknown) constant multiplies the value, we can write this as an equality with an unknown constant C:

$$t = C\sqrt{\frac{h}{g}} \tag{3.23}$$

So, since the time is proportional to the square root of the height, we can tell than if we drop an object first from 2 meters, and then from 8 meters, it will take twice as long to fall the second time, despite the distance being 4 times as long (because $\sqrt{4} = 2$).

3.1.4 An experiment

This is then put to the test in the lecture, by dropping apples, and timing their fall. One drop was from 3 meters, ± 0.003 meters, while the second was from 1.5 meters, also with ± 0.003 meters as the uncertainty.

The ratio between the two is easily calculated as 2, but what about the uncertainty? If the numerator were 3.003 m and the denominator 1.497 m, those would give the largest ratio possible with the uncertainty of ± 0.003. The result of that division is 2.006, so we consider the uncertainty to be 0.006:

$$\frac{h_1}{h_2} = \frac{3.000 \pm 0.003\text{m}}{1.500 \pm 0.003 \text{ m}} = 2.000 \pm 0.006 \tag{3.24}$$

Note that because this is a ratio between two lengths, the end result has no dimension and thus no unit.

Knowing this ratio, we can now predict the ratio between the fall times. Since the ratio between the heights is 2, and the time is proportional to the square root of the height, the ratio between the fall times should be about $\sqrt{2}$. Then there's that uncertainty again. We can use the same method to find the smallest possible and the largest possible result by calculating $\sqrt{2 + 0.006}$ and $\sqrt{2 - 0.006}$ and will find an uncertainty of about ± 0.002. That gives us

$$\frac{t_1}{t_2} = \sqrt{\frac{h_1}{h_2}} = 1.414 \pm 0.002 \tag{3.25}$$

So, the above is our *prediction*, and we have a set-up with the apple fall times being measured automatically. Let's see the results!

The apple falling from 3 meters ± 3 mm took 0.781 ± 0.002 seconds to fall. The apple falling from 1.5 meters ± 3 mm took 0.551 ± 0.002 seconds to fall.

If we then calculate the ratio between the two times, we find

$$\frac{0.781 \pm 0.002}{0.551 \pm 0.002} = 1.417 \pm 0.008 \tag{3.26}$$

... which is in agreement with the prediction in (3.25) when we consider the uncertainties in our measurements. *Physics works*, as Prof. Lewin would say.
As far the uncertainty of the above result goes, I get ± 0.009 when calculating the same way as before. However, as mentioned before, this method is not truly correct, and the truly correct way is out of the scope of this course, so such a small difference does not matter.
As long as the uncertainty is 0.001 or more, the results can agree with each other.

3.2 Lecture 2: Introduction to Kinematics

3.2.1 Distance vs displacement and velocity vs speed

In everyday English, speed and velocity are usually used as synonyms. In physics, however, the two are very different, and it's important to understand the difference.
We can define the two as

$$\text{speed} = \frac{\text{distance traveled}}{\text{time taken}} \tag{3.27}$$

$$\text{velocity} = \frac{\text{displacement}}{\text{time taken}} \tag{3.28}$$

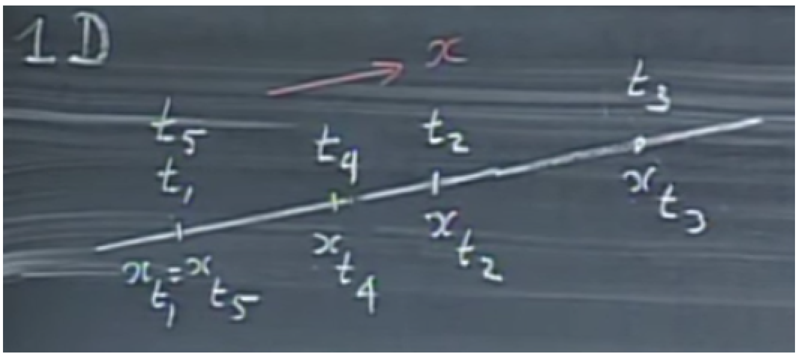
On first glance, the two may appear to say the same thing, but they don't.
There is an important difference between the terms *distance traveled* and *displacement*. The first is a scalar, and is always positive (if not zero, if you have been standing still the entire time) and is equal to what a car's odometer would display.
Displacement, however, is a *vector* (see Part I on vector mathematics, or lecture 3). It is the distance between the starting point and the ending point - which may be zero, if you've traveled back to the start. The displacement vector, like all vectors, also has a direction, which is defined as pointing from the starting point to the ending point.
With this in mind, it should be clear that the distance traveled must *always* be greater than or equal to the displacement. Anything else would require teleportation!

Now, consider the case where we travel 1 km due north, turn around, and travel 1 km due south. We have traveled 2 km, but we are still standing exactly where we started! In other words, the displacement is zero. Using the above definitions, our average *velocity* for the entire journey was *zero* – zero displacement divided by any measured time is zero.
The average *speed*, on the other hand, is guaranteed to be positive, and can be found by dividing the 2 kilometers traveled by the time the journey took.

There is another difference between the two: speed is a scalar, that is, a regular number like any other. Velocity, on the other hand, like displacement, is a vector.
The average speed for the first half of the journey (right where we turned around) might have been 10 m/s, while the average velocity at that point might have been 10 m/s to the north.

Vectors are introduced properly in the first part of these notes, and in the next lecture of the course as well.

3.2.2 Kinematics



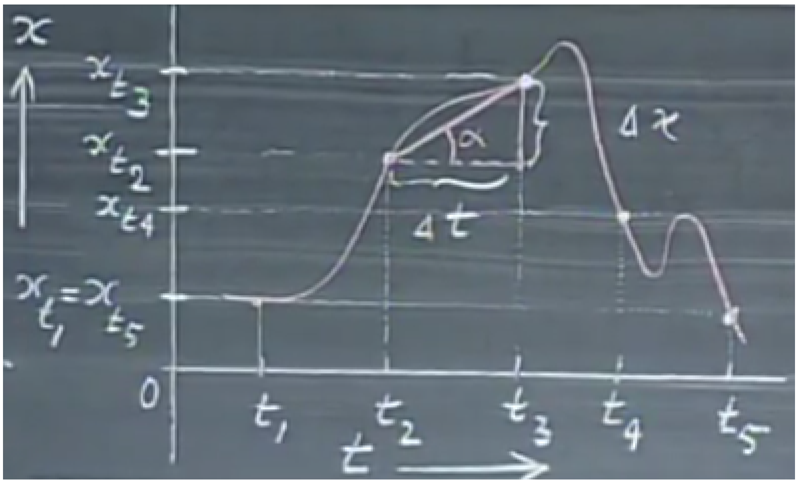
The increasing direction of x is as shown. An object moves along this line, first towards the right, then shortly after reaching the point x_{t3} at $t = t_3$, it reverses and moves back, until it is at $x_{t1} = x_{t5}$, where it started.

We can now introduce a definition for the *average velocity* of this object between two times t_1 and t_2 as the following:

$$\bar{v}_{t_1 t_2} = \frac{x_{t2} - x_{t1}}{t_2 - t_1} \tag{3.29}$$

This should make some intuitive sense – the numerator is just the distance between the two points (the *displacement*), while the denominator is the time that has passed. Displacement over time gives us the *average velocity.*

If we consider the average velocity between times t_1 and t_5, the answer is zero, because the position is the same for the two times, and so we have zero divided by the time taken, which is of course simply zero.
Between e.g. times t_2 and t_4, the velocity is *negative*, which indicates we have moved in the *opposite direction* of the positive x axis.
The average velocity can be positive, zero or negative, depending on the positions involved.
The average *speed*, however, is *always* positive or zero. The average *velocity* is still zero, because the distance between the starting point and the ending point is zero.



Here, we see a different way of notating what is really the same thing as we have above. If we call a difference in time Δt, and a difference in x position Δx, we can find the average velocity as

$$\bar{v} = \frac{\Delta x}{\Delta t} \tag{3.30}$$

This is simply a different way of notating what we already had. Be careful with signs, though - if you take the first x value in the middle, and the second from the right, be sure to take the t values in the same order, or you will get the negative of the correct answer. In other words, your average velocity will be in the opposite direction of the actual movement.

Also shown above is the angle α, that we can find between two arbitrary points. When $\alpha > 0$, as above (it's pointing upwards), the average velocity is positive. If it is instead negative, as it would be between t_4 and t_5, the average velocity is negative.

Instantaneous velocity

Since the definition of velocity we've seen thus far is only an average between two points in time, what is the meaning of instantaneous velocity (which is usually what we mean by "velocity" unless otherwise specified)?
Conceptually, the answer is that it is still an average, only that we move the two position measurements closer and closer together in time, until the time between them is zero.

Mathematically, velocity is the first *derivative* of position. We could write it as

$$v_t = \lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{\Delta t} = \frac{dx}{dt} = x'(t) = \dot{x} \tag{3.31}$$

The last three are just three different ways of writing the same thing: the first derivative of x with respect to t. Leibniz' notation looks like a fraction; Lagrange's notation uses the prime symbol (apostrophe) to indicate a derivative, and finally Newton's notation uses a dot above to signify the first time derivative. (In other words, the dot notation is used almost exclusively when the function is differentiated with respect to time, so the t is implicit.)

As for speed, we can simply define instantaneous speed as the absolute value of the instantaneous velocity. In other words, if the velocity has a minus sign, remove it. If not, the two are equal.

3.2.3 Calculating the average speed of a bullet

Using an electronic, an experiment was set up to measure the speed of a bullet. The bullet is fired from a rifle, and breaks a wire, at which point the timing starts. Soon thereafter, the same bullet breaks another wire, at which point the timing stops.

Using a measurement of the distance, and a measurement of the time taken, we can calculate the average speed.

The distance was measured to be 148.5 ± 0.5 cm, that is, 1.485 ± 0.005 m.
The time taken was measured as 5.8 ± 0.1 ms, which equals, 0.0058 ± 0.0001 s.

The average speed is then

$$v_{avg} = \frac{1.485\,\mathrm{m}}{0.0058\,\mathrm{s}} = 256\,\mathrm{m/s} \tag{3.32}$$

The relative error in the timing can be calculated as

$$\text{relative error} = \frac{0.1\,\mathrm{ms}}{5.8\,\mathrm{ms}} \cdot 100\% = 1.7\% \tag{3.33}$$

The uncertainty in the average speed can then be estimated. As the lecture question hints, we will ignore the uncertainty due to error in the distance measurement, because the timing error is much greater.

We can use the simple way introduced previously to find an approximate uncertainty:

$$\text{error} = \frac{1.485 \text{ m}}{0.0058 - 0.0001 \text{ s}} - 256\,\mathrm{m/s} = 4.5\,\mathrm{m/s} \tag{3.34}$$

(We would add +0.005 m at the top, if we didn't choose to ignore the uncertainty it that measurement.)
Alternatively, we could have simply used the 1.7% relative error we found above.
So in short, we can specify the average speed of the bullet as

$$v_{avg} = 256 \pm 4.5 \text{ m/s} \tag{3.35}$$

3.2.4 Acceleration

Just as velocity is the change in position, acceleration is the change in *velocity*. We can use an equation that looks extremely similar to find the *average* acceleration a:

$$\bar{a}_{t_1 t_2} = \frac{v_{t2} - v_{t1}}{t_2 - t_1} \tag{3.36}$$

The dimension of acceleration, as mentioned previously, is length per time^2, or [L] [T]$^{-2}$, with m/s^2 being the most common unit, at least in this course.

Just as before, we can simplify this by using delta notation, with the same caveat: make sure to use the correct signs, or the result may end up incorrect.

$$\bar{a} = \frac{\Delta v}{\Delta t} \tag{3.37}$$

As an example, let's use the following lecture question. We define the direction of increasing x as upwards (towards the sky). A tennis ball is thrown towards the ground at a velocity of about $-5\,\mathrm{m/s}$ - i.e. the speed is 5 m/s, downwards. It is in contact with the ground for about 1/100 second, after which it is moving at $5\,\mathrm{m/s}$, i.e. upwards.
What is the average acceleration of the tennis ball?

Well, we have the formula above, so this should be fairly easy!

$$\overline{a_{ball}} = \frac{5\,\mathrm{m/s} - (-5\,\mathrm{m/s})}{0.01\,\mathrm{s}} = \frac{10\,\mathrm{m/s}}{0.01\,\mathrm{s}} = 1000\,\mathrm{m/s^2} \tag{3.38}$$

Keeping the signs in mind, we end up with a positive value for the acceleration, which has a ridiculous magnitude - over 100 times the Earth's gravitational acceleration.

The professor adds another example of acceleration. There is a limit to the amount of acceleration things can tolerate before they break. He used examples of tomatoes and eggs, also thrown to the ground at 5 meters per second.
The impact time will probably be much greater (perhaps 1/4 second), and the change in velocity will be only 5 m/s rather than 10, as neither the egg nor the tomato would bounce back up.
Despite that, clearly, both would break, even though the acceleration is a more modest 20 m/s^2 or so.

The next example was that of a human skull on a marble floor, from a Sherlock Holmes movie. Even at a relatively small velocity, a skull hitting such a hard floor, with no "give", the impact time would be extremely short. Since the impact time is in the denominator, a shorter time will result in a higher acceleration, and a skull can break despite the low speeds involved.

Instantaneous acceleration

Just as we did with velocity, we now want a way to calculate the acceleration at a given instant, rather than the average between two measurements. We do this in exactly the same way: we find the first time derivative of the velocity.

$$a_t = \lim_{\Delta t \to 0} \frac{v_{t+\Delta t} - v_t}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} \tag{3.39}$$

We could also write this as $x''(t)$ or $v'(t)$, but I wanted to reduce the amount of clutter above somewhat. Since acceleration is the time derivative of velocity, and velocity itself is the time derivative of position, acceleration is the *second* time derivative of position, as shown above.

3.2.5 General forms for one-dimensional motion

We can write equations for the position and velocity in one-dimensional motion in such a way that they can be used for *any* one-dimensional motion with a constant acceleration:

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_x t^2 \tag{3.40}$$
$$v(t) = v_0 + a_x t \tag{3.41}$$

... where x_0 is the initial x position, v_0 is the initial velocity, and a_x is the acceleration in the positive x direction.
Each of these three numbers can independently be negative, zero, or positive, and produce valid physical situations.
The same formula can be used in a situation with a constant velocity: simply set $a_x = 0$, and the equations are valid. (The second equation becomes trivial, of course, so only the equation for $x(t)$ will be useful.)

3.3 Lecture 3: Vectors

Because I had already written the part on vector mathematics in these notes long before this course started (I did it about a year ago, when I considered taking 8.01 via MIT OCW), most notes from this lecture are in that part instead.

However, I do find it useful to talk about one thing that is more specific to physics than the vector mathematics part, and that is vector decomposition. Perhaps not decomposition in itself, but the consequences of it for simplifying 2- or 3-dimensional kinematics.

If we have a position vector $\vec{r_t}$, which changes with time:

$$\vec{r_t} = x_t \hat{x} + y_t \hat{y} + z_t \hat{z} \tag{3.42}$$
$$\vec{v_t} = \frac{d\vec{r_t}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \tag{3.43}$$
$$\vec{a_t} = \frac{d\vec{v_t}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} \tag{3.44}$$

... using Newton's notation, with a dot representing the first time derivative, and two dots representing the second time derivative.

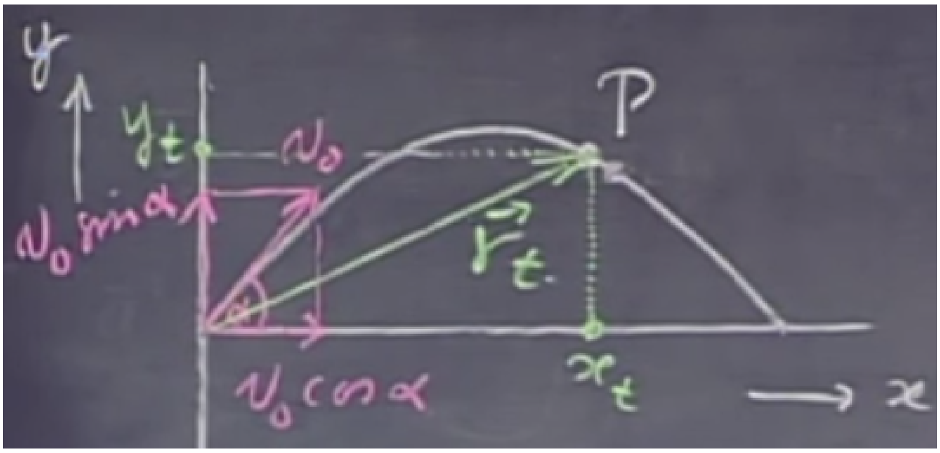
We could use these three equations as they stand, to calculate the position, velocity and acceleration of a particle in three dimensions. However, that would likely get complex very quickly.

What we can instead do is break the three-dimensional motion into multiple one-dimensional motions.
Imagine we throw a ball, sideways. Its motion will be constrained to two dimensions, if we neglect wind and air drag: it will start accelerating downwards due to gravity, and it will move horizontally in the direction we threw it at constant velocity. The latter is important: gravity only accelerates the ball downwards. If we neglect wind and air drag, as mentioned, there is no force acting on the ball parallel to the ground. Because of Newton's first law (which we have not yet introduced, but will next week), that means the velocity must be constant in that direction.

Thus, we have reduced a fairly complex problem of three-dimensional motion into *two* problems of one-dimensional motion. One horizontally, where the velocity is constant, and one vertically, where gravity acts as a constant acceleration downwards.

3.3.1 3-dimensional motion to two independent 1-dimensional motions

Let's examine the problem of a ball (or a similar object) being thrown diagonally upwards. If there is no air, and thus no wind that could cause the ball to curve, the motion will be constrained to two dimensions, despite moving in three-dimensional space.
We can therefore think of this as a 2D problem, where the ball moves along this trajectory:



The ball moves along the trajectory shown in white. It is launched (thrown) at an initial velocity v_0 (in magenta), which is a vector pointing at an angle α from the ground. Also in magenta, we have the initial velocities for the x and y directions, found via vector decomposition:

$$v_{0x} = v_0 \cos\alpha \tag{3.45}$$
$$v_{0y} = v_0 \sin\alpha \tag{3.46}$$

Because there is no force acting on the ball in the x direction, this is the velocity it will have in the x direction until it hits the ground.
In green, the ball's position vector at a later point is shown, together with its x and y components, all three dependent on time t.

We can now apply the equations we found earlier, for one-dimensional motion under either constant acceleration or constant velocity. That is, these:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \tag{3.47}$$
$$v_x(t) = v_{0x} + a_x t \tag{3.48}$$

The same equations can of course be used for y (and z) by simply replacing all x terms with y (or z).

With these equations in mind, we can now calculate the object's x position at any moment in time as

$$x(t) = (v_0 \cos\alpha)t \tag{3.49}$$

... since we are free to choose $x_0 = 0$, and there is no acceleration in the x direction ($a_x = 0$).

This simple equation describes the x position completely, from $t = 0$ when it is launched, to whenever it hits the ground. To find out when that is, we need to calculate the y position over time.

We use the same equations, with $y_0 = 0$ (again, we are free to choose where we place our zero coordinate), and $a_y = -g$, that is, the gravitational acceleration of the Earth. g is always positive, however, and in the diagram, we have chosen increasing y to be upwards. Therefore we need to be careful and write $-g$ in this case, or we would be saying that gravity would accelerate the ball towards the sky!

We make the substitutions for the values (y_0 and a_y as mentioned above, and the initial velocity in the y direction is $v_0 \sin\alpha$ as we saw before)

$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \tag{3.50}$$

$$v_y(t) = v_0 \sin\alpha - gt \tag{3.51}$$

Note the minus sign for the acceleration.
The last three equations completely describe the ball's x position, y position, and y velocity. The x velocity is known to be constant.

Together, we can use the $x(t)$ and $y(t)$ equations to describe the trajectory:

$$x(t) = (v_0 \cos\alpha)t$$

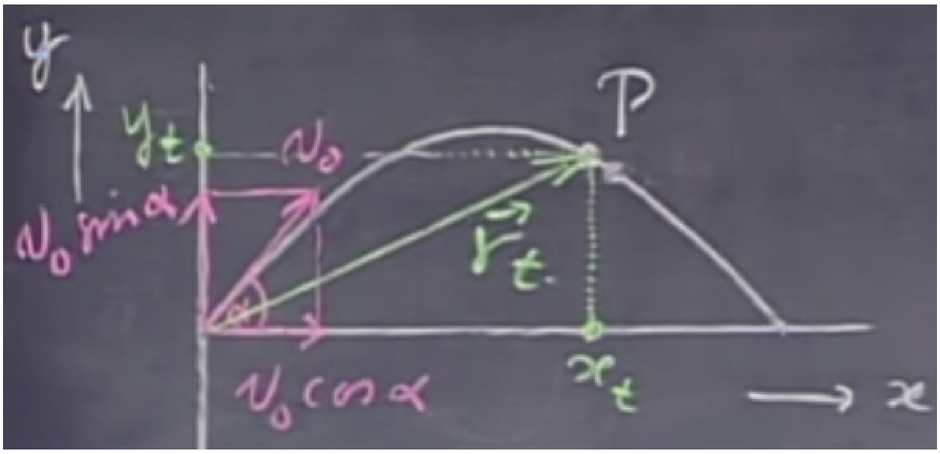
$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

This is then demonstrated in lecture, by firing a golf ball straight up (as seen by the launcher), from a cart moving on a rail.
For an outside observer, such as us, the ball moves in a parabolic trajectory, and returns to the launcher a few seconds later, as they moved together at the constant x velocity.
The successful demonstration concludes this lecture.

Chapter 4: Week 2

4.1 Lecture 4: The motion of projectiles

This lecture doesn't really contain anything new, and instead mostly consists of applications of the material covered last week.
Let's revisit this trajectory:



For now, ignore the green parts, which are the location of the position vector after a certain time has passed. Relevant equations for this trajectory can be written as

$$x(t) = (v_0 \cos\alpha)t \tag{4.1}$$
$$v_x(t) = v_0 \cos\alpha \tag{4.2}$$
$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \tag{4.3}$$
$$v_y(t) = v_0 \sin\alpha - gt \tag{4.4}$$

where v_0 is the initial velocity diagonally, at angle α to the ground, and $g = +9.8\,\mathrm{m/s^2}$. Since g is a positive number, we need to use a minus sign here: we have defined increasing y to be upwards, but gravity accelerates downwards.
The first and third equations could have x_0 and y_0 terms, respectively, but we can choose the origin of our coordinate system to be the exact point from where the ball is thrown, which means we choose them both to equal zero.

We can write equation (4.3) above in terms of x, instead of t, by solving (4.1) for t and making the substitution:

$$x(t) = (v_0 \cos\alpha)t \tag{4.5}$$
$$t = \frac{x(t)}{v_0 \cos\alpha} \tag{4.6}$$

$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \tag{4.7}$$
$$y(t) = (v_0 \sin\alpha)\left(\frac{x(t)}{v_0 \cos\alpha}\right) - \frac{1}{2}g\left(\frac{x(t)}{v_0 \cos\alpha}\right)^2 \tag{4.8}$$
$$= x(t)\tan\alpha - \frac{1}{2}g\frac{x(t)^2}{v_0^2 \cos^2\alpha} \tag{4.9}$$

We can then see than this has the form of x times a constant, minus x^2 times another constant.

$$y(t) = C_1x - C_2x^2 \tag{4.10}$$

In other words, the trajectory has the shape of a parabola.

We can calculate when the object reaches the maximum height (the apex of the trajectory), by setting the $v_y(t)$ equation equal to zero. The object is launched with an initial velocity, and will only ever stand "still" (on the y axis) when it changes from going upwards to going downwards, since the equation doesn't capture what happens when it hits the ground, etc.

$$v_0 \sin\alpha - gt_p = 0 \tag{4.11}$$
$$gt_p = v_0 \sin\alpha \tag{4.12}$$
$$t_p = \frac{v_0 \sin\alpha}{g} \tag{4.13}$$

In other words, the time to reach the peak height is simply the initial velocity in the y direction, divided by the acceleration that opposes that motion.
We can then find the highest point h that it ever reaches, by substituting the time found above into the $y(t)$ equation:

$$h = (v_0 \sin\alpha)\frac{v_0 \sin\alpha}{g} - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2 \tag{4.14}$$
$$h = \frac{v_0^2 \sin^2\alpha}{g} - \frac{v_0^2 \sin^2\alpha}{2g} \tag{4.15}$$
$$h = \frac{v_0^2 \sin^2\alpha}{g}\left(1 - \frac{1}{2}\right) \tag{4.16}$$
$$h = \frac{(v_0 \sin\alpha)^2}{2g} \tag{4.17}$$

Next up: at what time does the object return to $y = 0$? We could simply set up that equation, solve it, and pick the larger solution (since it will be true at $t = 0$ as well), but we can do it a bit faster. Because the curve is symmetric, the time must be exactly twice that to reach the curve's apex at t_p. (The lecture labels the points O at the origin, P at the peak and S where the object lands, but the illustration I used doesn't have them written down.)

$$t_s = 2t_p \tag{4.18}$$
$$t_s = \frac{2v_0 \sin\alpha}{g} \tag{4.19}$$

Finally, we can calculate the distance OS, which is the horizontal distance traveled. (Not the entire distance of the parabola, i.e. the arc length!)
This distance is simply v_{0x} times t_s, but that expression is slightly hairy. Let's write it down and then simplify it:

$$\text{OS} = (v_0 \cos\alpha)\left(\frac{2v_0 \sin\alpha}{g}\right) \tag{4.20}$$
$$\text{OS} = \frac{2v_0^2 \sin\alpha\cos\alpha}{g} \tag{4.21}$$

$2\sin\alpha\cos\alpha = \sin 2\alpha$ via a trigonometric identity, so:

$$\text{OS} = \frac{v_0^2 \sin 2\alpha}{g} \tag{4.22}$$

If we want to maximize the horizontal distance, what angle should we fire it at? Well, we could use calculus and attempt to maximize the function, but it can be done much faster (and easier) by simply looking at the equation. $\sin\alpha$ is maximized when $\alpha = 90°$, so $\sin 2\alpha$ must be maximized when $2\alpha = 90°$ or $\alpha = 45°$.

I would not call that immediately obvious, but it is obvious that the answer must be somewhere between 0 and 90 degrees, excluding both of those extremes. At 90 degrees, OS is zero, since the entire trajectory will be one-dimensional in y.
It cannot be 0 degrees, either, because in that case, it hits the ground instantly after it is fired, so OS is again zero! It's fired completely parallel to the ground, so that y never goes above 0 – but gravity starts pulling it downwards instantly.
The angle must be somewhere between 0 and 90 degrees, and as it turns out, it's exactly in between.

4.1.1 Trajectory demonstrations

We will now try to validate these results in real life, by firing a small projectile (a small metal pellet) at various angles, recording the horizontal distance traveled – keeping the uncertainties in mind.

First, we measure the maximum height that the pellet can be fired to, a few times. We make an estimate of the height, with an uncertainty, and can use that together with g to calculate v_o^2. That is then used in the equation for OS as shown above, to calculate where the pellet should land.

The measurement, along with many others the professor did in preparation, showed the height to be approximately $h_{max} = 3.07 \pm 0.15\,\text{m}$, so the error is about 5%.
We can solve for v_0^2:

$$h = \frac{(v_0 \sin\alpha)^2}{2g} \tag{4.23}$$

$$h = \frac{v_0^2 \sin^2\alpha}{2g} \tag{4.24}$$

$$v_0^2 = \frac{2gh}{\sin^2\alpha} \tag{4.25}$$

With the value of h_{max} and the error, we find $v_0^2 = 60.2 \pm 3.0\,\text{m}^2/\text{s}^2$. A strange unit, but this is the value we need, rather than v_0 itself.

Next, we need to set the angle. There will be uncertainty here, as well – the professor assumes $\pm 1°$ in his ability to set it up. Since the answer depends on the sine of twice the angle, we may be off by about 2 degrees. However, $\sin 88° \approx \sin 90°$: the error is about 0.06%, which pales in comparison to the 5% error above, so we can completely ignore this source of uncertainty.

Now, using equation (4.22) for OS, we can calculate the predicted horizontal travel distance:

$$45°\ \text{OS} = \frac{60.2\,\text{m}^2/\text{s}^2 \sin 90°}{9.81\,\text{m}/\text{s}^2} = 6.14 \pm 0.31\,\text{m} \tag{4.26}$$

The pellet is fired, and it indeed hits inside the uncertainty range, by the looks of it at more or less 6.05 meters.

Next up, we want to do the same, but at a 30 degree angle, instead. This time, the uncertainty due to the sine of the angle is no longer negligible. What previously was a 0.06% error suddenly becomes about a 2% error – the sine function is roughly "flat" around 90 degrees, but far from it around 60 degrees ($2 \times 30°$).

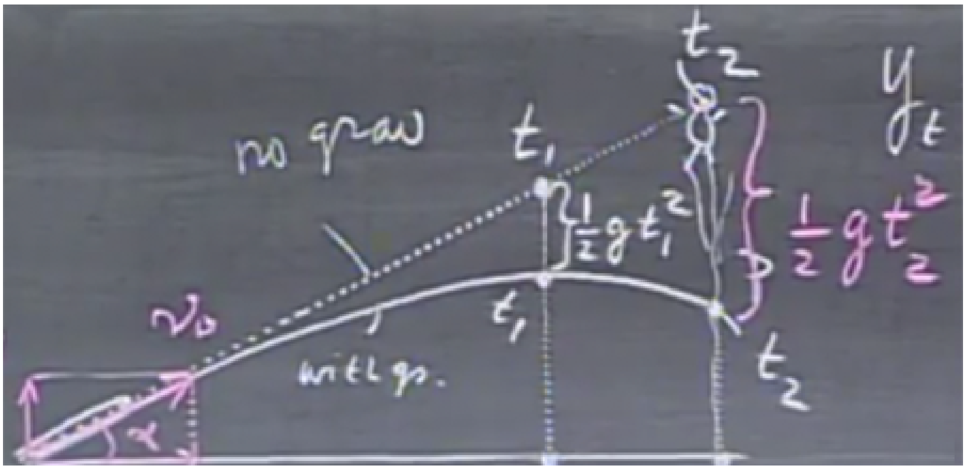
Making the same calculation as we did above, but with the different angle, we find 5.31 m, but with an uncertainty of 7% (using an easy but perhaps not 100% correct way of calculating uncertainties). That gives us a prediction of $5.31 \pm 0.37\,\mathrm{m}$ for the 30 degree angle.
Not only did the pellet hit within the uncertainty range, but it actually hit the indicator at the 5.31 meter mark!

Next up: 60 degrees. It turns out that the horizontal distance traveled should be exactly equal to that at 30 degrees, because $\sin(2 \cdot 30°) = \sin(2 \cdot 60°)$. v_0^2 and g certainly didn't change, so this should indeed be true.
The pellet yet again lands within the uncertainty range, though fairly close to being short. This is likely more indicative of the pellet gun's uncertainty and the exact angle than anything else, however – but it's important to keep in mind that while it was close, it still was *within* the uncertainty. This was certainly no failure.

4.1.2 A story about a monkey

No monkeys were hurt in the making of this demonstration!

Imagine a monkey, sitting in a tree. A short bit away, a hunter places a golf ball cannon, aimed directly at the monkey (dotted line, below).



We already know that unless the golf ball's velocity is very high, gravity will pull it down in a parabola such that it misses the monkey. Only if the vertical distance traveled due to gravity is smaller than the height of the monkey can it hit.
Because the horizontal velocity is the same regardless of whether there is gravity, we know that at a certain time t, the golf ball will be at the same x position regardless; only the height will differ.

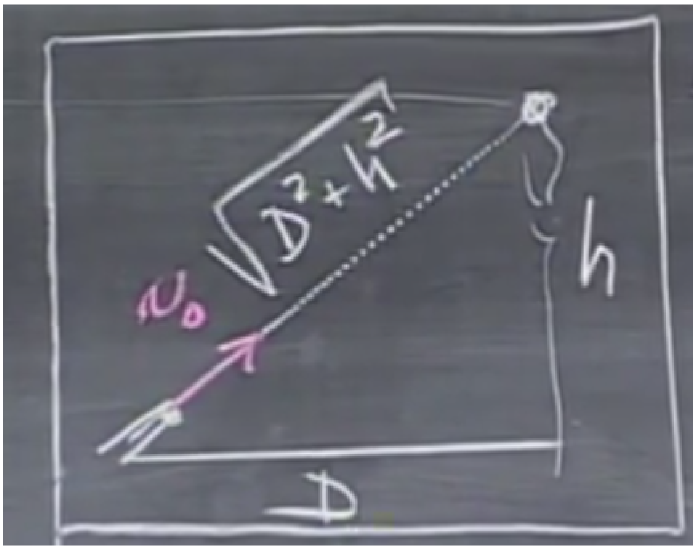
The dotted line above shows how the ball would travel in the absence of gravity, while the filled line shows the parabolic trajectory it would take on Earth. As we can see, it falls a distance of $\frac{1}{2}gt_1^2$ during a time interval t_1 after being fired – basic 1D kinematics.

However, that is true at all times t after $t = 0$, assuming it has not yet hit the ground (or anything else, for that matter).

Now, there's an additional crux in this problem: as soon as the monkey sees the cannon fire, he lets go and starts falling. The monkey will fall with exactly the same acceleration as the golf ball, and since they started falling at the same time, the golf ball will hit the poor monkey despite his attempt to flee. Had he instead stayed where he was, all would probably be well!

Note that this fact is independent of the golf ball's velocity, as long as it doesn't hit the ground before reaching the monkey's x coordinate. High velocity or low velocity, the gravitational acceleration is the same regardless, and so the ball and the monkey will both fall the same vertical distance in a given amount of time.

Now, let's imagine that all of this happens inside an elevator, which is in free fall. Both the gun and the monkey (and the tree) accelerate downwards at $-g$.



From the monkey's point of view, because he falls at the same acceleration and velocity as the gun, the golf ball comes straight at him, without any arcing. As shown above, as far as the monkey can see, the distance the ball has to travel is $\sqrt{D^2+h^2}$, the hypotenuse of the triangle created by the horizontal distance to the cannon and the (vertical) height of the tree.

Considering the golf ball's velocity, from this point of view, the monkey will get hit in

$$t_{kill} = \frac{\sqrt{D^2+h^2}}{v_0} \tag{4.27}$$

seconds. However, from a different perspective (see the previous image above), we would instead calculate it as

$$t_{kill} = \frac{D}{v_0 \cos\alpha} \tag{4.28}$$

since $v_0 \cos\alpha$ is the ball's velocity in the horizontal direction.
How come the two are not the same? Surely they must both agree? And they do. We can use the definition of $\cos\alpha$ in the above diagram:

$$\cos\alpha = \frac{D}{\sqrt{D^2+h^2}} \tag{4.29}$$

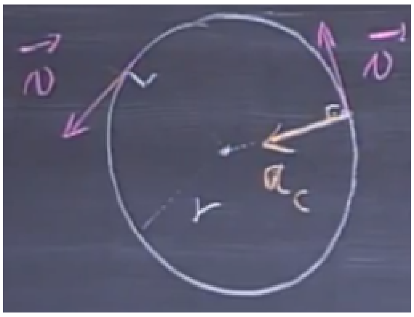
... and substitute that into what we had just above:

$$t_{kill} = \frac{D}{v_0 \frac{D}{\sqrt{D^2+h^2}}} = \frac{\sqrt{D^2+h^2}}{v_0} \tag{4.30}$$

and so the two agree on the timing of the monkey's unfortunate demise.

4.2 Lecture 5: Uniform circular motion

Consider an object moving at a constant speed v around a circle of radius r:



We can define a few variables that relate to this motion. First out is T, the *period* in seconds it takes the object to travel along the entire circumference once. Second is the *frequency* f, which measures how many times it travels around the circle per second. The two are then inverses, so that $f = 1/T$ and $T = 1/f$. The SI unit for frequency is Hertz; the dimension is then dim Hertz $= \dfrac{1}{[T]}$, and $1\,\text{Hertz} = 1\,\text{s}^{-1}$.

We can consider how fast it moves in a different way, in measuring velocity in *radians per second*, instead of meters per second (or other units of "regular" velocity). We call this *angular velocity*, symbol ω (Greek letter lowercase omega). Since there are 2π radians in a circumference, this implies that

$$\omega = \frac{2\pi}{T} \tag{4.31}$$

As for the speed v (not the velocity $\vec{v}$ just yet), we can write

$$v = \frac{2\pi r}{T} = \omega r \tag{4.32}$$

considering the relation shown in the previous equation. These two last equations are important to remember.

4.2.1 Centripetal acceleration

Note that as the object moves around in a circle, the direction of the velocity vector in constantly changing. This can only happen if there is a nonzero acceleration. This acceleration is called the *centripetal acceleration*, often denoted by $\vec{a_c}$. This acceleration vector always points towards the center of the motion. Because the velocity vector is always tangent to the circle at any given point, the acceleration vector is always perpendicular to the velocity, assuming a constant *speed* around the circle. (If the speed is *not* constant, there will also be a tangential acceleration component, which means the net acceleration vector will not be exactly perpendicular to the velocity vector; the centripetal acceleration by itself is however always perpendicular to the velocity vector.)

The magnitude of the centripetal acceleration can be stated as

$$|a_c| = \frac{v^2}{r} = \omega^2 r \tag{4.33}$$

Proportionality of r

Be careful when it comes to the proportionality of r, though! If T is held constant, v is a function of r, being equal to

$$v = \frac{2\pi r}{T} \tag{4.34}$$

and so increasing r will also increase v, and thus in the end increase a_c:

$$|a_c| = \frac{v^2}{r} = \frac{4\pi^2 r^2}{rT^2} = \frac{4\pi^2 r}{T^2} \tag{4.35}$$

Here, it's obvious that increasing r will increase a_c, assuming T is held constant. This should come as no surprise, as we are increasing our speed v by moving a longer distance in the same amount of time.

However, let's not fool ourselves into believing that $a_c \propto r$ is always a correct view! Let's now look at the case where we hold the velocity constant (thus changing T) while changing the radius. To get a nice look of how this works, we use the simple equation

$$|a_c| = \frac{v^2}{r} \tag{4.36}$$

Here, holding v constant, it's clear that the centripetal acceleration goes *down* as we increase the radius of the circle we travel in.

The cause of acceleration

Something must be causing this acceleration, however. We will introduce *force* next lecture, but for this lecture, we will talk about "push" and "pull" instead. Suppose you sat on a chair bolted onto a spinning disk. You would feel a "push" from the back of the chair, pushing you towards the center of the disk. If you instead tied a rope to a stick at the center, or just held on to a bolted-down stick in front of you, you would feel the rope/stick pulling you inwards. Note that in either case, the pull or push is towards the center.

If the pull/push was suddenly removed somehow, the object would simply continue forward in a straight line, along its velocity vector, unless there are other forces (pushes/pulls) acting on it, such as gravity. This is demonstrated by a spinning disk with a ball tied to a string. The string is cut as the ball's velocity vector points straight upward, and the ball flies several meters straight up in the air, and then falls straight down again.
Had it been cut at a different location, it would have followed a parabolic trajectory, as we have studied previously, with $v_0 = v$ and horizontal and vertical components being found by multiplying v_0 with the cosine and the sine, respectively, of the angle made with the ground.

4.2.2 Planetary orbits

Let's now have a quick look at the orbits of planets. We will look at them much closer in a few weeks, but until then, let's assume (incorrectly) that orbits are circular. (In reality, they are slightly elliptical.)
First out, we have a lecture question:

"The radius of Earth's orbit is 150×10^6 km. Assuming that the orbit is circular, what is the centripetal acceleration of the Earth?"

They want the answer in km/yr^2, so we shouldn't have to do any ugly conversions.
Let's see. The period is one year, by definition (not exactly 365.00 days, but that's another story).

Because $\omega = \frac{2\pi}{T}$ and $T = 1$ year, we find $\omega^2 = 4\pi^2$ radians per year, and $\omega^2 r = (4\pi^2)(150 \times 10^6) = 5.92 \times 10^9\,\text{km/yr}^2$.

Just to make sure, let's also calculate it using v^2/r.

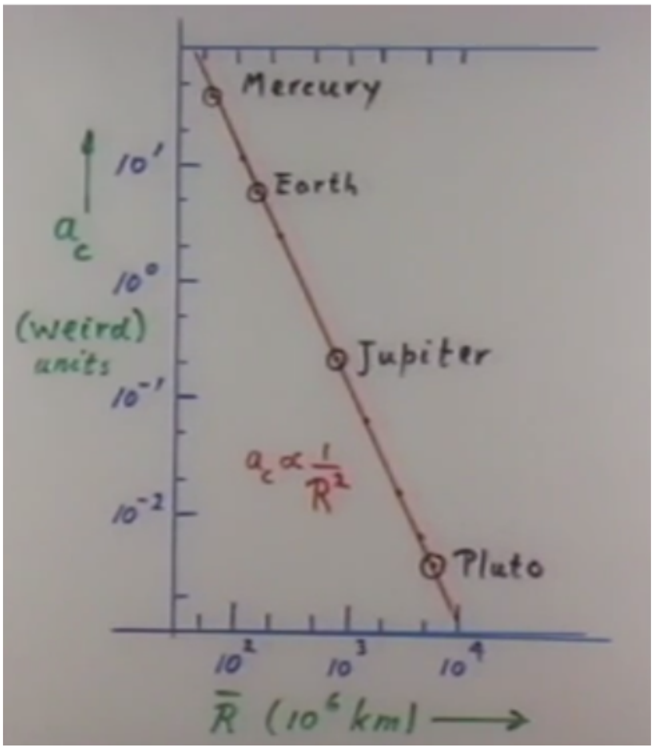
v is found by dividing the circumference of the orbit by the time (1 year), which in then equal in value (but obviously not units) to just the circumference. We then square that, and divide by the radius again; a bit redundant to divide out the radius, but let's go with it for simplicity:

$$v = \frac{2\pi r}{T} = \frac{2\pi r}{1 \text{ year}} \tag{4.37}$$

$$a_c = \frac{v^2}{r} = \frac{4\pi^2 r^2}{r} = 4\pi^2 r = 5.92 \times 10^9 \text{ km/yr}^2 \tag{4.38}$$

Unsurprisingly, we get the same answer. Still, we have now double-checked, and have also gained a bit of practice doing in two different ways.

Now, let's have a look at the orbits of various planets – their mean distance to the sun (mean, since the orbits are not truly circular) and periods, and let's compare the centripetal acceleration of various planets. What we find can be seen on this plot below:



(It's a bit fun to note that Pluto was still considered a planet when the lecture was recorded! Little has changed in classical mechanics since then, but that one thing certainly has.)

Here, we see the centripetal acceleration on the vertical axis, and the mean distance to the sun on the horizontal. It's clear that the $1/R^2$ fit is rather brilliant! The closer a planet is to the sun, the stronger the centripetal acceleration is, and it falls off following the inverse square law.

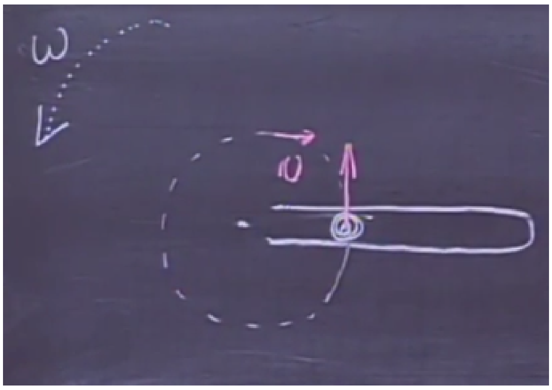
We've seen centripetal acceleration being proportional to r, and inversely proportional to r, but now it's inversely proportional to r^2! What gives? We will talk more about gravity and planetary orbits soon later in the course. Admittedly, I'm not sure about the exact answer, but looking up data on planetary orbits, I found that planets with larger r also have smaller v; the further out you go, the slower planets move through space, in addition to having a lot more distance to cover.

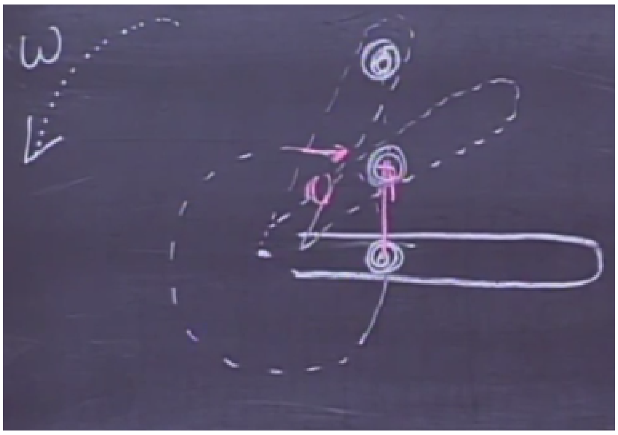
As with the previous examples of centripetal force, if we simply removed the sun (or somehow else removed its gravitational influence), the planets would simply continue on in straight lines, based on their previous velocity vectors.
We will discuss gravity further in the coming weeks, but let's leave it here for now: as the distance to the sun is increased by a factor x, the effect of gravity is x^2 times less. The same is true for e.g. Earth's gravity

too, of course, which is why the gravity is weaker further from the surface. (This is however *not* the reason astronauts experience weightlessness, as gravity is still about 90% as strong on the space station as it is on the surface. They are instead in constant free-fall around the Earth, which is essentially the meaning of an orbit!)

4.2.3 Centrifuges and more on centripetal acceleration

Let's now look at the rotation of a glass tube, with a marble inside. The glass tube starts out horizontal, with the marble inside it (see the first picture below):



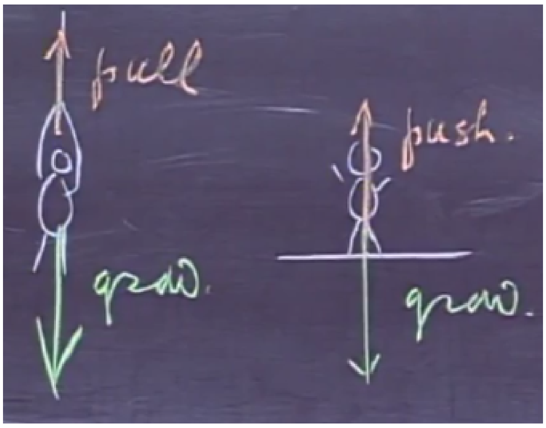


Because the glass and the marble are both very smooth, the glass can neither push nor pull on the marble, and so cannot provide any centripetal acceleration. What happens? Well, the glass tube will still rotate, of course – we assume it's powered by a motor of some kind. The marble, on the other hand, will continue on moving according to its still unchanged velocity vector.

A moment later in time (second picture above), the tube has rotated such that the marble's velocity will take it towards the end of the tube, where we know from experience it will also stay, as long as the tube rotates quickly enough.

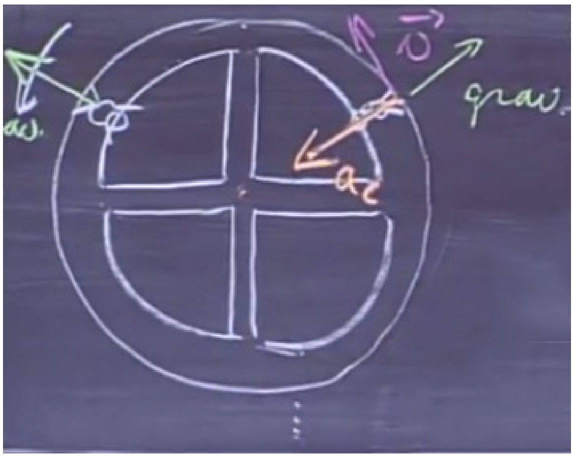
4.2.4 Artificial gravity through centripetal acceleration

Let's now look at "perceived gravity", or artificial gravity.



As the illustration shows, we will always experience gravity opposite to any pull or push. The same is true if we somehow hang on to a rope and spin around – or ride a merry-go-round or something to that effect. We will have a centripetal force inwards, and feel a "pull" inwards, but perceive gravity in exactly the opposite direction, as if we were drawn outwards.

Let us now consider a large, circular space station, which experiences almost no gravity (as it is in orbit, essentially in perpetual free fall). It is a big "wheel", with a radius of 100 m. We want the centripetal acceleration to be about $10\,\mathrm{m/s^2}$ for a person standing on the outer wall. How fast should it rotate (what should be the period)?



We can use $\omega^2 r$ here; it should equal $10\,\mathrm{m/s^2}$, so we find

$$(\omega^2)(100\,\mathrm{m}) = 10\,\mathrm{m/s^2} \tag{4.39}$$

$$\omega^2 = \frac{10\,\mathrm{m/s^2}}{100\,\mathrm{m}} \tag{4.40}$$

$$\omega^2 = 0.1\,\mathrm{rad^2/s^2} \tag{4.41}$$

$$\omega = \sqrt{0.1}\ \mathrm{rad/s} \tag{4.42}$$

And, because $\omega = \frac{2\pi}{T}$:

$$\frac{2\pi}{T} = \sqrt{0.1} \tag{4.43}$$

$$T = \frac{2\pi}{\sqrt{0.1}} \approx 20\,\mathrm{s} \tag{4.44}$$

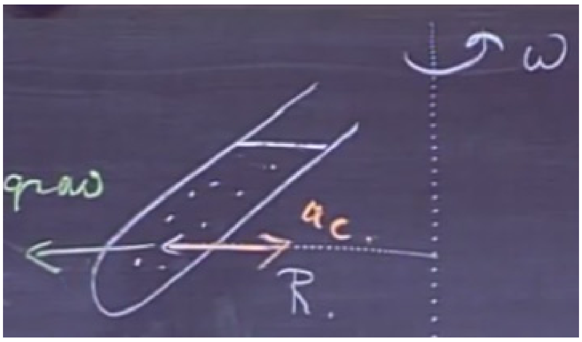
So if we rotated the space station with a period of about/just over 20 seconds, we would perceive it as if we had Earth's gravity.

Now consider how the station might be arranged. The centripetal force is proportional to the distance to the center, so it is strongest at the outer wall. In the exact center of the station, there will be no perceived gravity at all. How does one get there, though? Considering the fact that gravity is perceived as being radially outwards, walking to the center is the same concept as walking to the ceiling in a regular house. You will simply need stairs, ladders, or something similar.

For the same reason, you would have to use the stairs when going back "down" as well! The gravitational acceleration may be zero at the center, but it grows as you come closer to the outer edge. If you were to "jump" down the shaft, you would end up crashing into the outer wall at a velocity great enough that you may well be killed!

4.2.5 More on centrifuges

Let's have another look at centrifuges.
Say we have a liquid filled with very tiny and very light particles; tiny and light enough that they don't sink to the bottom.



When we spin it around at a high speed, causing a high centripetal acceleration, the light particles are not so light any longer (as we will soon see, weight is the product of mass and acceleration, the latter of which just increased by a lot), and they sink to the bottom – where the centripetal acceleration is the greatest.

Let's make a quick calculation based on a lecture question:

"The frequency of a centrifuge is 60 Hz and its radius is 0.15 m. What is the centripetal acceleration of an object in the centrifuge at a distance of 0.15 m from the center?"

60 Hz is 3600 rpm, so it's spinning rather quickly. We can once again use $\omega^2 r$:

$$|a_c| = \omega^2 r = (2\pi(60\,\text{Hz}))^2 \times 0.15\,\text{m} = 21318.3 \approx 21 \times 10^3\,\text{m/s}^2 \tag{4.45}$$

That's over 2000 *times* the acceleration due to gravity, so the particles now experience such an acceleration that they weigh over 2000 times as much as they do in regular gravity!

This is then put to the test in a demonstration. A solution of silver nitrate and sodium chloride is mixed, to produce a milky-white liquid of (most importantly) suspended silver chloride. A part of the solution is put in the centrifuge mentioned above, with a counterbalance of water on the opposite side, to provide stability.
The centrifuge is started, and we temporarily move on to other type of centrifuge.

Say we have a bucket of water attached to a rope. We swing it around such that it will be upside down at the top of its motion, with a velocity high enough that $|a_c| > g$. In other words, the bucket experiences an inwards pull due to the centripetal acceleration, which translates into it experiencing gravity in the opposite direction, outwards.
Therefore, if we spin it fast enough, the water will be forced into the bottom of the bucket, even when upside down, and no water will come out.

How fast do we have to swing it, if the rope is 1 meter long? (The question also states that the water's mass is 4 kg, which we didn't need to know to solve the problem.)

$$\frac{v^2}{1\,\text{m}} > 9.8\,\text{m/s}^2 \tag{4.46}$$

$$v^2 > 9.8\,\text{m}^2/\text{s}^2 \tag{4.47}$$

$$v > \sqrt{9.8}\ \text{m/s} \tag{4.48}$$

We need to swing it faster than about 3.13 meters per second, or the water will start falling.
Quickly returning to the silver chloride centrifuge, everything worked out as planned: the liquid is now clear, and there is a collection of white particles at the bottom of the tube, rather than spread out everywhere.

Finally, the water bucket swing is put into practice, and it indeed works.

4.3 Lecture 6: Newton's first, second, and third laws

In this lecture, we will introduce the concept of *force*, an extremely important quantity in physics. Last lecture used forces, but referred to them as "pushes" or "pulls". We will now start using the correct terminology.

4.3.1 Newton's first law

Newton's first law essentially dates back to Galileo Galilei, in his "law of inertia".
Newton wrote it as

> Every body perseveres in its state of rest, or uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

We have seen this in effect already, when decomposing 2D motion: the motion along the horizontal axis has thus far had a constant velocity, since we have ignoring air drag, which acts as a force to slow the object down. Along the vertical axis, on the other hand, we've always had gravity accelerating things downward.

Newton's first law, however, is not valid in all reference frames. It is only valid in inertial references frames, the definition of which is a reference frame where the motion of a particle not under any forces moves in a straight line at constant speed. In particular, this is not true in a reference frame which is being accelerated in any way.

So the question is: can we find an inertial reference frame? Is the lecture hall an inertial reference frame, for example?
In can't be. The Earth spins, causing a centripetal acceleration. That's an acceleration, so the answer is already no. There are many additional reasons, though: the Earth moves around the Sun, again causing centripetal acceleration. The Sun moves around the galaxy's center, the galaxy itself has an orbit, and so on.

The Earth has, as mentioned, a centripetal acceleration. We can estimate it by calculating $\omega^2 R_{earth}$, which turns out to be about $0.034\,\mathrm{m/s^2}$, which is of course much, much lower than the acceleration due to gravity. If the Earth spun much, much faster, the centripetal acceleration might start to cancel out gravity noticeably, however.

Let's then calculate the centripetal acceleration due to the orbit around the Sun. The radius of the orbit is about $150 \times 10^9\,\mathrm{m}$, and the period is of course one year.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{365 \cdot 24 \cdot 60 \cdot 60} = 1.992 \times 10^{-7}\,\mathrm{rad/s} \tag{4.49}$$

$$a_c = \omega^2 r = (1.992 \times 10^{-7}\,\mathrm{rad/s})^2 \times 150 \times 10^9\,\mathrm{m} = 5.95 \times 10^{-3}\,\mathrm{m/s^2} \tag{4.50}$$

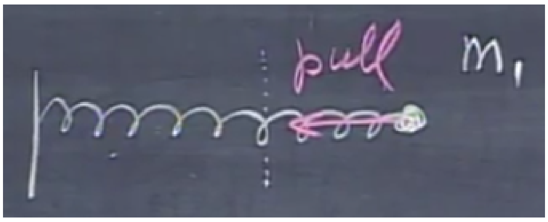
Because the centripetal accelerations calculated above are so tiny, we can consider the Earth as being very close to an inertial reference frame.

A mathematical statement of Newton's first law might be

$$\sum \vec{F} = 0 \Rightarrow \frac{d\vec{v}}{dt} = 0 \text{ (Newton's first law)} \tag{4.51}$$

4.3.2 Newton's second law

Say we have a spring (though the law isn't specific to springs), in the absence of gravity. We extend the spring, and attach a mass m_1 at the end of the spring.



Immediately after we let go of the mass, so that the spring's "pull" contracts the spring and pulls the mass with it, we measure the acceleration of the mass to be a_1.
We then replace the mass with another mass m_2, and measure the acceleration, in the same manner, to be a_2.

We will then find that $m_1 a_1 = m_2 a_2$. The product ma is the *force* (which we have called a "push" or a "pull" until now) exerted upon the mass by the spring. The spring's force is independent on the mass, but the acceleration caused on the mass is not; the acceleration is inversely proportional to the mass.

In equation form, Newton's second law – one of the most important equations in physics – reads

$$\vec{F} = m\vec{a} \text{ (Newton's second law)} \tag{4.52}$$

As shown above, force is a vector. The direction of the acceleration caused by a force is always in the same direction as the force.

There are other ways of stating it, such as the force being equal to the rate of change of momentum, but we have not yet introduced momentum and so will forget about that for now.
The SI unit for force is the newton, in honor of Newton himself, of course. Because the product ma is in units of $\text{kg} \cdot \frac{\text{m}}{\text{s}^2}$, 1 newton equals $1\ \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$.

Just as with the first law, we cannot truly prove Newton's second law. Like the first, it is only valid in inertial reference frames, and we cannot provide such a reference frame to conduct our experiments in.

Note that no statement is made regarding speed or velocity, only acceleration. The law holds equally well at 0 m/s, 5 m/s and 5000 m/s. However, once speeds start becoming noteworthy in relation to the speed of light, Newtonian mechanics becomes more and more inaccurate, and we instead need to use Einstein's relativity for accurate results. This tends to not be an issue in daily life, however, as the two agree very closely at speeds far lower than the speed of light. Even for speeds of 10000 kilometers per second, Newton's equations work quite well (to within about 0.1%). For speeds below 1000 km/s, in other words all everyday speeds, there is practically no difference at all.

4.3.3 Newton's third law

Let's now have a look at the gravitational force. Using the second law, we see that the force is equal to $m\vec{g}$. Double the mass, double the force, etc.
We assume that the lecture hall is an inertial reference frame. Consider an object that is at rest (relative to the lecture hall). We know from the above that there must be a gravitational force on the object, pulling it downwards. However, it is at rest, so there is no acceleration (in our reference frame). Therefore, the net force on it *must* be zero. This is only possible, of course, if there is an equal and opposite force – or sums of forces that adds up to exactly cancel the gravitational force out.

The above is the result of the third law, which can be stated as

> If one object exerts a force on another, the other exerts the same force in the opposite direction on the one.

In other words, if gravity pulls you down into your chair with a force of, say, 700 N, then the chair exerts a force of 700 N back on you. It can be stated more simply as action = −reaction.
We can therefore also write the law as

$$\vec{F_{12}} = -\vec{F_{21}} \text{ (Newton's third law)} \tag{4.53}$$

where $\vec{F_{ab}}$ means the force exerted *by* object a *on* object b. Note that some physics textbook authors use the reverse notation, which can get confusing.

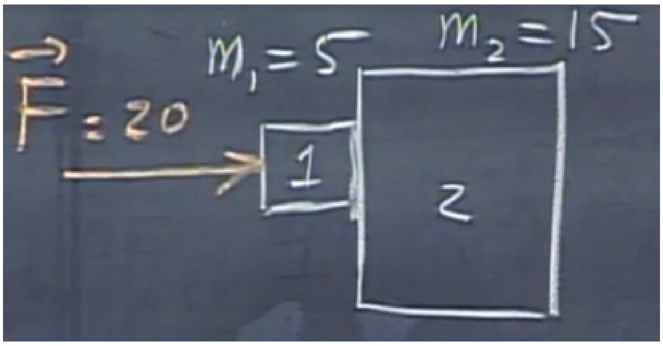
Unlike the first and second laws, the third laws always holds, including in accelerated reference frames.

Also unlike the first law, there are many intuitive examples of the third:

- A garden hose left on its own, with the water on, will start moving backwards. The hose sprays out water by a force, and so the water pushes back on the hose with a force of equal magnitude, and the hose moves backwards.
- You blow up a balloon, and then let it go. The balloon pushes the air out, so the air pushes on the balloon with an equal but opposite force, propelling the balloon backwards.
- When you fire a gun, the gun exerts a force on the bullet, and the bullet exerts a force back on the gun: recoil, causing the gun to move backwards unless held steadily.
- Even when you *walk*, you exert a backwards force on the Earth, which then exerts a force back on you, propelling you forward.

4.3.4 Examples of Newton's laws in use

Let's look at a few examples of Newton's law in practice.



There's a force of 20 newtons towards the right, as shown. Because the total mass is 20 kg, and the force is 20 newton, there will be an acceleration of $1\,\mathrm{m/s^2}$ via $\vec{a} = \dfrac{\vec{F}}{m}$ – Newton's second law. If not else, we know from intuition and daily life that both objects will move towards the right together, with the same acceleration (and thus velocity, since they started together), once they start moving.

The entirety of the force is on object 1. Since they move together, there must be a force between object 1 and object 2 ($\vec{F_{12}}$), towards the right, or object two could not accelerate.

Since we know that the force on object 1 from the left is 20 N, and we also know that $m_1 = 5\,\mathrm{kg}$ and $a_1 = 1\,\mathrm{m/s^2}$ to the right, we can use Newton's second law to find the net force on object 1 to be 5 N towards the right, despite the force on it from the left being 20 N.

How come? Well, the answer lies in object 2. We know that $m_2 = 15\,\mathrm{kg}$ and $a_2 = 1\,\mathrm{m/s^2}$, so the net force on object 2 *must* be 15 N towards the right. The *only* force on object 2 is $\vec{F_{12}}$, so that too must be 15 N towards the right.

What about object 1? Well, Because of $\vec{F_{12}}$ being 15 N to the right, there must be a force $\vec{F_{21}}$ of 15 N towards the left, back on object 1, which "cancels out" most of the 20 N, and leaves object 1 with a net 5 N force to the right. In math form:

$$\vec{F_1} = \vec{F} + \vec{F_{21}} = +20\hat{x} + (-15\hat{x}) = +5\hat{x} \tag{4.54}$$

... defining the increasing direction of x being towards the right.

Now, what about the sum of forces on object 2? Don't we have 15 N towards the right from object 1, and 15 N towards the left back to object 1, for a net zero force? No! The fact that $a \neq 0$ is enough to prove that this cannot be the case.

It's important to note and understand that the two forces $\vec{F_{12}}$ and $\vec{F_{21}}$ act on different bodies. They don't cancel each other out on an individual object. $\vec{F_{21}}$ is a force that object 2 exerts on object 1 – that fact does *not* in any way negate the force exerted by 1 on 2! If that were the case, object 2 could not accelerate, since its net force would be zero.

Newton's third law has an interesting, if immeasurable effect: not only do things we drop fall to the Earth, but the Earth always falls towards the things, as well. If we drop an apple from a certain height, there will be a gravitational force on the apple due to the Earth, causing a downward acceleration. However, the third law states that there must be an equal but opposite force on the Earth, due to the apple! The reason we never notice is that the Earth's mass is so extremely large, that the acceleration is on the order of 10^{-24} m/s^2 (or slightly less) or so in the case of an apple, with a total distance moved smaller than 10^{-23} m, even for an apple falling from 100 meters above the Earth's surface.
Such tiny movements and accelerations are impossible to measure, but they should occur.

4.3.5 Newton's laws: summary

Let's summarize Newton's laws, and point out a few possible pitfalls.

Newton's first law states that a body with no external forces (or no *net* external force) on it will remain as it is, either at rest or moving at constant velocity in a straight line. This only holds true in inertial reference frames! If you are in a car, moving at a *constant velocity* past a street lamp, from your (inertial) frame of reference, the street lamp moves with constant velocity – and there certainly shouldn't be any net force on it, so all is well.
If you accelerate, however, you will see the street lamp appear to accelerate without any net force on it. This is because your car is no longer an inertial reference frame, since it is accelerating with respect to the reference frame of the Earth (and the lamp), so the first law does not hold.

Newton's second law states (in one form) that the acceleration of a body is equal to the *net* force on that body, divided by the body's mass. The acceleration vector is in the same direction as the force vector. Mass is a measure of inertia, i.e. how much a body resists changes in motion. The larger the mass, the smaller the acceleration, for a given magnitude of force.

Newton's third law states that whenever an object a exerts a force on a body b (an "action force"), there is an equal but opposite force (a "reaction force") exerted by object b back an a.
This implies that when you are pulled downwards by the Earth, you also pull the Earth upwards.
However, it does *NOT* imply that when you sit on a chair, and the Earth pulls you down, the chair pushes you up! This is a *very* important distinction. An action-reaction pair *always* acts on different bodies, but note that the gravitational force and the chair's force both act on you!
The *second* law implies that if you don't move, the chair[1] must push you back up with a force of equal magnitude but opposite direction, because the net force on you must be zero if your acceleration is zero.

Another possible pitfall is to say that force is the cause of motion – not true! Force is the cause of *change* in motion, that is, acceleration. You can travel at any velocity with no forces on you whatsoever – in fact, the first law tells us that.
On a related note, keep in mind that while in daily speech, acceleration refers to increasing your speed, while in physics, acceleration simply means change in speed. You accelerate when stopping your car – the acceleration is in the opposite direction of the velocity (and will thus be negative if the velocity is positive), but it's an acceleration nonetheless.

[1] Or something else, or a combination of things, such that the net force is zero.

Finally, the kilogram is a unit of *mass*, not of weight. In daily speech, the two are the same, but in physics, they are distinct quantities, and it's very important to understand how they differ and how they are related.

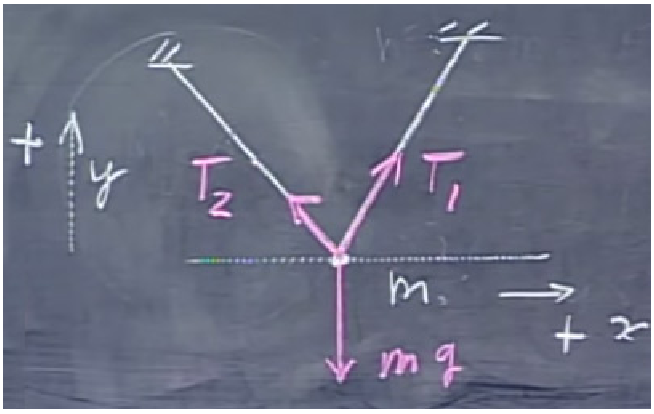
Mass is the measure of how difficult it is to change the motion of an object. Whether we think about pushing something to get it moving, or to try to stop something from moving does not matter: both will become more difficult as mass increases.
The mass of an object is independent of where it is located; it is a property of an object due to its makeup.

Weight, on the other hand, is the force exerted on an object by gravity. You can calculate your approximate weight on Earth as mg, where $g = 9.8\,\mathrm{m/s^2}$ is the approximate gravitational acceleration at the Earth's surface.
The weight of an object *changes* based on the local gravitational force – a person weighs much less on the Moon than they do on Earth, but their mass would be the same in either location.

A scale measures weight, not mass, but usually converts the measurement to a mass by dividing the measured force by g, which then via Newton's second law yields the mass.
This means that if you bring a regular bathroom scale to the Moon, and weigh yourself on it, it will report about 1/6 of your actual mass, as the force of gravity is so much smaller on the Moon, but the scale doesn't know that it has moved: it will still divide the measured weight in newtons by $9.8\,\mathrm{m/s^2}$, and find an incorrect answer. The weight it *measures* is correct, but the mass it reports is not.

4.3.6 Tension and another example of Newton's laws in use

Say we hang a mass m from two strings, suspended at different heights. The leftmost string makes an angle of 45 degrees with the roof above, while the rightmost string makes an angle of 60 degrees. We call the tension in the rightmost string T_1, and in the leftmost T_2. We consider increasing x to be towards the right, and increasing y to be upwards.



There will be a gravitational force of magnitude mg downwards. Because the object is in equilibrium, sitting still with no acceleration, the *net force* on the object must be zero – that is clear from Newton's second law.
Therefore, we conclude that the two tensions T_1 and T_2 perfectly balance the gravitational force mg, so that the net force on the object is zero.

Force is a vector, so we can decompose this into two one-dimensional problems. We don't need to decompose the gravitational force, of course: it is already only in the $-y$ direction. Let's decompose the tension vectors, though.

Let's start with T_1. It makes a 60 degree angle with the horizontal, so by using vector decomposition, we find

$$T_{1x} = T_1 \cos(60°) = \frac{T_1}{2} \tag{4.55}$$

$$T_{1y} = T_1 \sin(60°) = \frac{T_1\sqrt{3}}{2} \tag{4.56}$$

As for T_2, it makes a 45 degree angle, so the sine and cosine are both one over the square root of two. (It makes sense that the force is equal in both directions, since the angle is exactly in the middle of a 90 degree angle, so to speak.)

$$T_{2x} = T_2 \cos(45°) = \frac{T_2}{\sqrt{2}} \tag{4.57}$$

$$T_{2y} = T_2 \sin(45°) = \frac{T_2}{\sqrt{2}} \tag{4.58}$$

What then? Well, we know that the net force must be zero, since there is no acceleration. The same can be said for each axis independently, too: $\sum F_x = 0$ and $\sum F_y = 0$. We can set up equations representing this:

$$T_{1x} + T_{2x} = 0 \tag{4.59}$$

$$\frac{T_1}{2} - \frac{T_2}{\sqrt{2}} = 0 \tag{4.60}$$

$$T_1 = \frac{2T_2}{\sqrt{2}} = \sqrt{2} \cdot T_2 \tag{4.61}$$

Note that because T_2 points towards the negative x direction, the sum of these two forces becomes a subtraction.

As for the y axis:

$$T_{1y} + T_{2y} = mg \tag{4.62}$$

$$\frac{T_1\sqrt{3}}{2} + \frac{T_2}{\sqrt{2}} = mg \tag{4.63}$$

$$T_2 = \sqrt{2}\left(mg - \frac{T_1\sqrt{3}}{2}\right) \tag{4.64}$$

Alternatively, we could have written $T_{1y} + T_{2y} - mg = 0$ (minus mg since it is in the opposite direction) to show that the sum is zero, rather than saying that they must be equal. This is of course the same thing algebraically.

We now have two equations with two unknowns. Let's substitute the value of T_1 into the second equation from (4.61) and find T_2 as a function of only m and constants:

$$T_2 = \sqrt{2}\left(mg - \frac{\sqrt{2}T_2\sqrt{3}}{2}\right) \tag{4.65}$$

$$T_2 = \sqrt{2}mg - T_2\sqrt{3} \tag{4.66}$$

$$T_2 + \sqrt{3}T_2 = \sqrt{2}mg \tag{4.67}$$

$$T_2(1 + \sqrt{3}) = \sqrt{2}mg \tag{4.68}$$

$$T_2 = \frac{\sqrt{2}mg}{1 + \sqrt{3}} \tag{4.69}$$

Since $T_1 = \sqrt{2}T_2$, T_1 is simply

$$T_1 = \frac{2mg}{1+\sqrt{3}} \tag{4.70}$$

We can finally substitute in some values. The lecture used $m = 4\,\mathrm{kg}$, so let's try that. We find

$$T_1 = \frac{(2)(4\,\mathrm{kg})(9.8\,\mathrm{m/s^2})}{1+\sqrt{3}} \approx 28.7\,\mathrm{N} \tag{4.71}$$

$$T_2 = \frac{T_1}{\sqrt{2}} \approx 20.3\,\mathrm{N} \tag{4.72}$$

The professor's answers differ slightly, but match up perfectly if we use $g = 10\,\mathrm{m/s^2}$, so he most likely used that approximation.

As a sanity check, and additional practice, let's just make sure that the forces indeed balance out.

$$T_1\cos(60^\circ) - T_2\cos(45^\circ) \stackrel{?}{=} 0 \tag{4.73}$$

$$14.35 - 14.35 = 0 \tag{4.74}$$

... so that indeed works out in the x direction. Let's check y:

$$T_1\sin(60^\circ) + T_2\sin(45^\circ) \stackrel{?}{=} mg \tag{4.75}$$

$$24.85 + 14.35 = 39.2 \tag{4.76}$$

It works out perfectly!

Chapter 5: Week 3

5.1 Lecture 7: Weight, perceived gravity, and weightlessness

This lecture will discuss weight, its relation to mass, and other related topics.

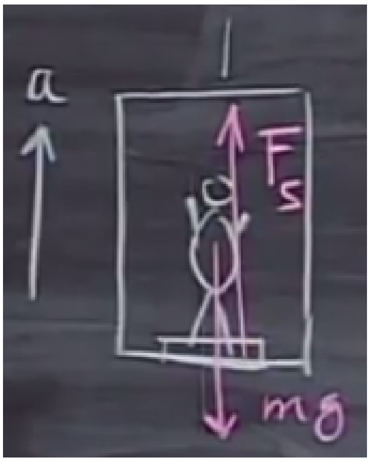
A regular scale, say a bathroom scale, measures weight – which is a *force*. Therefore, it measures in newtons, if we stick to SI units. We can only use a scale to find *mass* in kilograms by knowing the local gravitational acceleration, and dividing that out of the measured result.

As a result of this, such a bathroom scale would measure a mass only 1/6 as great on the moon, where the local gravitational acceleration is about 1/6 of Earth's. This is because the *weight* of the object being weighed has decreased, since the gravitational force is weaker. However, the *mass* of the object has not changed. If the scale made the calculation using the local gravitational acceleration, the measured/calculated mass would be correct. Most don't, of course, and assume the Earth's gravitational acceleration of about $g \approx 9.8\,\mathrm{m/s^2}$, and so display a much too small value for the mass when they divide the smaller weight by the incorrect gravitational acceleration, which is only valid on Earth's surface.

In other words, if the scale displayed weight in newtons, it would display the correct value everywhere, only that the correct value would differ based on location. The person's mass would not change with location, however, so a scale that is used to measure mass should always display the same value for a particular person.

So, you stand on a bathroom scale. Gravity is acting on you to pull you downwards, and as you are not being accelerated, there must be a net force on zero on you. Therefore, we conclude that the scale is pushing you up, with a force of the same magnitude, $F_S = mg$. This force that the scale exerts on you is the definition of your *weight*.

Now, we move the scale and you into an elevator.



Again, gravity acts on you with a force mg downwards. The scale pushes back up with force F_S. However, now, the elevator is being accelerated upwards. The net force must now be positive (upwards), not zero, since you could not have a nonzero acceleration with zero net force.

We find that $F_S - mg = ma$, so $F_S = m(a + g)$. The reading of the scale has increased, and increases linearly with increasing acceleration upwards. If the elevator accelerates upwards at $2g \approx 20\,\mathrm{m/s^2}$, your weight would be three times as high as usual. Only when $a_y = 0$ is your measured weight as it usually is.

Let's now reverse the situation. We now consider increasing a to be downwards, and the elevator is now accelerating downwards. In other words, $a > 0$.

Again, you have gravity acting downwards with a magnitude mg. If that were the only force, you would be in free fall with acceleration g, so there must be some upwards acting force. On the other hand, $|mg| > |F_S|$

or there would be no acceleration at all, so while $|F_S|$ is smaller than the force gravity exerts on you, it's still there.

Back to Newton's second law.

$$mg - F_S = ma \tag{5.1}$$

Reading that out loud, it does make a lot of sense: if $F_S = mg$, then ma is zero, and we are not accelerating. If mg is dominant, we are accelerating downwards (since $a > 0$ means downwards acceleration).

Rearranged,

$$F_S = m(g - a) \tag{5.2}$$

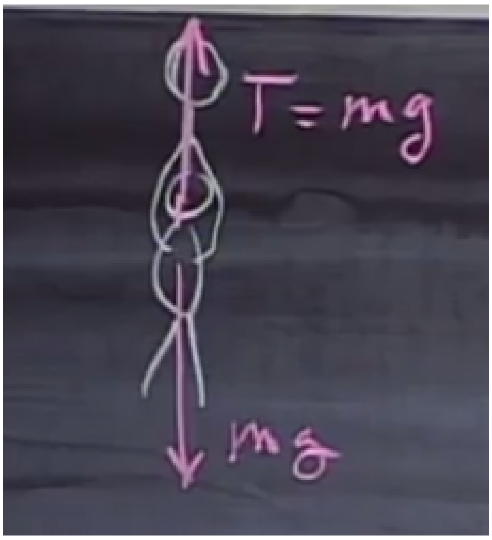
The larger the downwards acceleration, the *less* you weigh. I think most of us have experienced this (and the reverse situation) in fast elevators (that accelerate quickly).

Now, imagine we cut the cable of the elevator. What happens? Well, our equation has the answer. The net acceleration a will be equal to the acceleration from gravity g, so $F_S = 0$. That is, the scale will show you to have zero weight – and you will, because you are now in free fall. You are falling downwards, but other than that, you wouldn't notice gravity the same way we do now. The things falling with you wouldn't care about up or down – a glass filled with water would act the same whether upside down or not.

This is very similar to how things work in the space shuttle and on the International Space Station (ISS). Their orbits around the Earth keep them in constant free fall, only they never hit the surface, as they are going sideways with great velocity (about 7.7 km/s!). We will talk much more about orbits later in the course.

In short, weightlessness is when the forces acting on you are exclusively gravitational. You're not being held up by any floors, ropes, seats, etc, just falling due to gravity pulling on you.

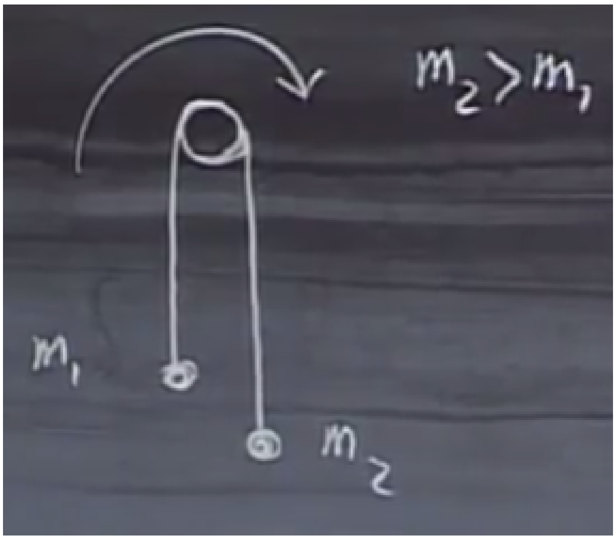
Let's now look at another type of scale:



The scale in this case is a tension meter, inside the string we are hanging from. Because we are hanging still, not accelerating, the tension in the string must equal the force of gravity pulling us down. Therefore, the tension meter reads mg, same as it would on a regular scale standing on the floor.
Thus we see that it makes no difference whether we measure the force a scale is pushing us up with, or the force a rope/tension meter *pulls* us up with.

Let's now accelerate this system. We accelerate it upwards, which must mean the tension in the string goes up – it must be greater in magnitude than mg or we wouldn't accelerate upwards.
Just as with the elevator, we find $T - mg = ma$ or $T = m(a + g)$. Same as before, only we use T for tension instead of F_S for force exerted by the scale.
If we instead accelerate downwards, we again find the same result as before, as do we if we simply cut the string and go into free fall.

Say we have the following system:



The string is massless, the pin/pulley is massless, but the two objects hanging an each end are not; the right one at mass m_2 has a greater mass than that of m_1 to the left.

What happens? Well, we know from intuition and experience that the system will accelerate as shown. m_2 will fall down, while m_1 will be pulled up.

Because we consider the case where there is no friction, and the string is massless, the tension in the left side must equal the tension on the right side. Only in the case of no friction and a massless string is this true, however.

Why is that, though? It's relatively easy to show. Consider a small piece of the string on the left side. It has gravity pulling the mass m_1 down, and a force upwards because of the mass m_2 on the other side of the pulley. If the two forces were not equivalent, the massless string would experience an acceleration $a = \frac{F}{0}$ – that's clearly impossible.

The same argument can be used for any part (of any length) of the string. The tension must be equal everywhere.
Again, this is only true because we consider the string massless, and the pulley frictionless.

Now, we earlier showed that the tension in the string is an indicator of the weight hanging from it. That means that while this acceleration is taking place, m_1 and m_2 have the same weight! The obviously don't have the same *mass*; that's different, by definition of $m_2 > m_1$. Because weight is mass times acceleration, however, the *weight* of m_1 has increased, as it is being accelerated upwards (think about the elevator and the bathroom scale), while the weight of m_2 has *decreased*, since it is accelerating downwards (falling, though not in free fall, so it still has a weight larger than zero).

Let's calculate the acceleration and tension for these objects and strings.
m_1 accelerates upwards. The tension in the string is $T = m_1 g$ while it is in equilibrium, but that cannot be the case now. The force upwards must be greater than mg, or it cannot accelerate upwards. Therefore, the tension in the string must be the sum of the downwards force due to gravity, plus the extra "perceived" gravity from the upwards acceleration. In total, we have $T = m_1 a + m_1 g = m_1(a + g)$.

For the second object, gravity pulls downwards with force $m_2 g$, while the string tension pulls back up.

This object is accelerating downwards, however, so $m_2 g$ must be greater in magnitude than the tension. However, remember that the tension must be the same as in the case of the first mass! It's the same string, and as we showed earlier, we get an impossible situation if the tension differs throughout the string.

To avoid sign confusion, we now denote *downwards* acceleration to be positive. We write the tension out as an equation, and find $T = m_2(g - a)$.

We now have two equations, so we can set them equal and solve for the acceleration a:

$$T = m_1(a + g) \tag{5.3}$$
$$T = m_2(g - a) \tag{5.4}$$
$$m_1 a + m_1 g = m_2 g - m_2 a \tag{5.5}$$
$$m_1 a + m_2 a = m_2 g - m_1 g \tag{5.6}$$
$$a(m_1 + m_2) = g(m_2 - m_1) \tag{5.7}$$
$$a = \frac{g(m_2 - m_1)}{m_1 + m_2} \tag{5.8}$$
$$\tag{5.9}$$

We can then substitute that value into $T - m_1 g = m_1 a$ that we found earlier:

$$T - m_1 g = m_1 \frac{g(m_2 - m_1)}{m_1 + m_2} \tag{5.10}$$
$$T = \frac{2gm_1 m_2}{m_1 + m_2} \tag{5.11}$$

The algebra isn't shown, but this is indeed the case.
These two results make intuitive sense. If we set $m_1 = m_2 = m$, we find $T = mg$ and $a = 0$. All is as it should: the tension on a string with a weight m hanging from it better be mg, if it's not accelerating! Likewise, a better be zero, since both masses and weights are equal, so there is no net force on either mass.

If we even consider as $m_2 \to 0$, we find that the tension goes to zero, and the acceleration goes to g (in magnitude). This is because m_1 is now in free fall, and since m_2 is massless and the string is massless, there is nothing left to cause tension in the string. The acceleration is $-g$ since it is simply in free fall – nothing is holding it up.

We can also show that if $m_2 > m_1$, then the relationship $m_1 g < T < m_2 g$ will hold for this system. As they are being accelerated, the tension will be equal for both masses. Therefore, m_1 must gain weight, and m_2 must lose weight. (Keep in mind that m_1 accelerates upwards, so it gains weight, like in an elevator, and m_2 accelerates downwards, and so it loses weight.)

5.1.1 Weightlessness

We talked a bit about perceived gravity and so on last week. Let's expand on it.

Consider the case where we are swinging an object around a rope of length R in the vertical plane. R is then the radius of the circle the object traces out.
Gravity with force mg acts downwards at all times. We spin the object with angular velocity ω.

The string will have a tension T, which will change in direction as the thing spins, of course. The magnitude should also change – if the angular velocity is low, just on the edge of this working out, the tension should be zero at the top. Any slower, and the string will slack off and the object starts falling down.

Let's calculate the tension. First, we know there will be a centripetal acceleration $|a_c| = \omega^2 R$. That must be the case, or the object cannot travel around in a circle like this, period.

At the lowest point of the circle, gravity acts downwards with force mg, while the tension upwards is T. There's also the centripetal force ma_c. We have not really used the term centripetal force yet, but it's a force, so it's found by the centripetal acceleration times the mass in question.

All in all, we find $T - mg = ma_c$, so that $T = m(a_c + g)$. This equation looks almost exactly like the one for acceleration in an elevator, which we found earlier. If $a_c = g$, you tension of the string would be twice the weight of the object.

Now, let's look at the top of the circle instead. As we did earlier, we will now reverse out sign convention so that downwards forces and accelerations are positive.
As before, the tension and the force of gravity add up to the centripetal acceleration, so we find $T + mg = ma_c$, so that $T = m(a_c - g)$.

This is just about exactly the same equation as we had for the elevator accelerating downwards, losing weight.
If $a_c = g$, the tension will be zero, and the object will be weightless. If $a_c > g$, there will be tension in the string, equal to the object's weight.
a_c cannot be smaller than g, however. That would give us negative tension, which can't happen. What this implies is that this situation simply would never happen: if $a_c < g$, then the object would never reach the top while tracing out a circle, but would have started falling prior to reaching the top.

The rest of the lecture is mostly demonstrations and related talk, which I didn't find it very useful to take notes for, but it's certainly worth watching, of course.
One thing is worth writing down though. The lecture talks about parabolic plane flights – they start by upwards at a ≈ 45 degree angle, then follow a parabolic trajectory (free fall) with the engines off, and then re-start the engines and repeat.

This causes the plane to be in free fall for about 30 seconds at a time, during which time any traveler would experience weightlessness – but *not zero gravity*. There are absolutely similarities between free fall and zero gravity (which you could almost experience far, far from any planets or stars, but certainly not on Earth), but they are not the same.

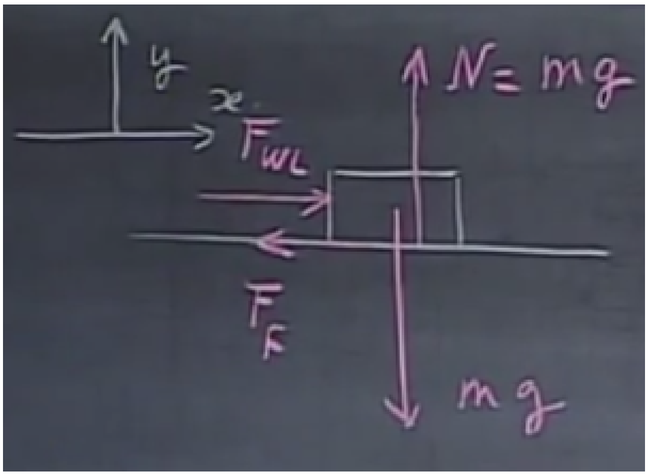
Zero gravity implies that the Earth's gravity somehow stops acting on the people in the plane, despite the fact that Earth's gravity is almost as strong up there as it is at the surface. Even at the ISS, in orbit at an altitude of about 350 km (which is quite close to the surface for a "space" station, all things considered), the gravitational pull is about 90% of the strength it is near the surface.
The astronauts are then weightless for exactly the same reason: they are constantly falling "towards" the Earth, only they have such a huge sideways velocity, that they never hit it.

We will talk more about gravity later in the course, including how g is calculated etc.

5.2 Lecture 8: Frictional forces

Let's talk about friction. Thus far, we have ignored friction (and air drag) in all problems we've solved; that will now start to change.

Let's look at a simple case to begin with. We have an object at rest, on a flat surface:



There is a gravitational force of magnitude mg pulling the object down, and a *normal force* $N = mg$ (also in magnitude) from the surface on the object, or it could not be at rest; Newton's second law. (Again, however, note that this is NOT a case of Newton's third law; there two forces both act on the object, and so they are not an action-reaction pair.)

We (the professor) exerts a force F_{WL} (WL for Walter Lewin) in the $+x$ direction. Because of friction, the object will not start to move unless the force is great enough. (Without friction, any force, no matter how small, causes an acceleration, even if it's tiny.)

There is a frictional force F_F in the $-x$ direction that exactly cancels out the force we apply. We push harder and harder, and eventually the frictional force reaches its maximum value, at which point we overcome it and the object starts to accelerate.

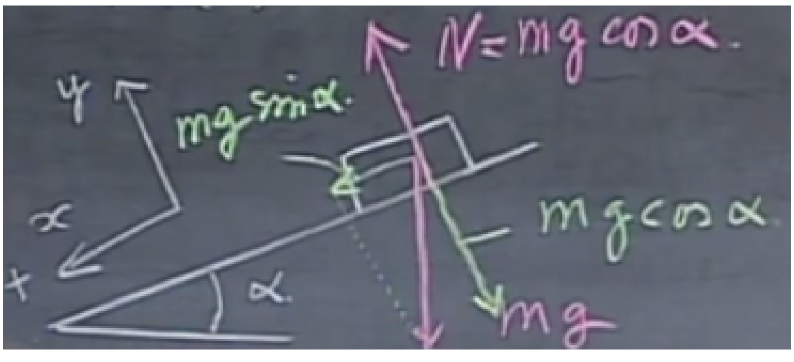
It is then experimental fact that this maximum, F_{Fmax}, is

$$F_{Fmax} = \mu N \tag{5.12}$$

where μ is a friction coefficient.
We can differentiate between the *static* friction coefficient μ_s and the *kinetic* (or dynamic) friction coefficient μ_k.

$\mu_s N$ is the frictional force we need to overcome to get a resting object to start moving, while $\mu_k N$ is the force we need to overcome to keep accelerating it. (If $\mu_k N = F$, there is zero net force, and so no acceleration. If $\mu_k N > F$, the object will slow down.)
We know from experience that it takes more force to get something to move in the first place, so $\mu_s > \mu_k$ – "always", the lecture says, but there does appear to be some strange exceptions to this rule. I'm assuming this course will not cover (or mention) them, however.

We can calculate a friction coefficient by putting an object on an incline, and measure the angle of incline required to get the object to move (due to the gravitational force downwards, of course).



Note the choice of coordinate system, which is tilted such that the acceleration (and movement) in the y direction will always be zero, but more importantly such that the y axis is exactly perpendicular to the surface of the incline.
The downside to this approach is then that we need to decompose the gravitational force, since it is no longer strictly in one axis.

Because of how the angle α is defined, the strength of the gravitational force in the y direction is $mg \cos \alpha$ (if $\alpha = 0$, $\cos \alpha = 1$ and so it is strictly in the y direction), while that in the x direction is $mg \sin \alpha$. This is a bit opposite to how it usually is (x tends to use the cosine, while y uses the sine), but it should make sense why this is.

There is a normal force N opposing $mg \cos \theta$, which must be equal in magnitude – there is no acceleration in the y direction, so the net force along that axis must be zero, via Newton's second law.

There is a frictional force F_F in the negative x direction, equal in magnitude to $mg \sin \alpha$ (the gravitational force along the slope), since the object is still not moving. We gradually increase the angle, which increases F_f, but of course also the gravitational force in the x direction (downwards along the slope). Sooner or later, the frictional force reaches its maximum, and gravity "wins".

How do we then calculate μ_s, the static friction coefficient, in terms of the angle α_{max} (the maximum angle possible before the object starts to slide)?

Well, it should be easy! We know the strength of the force pulling the object: $mg \sin(\alpha_{max})$. We know that $F_F = \mu_s mg \sin(\alpha_{max})$ must exactly equal this force in magnitude to keep it standing still.
We also know that $F_F = \mu_s N$, as we mentioned earlier.

Therefore, we set the two equal, and we can solve for μ_s. $N = mg \cos \alpha$, so

$$\mu_s mg \cos(\alpha_{max}) = mg \sin(\alpha_{max}) \tag{5.13}$$

$$\mu_s = \frac{mg \sin(\alpha_{max})}{mg \cos(\alpha_{max})} \tag{5.14}$$

$$\mu_s = \frac{\sin(\alpha_{max})}{\cos(\alpha_{max})} = \tan(\alpha_{max}) \tag{5.15}$$

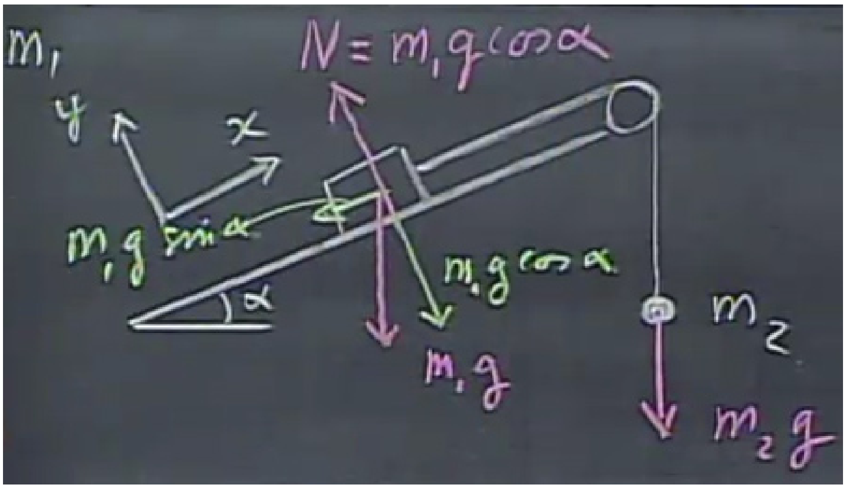
So finding the static friction coefficient is truly simple: measure the maximum angle possible before the object starts to slide, take the tangent of that angle, and you're done! And, since the incline makes a triangle with the vertical height, horizontal length and the incline itself (the hypotenuse), we can measure this even without knowing angles; we can calculate the angle even if we can only measure distances.

Note that two seemingly important quantities are nowhere to be seen in this result: the mass of the object does not matter, and neither does the amount surface area that is in contact with the incline!
This means that two parked cars – a large truck and a small car,will start to slide at the same angle, if they were tilted together, so to speak. Not only does the mass not matter, but the width of the tires (or the number of tires) also does not matter.

These two facts are then demonstrated qualitatively, by sliding a few objects down a wooden plank. Indeed, adding a few times the mass to a plastic container didn't change the result by much (but by a little – because the plank is not exactly uniform, the containers may not be identical, etc.). Neither did it make a noticeable difference to slide down two small pieces of wood, one lying down (large contact area) and one standing on the edge (small contact area).

5.2.1 Friction on a block with a pulley

Let's look at a different, but related example. We have the following setup:



A block of mass m_1 is sitting on an incline, with the same gravitational force, decomposed as previously. However, this time, a second mass m_2 hangs on a massless and fixed-length string (increasing tension doesn't increase the length of the string), over a pulley (which itself is massless and frictionless).

As before, there is no movement of the block in the y direction, so we be sure that the normal force N must exactly cancel out the gravitational force of $m_1 g \cos\alpha$ in our rotated $-y$ direction.
The string has a tension, since mass m_2 is hanging on it. As before, since the string is massless and has a fixed length, the tension must be the same everywhere in the string.

Now... This situation has a tricky part that the previous one didn't: depending on the friction coefficient of the block, and the magnitude of masses m_1 and m_2, one of three things can happen: the block can accelerate "downhill", it can accelerate "uphill", or it can simply stand still. We need to consider all of these possibilities when solving this problem. Because of this, we do not know in which direction of the frictional force is; we only know that it always opposes the object's motion, which could be either $+x$ or $-x$.

We do know that the maximum magnitude of the frictional force is

$$F_{Fmax} = \mu_s N = \mu_s m_1 g \cos\alpha \tag{5.16}$$

The tension in the string, T, can be drawn as a vector opposing $m_2 g$, in the string above mass m_2.

We will now evaluate three different situations. In all cases, the system is at rest for now, but what is about to happen differs: there's the case where it's *just* about to start accelerating towards the left (downhill for the block), the case where it's *just* about to start accelerating in the other direction, and the case where it will remain at rest/in equilibrium.

Because the system is rest, the tension must be of magnitude $m_2 g$, so that it exactly balances the gravitational force on mass m_2, and as mentioned previously, that tension is the same in all parts of the string.

Case 1: About to accelerate uphill

We will first look at the case where m_2 "wins", and the block is *just* about to start moving uphill, but is *still at rest.*
In this case, the frictional force is at a maximum F_{Fmax}, which opposes the direction it's about to move it, so F_{Fmax} acts together with gravity in the $-x$ direction.
We can write Newton's second law for the system:

$$T - m_1 g \sin\alpha - F_{Fmax} = 0 \text{ (we substitute } T = m_2 g) \tag{5.17}$$

$$m_2 g = m_1 g \sin\alpha + F_{Fmax} \tag{5.18}$$

Tension acts "uphill', while the other two act "downhill", and they must sum to zero since a is still zero.

When this equation is true, the block is *just* about to move uphill. Therefore, we can write a criterion for the uphill motion:

$$m_2 g \geq m_1 g \sin\alpha + F_{Fmax} \tag{5.19}$$

If m_2 increases in mass by even the tiniest bit, the system will start to accelerate so that m_2 starts moving downwards.

Case 2: About to accelerate downhill

In this case, the frictional force is in the same direction as the tension, and thus in the opposite direction of gravity. We write Newton's second law again:

$$T + F_{Fmax} = m_1 g \sin\alpha \tag{5.20}$$

$$m_2 g = m_1 g \sin\alpha - F_{Fmax} \tag{5.21}$$

If m_2 is just a tiny bit *smaller* (or m_1 greater, or F_{Fmax} smaller), the system will start moving downhill. Therefore, the criterion for downhill motion is

$$m_2 g \leq m_1 g \sin\alpha - F_{Fmax} \tag{5.22}$$

Case 3: Neither case matches

In case neither of the two conditions are met, the system will simply sit in equilibrium. The frictional force will be "adjusted" so that it causes the net force in the x direction to equal zero.

Example case

Let $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $\mu_s = 0.5$ and $\mu_k = 0.4$, while using $g = 10\,\text{m/s}^2$ for our calculations, just to get an idea. Also, let $\alpha = 30°$.

What will happen in a system with these parameters? Well, we have three possible cases, with equations (or inequalities) we can look at. Since both conditions depend on the same three force terms, let's calculate their values:

$$m_2 g = 20\,\text{N} \tag{5.23}$$

$$m_1 g \sin\alpha = 5\,\text{N} \tag{5.24}$$

$$F_{Fmax} = \mu_s m_1 g \cos\alpha \approx 4.33\,\text{N} \tag{5.25}$$

Let's now look at each of the two conditions. For the block to start sliding uphill, substituting in the values, we must have $20\,\text{N} \geq 5\,\text{N} + 4.33\,\text{N}$. Since that is true, the block will indeed start sliding uphill. Let's just verify that the second case is false, just for the sake of argument. We need $20\,\text{N} \leq 5\,\text{N} - 4.33\,\text{N}$, which is certainly not the case. So indeed, only one case matches, and it says the block will start accelerating uphill.

Let's now attempt to calculate the magnitude of the acceleration, and the string tension. We know that it will start moving uphill, so the frictional force is then downhill, in the $-x$ direction. The magnitude of this force now changes, however: the block is in motion, and so we must now use the kinetic friction coefficient μ_k instead. Using that, we find

$$F_{Fmax} = \mu_k m_1 g \cos\alpha \tag{5.26}$$

We can again write Newton's second law for this case. The tension is uphill, gravity downhill, and friction downhill. Those forces must equal $m_1 a$, where a is the uphill acceleration. (Since the block is now being accelerated, the net forces no longer sum to zero.)

$$T - m_1 g \sin\alpha - \mu_k m_1 g \cos\alpha = m_1 a \tag{5.27}$$

We need a second equation, however. T is unknown – because m_2 is now being accelerated downwards, it is "falling" or "losing weight", so $m_2 g > T$ or the object could not accelerate down!
Since m_2 will never change, and we are still on Earth's surface, g can also not change. The only thing that *can* change is the tension. The tension *must* go down, or m_2 simply cannot accelerate downwards!

The second equation can be found by thinking about mass m_2. Because the string has a fixed length, the acceleration of this mass *must* be equal to that of the block sliding uphill. Anything else and clearly, the string would need to get either longer or shorter, depending on which acceleration was greater.

Because they are equal in magnitude, then, we can write a second law equation for mass m_2, using positive values for the downwards direction (so that this a has the same positive sign as the other a uphill):

$$m_2 g - T = m_2 a \tag{5.28}$$

Solving this system, we get a fairly complex answer, unless we substitute in the numbers early. If we do, we find $a \approx 3.85\,\mathrm{m/s^2}$ and $T \approx SI12.3N$. If we don't, we find, after simplification,

$$a = g\,\frac{m_2 - m_1(\mu_k \cos\alpha + \sin\alpha)}{m_1 + m_2} \tag{5.29}$$

And for the tension:

$$T = g m_1 m_2 \frac{1 + \mu_k \cos\alpha + \sin\alpha}{m_1 + m_2} \tag{5.30}$$

Two things are important to note from the numerical results we found. One is that the acceleration was a positive number. We had already calculated that the block should move uphill, and since the positive x direction is uphill, a negative acceleration would mean it should move backwards. We already know that is not the case, so the acceleration must be positive in this case.

Second, the tension must be smaller than $m_2 g$, or that mass couldn't possibly be accelerating downwards. If $m_2 g$ doesn't "win" over the tension, how could the mass be accelerating downwards?

Let's take a quick look at what would happen in the same system, if $m_2 = 0.4\,\mathrm{kg}$ instead. In that case, $m_2 g = 4\,\mathrm{N}$. Let's look at the conditions again. Is it true that $4\,\mathrm{N} \geq 5\,\mathrm{N} + 4.33\,\mathrm{N}$? No, certainly not. Is it then true that $4\,\mathrm{N} \leq 5\,\mathrm{N} - 4.33\,\mathrm{N}$? No, that's not it, either.

Since neither condition is met, the system will stay as it is, with $a = 0$. Note that the equations we derived just above, for a and T, are not valid in this case and cannot be used. They only hold in the case of accelerations upwards, since that is what we derived them for.

What will happen is that the frictional force will be adjusted so that together with the tension, it holds the object up.

Chapter 6: Week 4: Exam review only

This week had only one lecture, and it was all a review from weeks 1-2, so I only worked through the problems and didn't take any notes.
There is one interesting bit at the very end, though, which is worth watching even now that the exam has closed.

Chapter 7: Week 5

7.1 Lecture 10: Hooke's law, simple harmonic oscillator

Say we have a spring, in its "relaxed" state, i.e. in equilibrium. We choose to place $x = 0$ at the spring's end, and then extend the spring a distance x.

There will be a restoring force that attempts to pull the spring back to its original length. For many springs, it is approximately true that this restoring force F is proportional to the displacement x.
For an *ideal spring*, we can write the force as

$$F = -kx \tag{7.1}$$

where k is known as the spring constant, and the minus sign signifies that the force opposes the displacement. (If x is positive to the right, the force will be to the left, and vice versa.)
This also holds if the spring is compressed (shortened) instead of stretched.
The above relation is known as *Hooke's law*.

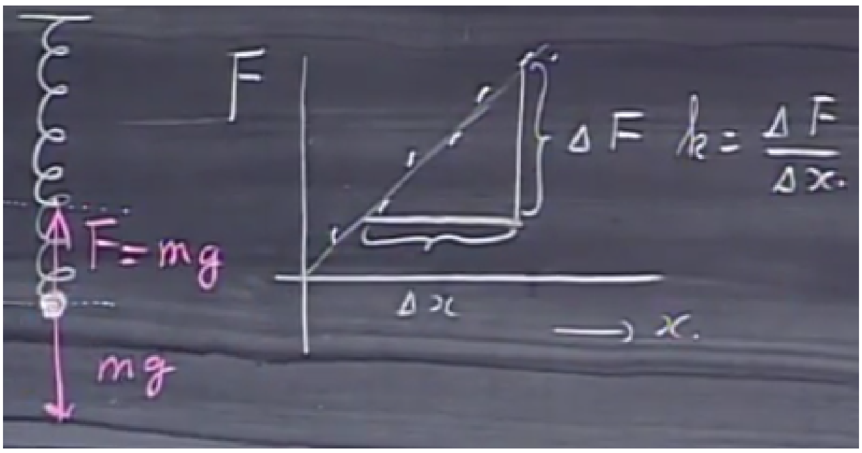
We can measure this spring constant in a few different ways. Perhaps the simplest would be to hang a mass from a spring and measure how far it extends due to the pull of gravity. When it is in equilibrium, we know that the upwards force from the spring must equal the downwards force due to gravity. Therefore, we can measure x and m, and we know g, so we can calculate the spring constant:

$$|F| = kx = mg \tag{7.2}$$

$$k = \frac{mg}{x} \tag{7.3}$$

Assuming we work in SI units, the units of the spring constant must then be in newtons per meter.

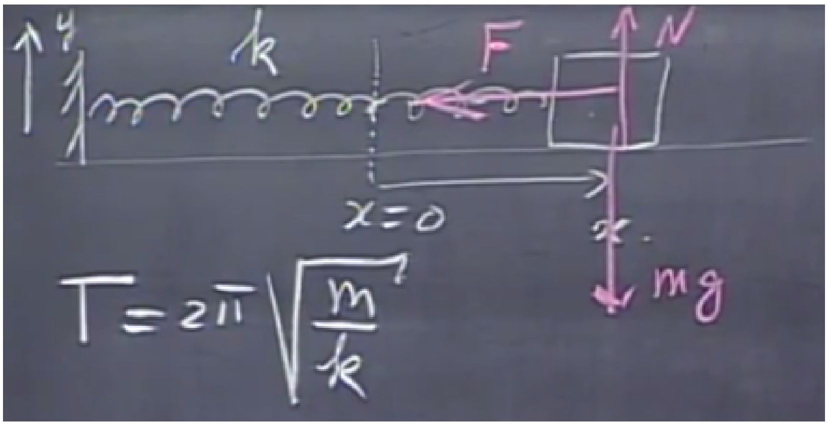
If we instead change the masses, we will get a plot that is a straight line, assuming Hooke's law holds. We can then find k as the slope of this line:



This is probably a more reliable test than the single calculation above, since it will show if Hooke's law doesn't hold for the particular spring, instead of silently assuming that it does.

Hooke's law has its limitations, as you might expect. It's possible to stretch a spring so far that it permanently changes its shape, in which case the restoring force will not increase linearly, but grow slower than Hooke's law would predict.

Let's look at a second way of measuring the spring constant of a given spring. Say we have another spring, again with $x = 0$ at the end of spring's relaxed length. We extend the spring further, and attach a mass m to the end of the spring. The mass rests on a frictionless surface.



When we let the mass go, this system will begin to oscillate. The spring force pulls the mass towards the left until it is relaxed, but when that happens, the mass is already moving towards the left, and has inertia in that direction. The spring will be compressed, and now push on the mass, which eventually comes to a halt, accelerates back towards the right, etc.

Because of the relationship shown, which will be derived shortly, we can either calculate a spring constant from a known mass (while using a stopwatch), or me can measure a mass, if we know the spring constant, even in the absence of gravity! Note that there is no relation to g in the formula. The only place where it appears is in the pull of gravity and the normal force, but since the surface is taken to be frictionless, neither force matters for the oscillation period.

It's interesting to note that the amplitude of the oscillation, i.e. how far it moves horizontally from the center point, does not affect the period at all. If the amplitude is small, it will move slowly back and forth, but if the amplitude is large, it will move at much greater speed, to keep the period constant – assuming Hooke's Law holds.

7.1.1 Simple harmonic oscillators: mathematical derivation

Let's have a look at the situation we have above. We apply Newton's second law to the system, and find

$$ma = -kx \tag{7.4}$$

Written in alternative notation, and divided through by m:

$$m\ddot{x} + kx = 0 \tag{7.5}$$

$$\ddot{x} + \frac{k}{m}x = 0 \tag{7.6}$$

$\dot{x}$ is used to signify the first time derivative of position (velocity), while $\ddot{x}$ is used for the second time derivative (acceleration).

Prof. Lewin calls this last equation "arguably the most important in all of physics". It is the equation that governs simple harmonic oscillators; there are many kinds of such oscillators.

First, it is demonstrated that the solution for $x(t)$ should be some form of sinusoid. Trying to keep this as general as possible, we can write

$$x = A\cos(\omega t + \varphi) \tag{7.7}$$

Here, A is the amplitude (how far it swings, from the center point), ω is the *angular frequency* (not to be confused with angular velocity), in radians/second, and φ is the *phase angle*, in radians.

As we have seen many times before,

$$T = \frac{2\pi}{\omega} \tag{7.8}$$

since if you increase t by T seconds, the argument to the cosine will have increased by 2π radians $= 360°$, and the function repeats.

We can write this in terms of frequency ("regular" frequency in Hertz, i.e. the number times something happens per second, rather than angular frequency):

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{7.9}$$

(We can think of the last equations as being in radians per second, divided by 2π radians; the "per second" is then all that remains.)

Next, we substitute our "trial answer" into the equation relating x and $\ddot{x}$. To do that, we must first find $\ddot{x}$, i.e. the second derivative of x with respect to time. Keeping in mind the chain rule, we find

$$x = A\cos(\omega t + \varphi) \tag{7.10}$$
$$\dot{x} = A\omega(-\sin(\omega t + \varphi) = -A\omega\sin(\omega t + \varphi) \tag{7.11}$$
$$\ddot{x} = -A\omega^2\cos(\omega t + \varphi) \tag{7.12}$$

Now, because $x = A\cos(\omega t + \varphi)$, we can also write

$$\ddot{x} = -\omega^2 x \tag{7.13}$$

All in all, our differential equation becomes

$$-\omega^2 x + \frac{k}{m}x = 0 \tag{7.14}$$

Because this must always hold for all x, it must be the case that

$$w^2 = \frac{k}{m} \tag{7.15}$$
$$\omega = \sqrt{\frac{k}{m}} \tag{7.16}$$

With the equation we already had for T, it turns out that

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{7.17}$$

... as shown in the figure prior to this derivation.

As we can see, the period is independent on the amplitude, and also independent on the phase angle φ. More on that now.

When we start this oscillation, we can decide two things: how far we stretch the spring before we let the mass go, and how much (if any) of a push we give it, i.e. initial velocity. The amplitude and phase angle will be decided by these *initial conditions.*

Say we give it a push, so that $\vec{v} = -3\hat{x}$ m/s, while it is at $x = 0$ at $t = 0$. With all these conditions, we can find both the amplitude A and the phase angle φ. We know the equation must hold true at $x = 0$ at $t = 0$, since that's a given, so we plug that in:

$$0 = A\cos(\varphi) \tag{7.18}$$

This equation can be true in two cases: $A = 0$, or $\cos(\varphi) = 0$. A cannot be 0, because we know there will be an oscillation with a nonzero amplitude. Therefore,

$$\cos(\varphi) = 0 \tag{7.19}$$

$$\varphi = \frac{\pi}{2}, \frac{3\pi}{2} \tag{7.20}$$

Either value of φ makes the cosine zero. (There are of course an infinite number of such angles, but we restrict them to $0 < \varphi < 2\pi$.)

Since we have a time-varying position, we can take the time derivative to find the velocity as a function of time, and relate that to the initial condition $v = -3$ m/s.

We calculate the time derivative of the equation (which we did earlier), and substitute in the values, including $t = 0$, and set it equal to -3 m/s:

$$x = A\cos(\omega t + \varphi) \tag{7.21}$$

$$\dot{x} = -A\omega\sin(\omega t + \varphi) \tag{7.22}$$

Keep in mind that $\dot{x} = v$.
In with the values, and solve:

$$-3 = -A\omega\sin(\pi/2) \tag{7.23}$$

$$-A\omega = -3 \tag{7.24}$$

$$A = \frac{3}{\omega} \tag{7.25}$$

Say the object has a mass $m = 0.1\,\text{kg}$, and the spring has a spring constant of $k = 10\,\text{N/m}$.

We find ω as

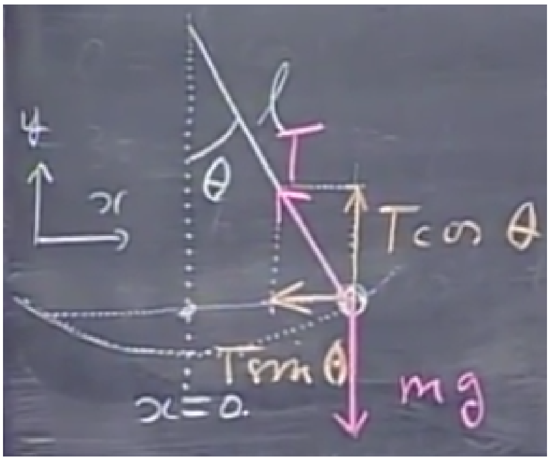
$$\omega = \sqrt{\frac{k}{m}} = 10 \text{ rad/s} \tag{7.26}$$

So $A = 0.3$ meters, and the full equation that explains this oscillation is

$$x(t) = 0.3\cos(10t - \frac{\pi}{2}) \tag{7.27}$$

7.1.2 Motion of a pendulum

Next up, we have a look at the equations that govern a pendulum's motion.



We have a mass m attached to a string of length ℓ, which is swinging back and forth. We choose a coordinate system with its origin at the pendulum's center, decompose the forces, and write down Newton's second law for the system.

Gravity is pulling downwards on the mass, while there is a tension in the string pulling upwards (but not straight upwards, of course). We decompose this tension, and write down the equation for the x direction:

$$ma_x = m\ddot{x} = -T(\theta)\sin\theta = -T(\theta)\frac{x}{\ell} \tag{7.28}$$

a is towards the right (since that is the positive direction of the coordinate system), but the horizontal component of the tension is towards the left at this moment. The second equality holds for trigonometric reasons.

For the y direction, we find

$$m\ddot{y} = T(\theta)\cos\theta - mg \tag{7.29}$$

We then have two coupled differential equations to solve, which is above this course's level... but we can simplify things a bit.
We start out by making some approximations. First out is the small angle approximation, which is usually used to imply that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ if $\theta \ll 1$ radian (it works quite well up to 0.2 rad $\approx 11.5°$ or so, at least, where $\cos(0.2) \approx 0.98$ and $\sin(0.2) \approx 0.1987$).

There is a second important approximation we can make if we assume the angle will always be small. Have a look at the diagram above, and note how the x amplitude is much greater than the y amplitude. For a 5 degree swing, the x motion is about 25 times as large as the y motion, and it is still 11 times as large at 10 degrees.
We can therefore approximate the acceleration in the y direction to be zero, so we find

$$0 = T(\theta) - mg \tag{7.30}$$
$$T = mg \tag{7.31}$$

The cosine disappears, since we approximated it to be one, and the left-hand side disappears since $a = \ddot{x} \approx 0$.

We substitute this into our differential equation for x, and find

$$m\ddot{x} = -mg\frac{x}{\ell} \tag{7.32}$$
$$m\ddot{x} + mg\frac{x}{\ell} = 0 \tag{7.33}$$
$$\ddot{x} + \frac{g}{\ell}x = 0 \tag{7.34}$$

Compare this to the spring-mass system, which obeyed $\ddot{x} + \frac{k}{m}x = 0$ – this is another simple harmonic oscillator!

Since the only difference between the differential equations is, practically, two variable names, the solution is of course the same. We find these equations for this system:

$$x = A\cos(\omega t + \varphi) \tag{7.35}$$
$$\omega = \sqrt{\frac{g}{\ell}} \tag{7.36}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \tag{7.37}$$

Keep in mind that these results are limited to the case where the angles are small, and the string can be considered massless in comparison to the mass at the end of the string.

Now, let's compare the results we found for the spring/mass system and the pendulum on a string.
For the oscillating spring, the period depends on the spring constant and the mass m. This can be explained simply, as follows: when you extend a spring, there is a restoring force, proportional to the distance you extended it. However, the force is not in any way dependent on the mass of the object you attach to the spring; therefore, the acceleration is inversely proportional to the mass, via Newton's second law:

$$|a| = \frac{|F_{spring}|}{m} = \frac{k|x|}{m} \tag{7.38}$$

If the acceleration is very low, clearly the period must increase.

As for the pendulum, the period is independent on the mass. Why?
Again, this can be shown quite easily. If m doubles, mg doubles, and so the tension $T = mg$ must also double (since the y acceleration is the same – approximately zero). The restoring force $T\sin\theta$ is proportional to T, so that doubles as well. If the mass doubles and the force doubles, the acceleration stays exactly the same, and so the period is not affected.

Next, k and g. If k is high, the period is short, which makes sense: the acceleration is proportional to $\sqrt{k}$, so for very large k (meaning a very stiff/strong spring – remember that it's in newtons per meter of displacement) the acceleration is high, and the period low.
As for the pendulum, it is *inversely* proportional to $\sqrt{g}$, so if g is low, the period is very large, and it goes to infinity as $g \to 0$. A pendulum could not work in weightlessness, where the perceived gravity is zero, since it relies on gravity to swing. (This too is easy to see: the restoring force is proportional to g, so with $g \approx 0$, there shouldn't be any restoring force, nor any string tension.)

"All" that remains in the lecture is one of the best demonstrations in this class, which means no notes taken, but careful watching instead!

7.2 Lecture 11: Work, energy and universal gravitation

Let's get started right away.
Work is a measure of the amount of energy a force uses when moving an object. In simple applications, it can be defined as $W = Fd$, where F is the magnitude of the force, and d is the distance the object moves.

A more useful definition, still in one dimension, is an integral, which then can take care of non-constant forces as well:

$$W_{AB} = \int_A^B F \ dx \tag{7.39}$$

... where A and B are the x coordinates where the object starts out, and ends up, respectively.

Work is a scalar quantity, and can be negative, zero or positive.
It is positive if the force and the displacement are in the *same* direction, and negative if they are in the *opposite* direction. It can be zero, e.g. if there is no displacement.

The SI unit of work is the joule, J, which from the definition clearly is the same as a force of 1 newton times a displacement of 1 meter. We rarely if ever write Nm for work; though Nm and J are mathematically equivalent, Nm is used for torque (which will be introduced later in the course).

Since $F = ma = m\dfrac{dv}{dt}$, and distance $dx = vdt$, we can rewrite this integral in terms of velocity:

$$W_{AB} = \int_A^B m\frac{dv}{dt}v\,dt = \int_{v_A}^{v_B} m\ v\,dv = \left[\frac{1}{2}mv^2\right]_{v_A}^{v_B} = \frac{1}{2}m\left(v_B^2 - v_A^2\right) \tag{7.40}$$

Here, we have also found the formula for kinetic energy, often notated as K_E, K_e or just K:

$$K_E = \frac{1}{2}mv^2 \tag{7.41}$$

If an object of mass m is moving at velocity v, the above formula can calculate its kinetic energy, i.e. how much energy is required to accelerate it to that velocity.

In other words, using the above two relations, we can see that

$$W_{AB} = \Delta K_E = K_{EB} - K_{EA} \tag{7.42}$$

This is known as the *work-energy theorem.* The difference in kinetic energy of an object is equal to the amount of work done on it by the net forces acting on it.
If the kinetic energy has increased when moving from point A to point B, the work is positive; if the kinetic energy has decreased, the work is negative, and if the kinetic energy is unchanged, the net work is zero.
Note that it's positive for multiple forces to work against each other, such that one provides positive work, a different force provides negative work, etc. such that the *net* work, and thus the change in kinetic energy, can be either positive, negative or zero, depending on the strengths and angles of the forces.

Let's try an applied example. Say an object is moving upwards, while gravity acts on it downwards as you'd expect.
We choose the positive y axis to be upwards, so gravity is $-mg\hat{y}$. The object has a velocity v_A where it starts out at point A, and moves upwards to point B while losing speed due to gravity.

We now want to calculate the distance h between points A and B, assuming that the object comes to (temporary) rest at point B.

We apply the work-energy theorem, with $K_{EA} = \frac{1}{2}mv_A^2$ and $K_{EB} = 0$.
The gravitational force is constant with a magnitude of mg, so the work gravity does is mgh. The direction of the force is downward, and the motion is upwards, so the work is negative.
We set the two equal and solve for h:

$$-mgh = 0 - \frac{1}{2}mv_A^2 \tag{7.43}$$

$$gh = \frac{1}{2}v_A^2 \tag{7.44}$$

$$h = \frac{v_A^2}{2g} \tag{7.45}$$

We have seen this result before, but we found it in a quite different way last time.

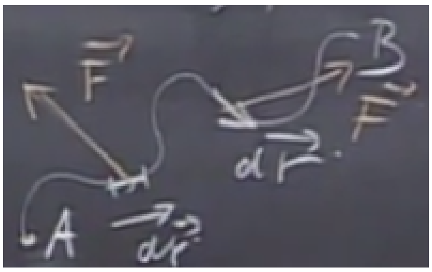
As a second example, say we lift an object against gravity, a height h above where it started out. It starts with 0 speed, and also ends up with 0 speed. Via the work-energy theorem, the net work must be zero, since the object's kinetic energy did not change.
Gravity still does its work of $|Fh| = |mgh|$, only that it's negative here: the force direction is down, and the motion is up. Since gravity does work $-mgh$ on the object, we, who lift it, must then provide positive work mgh in order to make the net work zero.

If we instead reverse the situation, and lower the object closer to the ground, the opposite thing happens. Gravity does positive work mgh, while we provide negative work $-mgh$ when lowering the object, and again the net work must be zero, if the object both starts out and ends up with no kinetic energy.

It's important to realize that work, as used in physics, is far from the same as we might think intuitively. If we lower a very heavy object from a height down to the ground, we will have provided negative work, $-mgh$, but we for sure have still *spent* energy burned in our muscles to provide it. We didn't get some sort of added energy reserve from doing so, even though the work is negative.
Likewise, we can get tired from holding an object perfectly still (try holding something heavy at arm's length for an extended period of time!), despite the fact that $Fd = 0$ and no work has been done.

7.2.1 Taking the step to three dimensions

We can extend what we have above to three dimensions. Say we apply a force $\vec{F}$ over a path. For each tiny point of this path, we can find a vector $\vec{dr}$, which represents a infinitesimal displacement along the line; so small that we can approximate it as a straight line, rather than some form of curve.



The net work done by the force over the entire path is

$$W_{AB} = \int_A^B = \vec{F} \cdot \vec{dr} \tag{7.46}$$

A dot product integral may look scary, but they're not too bad.
By the way, the above is a *line integral* (or *path integral*, or *contour integral*): it evaluates an integral along a path.

We can decompose this integral into three one-dimensional integral – surprise! – to make it easier to solve. In general terms, we can write the force vector and the displacement vector as sums of components:

$$\vec{F} = F_x\hat{x} + F_y\vec{y} + F_z\hat{z} \tag{7.47}$$
$$\vec{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z} \tag{7.48}$$

With this in mind, we can find dW, a small amount of work done by the force over a small distance dr, using the definition of the dot product:

$$dW = \vec{F} \cdot \vec{dr} = F_x\, dx + F_y\, dy + F_z\, dz \tag{7.49}$$

Integrate both sides, and we find

$$W_{AB} = \int_A^B \vec{F} \cdot \vec{dr} = \int_A^B (F_x\, dx + F_y\, dy + F_z\, dz) \tag{7.50}$$
$$= \int_A^B F_x\, dx + \int_A^B F_y\, dy + \int_A^B F_z\, dz \tag{7.51}$$

We now have three one-dimensional problems to solve, instead. Not only that, but we've already solved this integral in one dimension. We just need to add it up for the three dimensions:

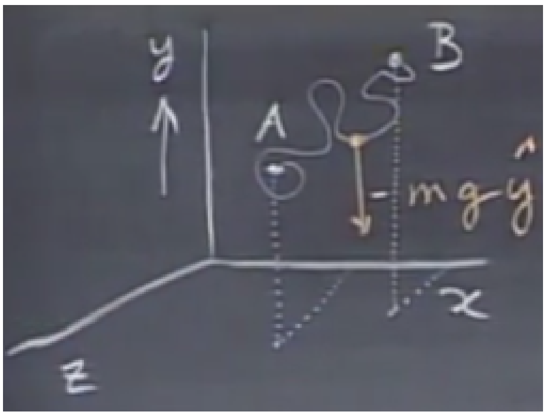
$$W_A B = \frac{1}{2}m\left(v_{Bx}^2 - v_{Ax}^2\right) + \frac{1}{2}m\left(v_{By}^2 - v_{Ay}^2\right) + \frac{1}{2}m\left(v_{Bz}^2 - v_{Az}^2\right) \tag{7.52}$$

Not pretty... but that's probably the last time we'll see it written like that. Here it is again, re-arranged, but exactly equal to the above:

$$W_A B = \frac{1}{2}m\left(v_{Bx}^2 + v_{By}^2 + v_{Bz}^2\right) - \frac{1}{2}m\left(v_{Ax}^2 + v_{Ay}^2 + v_{Az}^2\right) \tag{7.53}$$
$$= \frac{1}{2}m\left(v_B^2 - v_A^2\right) \tag{7.54}$$

... and so we find exactly the same result as we did in one dimension.

Let's as an example calculate the work done by gravity while moving an object around in 3D space.



We move from a point A to a point B, where $y_B - y_A = h > 0$, in other words, point B is higher up than point A.

The force due to gravity is $-mg\hat{y}$, so the integral would be

$$W_{gravity} = \int_A^B \vec{F} \cdot \vec{dr} = \int_A^B (-mg)\, dy = -mg \int_A^B dy = -mg(y_B - y_A) = -mgh \tag{7.55}$$

The x and z terms disappear, since $F_x = 0$ and $F_z = 0$ – gravity only acts along one axis, with the way we've defined our coordinate system.

We find that the work done by gravity is negative, as it should be – the force vector is down, but the object moved higher up. The net work of the force(s) that moved it upwards, in the y direction, is then $+mgh$.
(We can't say anything about the work along the x and z axes without more information, of course.)

Another thing is interesting about this result: it is independent of the path between points A and B. The only thing that matters, as far as gravity is concerned, is the difference in height. If you move an object up 10 meters and back down 9, the work done by gravity is exactly the same as if you'd just moved it up the one meter. The same goes for all and any movement in the x-z plane, which doesn't affect the work done by gravity whatsoever.

Any force that has the property that the work done is the same for any given pair of start/end points, regardless of the path moved in between them, is called a *conservative force*. As we have just shown, gravity is conservative. If a particle starts at one point, moves around in any path whatsoever, and comes back to that exact point, the work done by gravity is always zero.

7.2.2 Conservation of mechanical energy

We can apply the work-energy theorem to the equation above:

$$-mg(y_B - y_A) = K_{EB} - K_{EA} \tag{7.56}$$
$$-mg\ y_B - mg\ y_A = K_{EB} - K_{EA} \tag{7.57}$$
$$K_{EA} + mg\ y_A = K_{EB} + mg\ y_B \tag{7.58}$$

This is a very important result. The quantity mgy is what we call *gravitational potential energy*, often P_E or U. What the above result says, then, is that the sum of the kinetic and potential energies at point A must equal the sum of the kinetic and potential energies at point B.

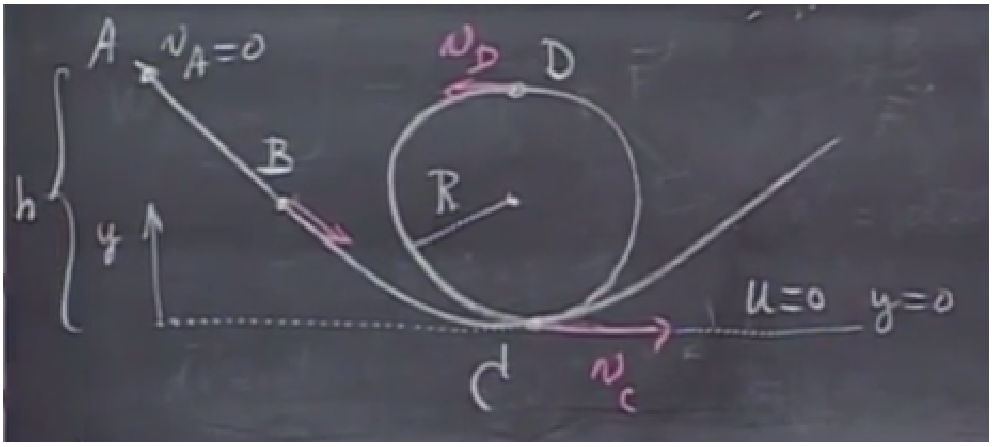
$$K_{EA} + U_A = K_{EB} + U_B \tag{7.59}$$

This is known as the *conservation of mechanical energy*, where the mechanical energy of an object is the sum of its kinetic energy and its potential energy. One can be converted into the other, but *as long as the forces involved are conservative*, the sum of the two must stay equal. This condition is an important one! Friction, for example, is *not* a conservative force, and this relationship will no longer hold if frictional forces are involved.
Spring forces *are* conservative, however.

The fact that frictional forces are not conservative should be fairly intuitive. The further we move something against friction, the more total work must be done to overcome the friction. You could move something back and forth on a table and the work done by the friction (and you, in moving it) would just increase and increase in magnitude.
If you did the same against gravity, moving something up, then back down, etc., the work done (by gravity, or by you) would *not* simply increase without bound.

With the definition of $K_E = \frac{1}{2}mv^2$, it's clear that kinetic energy is zero when the velocity v of an object is zero.
What about potential energy? That is zero where $y = 0$, but where is that? It is up to us to decide where to place that point. We are free to choose it, as long as g is the same at both point A and point B (or simply that g is close enough, so that we can neglect the difference).

Example problem



We place an object on this "roller coaster", at point A, which is h above the line where we choose $y = 0$ and $U = 0$. It gains speed, by converting gravitational potential energy to kinetic energy, until it reaches point C, where the velocity (and the kinetic energy) is at a maximum, while the potential energy is zero, by our definition.

At that point, it reaches the loop. The question is: from what height h must we let it go along the track, so that it manages to move around the loop without falling out before reaching the top?

We can apply the conservation of mechanical energy to this problem, assuming friction is negligible:

$$U_A + K_{EA} = U_B + K_{EB} = U_C + K_{EC} = U_D + K_{ED} \tag{7.60}$$

We let it go from rest at point A, so $K_{EA} = 0$. Our definition of potential energy as $U = mgh$, where h is the height above the plane where $U = 0$.
The vertical distance is must travel "upward" in going around the circle is twice the radius, so $2R$.

If we equate the total mechanical energy at A, $U_A = mgh$, with the total mechanical energy at any given point, we can find

$$mgh = mgy + \frac{1}{2}mv^2 \tag{7.61}$$
$$v^2 = 2g(h - y) \tag{7.62}$$

This equation should hold for any point, again assuming we can neglect friction and other forces. At point D, at the top of the loop, the constraint that $a_c > g$ must hold, or we know that it will fall out of the loop, instead of being pushed against it by the centripetal force.

Since centripetal acceleration is given by $a_c = \frac{v^2}{r}$, and we know v^2, we can write an inequality for this being larger than g, and then solve for h, to find the answer to our question about the minimum height we must release the object from.
$y = 2R$ at this point, so

$$\frac{2g(h - 2R)}{R} \geq g \tag{7.63}$$
$$2h - 4R \geq R \tag{7.64}$$
$$2h \geq 5R \tag{7.65}$$
$$h \geq \frac{5}{2}R \tag{7.66}$$

So if we find a point along the roller coaster where this condition is met, and perhaps add a small amount due to friction, the ball will "make the loop", so to speak. Without friction, air resistance etc., it would come up to the exact same height as it started rolling down from.

7.2.3 Newton's law of universal gravitation

Like the previous laws we've learned with Newton's name attached to them, the law of universal gravitational is quite simple.

Say we have two masses, often named m and M (where it's often the case that M e.g. a planet, and m is smaller; this is of course no requirement, however). The two point masses[1] are separated by a distance r. The force that m experiences because of M is then

$$F = \frac{GMm}{r^2} \tag{7.67}$$

in the direction of M (gravity is always an attractive force), where G is the gravitational constant, not to be confused with $g \approx \frac{GM_{Earth}}{R^2_{Earth}}$ (approximately because Earth's mass is not all at the center; Earth is not a perfect sphere, etc.).
G has a value of about $6.67 \cdot 10^{-11}$, with units that need to cancel out with the others, i.e.

$$[\mathrm{N}] = \frac{[\mathrm{G}][\mathrm{kg}]^2}{[\mathrm{meter}]^2} \tag{7.68}$$

$$\frac{[\mathrm{N}][\mathrm{meter}]^2}{[\mathrm{kg}]^2} = [\mathrm{G}] \tag{7.69}$$

So in more terse notation, the units of G are $\mathrm{N} \cdot \mathrm{m}^2\mathrm{kg}^{-2}$.

Because of $F = ma$, so $a = \frac{F}{m}$, we can find the gravitational acceleration due to a mass M to be the force, above, divided by the mass m, as already shown in the aside about g.

As the formula shows, as the distance from the source of the gravity doubles, the magnitude of the force (or the acceleration) is reduced by a factor of 4.

7.2.4 Gravitational potential energy

Let's now talk about gravitational potential in the general case. Previously, we've only used it where we are near the surface of the Earth, and g has a value that can be considered constant. This is of course not true in general, but there is a definition we can use that is valid everywhere in the universe.

If it is to be valid everywhere, where do we place the zero? There are only two plausible choices, and one (at $r = 0$) turns out to not make much sense. We end up with only one possibility, which is that $U = 0$ is at $r \to \infty$. This has the strange consequence of making all potential energy values negative, but that does not matter: it is still the *difference* in potential energy that matters, and we could have found negative values with the previous method too, for some choices of the point where $U = 0$.

A general definition of the gravitational potential energy is that U is the amount of work *you* must do to move a mass from infinity to some point P, located some distance from a mass M. Alternatively, but equivalently, it's the work gravity does in moving the same mass *from* some point P *to* infinity.

In either case, it's clear that the value must be negative: if the mass is attracted to the point, you don't have to do *any* positive work to move it there, but rather negative work.

[1] This also works for masses of nonzero size in some cases, more on that at a later time.

In the second formulation, gravity doesn't do *any* positive work when you move the mass *away* from another mass, so again the value is negative.

Let's now calculate a general formula for gravitational potential energy. Using the definition above, where it equals the work we do in moving a mass from infinity to P (located R away from the mass M),

$$U = \int_{\infty}^{R} \frac{GMm}{r^2} dr = GMm \int_{\infty}^{R} r^{-2} dr = \Big[-\frac{GMm}{r} \Big]_{\infty}^{R} = -\frac{GMm}{R} \tag{7.70}$$

Note how the value will always be zero "at" infinity, and grow larger (*less negative*, but still larger!) as we move away from a body.

Example calculation

Lecture question: A mass M is released at zero speed, $2 \cdot 10^6$ km from the center of the Earth. At what speed does it hit the Earth, ignoring the gravitational forces from all other objects in the solar system?

We could integrate the acceleration to find the answer, but that is dependent on time, so that seems more trouble than it's worth.

Let's instead attempt to do this by conservation of energy – I'm fairly sure that was the intention, anyway. If we assume mechanical energy will be conserved, which ought to be a fairly good approximation here, 100% of the change in the potential energy is converted to kinetic energy. What is the change in potential energy, ΔU? Using the formula for potential energy above, and using A for the initial location and B at the surface at the Earth, it must be

$$\Delta U = U_A - U_B = \left(-\frac{GM_{Earth}M}{2 \times 10^9 \,\text{m}} \right) - \left(-\frac{GM_{Earth}M}{R_{Earth}} \right) \tag{7.71}$$

$$= GM_{Earth}M \left(\frac{1}{R_{Earth}} - \frac{1}{2 \times 10^9 \,\text{m}} \right) \tag{7.72}$$

This must then be equal to the kinetic energy $\frac{1}{2}Mv^2$ when it hits. We equate the two, use $R_{Earth} = 6.378 \times 10^6 \,\text{m}$ and $M_{Earth} = 5.97 \times 10^{24} \,\text{kg}$:

$$\frac{1}{2}Mv^2 = GM_{Earth}M \left(\frac{1}{R_{Earth}} - \frac{1}{2 \times 10^9 \,\text{m}} \right) \tag{7.73}$$

$$v^2 = 2GM_{Earth} \left(\frac{1}{R_{Earth}} - \frac{1}{2 \times 10^9 \,\text{m}} \right) \tag{7.74}$$

$$v = \sqrt{2GM_{Earth}} \cdot 0.0003953 \approx 11.2 \,\text{km/s} \tag{7.75}$$

From the previous formula, we can find the kinetic energy as it hits, as that is equal to ΔU. It turns out to be about 3.1×10^{10} J – over 30 gigajoules.

7.3 Lecture 12: Resistive forces

This lecture introduces resistive forces and *drag* forces, such as air drag – a concept we have previously ignored while solving problems. In many cases, it cannot be ignored without yielding wildly incorrect results, something we may be equipped to handle sooner rather than later now.

Unlike friction, drag forces depend not only on the medium the object moves though (which we could perhaps liken to the friction coefficient), but also the object's shape, size and speed. In addition, the object's mass also matters for its movement, though the drag forces don't depend on it. More on that later.

The fact that it depends on the medium is obvious, perhaps unlike some of the above. We know from experience that the drag force is much greater in water than it is in air, as it's very hard to make fast movements underwater, compared to in air. Oil has an even greater drag force than water.

In general, we can write resistive forces as

$$\vec{F_{res}} = -\left(k_1 v + k_2 v^2\right)\hat{v} \tag{7.76}$$

In other words, it is always in the *opposite* direction of the velocity, *relative to the medium.* v is the speed, however, i.e. it is always a positive number, as is of course v^2.
k_1 and k_2 depend on the medium, the shape and size of the object, etc.

This lecture will focus exclusively on spheres for the shape, which means we can describe their size as a single variable, the radius r. We can write the magnitude of the drag force as

$$|F_{res}| = C_1 r v + C_2 r^2 v^2 \tag{7.77}$$

C_1 is referred to as the viscous term, as it has to do with the viscosity (essentially "thickness") of the medium. A low liquid of low viscosity flows easily; water is a good example. The higher the viscosity, the thicker a fluid is; oil has a higher viscosity than water, and honey an even higher viscosity. The units of C_1 is $\mathrm{kg}/(\mathrm{m}\cdot\mathrm{s})$, or $\mathrm{Pa}\cdot\mathrm{s}$ where Pa for pascal is the SI unit of pressure; $1\,\mathrm{Pa} = 1\,\mathrm{N}/\mathrm{m}^2$.

C_2 is referred to as the pressure term. It is closely related to the density of the medium; the units of C_2 is also in kilograms per cubic meter ($\mathrm{kg}\cdot\mathrm{m}^{-3}$). It is not identical to the density, but closely related.

7.3.1 Terminal velocity

If an object is in free fall, it will be accelerated downwards by gravity, with a constant force mg (assuming the fall is small enough that g can be considered constant). The resistive force upwards will not be constant, however, since it is a function of the speed. Since the resistive force will grow as the object falls, there comes a time where the resistive force is equal to the downwards force by gravity, and the net force on the object is zero. Since there is no net force, and thus no acceleration (Newton's second law), via Newton's *first* law, the object will maintain a constant velocity.
We call this velocity the *terminal velocity*; it is then the highest velocity (or speed) that object can achieve given a certain value of g, in that medium. We can then find that velocity by setting the two forces equal, and solving for v. In doing so, we get two answers (since the equation is quadratic), though one of the solutions is unphysical and must be ignored.

In many cases, one of these two terms is dominating. In the first case, which we will call regime I or the viscous regime, the first term dominates, so that

$$|F_{res}| \approx C_1 r v \tag{7.78}$$

In the second, regime II, the pressure term dominates, so that

$$|F_{res}| \approx C_2 r^2 v^2 \tag{7.79}$$

Let's look at the case where the two are the same; this happens at a certain velocity, called the *critical velocity*, v_{crit}. It occurs when

$$C_2 r^2 v_{crit}^2 = C_1 r v_{crit} \tag{7.80}$$

$$v_{crit} = \frac{C_1}{C_2 r} \tag{7.81}$$

In regime I, which implies $v \ll v_{crit}$, the terminal velocity v_{term} can be found as approximately

$$C_1 r v_{term} = mg \tag{7.82}$$
$$v_{term} = \frac{mg}{C_1 r} \tag{7.83}$$

If we drop an object of uniform density ρ (or ρ_{obj} to clarify that it is the density of the *object*, not the medium), we can write the mass as $m = \frac{4}{3}\pi r^3 \rho_{obj}$ (since we are working with spheres only so far), in which case the terminal velocity becomes

$$v_{term} = \frac{4}{3}\pi \rho_{obj} \frac{gr^2}{C_1} \tag{7.84}$$

So in other words, $v_{term} \propto r^2$.

In regime II, which implies $v \gg v_{crit}$, we instead ignore the viscous term, and concentrate on the second term, the pressure term.

$$C_2 r^2 v_{term}^2 = mg \tag{7.85}$$
$$v_{term} = \sqrt{\frac{mg}{C_2 r^2}} \tag{7.86}$$

If we now write the mass m as we did previously, we find

$$v_{term} = \sqrt{\frac{4}{3}\pi \rho_{obj} r^3} \sqrt{\frac{g}{C_2 r^2}} = \sqrt{\frac{4\pi \rho_{obj} g r}{3C_2}} \tag{7.87}$$

... so that $v_{term} \propto \sqrt{r}$ instead.

Much of this lecture is hard to take good notes of, but the course does provide handouts which are very useful, under each lecture video segment. I did not write anything down during the excellent demonstration of ball bearings falling through syrup, though I would recommend watching it and having a close look at the transparencies provided.

7.3.2 Trajectories with air drag

After the demonstration, which was exclusively in regime I, we start looking at motion through air (at standard temperature and pressure, i.e. indoor conditions).

Here, we find values of $C_1 \approx 3.1 \cdot 10^{-4}$ pascal-seconds, while $C_2 \approx 0.85$ kg/m^3.
We earlier found the critical velocity as $C_1/(C_2 r)$, so for air, it is about

$$v_{crit,air} \approx \frac{3.7 \cdot 10^{-4}}{r} \text{ m/s} \tag{7.88}$$

Clearly, then, for objects of noteworthy size, such as $r > 1$ cm, the critical velocity is on the order of a few centimeters per second, or less. In other words, for almost any motion though air, we are practically exclusively in regime II.

As a rule of thumb, liquids are usually in regime I, while air (and similar gases) are usually in regime II. It is of course always a good idea to test this assumption before you use it to solve a problem!

Let's take a quick look at how air drag changes the trajectory of e.g. a ball flying through air, in the presence of gravity. Since we can decompose the motion into x and y motions, both of which are through the same medium, it is clear that there will be resistive forces in *both* directions, in addition to gravity, that is slowing the ball down. Thus, the x velocity will no longer be constant. Not only that, but the trajectory will no longer be symmetric, either!

Think of what happens if we fire a small, lightweight ball (think ping-pong ball, or something of a similarly small mass) into the air. It has some initial velocity $\vec{v_0} = v_{0x}\hat{x} + v_{0y}\hat{y}$. The x velocity is constantly reduced by the drag force opposing the motion. That force is proportional to v_x^2, so the force is neither constant nor linear.
In the y direction, we have a similar situation, but we also have gravity which is constantly trying to pull the ball down.

Because of this asymmetry, it will take *longer* for the ball to fall down from its peak back to the ground (or back to the height from which it was launched, to be more precise), than it will for it to actually reach it. This can be seen fairly easily, at least once you know how to think about the problem.

We can neglect horizontal motion for a second, and only think about how it travels in the y direction, since that alone decides whether when it reaches the peak / hits the ground. We launch it with an initial velocity upwards, say 10 m/s. Without air drag, it takes 1 second (using $g \approx 10\,\mathrm{m/s^2}$) to reach the peak at 5 meters up ($y = y_0 + v_0 t - 0.5gt^2 = 0 + 10 \cdot 1 - 0.5 \cdot 10 \cdot 1^2$), after which it falls down, and hits the ground at 10 m/s again, having accelerated at g for one second.

With air drag, it doesn't reach as high, since there is an additional downwards force now, due to air drag. Say it reaches 3 meters instead, so that at $y = 3$ m, the velocity is zero. In then begins to fall downwards, with gravity pulling it down, and air drag pushing it upwards (opposing the motion relative to the air). It is clear, then, that the net acceleration must be *less* than g, so the velocity it hits the ground with is also *less* than the 10 m/s we launched it at. Even if it *did* accelerate at g, it only has 3 meters to do so at, and so the maximum possible velocity, if we *neglect* air drag for the downwards portion only, can be found by solving the equation

$$3\,\mathrm{m} - \frac{1}{2}gt^2 = 0 \tag{7.89}$$

$$t = \sqrt{\frac{6\,\mathrm{m}}{g}} \approx 0.77\,\mathrm{s} \tag{7.90}$$

Accelerating at g for 0.77 seconds yields a speed of about 7.7 m/s, clearly less than the initial velocity of 10 m/s. Since the *speed* for the downwards fall was less than the speed going up, the downwards portion *MUST* take longer. We even neglected air drag on the way down, so the real effect is even more significant than this quick calculation shows!

Chapter 8: Week 6

8.1 Lecture 13: Equation of motion for simple harmonic oscillators

The lecture begins with what is really a review of gravitational potential energy, which is still certainly worth watching, to make sure that you everything everything clearly. In addition, the explanation for the potential energy (below) is compared to gravitational potential energy.

First, the potential energy of a spring is derived, which I did in homework 4 (week 5), problem 5.

As a quick refresher: we set $x = 0$ at the relaxed length of the spring, and extend it a distance x further. The spring force is $-kx$, and the force we need to provide to overcome that is $+kx$. The work we do in extending the spring is all stored as potential energy in the spring, so

$$U_{spring} = W = \int_0^x kx\,dx = \left[\frac{1}{2}kx^2\right]_0^x = \frac{1}{2}kx^2 \tag{8.1}$$

It follows, then, that $U_{spring} = 0$ at $x = 0$. As usual, we can define this however we want, but any other definition would only cause problems in most cases, and therefore be silly to make.

As is the case with gravity, as stated in the beginning of the lecture (of which I took no notes), the force is always in the direction *opposite* that of increasing potential energy. For gravity, this turns out to be an always-attracting force. For springs, this turns out to be a restoring force: if you stretch the spring, the force is always such that it pulls to spring back together. If you instead compress the spring, the force reverses, and now tries to push it back to its original length. In both these cases, the force is in the opposite direction of increasing potential energy, since potential energy increases both when the spring is compressed and when it is extended.

Let's now look at the reverse situation. Can we go from knowing only the potential energy, to finding the spring's force? Yes, we can, and it's very easy: we take the derivative of the potential energy, with respect to x:

$$U = \frac{1}{2}kx^2 \tag{8.2}$$

$$\frac{dU}{dx} = +kx = -F_{sp} \tag{8.3}$$

$$\frac{dU}{dx} = -F_x \tag{8.4}$$

Since the force is one-dimensional, we can write F_x for the force. The minus sign is important, and means what we mentioned above: the force is in the direction *opposite* the increase in potential energy. If $\frac{dU}{dx}$ is positive, you are moving in that direction, and you get a minus sign for the force – it is opposite our motion, since the motion is towards increasing potential energy.
If $\frac{dU}{dx}$ is negative, we are moving towards decreasing value of potential energy, and the force is positive (in the same direction as the x motion).

In multiple dimensions, we can find a similar result. If we know the potential energy as a function of x, y and z, we can find the force components along each axis by taking partial derivatives. So given $U(x, y, z)$, we can find force components F_x, F_y and F_z:

$$\frac{\partial U}{\partial x} = -F_x \tag{8.5}$$
$$\frac{\partial U}{\partial y} = -F_y \tag{8.6}$$
$$\frac{\partial U}{\partial z} = -F_z \tag{8.7}$$

Partial derivatives are quite simple; you calculate them for one function argument at a time. So you essentially first find $\frac{d}{dx}U(x, y, z)$, while *treating y and z as constants*; that gives you the negative of the force along the x axis.
In other words, as an example:

$$\frac{\partial}{\partial x}\left(2x^2 - 3y + 2xy - 3z\right) = 4x + 2y \tag{8.8}$$

The $-3y$ term disappears, since we consider it constant. Likewise, $2xy$ becomes $2y$, since $\frac{d}{dx}(2xy) = 2y\frac{d}{dx}(x)$ if y is a constant. Similarly, the z term disappears, since we treat it as a constant.
If this polynomial was $U(x, y, z)$, then we just found $-F_x = 4x + 2y$, so $F_x = -4x - 2y$.

You then simply repeat the process for the y and z components, keeping the other two axes constant, and you are done.

Two more realistic examples are covered in the lecture. First, for one-dimensional gravitational potential energy:

$$U = +mgy \text{ (with +y upwards)} \tag{8.9}$$
$$\frac{dU}{dy} = mg \tag{8.10}$$
$$F_y = -\frac{dU}{dy} = -mg \tag{8.11}$$

And indeed, the gravitational force is $-mg$, assuming increasing y is upwards.

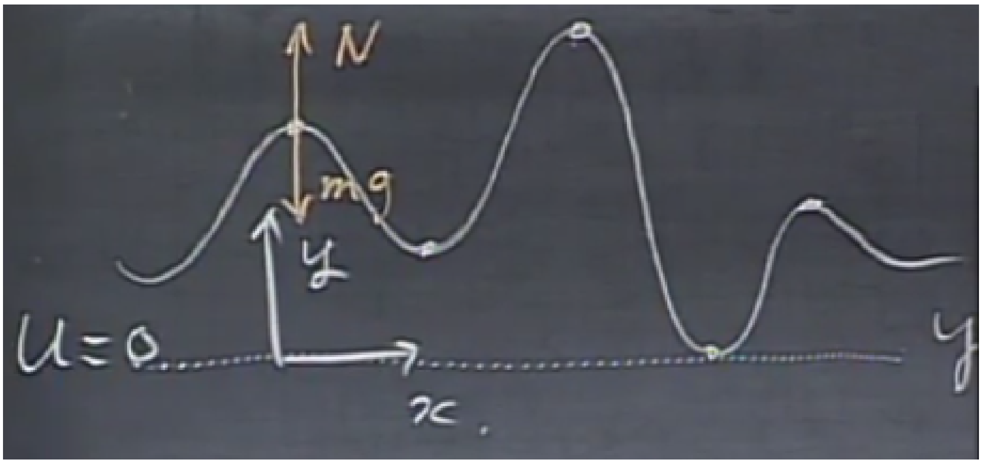
Next, another one-dimensional problem of gravitational potential energy, this time in general, rather than very close to Earth's surface:

$$U = -\frac{MmG}{r} \text{ (where } M \text{ is the mass of the Earth)} \tag{8.12}$$
$$\frac{dU}{dr} = +\frac{MmG}{r^2} \tag{8.13}$$
$$F_r = -\frac{dU}{dr} = -\frac{MmG}{r^2} \tag{8.14}$$

Again, we find a familiar result.

8.1.1 Stable and unstable equilibrium

Next up, let's look at equilibrium. Say we have a surface, that may look like this:



The points along the curve have a gravitational potential energy which is $U = mgy$, since we defined $U = 0$ at $y = 0$ (the $= 0$ part is just outside the screen grab above). Since the plot is of $y = f(x)$, for some function f, we can also write that $U = mgf(x)$.

There are then points along this curve where $\frac{dU}{dx} = 0$. Those points occur where the curve's slope (or derivative) are zero, by definition, which is at the top of each peak, and at the bottom of each valley, as signified by a dot in the above figure.

From the definition we found before, that then means that $-F_x = 0$, so the net force in the x direction is zero.
At such a point, there is a force $-mg$ in the y direction, and a normal force $N = +mg$, so that there is no net force there, either.

Since there is no net force on the object at one of these points, and we can put it in such a situation at rest, it will stay exactly where it is.
However, there is an important difference between these two types of points (peaks and valleys). If we try to balance a marble at the top of such a peak, just about any tiny vibration, small amount of wind etc. will get it moving. Being at the top of a large downwards slope, in either direction, it will then clearly begin to accelerate downwards – and again, the force is in the direction of decreasing potential energy (which of course is the same thing as being in the *opposite* direction as *increasing* potential energy).

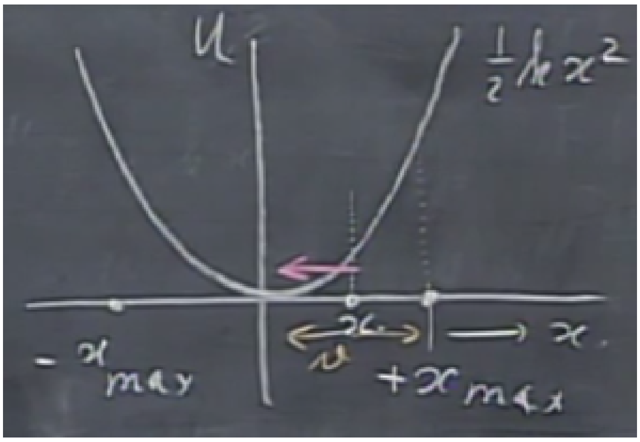
However, if we put a marble in one of the valleys, what happens? If there is a small force, causing a motion in any direction, it will be forced back into the valley. The force is yet again in the direction opposing the increase in potential energy, and potential energy increases both to the left *and* to the right! Therefore, the force is such that the marble is returned to the middle of the valley again, to the point of lowest potential energy.

The difference between these two zero points are that the peaks provide *unstable equilibrium*, while the valleys provide *stable equilibrium*. If there is a disturbance in the first case, it goes out of control. In the second case, in the valleys, any small disturbance is automatically countered, and the object goes back to where it was, at the bottom.

We can find out which of these two cases a point is mathematically, by looking at the second derivative. If the second derivative of potential energy with respect to x is positive, it's a point of stable equilibrium. If it is negative, it's instead a point of unstable equilibrium.

8.1.2 Another look at a spring oscillator

Let's have another look at the oscillation of a mass on a spring, this time from an energy perspective. We know that $U = \frac{1}{2}kx^2$, so a plot of $U(x)$ would be a parabola:



Say we have a mass attached to a spring, as usual, and we extend it to x_{max}, and let it go, with zero speed.

We know that it will oscillate between $+x_{max}$ and $-x_{max}$, but we can now gain a second insight into this oscillation (albeit one mentioned earlier). Say we release the mass at an extension x_{max} beyond the spring's natural length. That means the potential energy in the spring at that time is

$$U_{initial} = \frac{1}{2}kx_{max}^2 \tag{8.15}$$

Since we know that the force will be in the direction opposing the increase in potential energy, the mass will be pulled inwards, towards $x = 0$. Once it crosses the zero point, the force switches directions, since the current *velocity* vector is towards increasing potential energy (the spring is being compressed to be shorter than its natural length). That means the force (and thus the acceleration) instantly flips over, and the mass starts slowing down. The new force is once again in the direction opposing the increase in potential energy, which is again towards $x = 0$, which is now towards the right in the figure.

Because spring forces are conservative (for ideal springs), we can use conservation of energy to write an equation for this system. The *total* energy in the system must equal the spring's stored potential energy at $t = 0$, plus the mass's kinetic energy at $t = 0$. The latter is zero, since we release it at rest (zero speed), so $E_{total} = U_{initial}$. That energy must be held constant – conservation of energy. Therefore, the sum of the mass's kinetic energy $\frac{1}{2}mv^2$ and the spring's stored energy $\frac{1}{2}kx^2$ must always equal that initial energy. We can set up an equation for this:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{max}^2 \tag{8.16}$$

This equation must *always* hold for this system, unless there are other forces, such as friction, which we ignore for now.
Because $v = \dot{x}$, we can rewrite this equation a bit, by making that substitution, and getting rid of all of the one-halves, and dividing through by m:

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{max}^2 \tag{8.17}$$

$$\dot{x}^2 + \frac{k}{m}x^2 - \frac{k}{m}x_{max}^2 = 0 \tag{8.18}$$

We can then take the time derivative of this. Keep in mind that since the equation is in terms of x, we need to use the chain rule for most terms.

$$\frac{d}{dt}\left(\dot{x}^2 + \frac{k}{m}x^2 - \frac{k}{m}x_{max}^2\right) = \frac{d}{dt}(0) \tag{8.19}$$

$$2\dot{x}\ddot{x} + 2\frac{k}{m}x\dot{x} - 0 = 0 \tag{8.20}$$

We can simplify this equation by dividing through by $2\dot{x}$:

$$\ddot{x} + \frac{k}{m}x = 0 \tag{8.21}$$

Isn't it remarkable? We get the equation for simple harmonic motion, and so we find the same old solutions:

$$x = x_{max}\cos(\omega t + \varphi) \tag{8.22}$$

$$\dot{x} = -\omega x_{max}\sin(\omega t + \varphi) \tag{8.23}$$

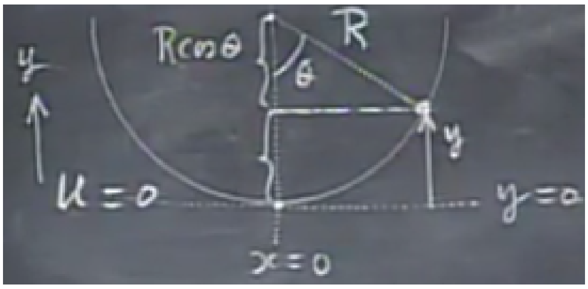
$$\ddot{x} = -\omega^2 x_{max}\cos(\omega t + \varphi) = -\omega^2 x \tag{8.24}$$

$$\omega) = \sqrt{\frac{k}{m}} \tag{8.25}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{8.26}$$

8.1.3 Motion of a ball along a circular track

Say we have a circular (or at least semicircular) track of radius R. We define $y = 0$ and $U = 0$ to be at the bottom of the track.



When the ball is at some random location y, we can find the angle made with the vertical, θ, via trigonometry.
First, we find that the radius R acts as the hypotenuse of a right triangle, where the x component $R\sin\theta$ is at the bottom, and the left side has height $R\cos\theta$. Note that the y coordinate fits $y = R - R\cos\theta$, so that $y = R(1 - \cos\theta)$.

With that in mind, we can write U as a function of the angle θ now:

$$U = mgy = mgR(1 - \cos\theta) \tag{8.27}$$

Notice that at $\theta = 0$, $U = 0$, as we defined.
At $\theta = \frac{\pi}{2}$, $U = mgR$, since it is a height R above $y = 0$.

Using the definition of a radian as the arc length subtended by an angle, where dS is the arc length and $d\theta$ the angle, we find

$$\frac{dS}{R} = d\theta \tag{8.28}$$

$$dS = Rd\theta \tag{8.29}$$

Taking the time derivative of both sides, we find

$$\frac{dS}{dt} = R\frac{d\theta}{dt} = R\dot{\theta} \tag{8.30}$$

The left-hand term is just the distance moved per unit time, so $\frac{dS}{dt} = v = R\dot{\theta}$.

In most cases, we use $\omega = \dot{\theta} = \frac{d\theta}{dt}$, but in most cases, ω is also a constant. In this case, it is a function of the angle; the angle will change the fastest near $\theta = 0$ (at the bottom), where the speed is at a maximum, while it will change slower as the ball climbs up the "edges" of the circle (I think of it as a "two-dimensional bowl"), as it is about to come to a halt, and change direction.

As a short aside, we can, as a small-angle approximation, use

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \tag{8.31}$$

This approximation uses the first two terms of a Taylor expansion for $\cos\theta$. If you are unfamiliar with Taylor expansions, you could look them up (even the basics are a bit too much to cover in what is already an aside). In short, they provide for a way to approximate about any function as a polynomial, or – with an infinite amount of terms – exactly equal those functions.

Last time we used such an approximation, we used only the first term, $\cos\theta \approx 1$. That's too inexact for this case, though – we would end up with $U = mgR(1-1) = 0$ for all θ!

Even for angles of about 11.5 degrees, the error caused by this approximation is way, way less than 1% (less than 0.01%, actually). In fact, for as much as 30 degrees, we have $\cos(30°) \approx 0.8660254$, while the approximation gives 0.862922. It's off by about 0.3% – still not a lot, all things considered.

Let's return to the problem at hand. Using this approximation, we apply the conservation of mechanical energy to this system. The total mechanical energy must be a constant. If we release the object as zero speed, and thus zero kinetic energy, the total energy (kinetic + potential) must always equal that value:

$$M_E = \frac{1}{2}mv^2 + mgR(1-\cos\theta) \tag{8.32}$$

Since $v = R\dot{\theta}$, $v^2 = R^2\dot{\theta}^2$. Let's also apply our approximation for the cosine. What we end up with is

$$M_E = \frac{1}{2}mR^2\dot{\theta}^2 + mgR(1-(1-\frac{\theta^2}{2})) \tag{8.33}$$

$$M_E = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\frac{\theta^2}{2} \tag{8.34}$$

We can now take the time derivative of this. M_E is a constant, so that becomes zero. As far the rest, we use the chain rule again:

$$0 = \frac{1}{2}mR^2(2\dot{\theta}\ddot{\theta}) + \frac{mgR}{2}2\theta\dot{\theta} \tag{8.35}$$

$$0 = mR^2\dot{\theta}\ddot{\theta} + mgR\theta\dot{\theta} \tag{8.36}$$

$$0 = R^2\ddot{\theta} + gR\theta \tag{8.37}$$

We can rearrange that as

$$\ddot{\theta} + \frac{g}{R}\theta = 0 \tag{8.38}$$

... and it is then again obvious that we have as simple harmonic oscillator! We know the solution to this differential equation, so we can write down

$$\theta = \theta_{max}\cos(\omega t + \varphi) \tag{8.39}$$

$$\omega = \sqrt{\frac{g}{R}} \tag{8.40}$$

$$T = 2\pi\sqrt{\frac{R}{g}} \tag{8.41}$$

Note that this ω is completely unrelated to the $\frac{d\theta}{dt}$ we had earlier in the derivation – it's a good thing we didn't call that ω! This one is a constant, while the other one changed with time.

Note how these equations are identical to the ones for a pendulum, that we derived earlier, also using a small-angle approximation. This time, however, it is our approximation which caused the similarity – we made the equation quadratic in θ by doing that. The spring oscillation was quadratic in x from the beginning.

Finally, on to an important detail. Nowhere in this derivation did we consider the normal force from the track on the ball. Is it really safe to ignore it? Why?

It turns out that yes, we can ignore it, because in the case of this circular track, it is always perpendicular to the direction of motion. A force perpendicular to a motion *cannot* do work, because of the definition of the dot product: an angle of 90° between force and displacement always means zero work.

Next, a very interesting demonstration follows, that might cause some sleeplessness until we find the answer to what's going on, likely in two weeks or so.

8.2 Lecture 14: Orbits and escape velocity

As we know, the gravitational force has infinite range. Its strength at a distance is limited, though, due to the inverse square relationship. Because of this, there is a speed, the *escape velocity*, that lets you escape from a body's gravitational field. That is, if you start out with that speed, you will escape it forever, even with no additional outwards force (no engines required). Of course, if you *do* have engines, you certainly don't need to stay above the escape velocity the entire time to get away; all you need to do is overcome the force of the gravitational pull.

We can find this velocity for a given body (such as the Earth) quite easily, using conservation of energy. The kinetic energy at launch must be $\frac{1}{2}mv_{esc}^2$, and since the problem definition is that in never adds to that kinetic energy (no engines). That must therefore be the total energy of the object, at all times. The total energy at any given time is the sum of the kinetic energy at some point r, which we call $\frac{1}{2}mv_r^2$, and the potential energy at that point, $-\frac{GMm}{r}$, with M being the mass of the Earth (or the body), and m the mass of the object trying to escape.

$$\frac{1}{2}mv_{esc}^2 + \left(-\frac{GMm}{R_{Earth}}\right) = \frac{1}{2}mv_r^2 + \left(-\frac{GMm}{r}\right) \tag{8.42}$$

On the left side, we have the total energy as we start out our journey, and on the right, the total energy some distance r away.

However, since the goal is for the energy to be enough to escape to an infinite distance, the kinetic energy "at" infinity (let's just say extremely, extremely far away, since being "at" infinity is meaningless), the potential energy is zero, by definition. The kinetic energy is also zero, *if* we gave it *just* enough energy, and not any more than required (we know that the Earth's gravity will reduce the speed, and thus the kinetic energy, as time goes on).

Because of this, we can set the entire right side of the equation equal to zero, which is valid "at" infinity (or just so far away that the gravitational pull of the Earth is now completely negligible), and solve for the escape velocity:

$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{R_{Earth}} = 0 \tag{8.43}$$

$$v_{esc}^2 - \frac{2GM}{R_{Earth}} = 0 \tag{8.44}$$

$$v_{esc} = \sqrt{\frac{2GM}{R_{Earth}}} \tag{8.45}$$

where, again, M is the mass of the Earth. For Earth, this value is then about 11.2 km/s. So if we neglect air resistance, which will surely make these results valid if we are at the Earth's surface, if we could fire a cannon ball at more than 11.2 km/s, it would never fall back to Earth.

If the initial velocity is greater, then you will still have kinetic energy (and thus speed) left when you've escaped. In the case you do "escape", with the condition $E_{init} \geq 0$, you are in an *unbound orbit*. In the case that $E_{init} < 0$, you enter a *bound orbit*, and will never escape the gravitational pull of the Earth (or the body in question).

8.2.1 Circular orbits

Elliptical orbits will be covered later in the course, along with Kepler's laws and other fun stuff, but for now, let's introduce circular orbits, as a simplified case.

Say we have a mass m orbiting the Earth, with Earth's mass being M, and say that $m \ll M$.
It moves in a circle around the Earth at constant (tangential) speed, but not constant velocity – there is a constant centripetal acceleration, or it wouldn't be moving in a circle. Centripetal acceleration is provided by centripetal force, which in this case is the attractive force of gravity of the Earth on the mass.

We know how to find the gravitational force using the Newton's law of universal gravitation, and we can set that equal to the centripetal force $\frac{mv^2}{r}$ (which is just $a_c m$, via $F = ma$):

$$\frac{GMm}{r^2} = \frac{mv_{orbit}^2}{r} \tag{8.46}$$

$$\frac{GM}{r} = v_{orbit}^2 \tag{8.47}$$

$$\sqrt{\frac{GM}{r}} = v_{orbit} \tag{8.48}$$

where r is the radius of the orbit, which has nothing to do with the radius of the Earth itself. v_{orbit} is then the tangential speed of the object that is in orbit. Knowing these facts, we can now find the period of the orbit:

$$T = \frac{2\pi r}{v_{orbit}} = 2\pi\frac{r^{3/2}}{\sqrt{GM}} \tag{8.49}$$

If we plug in the Sun's mass, and $r = 149.6 \times 10^9$ m, the approximate average distance to the sun, we find Earth's orbital period $T \approx 365.33$ days. Not bad at all, since this is only an approximation (it ignores the several things that matter, including the Earth's elliptical orbit).

As a different example, we can take the space shuttle, or the space station, which orbit at 250-400 km above the Earth's surface. If we make the calculation for 400 km, so that $r = R_{Earth} + 400$ km, we find $v_{orbit} \approx 8\,\text{km/s}$ and $T \approx 90$ minutes(!).

Note that the orbital parameters are independent on the mass of the orbiting object. It only depends on the mass of the object you orbit, and the distance from it (i.e. the radius of the orbit), times some constants.

Also note that $v_{esc} = \sqrt{2} \times v_{orbit}$, for a given point. (In v_{esc}, we used the radius of the Earth, because we wanted to calculate the escape velocity from the surface.)

The total mechanical energy at some radius r, at orbital velocity v_{orbit}, is

$$E = \frac{1}{2}mv_{orbit}^2 - \frac{GMm}{r} \tag{8.50}$$

We can substitute the value for v_{orbit}^2 in there, though:

$$E = \frac{1}{2}m\frac{GM}{r} - \frac{GMm}{r} \tag{8.51}$$

$$E = -\frac{1}{2}\frac{GMm}{r} = \frac{1}{2}U = -K_E \tag{8.52}$$

Quite an interesting result. In words, then, the total energy of an orbiting object is always half its gravitational potential energy, and also the negative of its kinetic energy.

Now, for something completely different (more on orbits in a few weeks).

8.2.2 Power

Power is energy per unit time – or work per unit time, since energy and work are closely related, and share the same dimension. The SI unit for power is joules per second, or watts, W; not to be confused with W for the quantity of work! If we have $W =$ something then it's work; if we have $P = 10$ W, then it's watts.

Stated differently, it is then just the derivative of work – that is, $P = \dfrac{dW}{dt}$.
Since $dW = \vec{F} \cdot \vec{dr}$, we can take the time derivative of both sides:

$$\frac{dW}{dt} = \vec{F} \cdot \frac{\vec{dr}}{dt} \tag{8.53}$$

... and since the rate of change versus time of $\vec{dr}$ is simply the velocity:

$$P = \vec{F} \cdot \vec{v} \tag{8.54}$$

Power in riding a bicycle

Let's look at an example: riding a bicycle. We try to keep a constant velocity, which means there should be no net force on the bike. However, there *is* air drag, and the force opposing your motion, $F_{res} \propto kv^2$. In order for there to be no *net* force, your pedaling must then provide an equally great force in the forwards direction, in order for you to keep a constant speed.

As an aside, how does pedaling provide this force? First, you push down on the pedals, and the pedals push back on you with equal force via Newton's third law. This causes no net force on the bike, and we call these forces *internal forces*.

The pedals push on the chain, and the chain pushes on the wheel, all of which cancels, but finally, the wheel now wants to rotate, because of the force exerted by the chain.

The wheel pushes backwards on the ground, which leads to a reaction force such that the ground pushes the wheel forward. Finally something useful! This only works because of friction, of course – without friction, it would simply start spinning, and there would be no forward force on the bicycle.

Now, let's look at the amount of power you must provide to overcome air resistance. We can model this as a regime II problem, so the drag force is proportional to k_2v^2. Say that the power you must provide at 10 miles/hour is 15 watts – which is a given, and not something we actually show.

Now, the power we must provide is $P = \vec{F} \cdot \vec{v}$, as we showed earlier. Since the force and the velocity are in the same direction, $P = Fv$. Since $F = k_2v^2$, we find $P = k_2v^3$! It is proportional not to v^2, but v^3.

If you then want to speed up to 25 mph, 2.5 times the original speed, you need to provide $2.5^3 \approx 15.6$ times the power, about 230 watts! For 30 mph, 3 times the original speed, you need $3^3 = 27$ times the power (over 400 watts)! Needless to say, we reach the limits of human physiology rather quickly if we keep going like this. At 50 mph, it would take over 240 times the power (over 1800 watts – far above what any human could do, except for elite athletes for a period of seconds or less)!

8.2.3 Heat energy

First, a few definitions. We use the symbol Q for heat energy, often in the unit of calories. A calorie is the energy required to raise the temperature of 1 gram of water by 1 degree Centigrade (or 1 Kelvin, which is the same thing). There are, unfortunately, a ton of different definitions for a calorie, but all are close to 4.2 joules. (Some are defined as the energy required to heat 1 g of water from 3.5°C to 4.5°C, others from 14.5 to 15.5, 19.5 to 20.5, etc.)

Next, there is the *specific heat* C, which is a constant (for a given material) that specifies the amount of energy required to raise the temperature of that material by 1 degree centigrade, per unit mass. That is, it's in cal/(g°C).
If we want the unit to use kilograms instead of grams, which is always a nice thing when using the MKS (meter-kilogram-second) system, we simply multiply the constant by 1000.

The amount of heat energy Q, is then

$$Q = mC\Delta T \tag{8.55}$$

in calories, if m is in grams, C in the units stated above (per gram, not per kilogram), and ΔT in either Kelvin or degrees centigrade (they are equivalent; the zero point is the only difference).
As a reference, ice has a specific heat of about 0.5, compared to liquid water's 1. Aluminium has a specific heat of about 0.2, and lead a very low 0.03.

James Joule first found that mechanical work and heat energy are equivalent in 1845, though he was not the first to begin research on the topic. This research, among other things, led to the naming of the joule in his honor.

8.2.4 Power and the human body

The human body radiates heat, infrared radiation, at a rate of about 100 watts – 100 joules per second. That is about 10^7 joules per day, which then is about 2.4 (or ≈ 2) million calories.
Clearly, then, we need to input an equivalent amount of energy, or we would run dry sooner rather than later! We get this energy from food, of course.
Food labels are usually in kilocalories, which is sometimes written as either Cal (capital c) or kcal. They often mention the equivalent value in kilojoules as well.

So when a food label says 400 (kilo)calories, that that is enough for a power output of 100 watts for about 4 hours.

Normal, daily activities use almost no energy at all, compared to the rate of energy use of the body that occurs either way. Walking up 10 meters (vertically) of stairs, 5 times a day, is an average power use of about 1 W, if spread out over over 10 hours or so. That's only about 1% of the heat energy produced by the body when essentially at rest.

On the other hand, if you climb a mountain of 5000 feet (about 1500 meters), you might do about a million joules worth of mechanical work just to overcome gravity, which is not negligible compared to the 10^7 joules daily. In other words, you need to eat more, in order to stay the same weight (or mass, rather!). Because the conversion from food energy to mechanical work is quite inefficient, eating 10% more won't do it, though; you may have to eat 40%+ more in order to account for the increased energy use.

8.2.5 More heat, and electric energy

Consider taking a bath: we might need to heat 100 kg of water, by about 50 degrees C. (I'm not so sure I agree with that number, though! Even if the water was 0°C to begin with, it would probably be painfully hot! Anyway, let's work with the number from the lecture.)

With $C = 1$ cal/(g°C) for water, the answer is then simply the mass (in grams!) times $C = 1$ times 50 degrees C, or about $5 \cdot 10^6$ calories, which is about 2×10^7 J.

It's fairly difficult to produce 120 watts of work in turning a crank on a generator that then produces electric energy; a student was unsuccessful, and the professor says he is also unable to do so. Still, you would need to produce those 120 watts for 48 hours in order to heat up the water in that bath by 50 degrees!

Next, a demonstration of a simple battery is shown; four cells consisting of a zinc anode and a copper cathode in a sulfuric acid solution are wired in series to power a small light bulb.

After that, some more numbers: the global energy consumption (in 1999, when the lectures were recorded) is/was about 4×10^{20} J per year.
The USA consumes about 1/5 of that, with 1/30 of the world's population.

The Sun has a power output of about 4×10^{26} W, radiated in mostly visible light and infrared. Of course, it radiates a roughly equal amount in all directions, so only a small fraction of that reaches the Earth. We can calculate how large Earth's cross section is, and find the ratio of that divided by $4\pi R^2$, with R being the mean distance between the Earth and the Sun. We find about 1400 watts per square meter, that reaches the Earth's atmosphere.
The measured value at the ground varies greatly, for many reasons: the Sun's altitude above the horizon (i.e. day/night cycle, seasons, location on the Earth), cloud cover, and more.

If we try put solar panels on a horizontal roof, then clearly they will not do anything useful when the sun is just at the horizon.

Taking all this into account, along with solar cell inefficiency, we could perhaps provide enough energy to power the planet continuously by having a 400 mile by 400 mile solar grid – not a small area! That area is three times that of England. Clearly, we cannot sustain our current energy use by solar power alone.

Lecture question time:
"When the sun is in the zenith and the sky is perfectly clear, the solar power we receive on the surface of the earth is roughly 1 kW per square meter (on a horizontal surface that is normal to the direction of the sun).

Calculate the average solar power per square meter (in Watts) on a horizontal surface for a day when the sun goes through the zenith at noon.

The sun goes through the zenith exactly twice a year on latitudes that are close to the equator. Take the angle to the sun into account."

Okay, let's see. Let's imagine one single square meter somewhere on the Earth. First we have sunrise, where the sun is at $-\pi/2$ radians compared to the zenith. At noon, it is at 0 rad (straight above), and at sunset, at $+\pi/2$ radians. The power at a given angle θ should be $P = (1000\,\mathrm{W})\cos\theta$. Next, we need to find an average.

We can do find the average by an integral:

$$\overline{P} = \frac{1}{b-a}\int_a^b 1000\cos\theta d\theta \tag{8.56}$$

where $a = -\pi/2$ and $b = \pi/2$. However, that is only exactly half a day – the other 12 hours, there is zero sunlight, so we need to divide our answer by two, which causes the additional factor of 1/2 below. We find

$$1000\frac{1}{\pi}\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{1000}{2\pi}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{1000}{2\pi}\Big[\sin\theta\Big]_{-\pi/2}^{\pi/2} \tag{8.57}$$

$$= \frac{1000}{2\pi}\times 2 = \frac{1000}{\pi} \approx 318.3\,\mathrm{W} \tag{8.58}$$

I do believe I'm missing a simplification, but at least this gets us the correct answer.

Finally, a small note on the current energy crisis.
We are using fossil fuels at a rate which is about one *million* times greater than nature's production. At this rate, we will run out in less than 100 years. Fossil fuels account for a bit over 80 percent of human energy consumption, so clearly we need to start producing a *lot* of energy from other sources, or we will simply run out. While solar power is very useful, we have already ruled it out as a full replacement. We can combine many energy sources, though, and have solar power be one of them.

Nuclear fusion is often touted as the "energy source of the future". And indeed, if we can create an efficient reactor (current designs are not usable in practice, and mostly use more power than they provide back), it could provide practically limitless energy. One possible fuel source is deuterium and tritium, isotopes of hydrogen, present in sea water. We have enough water to provide the current world's energy usage for about 25 billion years, so if it could work, the energy problem would be solved.

Not only that, but fusion is both clean and intrinsically safe. Most of us know of Chernobyl or Fukushima – rare accidents, but they do happen, and can be very problematic.
(Though as of late 2013, the death toll due to Fukushima is still zero (but the lifetime rate of cancer has likely increased in some people), and fission energy has a lower death rate per unit energy produced than even wind and solar energy. I'm not trying to sell some propaganda here, but I do feel that nuclear fission has a worse reputation than it deserves, even though there obviously are risks.)

If a tsunami or an earthquake were to hit a *fusion* reactor, however, not a lot can happen. In magnetic containment fusion, the magnetic field would collapse if power was cut, and the reactor would automatically shut down. Not automatically as in a safety protocol, but rather without the containment, the reaction cannot be sustained, and stops all by itself. That's in contrast with a fission reactor, which must rely on safety measures to stop the reaction, so that it doesn't go out of control.
(Newer fission reactor designs are safer than ones in use, but few new designs are actually put into use, likely in part due the rather vocal opposition to nuclear energy.)
In addition, a fission reactor usually has many tons of fuel inside, while a fusion reactor can be powered off just grams of fuel, and use well below 1 ton of fuel in a year – and the fuel is just hydrogen isotopes. Granted, tritium is radioactive, and there are some radioactive byproducts, but nowhere near the many tons a year of poisonous, radioactive material that fission reactors produce per year.

Chapter 9: Week 7

9.1 Lecture 15: Momentum and its conservation

We will now introduce the concept of momentum.
Momentum is a vector: the product of mass and velocity. The SI units are then kg · m/s or N · s; there is no named unit for this quantity.

It is usually written as $\vec{p}$, so

$$\vec{p} = m\vec{v} \tag{9.1}$$

Momentum is closely connected with force, and knowing the above, it is easy to show how. $F = ma = m\dfrac{dv}{dt}$. We can work backwards, assuming m is a constant so that $m\frac{dv}{dt} = \frac{d(mv)}{dt}$:

$$F = m\frac{dv}{dt} = \frac{d(mv)}{dt} = \frac{dp}{dt} \tag{9.2}$$

So force is the time derivative of momentum.
This also implies that in order for an object's momentum to change, a force must have acted on it. And, conversely, if a (net) force acts on an object, its momentum must change.

Next, the professor shows (in some detail) how momentum is conserved for a *system* of particles/objects, unless there is a net *external* force on them. What happens internally does not matter, since all such internal forces cancel out, when you consider the system as a whole. For example, if two particles collide, the momentum of both particle #1 and of particle #2 may change, but the momentum of #1 + #2 will stay a constant.

This then leads to the *conservation of momentum*, which is a very helpful concept in solving some kinds of problems. Let's solve a simple problem using this principle.

Say we have two objects of masses m_1 and m_2 respectively, both moving towards the right, with velocities v_1 and v_2, where $v_1 > v_2$. Eventually, the two will collide.

Momentum prior to the collision can be found by the sum $m_1v_1 + m_2v_2$; since it is a vector, it has a direction. Both velocities are towards the right, so the net momentum will be as well; we take right to be increasing value of x, so that the numbers are positive.

In the collision, the masses will stick together – pretend that we put glue on one of them. (Collisions where the two separate after colliding are covered later in the lecture, or in the next.)
Because they stick together, they will share a velocity later – and a momentum, as well. The momentum after the collision can be written as $(m_1 + m_2)v'$, if we call the new velocity v'.

Via conservation of momentum, the two must be equal, if there are no external forces (so this collision happens on a frictionless table, with no air drag etc). We set them equal, and solve for v':

$$(m_1 + m_2)v' = m_1v_1 + m_2v_2 \tag{9.3}$$

$$v' = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \tag{9.4}$$

To get a feeling for this, say $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $v_1 = 5\,\text{m/s}$ and $v_2 = 3\,\text{m/s}$. Mass 1 has a momentum of 5 kg m/s, while mass 2 has 6 kg m/s. The net momentum prior to the collision is then 11 kg m/s, since the two are in the same direction.

Both velocities are towards the right and thus positive, so v' will also be positive.
Plugging the numbers in, we find $11/3\,\text{m/s}$ as the new velocity for the masses, moving together.

What is now the momentum? It is the new velocity times the total mass, which is $(11/3) \times 3$, so indeed we find that momentum was conserved. Not a huge surprise, since we derived the equation from that assumption!

What may be surprising is instead what happens to the kinetic energy.
Prior to the collision, the total kinetic energy was 12.5 J + 9 J = 21.5 J.

After the collision, the total kinetic energy is only $20.1\bar{6}$ J – we lost one and a quarter of a joule. Not a whole lot, perhaps, but let's look at a second situation.

Say we have the same values for the masses and velocities, only that v_2 is now negative, i.e. to the left, and so the two hit each other head-on.

Mass 1 still has the same momentum of $5\,\text{kg m/s}$, but v_2 now has $-6\,\text{kg m/s}$. The net momentum is now $-1\,\text{kg m/s}$ instead of $+11$.

Using the same formula, we now find $v' = -1/3$ m/s. The initial kinetic energy is unchanged – the masses are the same, and the *speeds* are the same – but the kinetic energy after the collision is now a tiny 1/6 of a joule!

In short: in the absence of (net) external forces, the momentum of a *system* of two or more objects is always conserved; kinetic energy, however, is not.

Let's have a look at a two-dimensional problem from a lecture question:

"In a scattering experiment, an incident alpha particle of mass $M_1 = 4u$ interacts with a static proton of mass $M_2 = u$. The incident particle is initially moving along the x-axis with a velocity $\vec{v_1} = v_{1x}\hat{x} = 0.05c\hat{x}$, and a final velocity (after collision) $\vec{v_1}' = v'_{1x}\hat{x} + v'_{1y}\hat{y} = 0.044c\hat{x} + 0.008c\hat{y}$, where c is the speed of light ($c = 3 \times 10^8\,\text{m/s}$).

What is the speed of the proton after the collision?
What is the direction of the proton after the collision? (give the angle with respect to the x-axis in radians)"

Haha, honestly, I was stuck for a while, since I found one equation with two unknowns, for each component, so total two equations, four unknowns. It turns out that half of those "unknowns" are part of the question, only I didn't realize at first. Duh!

To solve this, we apply conservation of momentum on the two axes, independently of each other.
Before and after the collision, in the x direction:

$$4uv_{1x} = 4uv'_{1x} + uv'_{2x} \tag{9.5}$$
$$4(v_{1x} - v'_{1x}) = v'_{2x} \tag{9.6}$$

where v'_{2x} is the x component of the proton's velocity after the collision. Plugging in the numbers given, we find $v'_{2x} = 4(0.05c - 0.044c) = 0.024c$.

Next, the y direction. Neither particle has any y component whatsoever at the moment, so the net momentum prior to the collision is clearly zero. That also means that the net momentum *after* the collision must be zero.

$$4uv'_{1y} + uv'_{2y} = 0 \tag{9.7}$$
$$v'_{2y} = -4v'_{1y} \tag{9.8}$$

So we find $v'_{2y} = -0.032c$.

With that in mind, we can now calculate the speed as $v_2' = \sqrt{(v_{2x}')^2 + (v_{2y}')^2} = 0.04c$.

Next up, the angle made with the y axis. If we consider the components, it must be angled downwards to the height. Drawing it out, we find that

$$\theta = \arctan \frac{v_{2y}'}{v_{2x}'} = \arctan \frac{-0.032}{0.024} = -53.13° \tag{9.9}$$

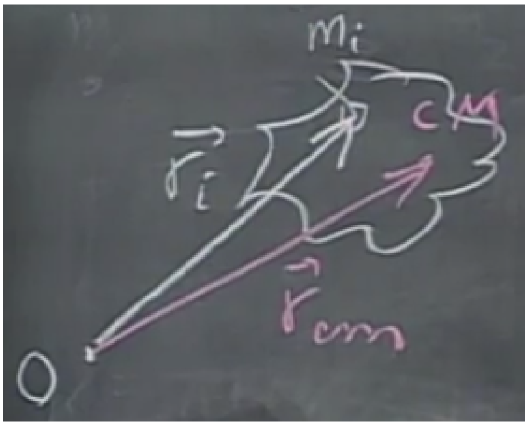
This angle would put it in the correct quadrant, and since the magnitude of the y component is slightly greater than that of the x component, it makes sense that the angle is a bit more than a 45 degrees down from the axis.

Next, there is a great demonstration of the conservation of momentum. I didn't take any notes of it, however.

9.1.1 Center of mass

Every object, regardless of shape or size, has a center of mass; a single point, which has some very interesting and useful properties.

We take any object, of any size (greater that zero, however, or the entire point is lost), and think of it as being composed by a practically infinite amount of small masses m_i. Each mass has a position vector $\vec{r_i}$ from the origin, which we are free to choose.



It is then true that the

$$M_{tot}\vec{r_{cm}} = \sum_i m_i\vec{r_i} \tag{9.10}$$

$$\vec{r_{cm}} = \frac{1}{M_{tot}} \sum_i m_i\vec{r_i} \tag{9.11}$$

In the limit as the masses become infinitesimally small, this becomes an integral:

$$\vec{r_{cm}} = \frac{1}{M_{tot}} \int \vec{r}\, dm \tag{9.12}$$

x, y and z components of this can be found in the same way, with three separate integrals.

For the simple case of two particles along a single axis:

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{9.13}$$

This gives a result where the center of mass is closer to the more massive of the two objects. If they are equally massive, the center of mass is at the midpoint between the two.

Returning to the first equation solved for $\vec{r_{cm}}$, we can take the time derivative of both sides of this equation. $\vec{r_{cm}}$ becomes $\vec{v_{cm}}$, and $\vec{r_i}$ becomes $\vec{v_i}$, since velocity is the time derivative of position.

$$\vec{v_{cm}} = \frac{1}{M_{tot}} \sum_i m_i \vec{v_i} \tag{9.14}$$

However, note that $\sum_i m_i \vec{v_i}$ is the sum of the mass-velocity products: it is the total momentum of the system. In other words,

$$\vec{v_{cm}} = \frac{1}{M_{tot}} \vec{p_{tot}} \tag{9.15}$$

$$\vec{p_{tot}} = M_{tot} \vec{v_{cm}} \tag{9.16}$$

This second result is an important one: the total momentum of a system can be found by knowing its total mass and the velocity of its center of mass, and is the same regardless of what the rest of the system is doing.

Not only that, but we can take the time derivative of this. The time derivative of momentum is force (or net external force, to be more precise), while the derivative of velocity of the center of mass is the *acceleration* of the center of mass:

$$F_{ext} = M_{tot} \vec{a_{cm}} \tag{9.17}$$

This is a very interesting result. Regardless of the shape of an object, if we know the external force and the total mass, we can predict how the center of mass moves in a simple way, even though the motion of the object as a whole may be very complicated and involve tumbling/spinning at varying speeds, etc.

So if the (net) external force is zero, the center of mass will continue to move in a straight line, forever, regardless of what the rest of the object is doing.

9.2 Lecture 16: Elastic and inelastic collisions

Last lecture was focused on inelastic collisions; we will now consider general collisions, including elastic ones. Again, let's start with a one-dimensional example.

A mass m_1 is moving towards the right with speed v_1, towards a mass m_2 which is at rest. We take increasing values of x to be towards the right.

After the collision, $v_1^{'}$ can be either positive or negative (to the right or to the left), while $v_2^{'}$ is certainly towards the right. (If an object hits it from the left, how could it start moving towards the left?)

To find the velocities after the collision, we can apply conservation of momentum. Only the first mass had any momentum prior, so that must be the sum of the momentum after the collision as well:

$$m_1 v_1 = m_1 v_1^{'} + m_2 v_2^{'} \tag{9.18}$$

Unfortunately, the equation has two unknowns; we need a second equation to get anywhere.

In order to find a second equation, we can use the conservation of energy. *Kinetic* energy is not necessarily conserved in collisions, which we saw last lecture. However, the kinetic energy that is lost must be converted to some other form of energy, such as heat energy.

If we use an extra Q to denote the rest of the energy, we can write an equation of the form $K + Q = K'$, where K is the kinetic energy prior to the collision, and K' is the kinetic energy after.

There are three possible cases here.

- $Q > 0$: we call this a *superelastic* collision; the amount of kinetic energy has *increased* (as demonstrated with a spring's stored energy as the source in the previous lecture; an explosion or such could also cause this to happen).
- $Q = 0$: this is an *elastic* collision (or "completely elastic"; the modifier "completely" is really not necessary, however). Kinetic energy is *conserved* in this special case.
- $Q < 0$: this is an inelastic collision. Kinetic energy is lost in the collision (in any amount from almost none lost, to *all* kinetic energy lost), and is mostly turned into heat (but perhaps also noise and vibration, etc).

Let's focus on the special case of elastic collision, so that $Q = 0$ and $K = K'$. In this case, we can write an equation relating the initial kinetic energy to the final kinetic energy, as follows:

$$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 (v_1^{'})^2 + \frac{1}{2}m_2 (v_2^{'})^2 \tag{9.19}$$

$$m_1 v_1^2 = m_1 (v_1^{'})^2 + m_2 (v_2^{'})^2 \tag{9.20}$$

Combining this with the equation that relates the momentum of the system, we can find expressions for $v_1^{'}$ and $v_2^{'}$ as follows, by solving the system of equations:

$$v_1^{'} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 \tag{9.21}$$

$$v_2^{'} = \left(\frac{2m_1}{m_1 + m_2}\right) v_1 \tag{9.22}$$

So the above is valid under three conditions: the initial velocity $v_2 = 0$ initially, $Q = 0$ and momentum is conserved (i.e. there is no net external force on the system).

We will now look at what will happen in three special cases: $m_1 \gg m_2$, $m_1 \ll m_2$ and $m_1 = m_2$.

First out is $m_1 \gg m_2$, which turns out to be the same as $m_2 \to 0$. What happens in the above equations as $m_2 \to 0$?

Well, $v_1^{'} = v_1$ – which comes as no surprise. If a very massive object runs in to one that has practically zero mass, it will just continue on its way.
What happens to the smaller object is more interesting: its velocity goes to $v_2^{'} = 2v_1$. As long as $m_1 \gg m_2$, no matter what the actual masses or velocities are, it will zoom away at twice the speed of the object that hits it.

Next, the opposite case, where $m_1 \ll m_2$, which means $m_1 \to 0$. Since m_1 is the one that moves initially, I would expect it to change direction and move backwards at some speed, while m_2 does almost nothing. Let's plug it in: we find $v_1^{'} = -v_1$ and $v_2^{'} = 0$, as predicted. It turns out that m_1 simply bounces back with the *same* speed, only in the opposite direction. Considering what happened to the tiny mass in the previous case, this might not be obvious – it was twice the speed in the previous case!

Finally, what happens if the masses are about the same? Plugging it in, we find $v_1^{'} = 0$, and $v_2^{'} = v_1$: the first object stops, and the second moves as the first one did prior to the collision.
Most have seen this in action, perhaps while playing pool, or in a "Newton's cradle" where several balls (usually at least 3, but 2 works) hang suspended as pendula; you raise one up, and when it whacks on the rest, only the one at the other end starts moving; the others merely "relay" the momentum through until it reaches the last ball.

The cases where $m_1 = m_2$ and $m_1 = 0.5m_2$ are then demonstrated, with very convincing results!

What if $v_2 \neq 0$? More specifically, $v_2 > 0$, so that they are both moving towards the right, with $v_1 > v_2$ so that they will eventually collide. Again, we assume an elastic collision.

If we set up the system of equations, with the change that both masses now have momentum towards the right, and both have initial kinetic energy, we find

$$v_1^{'} = \frac{m_1 v_1 - m_2 v1 + 2m_2 v_2}{m_1 + m_2} \tag{9.23}$$

$$v_2^{'} = \frac{2m_1 v_1 - m_1 v_2 + m_2 v_2}{m_1 + m_2} \tag{9.24}$$

With $m_1 = m_2$, this yields a funny result: the two essentially trade speeds with one another. $v_1^{'} = v_2$ and $v_2^{'} = v_1$.

9.2.1 Elastic collisions seen from the frame of the center of mass

We can choose our reference frame such that the center of mass has zero velocity in our frame. This is referred to as the "center of mass frame", "center of momentum frame" or COM frame.

In this frame, total momentum is always zero. We found last lecture that $\vec{p_{tot}} = M_{tot}\vec{v_{cm}}$, and *in the COM frame*, $v_{cm} = 0$ by definition. That definition is what makes it the COM frame.

Using the definition for the velocity of the center of mass

$$v_{cm} = \frac{1}{M_{tot}} \sum_i m_i v_i = \frac{\sum_i m_i v_i}{\sum_i m_i} \tag{9.25}$$

and a Galilean transformation for the velocities of two particles,

$$u_1 = v_1 - v_{cm} \tag{9.26}$$

$$u_2 = v_2 - v_{cm} \tag{9.27}$$

where the u notation is used for the particle's velocities in the COM frame, we can show that the total momentum must be zero in this frame, in a different way.

The sum $\sum_i m_i u_i$ is the net momentum in the COM frame. $u_i = (v_i - v_{cm})$, so we substitute that in and find $\sum_i m_i(v_i - v_{cm})$ as the total momentum. Using the definition of v_{cm} above, that is

$$\sum_i m_i \left(v_i - \frac{\sum_i m_i v_i}{\sum_i m_i} \right) = \sum_i m_i v_i - \sum_j m_j \frac{\sum_j m_j v_j}{\sum_j m_j} = 0 \tag{9.28}$$

The denominator in the fraction becomes 1, and we then have the subtraction of two equal quantities that equals the total momentum, as seen from the COM frame, since the indices i and j are equivalent.

Now that we can hopefully accept the above as being true (I had trouble seeing it until I did the above)... Say we are in this frame, and there are two particles with velocities inward toward the center of mass; one mass m_1 with speed u_1, and one mass m_2 with speed u_2 – again with those speeds being in the COM frame. Also say $Q = 0$ for this collision, i.e. it is an elastic collision.

Momentum is zero both before and after the collision. In addition, because this is an elastic collision, we can also write down an equation relating kinetic energy before and after the collision. Altogether, we have:

$$m_1u_1 + m_2u_2 = 0 \tag{9.29}$$

$$m_1u_1' + m_2u_2' = 0 \tag{9.30}$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1(u_1')^2 + \frac{1}{2}m_2(u_2')^2 \tag{9.31}$$

With all this information, we can find (through some tedious algebra) two very simple answers: $u_1' = -u_1$ and $u_2' = -u_2$.

That is, *as seen from the center of mass frame*, which is moving, both simply reverse direction, but keep moving at the same speed. This happens regardless of the masses and speeds, so this is clearly only possible in this very special frame.

In this simple case with only two objects, the definition we have above for v_{cm} is fairly simple:

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \tag{9.32}$$

We can then follow the process of first transforming our velocities into velocities as seen from the center-of-mass by subtracting v_{cm}, do the collision calculations knowing that momentum is zero both before and after the collision (this holds for all collisions; inelastic, elastic and superelastic), and then transforming back to the external frame by *adding* v_{cm} back.

Since the center of mass moves at a constant velocity in the absence of external forces, we need not worry about it having changed during the collision (unless we are making an incorrect assumption that the external forces are zero).

Let's try an example (from a lecture question).

"Before a 1-dimensional collision, two masses $m_1 = 3$ kg and $m_2 = 5$ kg have velocities $v_1 = -5$ m/s and $v_2 = 3$ m/s with respect to their center-of-mass frame.

What are their velocities (in m/s) in the laboratory frame after an elastic collision? (The velocity of the center of mass is $v_{cm} = 2$ m/s)"

Alright, so in the center of mass frame, $v_1' = 5$ m/s and $v_2' = -3$ m/s, since all they do in that frame is reverse direction. To convert this to the lab frame, we need to *add* v_{cm} to these numbers, so we find 7 m/s and -1 m/s, respectively. That was certainly very easy.

9.2.2 Inelastic collisions seen from the center of mass frame

The center of mass frame has another interesting property. In the case of a completely inelastic collision, i.e. the two masses that collide stick together, both velocities go to zero (as seen from the center of mass; this would be true even if they were sliding together according to an outside observer).
Zero velocity means zero kinetic energy, so *all* kinetic energy will be lost in this frame.

This kinetic energy, as seen from the center of mass frame, is called the *internal energy*; it is the maximum energy that can be converted to heat in a collision.

Let's first calculate the amount of kinetic energy lost in a completely inelastic collision, as seen from the "lab frame" (one that is fixed to the room you're in, i.e. what at least I personally would consider the default frame).

We take the case where a mass m_1 moves with speed v_2 towards a second mass, m_2, that is at rest (with respect to the lab frame). It's a completely inelastic collision, so they stick together after the collision.

After the collision, we call the velocity v' (which is just a speed in the same direction as v_1, since momentum is conserved), and the total mass is then $m_1 + m_2$.
Conservation of momentum gives

$$m_1 v_1 = (m_1 + m_2)v' \tag{9.33}$$

$$v' = \frac{m_1 v_1}{m_1 + m_2} \tag{9.34}$$

$v_{cm} = v'$; that can be seen very easily by looking at the equation for v_{cm}, and considering the case where $v_2 = 0$, as it is here. v_{cm} equals exactly the above expression in that case.

Next, we can calculate the difference is potential energy before (K) and after (K') the collision. This is the Q we had in a previous equation (see the note just below; I made a small mistake):

$$\begin{aligned}
Q = K' - K &= \frac{1}{2}m_1 v_1^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1 v_1}{m_1 + m_2}\right)^2 && (9.35)\\
&= \frac{1}{2}m_1 v_1^2 - \frac{m_1^2 v_1^2}{2(m_1 + m_2)} && (9.36)\\
&= \frac{m_1 v_1^2 (m_1 + m_2)}{2(m_1 + m_2)} - \frac{m_1^2 v_1^2}{2(m_1 + m_2)} && (9.37)\\
&= \frac{m_1 v_1^2 (m_1 + m_2) - m_1^2 v_1^2}{2(m_1 + m_2)} && (9.38)\\
&= -\frac{m_1 m_2}{2(m_1 + m_2)} v_1^2 \text{ (see below re: minus)} && (9.39)
\end{aligned}$$

Phew! This is then, from the external reference frame, the amount of kinetic energy lost in the collision. As it turns out, I accidentally calculated $K - K'$ instead. The only difference is a minus sign, of course, so the actual answer should be *minus* what I actually found; I added the sign in the last step above, instead of re-writing the code for this mess; sorry about that.
So the last line in the equation above is correct.

Next, we do the same calculation in the center of mass frame. We know v_{cm}, v_1 and v_2, so we can jump straight to converting the velocities. Using u_i for the velocities as seen from the center of mass frame,

$$\begin{aligned}
u_1 = v_1 - v_{cm} &= v_1 - \left(\frac{m_1 v_1}{m_1 + m_2}\right) && (9.40)\\
u_2 = v_2 - v_{cm} &= -\left(\frac{m_1 v_1}{m_1 + m_2}\right) && (9.41)
\end{aligned}$$

The first equation simplifies:

$$\begin{aligned}
u_1 &= \frac{v_1(m_1 + m_2) - m_1 v_1}{m_1 + m_2} && (9.42)\\
u_1 &= \frac{v_1 m_1 + v_1 m_2 - m_1 v_1}{m_1 + m_2} && (9.43)\\
u_1 &= \frac{v_1 m_2}{m_1 + m_2} && (9.44)
\end{aligned}$$

And we can then write them as

$$\begin{aligned}
u_1 &= \left(\frac{m_2}{m_1 + m_2}\right) v_1 && (9.45)\\
u_2 &= -\left(\frac{m_1}{m_1 + m_2}\right) v_1 && (9.46)
\end{aligned}$$

We can then calculate the total kinetic energy *prior* to the collision as

$$K = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 \tag{9.47}$$

$$K = \frac{1}{2}m_1 \left(\left(\frac{m_2}{m_1+m_2}\right)v_1\right)^2 + \frac{1}{2}m_2\left(-\left(\frac{m_1}{m_1+m_2}\right)v_1\right)^2 \tag{9.48}$$

$$K = \frac{1}{2}m_1\left(\frac{v_1 m_2}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{v_1 m_1}{m_1+m_2}\right)^2 \tag{9.49}$$

$$K = \frac{v_1^2 m_1 m_2^2}{2(m_1+m_2)^2} + \frac{v_1^2 m_2 m_1^2}{2(m_1+m_2)^2} \tag{9.50}$$

$$K = \frac{v_1^2 m_1 m_2^2 + v_1^2 m_2 m_1^2}{2(m_1+m_2)^2} \tag{9.51}$$

$$K = \frac{(m_1+m_2)m_1 m_2 v_1^2}{2(m_1+m_2)^2} \tag{9.52}$$

$$K = \left(\frac{m_1 m_2}{2(m_1+m_2)}\right)v_1^2 \tag{9.53}$$

Again, phew! Note that this is exactly the same as the answer we found earlier, for the kinetic energy lost in the lab frame.
In this frame, that energy is *all kinetic energy that exists to begin with.* After the collision, kinetic energy will be zero, since the wreck sticks together (this entire section is about a completely inelastic collision), and neither body will move with respect to the center of mass after the collision.
In other words, the *change* in kinetic energy is the same in both reference frames, even though the initial and final energies are different in the different frames.

Lecture question time.

"Just before an inelastic head-on collision, two cars have a relative speed of $v = 40$ km/h (25 mph). The cars have masses $m_1 = 1300$ kg and $m_2 = 1600$ kg.
How much kinetic energy is lost during the collision?"

Hmm. Well, we can use the equation above. Assume $v_2 = 0$, and then it's just a matter of sticking the numbers in there, which yields about 44300 J.

9.3 Lecture 17: Momentum of individual objects

Previously, we measured the speed of a bullet, simply by measuring how long it took the bullet to move a certain distance. This was only possible because of the electronic timer, which both started and stopped automatically, as the bullet was shot through two wires.
We will now calculate the speed, by a more manual, indirect method, of firing the bullet into a block hanging as a pendulum. This way, we can find the velocity (with a fairly large uncertainty, but still) with nothing but a small meter stick and knowledge of physics.

The way in which we do this is fairly complex, but let's start simple. We have a solid block of mass M hanging from a string of length ℓ; this forms a *ballistic pendulum.*

The bullet of mass m comes in with a velocity v, and "merges" with the block (gets stuck inside), so we can model this is a completely inelastic collision. The block moves from its equilibrium position (straight down), towards the right and slightly upwards (since it is a pendulum!), with velocity v'.

We can apply conservation of momentum to find v':

$$mv = (m + M)v' \tag{9.54}$$

$$v' = \frac{mv}{m + M} \tag{9.55}$$

Soon thereafter, v' will have gone to zero, as the pendulum reaches its highest point. Here, we know that kinetic energy is zero, and all kinetic energy has been converted into gravitational potential energy.
If we define $U = 0$ at the equilibrium position, the change in gravitational potential energy was $(m+M)gh$, where h is the amount the block moved upwards. This energy must have come from the kinetic energy, so via conservation of energy, we can relate the block's initial kinetic energy (as the bullet is absorbed) and the gravitational potential energy as it stops:

$$\frac{1}{2}(m + M)(v')^2 = (m + M)gh \tag{9.56}$$

$$v' = \sqrt{2gh} \tag{9.57}$$

With this in mind, we could theoretically fire a bullet into the block, measure how far it moves up, and calculate the speed of the bullet. However, the upwards movement is minuscule, less than a single millimeter; we still cannot measure that with any useful accuracy. We *can* measure how far it travels towards to the side, though, since that excursion is must greater. (Remember how we even neglected the upwards motion of a pendulum completely when we derived an equation for it using simple harmonic motion?)

So if we set the origin at the equilibrium position, we can call the maximum horizontal displacement of the pendulum x. Via trigonometry, we can find that $x = \ell \sin\theta$, and $h = \ell - \ell\cos\theta = \ell(1 - \cos\theta)$.

Using the same small-angle approximations we have used previously, that $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$ and $\sin\theta \approx \theta$ (both only valid for radians), $h \approx \ell\dfrac{\theta^2}{2}$.

For $\ell = 1$ meter and $\theta = 2°$, we find that $h \approx 0.6$ mm, far to small to measure with any useful accuracy. However, $x \approx \ell\theta \approx 3.5$ cm, which is much more reasonable.

Since we now know x as a function of θ, we can write h as a function of x by combining the two equations:

$$h \approx \ell\frac{\theta^2}{2} \approx \frac{\ell}{2}\left(\frac{x}{\ell}\right)^2 = \frac{x^2}{2\ell} \tag{9.58}$$

With this in mind, we can find v' as a function of x

$$v' = \sqrt{2g}\left(\frac{x}{\sqrt{2\ell}}\right) = x\sqrt{\frac{g}{\ell}} \tag{9.59}$$

... and then finally the bullet's original velocity v as a function of v', by using the old conservation of momentum equation we had:

$$v = \frac{v'(m + M)}{m} = x\frac{m + M}{m}\sqrt{\frac{g}{\ell}} \tag{9.60}$$

Let's now look at some numbers. The mass of the bullet is $m = 2.0 \pm 0.2\,\mathrm{g}$; $M = 3.20 \pm 0.02\,\mathrm{kg}$, and $\ell = 1.13 \pm 0.02\,\mathrm{m}$.

With these numbers, we find $v = 4.7 \times 10^3 x$. The total uncertainty is somewhere around 15%, in large part because of the large uncertainty in m.

When the experiment is carried out, $x \approx 5.2\,\mathrm{cm} = 0.052$ meters, so $v \approx 244\,\mathrm{m/s}$. Of course, with a 15% uncertainty, that last four may be rather meaningless. 15% less than this is just 207.5 m/s, and 15% more is 280.5 m/s.

Comparing the initial kinetic energy of the bullet $\frac{1}{2}mv^2 \approx 59.5$ J (with a large uncertainty) and the maximum kinetic energy of the block-bullet system $\frac{1}{2}(m+M)(v')^2 \approx 0.038$ J, we see that the vast majority of the kinetic energy was lost to heat, deforming the block (and perhaps bullet), etc. About 99.94% was lost, according to the lecture, which seems to match this calculation quite well.

9.3.1 Impulse

Impulse is a concept closely related to momentum. Any time a change in momentum occurs for an object, an impulse was imparted on that object.
It is a vector, and can be written as the time integral of force:

$$\vec{I} = \int_{t_0}^{t_1} \vec{F}\,dt = \int_{0}^{\Delta t} \vec{F}\,dt \tag{9.61}$$

using $\Delta t = t_1 - t_0$. In the simple case where the force is constant, $\vec{I} = \vec{F}\Delta t$.

However, force is also the rate of change of momentum. Therefore, the integral above can also be written as the integral of the derivative of momentum – clearly, the dimension here is going to be the same as that of momentum.
In fact, we can show that the impulse is just the difference in momentum at two different times; using $\vec{p_i}$ for the initial momentum and $\vec{p_f}$ for the final momentum,

$$\vec{I} = \vec{p_f} - \vec{p_i} \tag{9.62}$$

The units of impulse are then the same as those of momentum: kg m/s or newton-seconds ($N \cdot s$).

As an example, take the collision of a ball bouncing on the floor. In the simple case where the collision is elastic, and the ball bounces back to the same height (which is of course impossible, but we can come close), the ball hits the floor with momentum mv, if we take downwards to be the positive direction, and leaves with equal momentum in the opposite direction, that is, $-mv$.

The impulse is then found as $-mv - (mv) = -2mv$. The impulse is upwards in this case, and has the magnitude of the change in momentum $2mv$.

In the case of a completely inelastic collision (in other words, no bounce), as with a tomato hitting the floor, the impulse is smaller in magnitude at just mv – the colliding object loses all of its momentum to the floor, and ends up with zero speed and zero momentum.

Using the definition of impulse in terms of force, we can calculate the average force on a body during a collision as

$$\langle F \rangle = \frac{I}{\Delta t} \tag{9.63}$$

As an example, a ball (bouncy ball, super ball or what you may call it) with a mass $m = 0.1$ kg is dropped from a height of 1.5 meters. That gives it a speed of about 5.5 m/s (a bit less) as it hits the floor. Assuming an elastic collision, $I = 2mv = 1.1$ kg m/s.
We can then divide this by the impact time to find the average force. The impact time was measured (by high-speed photography) to be just 2 milliseconds. That gives an average force of

$$\langle F \rangle = \frac{1.1\,\text{kg m/s}}{0.002\,\text{s}} = 550\,\text{N} \tag{9.64}$$

Remember that our definition (at least one definition) of *weight* was the magnitude of the normal force exerted by e.g. the floor, to counteract gravity. (It could also be the tension in a rope, pulling you upwards.)

That means that this ball, during the short moment of the collision, has a weight about 550 *times* greater than it would otherwise ($550g$), since its "normal" weight is just $mg \approx 1$ N (a bit less, but we use $g \approx 10\,\text{m/s}^2$ for simplicity).

Lecture question time:
Car 1 of mass m_1 is moving along the $+x$-axis with speed v_1 towards car 2 of mass m_2 and speed v_2 moving along the -x-axis. They have a head-on collision that lasts a time interval Δt. After the collision the cars stick together. (Note: m_1 and m_2 include the drivers.)

The magnitude of the average force acting upon the driver of mass m_{dr1} in car 1 by her seat belt during the collision is given by:

$$F_{dr1} = \frac{m_1}{m_1 + m_2}(v_1 - v_2)\frac{m_{dr1}}{\Delta t} \tag{9.65}$$
$$F_{dr1} = \frac{m_1}{m_1 + m_2}(v_1 + v_2)\frac{m_{dr1}}{\Delta t} \tag{9.66}$$
$$F_{dr1} = \frac{m_2}{m_1 + m_2}(v_1 - v_2)\frac{m_{dr1}}{\Delta t} \tag{9.67}$$
$$F_{dr1} = \frac{m_2}{m_1 + m_2}(v_1 + v_2)\frac{m_{dr1}}{\Delta t} \tag{9.68}$$

If $v_1 = v_2$, considering the two are *speeds* in opposite directions, the driver will most certainty not experience zero force, so the two options with minus signs should both be wrong. Still, let's solve this the proper way.

Total momentum is conserved, and the cars stick together. Considering that v_2 is towards the $-x$ axis, its velocity is negative, so the sum of momenta becomes a subtraction. The velocity after the collision is given by

$$m_1 v_1 - m_2 v_2 = (m_1 + m_2)v' \tag{9.69}$$
$$v' = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} \tag{9.70}$$

The driver's initial and final speeds are the same as the car's, of course. With that in mind, we can calculate the impulse of the driver directly, as

$$I = m_{dr1} v' - m_{dr1} v_1 \tag{9.71}$$
$$I = m_{dr1}\frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} - m_{dr1} v_1 \tag{9.72}$$
$$I = -\frac{m_2(v_1 + v_2)}{m_1 + m_2} m_{dr1} \tag{9.73}$$

We wanted a magnitude, but got a negative number; that is simply due to the coordinate system choice. Since all variables must be positive (v_1 and v_2 are speeds and never negative), we can simply remove the minus sign to find the magnitude.

By definition, $\langle F \rangle \,\Delta t = I$, so we can write this in terms of an average force. We just make that substitution, and solve for the force by dividing both sides by Δt:

$$\langle F \rangle = \frac{m_2(v_1 + v_2)}{m_1 + m_2}\frac{m_{dr1}}{\Delta t} \tag{9.74}$$

Finally, we have one of the four possible answers, and it is indeed the correct one.

Next up, we have some talk about impact times, though nothing general enough to really write down. As always, no notes doesn't mean not worth watching – it's really a bit of the opposite.

9.3.2 Thrust and rockets

Consider the case of throwing tomatoes towards the floor again. Say we throw n tomatoes per second, and each tomato has a mass m. nm is (tomatoes/second) times (kilograms/tomato), so the dimension of this is in kilograms per second of "stuff" we throw.

The change in momentum for each tomato is mv; nmv gives us the dimension of impulse per time, so

$$nmv = \frac{\Delta p}{\Delta t} = \langle F \rangle \tag{9.75}$$

Since $\frac{dp}{dt}$ is the definition of force, the above yields a time-averaged force. The floor experiences a net downwards force from all these tomatoes.

In the form of a proper derivative, we have

$$F = \frac{dm}{dt}v \tag{9.76}$$

Similarly, in the case of a more horizontal case, we need to accelerate these tomatoes from a velocity of 0 to a horizontal velocity of v_x. The object they hit (a poor person's face, in the lecture) experiences a force in the same direction as the tomatoes' velocity vector, which should be quite intuitive. Why does this happen, though?
Well, the tomatoes come in with velocity v_x and momentum mv_x. They hit the person, and all of a sudden $v_x = 0$, and they have lost all of their momentum. Momentum is conserved (there is no relevant external force involved in the horizontal direction), so the momentum is imparted on the person. Since a change in momentum is a force by definition, the person experiences a force in the same direction as their gain in momentum – away from the tomato thrower.

However, for reasons of symmetry, when we *throw* the tomatoes, they start with zero velocity. It is up to us to give them that velocity v_x, and with that, the momentum mv_x. Momentum is conserved for us, too, so *we* must experience a change in momentum opposite to that of the tomatoes, so that the sum is zero. Again, a change in momentum is a force – we feel a force backwards!
This too should be fairly intuitive. Recoil from firing a gun is one example of this in effect.

This is, then, how a rocket works. It ejects massive amount of gas, at an extremely high velocity. Both $\frac{dm}{dt}$ and v are high, and the force generated is enormous. This then yields a forward (or upward) *thrust*, which is essentially the reaction force caused by ejecting all that matter.

Note that the thrust of the rocket is not dependent on the ejected matter hitting anything; it works just the same in the vacuum of space.
Helicopters work on the practically same principle, only that the air they eject is not stored as a fuel, but simply sucked in from above. Helicopters *do* have a stronger lift near the ground, due to an unrelated effect called the ground effect. With that said, helicopters don't depend on this effect to fly – if they did, they could only fly at very low altitudes. In fact, the effect is almost completely negligible at a height

where the rotor's distance to the ground is greater than the rotor diameter, so the effect becomes irrelevant at about 20 meters off the ground.

As on example, the Saturn V rocket, the exhaust velocity was on the order of 2.5 km/s (!), and about 15000 kg/s of material was spewed out. The net thrust is then the product of the two, about 37.5 million newtons. That sounds like an incredible lot, of course, but the thrust-to-weight ratio (which clearly must be greater than 1 to take off vertically, or gravity would win) was only about 1.2:1 at liftoff. That is still enough to have a net acceleration, though, so that it could reach a speed of 2.7 km/s in less than 3 minutes (and the speed only increased from there, to a bit over 7 km/s).

So this thrust then imparts a impulse on the rocket. The force (the thrust) acts for a certain time, the burn time. However, as the rocket accelerates, the mass of the rocket goes down, since the fuel is being burned and ejected. That in turn causes the acceleration to increase, and so it gains velocity faster. (If the force is constant, and the mass goes down, acceleration must go up. The force is not constant though, but increases; more on that later, I believe.)

9.3.3 Velocity change in a rocket

Let's look at calculating the change in velocity for a rocket, using an approach based on the conservation of momentum.

Consider the rocket at a time t. It is moving upwards with a velocity v (relative to an observer on the ground), and has a mass m.
A short time Δt later, the velocity is now $v + \Delta v$, and the mass $m - \Delta m$, since some of the fuel has been burned and ejected to create thrust.

If we use u to denote the exhaust velocity *relative to the rocket* (all other velocities are relative to the ground), the piece of exhaust is moving upwards with velocity $v - u$ as seen from the ground.

If the rocket's velocity is larger than the exhaust velocity, we see the exhaust moving upwards; if not, we see it moving downwards. Both are possible cases, and both are handled by the signs, with positive being upwards.

In the case where no external forces has acted on the system (we will look at gravity soon), momentum is conserved. The rocket's momentum will change for sure, but there will be an equal and opposite change in the exhaust's momentum, such that the net momentum is conserved.

At time t, the momentum is mv. At the later time $t + \Delta t$, the momentum is still mass times velocity, which is $P_{after} = (m - \Delta m)(v + \Delta v) + \Delta m(v - u)$. The last term is the momentum of the exhaust, which we must not forget!
Multiplied out, this is $P_{after} = mv + m\Delta v - v\Delta m - \Delta m\Delta v + v\Delta m - u\Delta m$.

$v\Delta m$ cancels out, and $\Delta m\Delta v$ is the product of two tiny numbers, so we neglect it. We find the momentum as

$P_{after} = mv + m\Delta v - u\Delta m$.
The net change in momentum must be zero, since momentum is conserved. $\Delta p = p_f - p_i$, so $\Delta p = m\Delta v - u\Delta m = 0$.

Considering the case where $\Delta t \to 0$, we can take the time derivative of the above equation, and find

$$0 = m\frac{dv}{dt} - u\frac{dm}{dt} \tag{9.77}$$

$$0 = ma - u\frac{dm}{dt} \tag{9.78}$$

Since we previously had the definition that $F_{thrust} = u\frac{dm}{dt}$, where u is the exhaust velocity relative to the rocket, what we really found is

$$F_{thrust} = ma \tag{9.79}$$

What happens if we consider gravity? In a still simplified case, we consider a fully vertical launch. The thrust and the force due to gravity mg are then in exactly opposite directions.
We would then find that $ma = F_{thrust} - mg$.

In a derivation not shown, we can then find the change in velocity $\Delta v = v_f - v_i$ (this Δv is the *total* change in velocity during the entire burn time of minutes (or so), and has nothing to do with the tiny Δv's in the derivation above, over a tiny time period Δt).

$$\Delta v = -u \ln \frac{m_f}{m_i} - mg \tag{9.80}$$

This then only holds in a fully vertical launch. Since $m_i > m_f$ (the fuel used up will cause the final mass m_f to be much smaller than the initial mass m_i), this equation will always be positive, assuming the thrust is greater in magnitude than the force of gravity. If it is not, then clearly, the rocket will either slow down in its upwards motion, or speed up in its fall back to Earth. We could rewrite the signs with this in mind, and flip the fraction inside the natural logarithm:

$$\Delta v = u \ln \frac{m_i}{m_f} - mg \tag{9.81}$$

If we remove the $-mg$ term, this equation is known as the rocket equation (or ideal rocket equation, or Tsiokovsky rocket equation).

According to this equation, the change in the velocity is fixed for a certain amount of fuel burned (assuming u is constant). However, the change in kinetic energy is *not* fixed. In other words, burning the same amount of fuel, in the same rocket, for a certain amount of time will cause a fixed increase in velocity, but the increase in kinetic energy will be *different* for different such burns, depending on the initial velocity!

Consider the increase in kinetic energy from a velocity of 0 m/s to 100 m/s; the increase is $\frac{1}{2}m100^2$ J. If the rocket instead already has a velocity of 1000 m/s, and we perform exactly the same burn – same amount of fuel, same exhaust velocity, same burn time and same increase in velocity of 100 m/s (so that the new velocity is 1100 m/s) – the increase in kinetic energy is now $\Delta K_e = \frac{1}{2}m1100^2 - \frac{1}{2}m1000^2 = \frac{1}{2}m(2.1 \times 10^5)$! The increase in kinetic energy is *21 times greater*, and the only difference was the initial velocity. Very non-intuitive!

There is one fairly intuitive way to think about this, though. Work is force times distance; consider the thrust as the force. As the rocket starts out, it is standing still, so the thrust does zero work to begin with. The faster the rocket moves, the greater the distance moved per unit time is (obviously, since that's the definition of speed), and so the amount of useful work is greater at higher speeds.

Chapter 10: Week 8: Exam review only

I didn't take any additional notes this week.

Chapter 11: Week 9

11.1 Lecture 19: Rotating rigid bodies, inertia and axis theorems

This week is mostly if not exclusively about rotation and related concepts such as rotational energy, moments of inertia, angular momentum, torques, etc.
To get started, we begin by finding some equations for rotational motion, very similar to the kinematics equations for linear motion that we first saw in week one of the course.

Say an object is moving along a circular path, at some angular velocity ω. It has a tangential speed v, which always points tangent to its position on the circle. So far, nothing has changed compared to uniform circular motion.

However, we can now allow v to change in magnitude. Previously, only the direction of the velocity vector $\vec{v}$ changed, in order to stay along the circle. The tangential speed v was always the the same in uniform circular motion. That is what we now change.

$v = \omega R$, as we have seen before. $\omega = \dot{\theta}$ (that is, the first time derivative of theta), so $v = \dot{\theta} R$ also.
If we take the time derivative of the angular velocity ω, we find the angular acceleration, $\alpha = \dot{\omega} = \ddot{\theta}$. We use the symbol lowercase alpha for angular acceleration, and the units are rad/s^2.

There are now two accelerations that this object experiences. One is the radial acceleration, that is, the inwards (centripetal) acceleration a_c required for it to change direction so that the motion is circular. There is also the tangential acceleration α, which changes the angular velocity of the object as it moves along the circle.
Note that the two have different units, however. The centripetal acceleration will, in MKS units, be in m/s^2, while the angular acceleration is in rad/s^2. In order for them to have the same set of units, we need to convert the angular acceleration tangential acceleration via $a_{tan} = \alpha R$. Only then can we add the two to find the net acceleration vector.

By making some very simple substitutions, we can use the same equations we used previously. We replace x with θ, v with ω and a with α, and that's it! These equations can be derived the same way as the kinematics equations, by assuming a constant angular acceleration α and integrating that with respect to time.

For the angular velocity, we find

$$\omega = \int \alpha \, dt = \omega_0 + \alpha t \tag{11.1}$$

ω_0 appears from the constant of integration. We can then integrate this to find the angle as a function of time:

$$\theta = \int (\omega_0 + \alpha t) \, dt = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{11.2}$$

Again we have a constant of integration, which is the initial angle θ_0.

We can then use these equations in all cases where there is a *constant* angular acceleration. In other cases, integrals are the way to go.

The direction of angular velocity (and acceleration) is found by using the right-hand rule; see the part on vector mathematics for more information. In short, you can curl the fingers of your *right* hand (the left hand will give the opposite answer! The convention to use the right hand for consistency) along the

rotation, and your thumb will point along the vector's direction, perpendicular to the actual rotational movement.
Alternatively, you can use the professor's preferred version, the right-hand corkscrew rule. Imagine turning a corkscrew clockwise: it goes into the screen. Turn it counterclockwise, and it goes out of the screen.

For accelerations, beware that you need to curl your fingers along the acceleration, not along the current rotation! The two are the same if the rotation speed is increasing, but opposite if it is decreasing!
For example, if the current rotation is counterclockwise at 100 rad/s, the angular velocity vector points "out of the screen" using the right hand rule.
If the rotation is speeding up, the acceleration vector is also in this direction.
However, if it is slowing down, so that it will eventually come to a halt and reverse, the acceleration is in the opposite direction. We curl our fingers opposite the motion, so clockwise in this case, which means you need to turn your hand (rather awkwardly) to curl your fingers, and your thumb then points inwards. In this case, the right-hand corkscrew rule is certainly easier.

11.1.1 Moment of inertia and rotational kinetic energy

Let's now calculate the kinetic energy stored in rotating objects. First, let's limit ourselves to a simple disk, rotating along a perpendicular axis.

The disk has a mass m, and a radius R, and rotates with angular velocity ω (that may or may not be constant).

In order to find the rotational kinetic energy, we add up the kinetic energy of each tiny portion of the disk. Say we divide it into tiny pieces, each with a mass m_i, at a distance r_i from the center of the disk. It is clear that the elements very near the edge of the disk move at a high velocity, while ones near the center barely move at all, making tiny tiny circles.

The kinetic energy of one such mass piece is simply $K_i = \frac{1}{2}m_i v_i^2$, where we can find the velocity as $v_i = \omega r_i$ – something that always holds for circular motion. Because of this relationship, we can re-write the kinetic energy in turns of ωr_i instead of v_i, and find

$$K_i = \frac{1}{2}m_i\omega^2 r_i^2 \tag{11.3}$$

This is a useful change, since v_i depends on the location of the element, as noted above. ω is a constant for the disk, however, so we now have the kinetic energy in terms of our elements m_i and their distances from the center r_i only.
The total kinetic energy is then the sum

$$K = \sum_i \frac{1}{2}m_i\omega^2 r_i^2 = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \tag{11.4}$$

We can factor out the $\frac{1}{2}\omega^2$, since it is the same for all elements. The sum we have above is known as the *moment of inertia*, I (not to be confused with impulse, which is unrelated).

$$I_C = \sum_i m_i r_i^2 \tag{11.5}$$

This is the moment of inertia about the center of the disk, which is why there is a C above; more on that in a second. However, now that we have a name for this sum, we can write the kinetic energy in a form very similar to the one we already know:

$$K = \frac{1}{2} I_C \omega^2 \tag{11.6}$$

ω takes the place of v, as we mentioned earlier regarding the kinematics equations, but note that I_C takes the place of the mass m. The mass (inertial mass) of an object is a measure of its inertia, that is, how hard it is to accelerate it. The greater the mass, the greater the force required for a certain acceleration. The same thing can be said about the moment of inertia, in the case of rotational motion. The higher the moment of inertia, the harder it is to change the angular velocity of an object about an axis of rotation, so the *torque* required is higher. (Torque is introduced later this week, but in short, it is sort-of the amount of twisting a force produces.)

We now know how to calculate kinetic energy, given we know the moment of inertia. The professor recommends looking those up in tables in books, rather than memorizing, since they depend not only on the shape of the object, but also on about which axis you rotate it, and whether that axis is centered or not.

Let's try to calculate that of the disk, though. Say we rotate it as mentioned, about the perpendicular axis, through its center (i.e. in the most obvious way there is). Also, let's actually model it as a cylinder, since the height may matter, so that we get a more general result. The height is h, radius R. The volume is then $\pi R^2 h$, and the density $\rho = \frac{M}{\pi R^2 h}$, assuming uniform density.

The derivation is fairly long if we don't skip any steps, so to be clear, I will spell many of them out. We begin with the definition of the moment of inertia, and take the limit to get an integral, via the definition of the integral:

$$I_C = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r_i^2 \, dm \tag{11.7}$$

dm is given by ρdV if ρ is the density, and dV a small volume element. If the density is uniform, we can find ρ from the total mass, divided by the volume:

$$\rho = \frac{M}{\pi R^2 h} \tag{11.8}$$

Meanwhile, dV can be written in terms of dr. We can find the volume of a cylinder by integrating infinitesimally thin cylindrical shells. They then have a thickness dr, and circumference $2\pi r$. $V = h \int 2\pi r \, dr$, so $dV = 2\pi r h \, dr$. We finally have all the parts, so we put them together, integrate and simplify:

$$I_C = \int r^2 \, dm = \int r^2 \rho \, dV = \int r^2 \rho (2\pi r h \, dr) = 2\pi \rho h \int r^3 \, dr \tag{11.9}$$

Make the substitution for ρ:

$$I_C = 2\pi \left(\frac{M}{\pi R^2 h} \right) h \int_0^R r^3 \, dr = \frac{2M}{R^2} \left(\frac{R^4}{4} \right) = \frac{MR^2}{2} \tag{11.10}$$

So in the end, we find the moment of inertia of a cylinder (or disk) with uniform mass density, rotating around its center on an axis perpendicular to the radius, is

$$I_C = \frac{1}{2} M R^2 \tag{11.11}$$

Never forget that this result is only valid for the conditions above, though! The moment of inertia for other shapes, or even the same shape but different axes or off-center rotation are all different, as we'll see rather soon. I'm starting to see why the professor didn't derive any examples in class!

What about for a sphere, again of uniform density, rotating around an axis through its center?
This derivation may seem very simple, but if you start from the definition for the moment of inertia of a point mass, it's actually a rather ugly triple integral. The reason is that the r_i in $\int r_i^2\,dm$ is not the distance from the sphere's center, but the distance from the *axis of rotation*. Consider a point near the "north pole" of the sphere. It is R from the center of the sphere, but much closer to the axis of rotation, so a simple integral doesn't give us the correct answer.

We can, however, derive it in terms of infinitely thin disks, now that we know the above result. We stack an infinite number of such disks, where the top disk has approximately 0 radius, and they grow up to R, and then go back down to 0 near the opposite pole again. The radius of each disk, call it x (since r could be confusing, see above), can be found using the Pythagorean theorem. I find it a bit difficult to visualize, but I did draw it out and found the relationship $z^2 + x^2 = R^2$, where z is the height above the sphere's center. That gives us $x^2 = R^2 - z^2$.

We then use a coordinate system centered on the sphere, and integrate from $z = -R$ to $z = +R$.

$$I_C = \int_{-R}^{R} \frac{1}{2} x^2\,dm \tag{11.12}$$

For a disk, $dm = \pi x^2 \rho dz$, where dz is the height of the disk. (The total height of the sphere is then $z = 2R$.)

$$I_C = \int_{-R}^{R} \frac{1}{2} x^2 (\pi x^2 \rho\,dz) = \frac{\pi\rho}{2} \int_{-R}^{R} x^4\,dz \tag{11.13}$$

Finally, using the relationship for x^2 above – since $x^4 = (x^2)^2$ – and integrating from 0 to R to simplify (the problem is symmetric, so this doesn't change the answer if we multiply it by 2 also)

$$I_C = \frac{\pi\rho}{2} \int_{-R}^{R} (R^2 - z^2)^2\,dz = \pi\rho \int_{0}^{R} (R^2 - z^2)^2\,dz \tag{11.14}$$

We substitute in $\rho = \dfrac{M}{(4/3)\pi R^3}$, which contains a divided by π that cancels in front of the integral:

$$I_C = \frac{M}{(4/3)R^3} \int_{0}^{R} (R^2 - z^2)^2\,dz = \frac{M}{(4/3)R^3} \int_{0}^{R} \left(R^4 - 2R^2 z^2 + z^4\right) dz \tag{11.15}$$

The integral equals $\dfrac{8R^5}{15}$, so

$$I_C = \frac{M}{(4/3)R^3} \left(\frac{8R^5}{15}\right) = \frac{24MR^2}{60} = \frac{2}{5}MR^2 \tag{11.16}$$

is the moment of inertia for a solid sphere of uniform density.

11.1.2 Parallel axis theorem

Note that the moments of inertia we've found so far are only valid along exactly one axis. That axis must always be exactly through the center of mass of the object. There are two useful theorems that we can use to find the moment of inertia about other axes.

First out is the parallel axis theorem, which we can use the find the moment of inertia for an off-center axis, that is parallel to the original one (thus the name!).
Unfortunately, the lecture video refuses to play properly (on the 8.01x site and on YouTube as well), so I can't grab a screenshot.

Imagine the disk rotating as before, around an axis through its center of mass. We move this axis a distance d from the disk's center, so that the disk is now wobbling back and forth as it rotates. The new moment inertia for this off-center axis is

$$I = I_C + Md^2 \tag{11.17}$$

where M is the total mass of the disk.
This theorem is not limited to disks, however, but works for a mass distribution of any shape, which makes it very powerful.

11.1.3 Perpendicular axis theorem

In the case where we have a very thin mass distribution, i.e. a practically 2-dimensional object, we can also use a second theorem: the *perpendicular axis theorem.*

Say we have three axes, x, y and z, each perpendicular to each other, going through a common point of the object. We can then relate the moments of inertia of rotations along these three axes.
We define the z axis to be perpendicular to the object's area (since it is 2-dimensional), while the x and y axes are in the plane of the object. It then holds that the moment of inertia of rotation along the z axis is the sum of the moments of inertia for the x and y axes:

$$I_z = I_x + I_y \tag{11.18}$$

This can then be used in a few different cases, depending on what you know and what you want to know.

11.1.4 Flywheels

We already know that rotating objects have kinetic energy. Unlike the case of linear motion, however, it is often fairly simple to "store" energy in a rotating object. Linear motion would clearly mean that the object needs to move, while a rotating object can remain in one place and still store vast amounts of kinetic energy.

A rotating disk or wheel that is used to store energy is known as a *flywheel.* The idea is that we can store energy and the use it up later. We will now look at one of many cases where a flywheel can be used: to store energy that is otherwise wasted as heat in the brakes of cars.

Say we are driving through the mountains, on a dangerous, narrow road, so that we must keep our speed low in order to not lose control. The car starts out 500 meters above a valley, that it is driving into (and later out of, back to another 500 meter high peak). The mass of the car is 1000 kg.

If we say the car's speed must not exceed 4 m/s (14 km/h), the kinetic energy of the car is roughly 8 kJ (or less).
The speed will of course increase by itself by driving downhill, so the driver constantly applies the brakes, which simply turn the kinetic energy into heat – wasting it, in other words, in addition to causing wear on the brakes.

By the work-energy theorem, the total increase in kinetic energy, almost all of which is ultimately wasted as heat, comes from the change in gravitational potential energy mgh. For the numbers given, $mgh = 5 \times 10^6$ J, or 625 times the car's maximum allowed kinetic energy, due the speed limit we set.

Now consider what would happen if we used that energy to start rotating a flywheel instead. The flywheel can use magnetic bearings and be mounted in a vacuum, so that the amount of friction practically goes to zero, so that almost no energy is lost, at least not over reasonably short periods of time.

Say we give the wheel a radius of $R = 0.5$ m, and a mass $M = 200$ kg – that gives it a moment of inertia of $I = \frac{1}{2}MR^2 = 25\,\text{kg m}^2$.
We then want the disk to store as much as possible of the 5 MJ of gravitational potential energy we could use up. We can set that equal to the kinetic energy of the disk, and find out what the angular velocity needs to be:

$$\frac{1}{2}I\omega^2 = mgh \tag{11.19}$$

$$\omega = \sqrt{\frac{2mgh}{I}} \tag{11.20}$$

For these numbers, $\omega \approx 632$ rad/s, which is about 100 revolutions per second, or very close to 6000 rpm.

Volvo announced such a system in 2013, with a 6 kg carbon fiber disc, with a 20 cm radius. It can spin at up to 60 000 rpm, however, so let's have a quick look at the maximum energy storage, considering the much smaller dimensions and lower mass. Kinetic energy goes with ω^2 so I would not be surprised if the net result was still similar to the above.

$$\frac{1}{2}\left(\frac{1}{2}MR^2\right)\left((1000\,\text{Hz})(2\pi\,\text{rad})\right)^2 = 2.37 \times 10^6\,\text{J} \tag{11.21}$$

It turns out that their system stores about half the amount of energy, though in a flywheel that is 3% the mass, and less than half the radius.

This type of braking is useful for driving on flat ground too, of course, only that the initial energy source will likely have to be the car's engine in that case. You should still be able to store energy in a flywheel when braking, and extract it at a later time, and perhaps use it to power an electric engine.

Designing such a system is certainly not an easy task, but it can be done, and has been demonstrated. There are other issues than simply finding an efficient way of extracting and storing the energy, though, including one we might understand better at the end of this week, or next week, regarding how the rotating wheel will very strongly resist changes in its motion.

"A car has a flywheel (a disk of radius $R = 0.2$ m and uniformly distributed mass $M = 100$ kg) that can convert 25% of the rotational kinetic energy into translational kinetic energy. The mass of the car is 1000 kg (including flywheel). Suppose the car is at rest, and the flywheel has an angular speed of 200 rad/sec. After all the rotational energy is converted to kinetic energy of the car, what is the speed of the car? Ignore air resistance."

Okay, so we can start out by finding the amount of stored kinetic energy. The moment of inertia is $2\,\text{kg m}^2$, and with $\omega = 200$ rad/s, that gives us 40 kJ worth of energy. Only 25% gets converted to kinetic energy, so that leaves 10 kJ. The speed for a given kinetic energy can be calculated easily:

$$\frac{1}{2}mv^2 = K \tag{11.22}$$

$$v = \sqrt{\frac{2K}{m}} \tag{11.23}$$

For $K = 10$ kJ and $m = 1000$ kg, $v \approx 4.47$ m/s. Not a great deal, but then again the amount of stored energy was fairly low, and could have been 50 times greater, in which case it would give a final speed of 31.5 m/s = 113 km/h. Not bad.

11.1.5 Rotational kinetic energy in celestial bodies

As we all know, the Earth rotates about its axis, with a period of one day. The Sun also rotates about its axis, with a period of about 26 (Earth) days. Because of their vast masses, they store huge amounts of rotational kinetic energy.
The moment of inertia for the Earth is about 1×10^{38} kg m^2, which translates into a rotational kinetic energy of about 2.5×10^{29} J.
As for the Sun, the moment of inertia is about 4×10^{47} kg m^2, and the rotational kinetic energy then is about 1.5×10^{36} J.

These are rather crude approximations, based on a uniform mass distribution. In both cases, the density is higher at the center, so this is not really the case.

Is it possible that the energy we receive from the Sun is little more than rotational kinetic energy that is converted into light? No, because the Sun's power output is about 4×10^{26} W, which means it would run out of rotational energy in a little less than 120 years, assuming the energy output is roughly constant. We know that the Sun's rotation does not slow down anywhere near as fast as would be required.
We know now, of course, that nuclear fusion is the source of the energy the Sun outputs, but the concept of nuclear fusion is fairly new, at about 80 years. Prior to that, other explanations were needed. At some point, this may have been one.

Can we use this process here on the Earth, though? Slow the Earth's rotation, and use the energy we could extract from that?
Well, the first big question is of course *how* we would do that... but let's put that craziness aside (this isn't a serious idea!), and look purely at the energy considerations. Will it be enough? We obviously can't slow the rotation too much, or the lengths of day and night would shift too much.

The world energy consumption is on the order of 5×10^{20} joules per year. At that usage rate, we could extract rotational energy for 500 million years before the Earth stopped rotating.

If we instead extracted enough energy for it to last for one year, how much would the rotation slow down? Well, let's see. The rotational kinetic energy goes down by 5×10^{20} J, so

$$\frac{1}{2}I\omega_{before}^2 - \frac{1}{2}I\omega_{after}^2 = 5 \times 10^{20}\ \mathrm{J} \tag{11.24}$$

One day would become 86400.0000817 seconds instead of the 86400 exactly I assumed in the calculation, so a day would become about 82 microseconds longer. I think we could deal with that – but as mentioned, there's no feasible way to put this into practice. Let's move on to something more realistic.

Another spinning celestial object is the Crab pulsar, located in the Crab nebula, named after its distinct shape (the pulsar is then named after the nebula). The pulsar was created in a supernova, that was first observed here on Earth in the year 1054. (The nebula was also created due to this supernova.)
The next lecture talks about this in more detail.

The Crab pulsar spins at a rate of about 30.2 Hz (compared to the approximately 10^{-5} Hz of the Earth, so about 2.6 million times faster) and has a tiny radius of about 10 to 15 km. The *mass*, however, is slightly greater than that of our Sun, so the density is just mind-bogglingly large, about 10^{14} grams per cubic centimeter (or, in more silly units, on the order of 10^{12} kilograms per teaspoon).

The moment of inertia is about the same as the Earth's – with such a small radius, the huge mass can't quite make up for the tiny radius, since $I \propto MR^2$.
The rotational kinetic energy, on the other hand, is off the charts. The rotational kinetic energy is

proportional to ω^2, and the pulsar spins at > 30 Hz, i.e. with a period of about 33 milliseconds. That gives it a rotational kinetic energy of more than 10^{42} J, a million times that of our Sun.

Unlike the Sun, the Crab pulsar *does* give off light energy that ultimately comes from its rotational kinetic energy. Much of it is in X-rays and gamma rays (and other frequencies/wavelengths). If we add that output power up, we find a number that is about 6×10^{31} W – about 150 000 times more than the Sun.

Because it is a pulsar, it by definition "blinks" (pulses) light at us, once every 33 milliseconds. We can measure its rotation extremely accurately by timing these pulses. When the lecture was recorded, in 1999, its period was $T = 0.0335028583$ s. This goes up with time, with a few hundred nanoseconds per day.

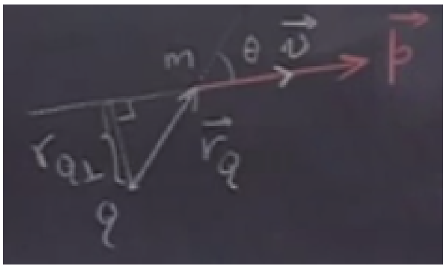
This means it is slowing down, and therefore losing rotational energy. The loss of rotational energy, as measured by the change in T and therefore ω, can be calculated to happen at a rate of 6×10^{31} W. That is exactly the number we found for its total energy output, two paragraphs up! For this reason (hopefully calculated in a more rigorous way than here!), we can say that the source of its power output is its rotational kinetic energy.

As a side note, not all pulsars are powered by rotational kinetic energy. Other types include accretion-powered pulsars, powered by the gravitational potential energy of matter that is accreted (matter that spirals in because of the gravitational attraction), and *magnetars*, powered by extremely strong magnetic fields that lose energy over time.

11.2 Lecture 20: Angular momentum

Say we have an object with mass m, moving at some velocity $\vec{v}$. It is clear that it has a momentum $\vec{p} = m\vec{v}$, which is valid for all points of origin in a given reference frame; the magnitude (and direction) can differ for different reference frames, however.

We can find the angular momentum of this object relative to any point of our choosing. The professor calls some point Q, and draws a position vector $\vec{r_Q}$ from Q to the moving object.



The definition is then that the angular momentum relative to the point Q is

$$\vec{L_Q} = \vec{r_Q} \times \vec{p} = (\vec{r_Q} \times \vec{v})m \tag{11.25}$$

The direction can be found via the right-hand rule (into the blackboard in this case), and the magnitude is

$$|L_Q| = mvr_Q \sin\theta \tag{11.26}$$

where, in shorthand notation, $r_Q \sin\theta = r_{\perp Q}$ may also be used. θ is, as usual in these cases, the smallest angle between the two vectors. This magnitude follows directly from the definition of the cross product, $\vec{a} \times \vec{b} = ab\sin\theta$. Also, $m(\vec{a} \times \vec{b}) = mab\sin\theta$; nothing strange there.

Things do start to get strange now, however. We can consider the angular momentum as seen from a different point, call it C, that is located anywhere along the line of the velocity vector (i.e. a point where the object has been, or will be, assuming the direction of the velocity does not change).

Here, because $\vec{r_C} \times \vec{v} = \vec{0}$, since the two are parallel and cross products have a $\sin\theta$ term in them, the angular momentum relative to C is zero.
If we instead choose a point D above this line, the angular momentum relative to this new point D even has the opposite direction as the angular momentum relative to point Q.

This is then clearly a major difference between angular momentum and "regular" momentum (linear momentum, or translational momentum; both names are used).
Linear momentum has a certain value that is fixed for a certain reference frame. (We can still find reference frames where it is zero, or even opposite in direction, but that is a different discussion.)
Angular momentum, however, depends not only on your reference frame, but also on your point of origin. If you consider the origin to be at yourself, and look at a moving object, its angular momentum depends on where you stand, not only on your reference frame.

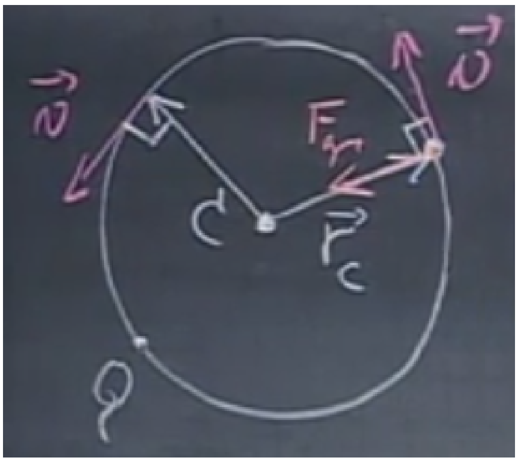
Consider the case of an object moving along a parabola (or a similar shape – with or without air drag). It starts out at a point C, and first moves up, and then falls back down, all the while it moves at constant velocity along the x axis.

At time $t = 0$, the object is located at point C, so the angular momentum at time $t = 0$ is clearly 0: $\vec{r_C} = 0$ makes the cross product zero.
At any later time, however, there is a nonzero position vector, and a velocity vector that is not parallel to the position vector. Therefore, angular momentum is constantly changing. This does make sense, since the velocity vector is constantly changing.

There are, however, cases where a constantly changing velocity does not imply that angular momentum is changing.

Consider the Earth, going around the Sun, in a circular orbit. (The true orbit is elliptical, but we have not introduced such orbits yet.)

We define the point C to be at the center of the orbit, i.e. at the center of the Sun. We then have the position vector $\vec{r_C}$ to the Earth, which itself has a velocity vector $\vec{v}$, which is always changing, to be tangential to the orbit. The orbital speed v is constant, however.



The direction of the angular momentum relative to point C, L_C, is easy to find via the right hand rule; it is out of the page. The magnitude is found as

$$|L_C| = mvr_C \sin\theta \tag{11.27}$$

However, the angle θ between the position vector and the Earth's velocity vector is always 90 degrees. Therefore, the sine of that angle is always 1, and $|L_C| = mvr_C$.
A while later in time, the exact same thing still applies. The velocity vector has changed direction, but the direction of the cross product remains constant. θ remains 90 degrees, and so the magnitude remains. The angular momentum, relative to point C, is constant.

What about relative to some other point Q, which is on the Earth's orbital circle? It is clearly changing with time. When the Earth is *at* that point, it must be zero, since $r_Q = 0$. When the Earth is *not* at that point, it must be nonzero, as $\vec{r_Q} \times \vec{v} \neq 0$.

In other words, angular momentum is *conserved* relative to point C, but is *not* conserved relative to any other point!

11.2.1 Torque

Let's now have a look at torque. If angular momentum is the rotational analogue of linear momentum, torque is the rotational analogue of force.

We can write down an expression for the angular momentum relative to some point Q (which can be any point whatsoever), and then take the time derivative of that expression. We will need to use the product rule, though that is not a particularly difficult task:

$$\vec{L_Q} = \vec{r_Q} \times \vec{p} \tag{11.28}$$

$$\frac{dL_Q}{dt} = \frac{d\vec{r_Q}}{dt} \times \vec{p} + \vec{r_Q} \times \frac{d\vec{p}}{dt} \tag{11.29}$$

The first of the two terms is the cross product between $\vec{v_Q}$ and the momentum vector of the Earth, but because that velocity and the momentum vector are always parallel ($\vec{p} = m\vec{v}$), that term is zero. What remains is a term that is the position vector cross the net force on the Earth, since $\vec{F} = \frac{d\vec{p}}{dt}$.

The quantity $\frac{d\vec{L_Q}}{dt}$ is known as the *torque* relative to point Q. We use the symbol τ (Greek letter tau) for torque:

$$\vec{\tau_Q} = \frac{d\vec{L_Q}}{dt} = \vec{r_Q} \times \vec{F} \tag{11.30}$$

Torque is also known as moment or moment of force, and may also be translated to something along the lines of "turning moment" in some other languages. M or N are other symbols used for torque, especially when it is called moment.
Note that torque, exactly as with angular momentum, is also relative to a point! We cannot, in general, talk about "the" torque on an object, without specifying our point of origin. Therefore, I used $\vec{\tau_Q}$ above, to show that we are talking about the torque relative to that same point Q.

The torque is the amount of "twisting" a force provides. Consider a nut and a wrench. The further out you grip the wrench, the easier it is to loosen a nut/bolt. That is because you are increasing the position vector $\vec{r_C}$, and the torque is proportional to this length. Needless to say, if you increase the amount of force you exert, the torque also increases.

The torque is also proportional to $\sin\theta$ (because of that term in the cross product $\vec{r} \times F$) which is quite intuitive. Only the force that is perpendicular to the wrench causes any turning of the nut. If the force is entirely parallel, you are just pushing or pulling on it the nut/bolt, and it will certainly not turn because of that force. For angles in between the extremes of 0 and 90 degrees, the closer you are to a perpendicular angle, the stronger the torque is.

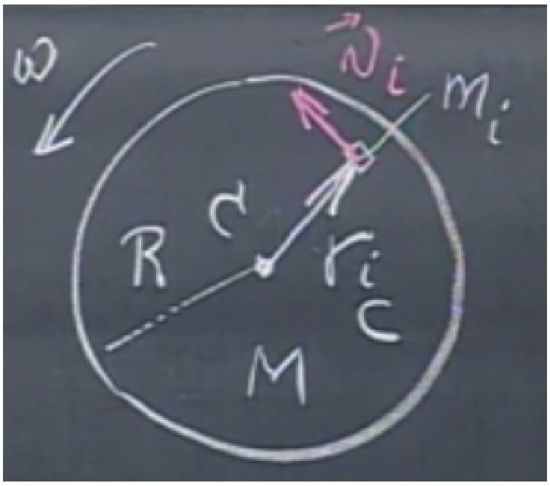
If there is a net torque on an object (relative to a point), angular momentum (relative to that same point) must change. With zero net torque, angular momentum is conserved.
There is a clear parallel to conservation of linear momentum here: if there is a net force on an object, its momentum must change. With no net force, momentum is conserved.

Now, consider the case of the Earth orbiting the Sun again. The force vector is always inwards, and it is therefore always anti-parallel to $\vec{r_C}$. The cross product $\vec{\tau_C} = \vec{r_C} \times \vec{F}$ is always zero! This is the same result as we found earlier, but now we know why that must be.

Of course, if we calculate the torque at point Q, or any other point for that matter, the torque will not be zero, and will also not be constant. We will come back to what is so special about this center point C.

11.2.2 Spin angular momentum

So far, we have only really talked about the angular momentum of a point mass, moving through space (even if one such "point mass" was the Earth!). We will now consider objects of nonzero size, that rotate around their center of mass. For example, a rotating disk. It has a radius R, a mass M, and rotates about point C which is its center of mass.



We can now, as we did for the moment of inertia among other things, split the disk up into small mass elements m_i. Each such element moves with velocity v_i. However, as before, we want to express this velocity in terms of ω, since the velocity is a function of the distance to the center, whereas ω is not. We can use $v_i = \omega r_{iC}$; clearly we need to have that radius in there, anyhow.

The magnitude of the angular momentum for this tiny mass element alone is $L_{Ci} = (r_i\vec{C} \times \vec{v_i})m_i = m_i r_{iC} v_i = m_i r_{iC}^2 \omega$. There is no $\sin\theta$ term since θ is always 90 degrees, and so $\sin\theta = 1$.
The direction can be found by the right-hand rule as usual, and is out of the blackboard (or page) in this case.

Now, in order to find the total angular momentum, we must sum up the angular momenta of all these tiny mass elements:

$$L_{disk_C} = \sum_i m_i r_{iC}^2 \omega = \omega \sum_i m_i r_{iC}^2 = I_C \omega \tag{11.31}$$

If we factor out ω of the sum, since it is common to all elements, all that remains in the sum is $\sum_i m_i r_{iC}^2$, which as we have seen previously is just the moment of inertia for the disk. Therefore, we can find the angular momentum of the disk relative to point C as $I_C\omega$.

What is now remarkable is that this value is the angular momentum relative to *all* points anywhere in space, not just relative to point C. This is true because of, and only in the cases where, this is a rotation around the center of mass.

$I_C\omega$ is referred to as the spin angular momentum, and is an intrinsic property of an object, regardless the origin you choose.
It is valid for objects of all shape, not only disks, as long as the rotation is about the center of mass. (Everything I find about the term "spin angular momentum" is about quantum mechanics, however. I'm unsure whether this is a common terminology or not, but it appears not.)

$I_C\omega$ for the disk then gives us $L_{disk_C} = \frac{\omega}{2}MR^2$.

This concept is of course very handy. In the case of an object spinning around its center of mass, we can now talk about *the* angular momentum of that disk, without having to specify any point of origin.

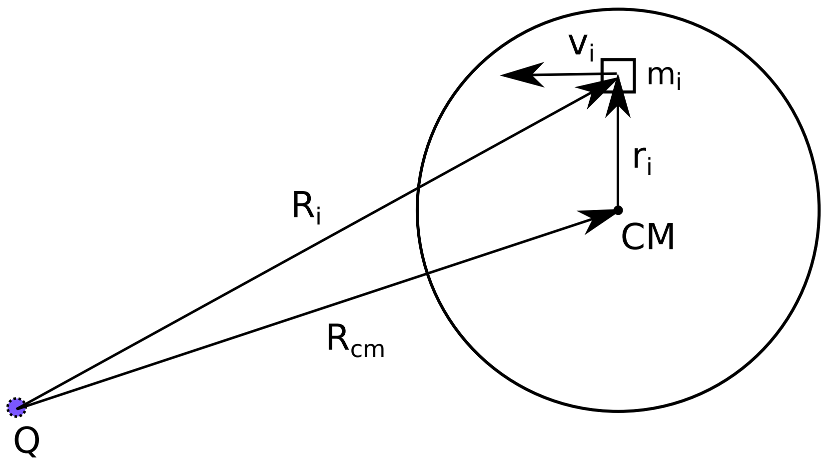
11.2.3 Derivation/proof of spin angular momentum

Let's prove that we can indeed don't need to measure spin angular momentum relative to some point, by calculating the angular momentum of an object spinning about its center of mass (where the center of mass is stationary, relative to the point). I call this point Q. In the end, everything relating to this point will have turned out to be zero.

First, let's make some definitions; this isn't as bad as it looks.

$\vec{R_{cm}}$ is the position vector from point Q to the center of mass (the center of the rotation)
$\vec{R_i}$ is the position vector from point Q to each mass element m_i
$\vec{V_i}$ is the velocity vector of each mass element m_i seen from point Q
$\vec{r_i}$ is the position vector from the center of mass to each mass element m_i
$\vec{v_i}$ is the velocity vector from the center of mass to each mass element m_i

So capital letters are vectors from point Q, and lowercase are from the center of mass/relative to the mass elements themselves. Keep that in mind and this shouldn't be that hard to read.



Via vector addition, we have

$$\vec{R_i} = \vec{R_{cm}} + \vec{r_i} \tag{11.32}$$

By taking the time derivative of the above equation, we find $\vec{V_i} = \vec{V_{cm}} + \vec{v_i}$.
The definition of angular momentum relative to point Q is the sum of the angular momenta of each tiny mass element as seen from point Q, which is found as $\vec{R_i} \times \vec{P_i}$.

$$L_Q = \sum_i \vec{R_i} \times \vec{V_i} m_i = \sum_i (\vec{R_{cm}} + \vec{r_i}) \times (\vec{V_{cm}} + \vec{v_i}) m_i \tag{11.33}$$

The entire point of this proof is that we have pure rotation about the center of mass, so $V_{cm} = 0$. Expanding the sum out and removing everything multiplied by V_{cm}, we find

$$L_Q = \sum_i \vec{R_{cm}} \times \vec{v_i} m_i + \sum_i \vec{r_i} \times \vec{v_i} m_i \tag{11.34}$$

$\vec{R_{cm}}$ doesn't change during the summation, since it points to the center of mass, not to each mass element (not to mention $V_{cm} = 0$, so it's a constant):

$$L_Q = \vec{R_{cm}} \times \sum_i \vec{v_i} m_i + \sum_i \vec{r_i} \times \vec{v_i} m_i \tag{11.35}$$

We can now see that the first term in zero: the sum of the momenta $\vec{v_i} m_i$ is zero in the center of mass frame (it is also called the center of momentum frame, or zero momentum frame). The term that remains can be shown to be equivalent to $I_{cm}\omega$, which is really derived in the section prior to this one (for a disk, at least).
For a rigid object seen from the center of mass, $\vec{r_i}$ and $\vec{v_i}$ are always perpendicular. We can get evaluate the cross product (since we only care about magnitude; the direction is certainly the same as the $\vec{\omega}$): $|\vec{r_i} \times \vec{v_i}| = r_i v_i \sin(\pi/2) = r_i v_i$.
Next, we apply $v_i = \omega r_i$:

$$L_Q = \sum_i r_i v_i m_i = \sum_i r_i (r_i \omega) m_i = \omega \sum_i r_i^2 m_i \tag{11.36}$$

The sum is now simply the moment of inertia about the center of mass:

$$L_Q = L_{cm} = \omega \sum_i r_i^2 m_i = I_{cm}\omega \tag{11.37}$$

And we are done! Everything specific to point Q has disappeared, and we end up with this simple result. Since there was nothing specific whatsoever about the point chosen, this holds for all points.

11.2.4 Back to spin angular momentum

The Earth has both a spin angular momentum and an orbital angular momentum due to its motion around the Sun. The spin angular momentum is quite easy to find as $I_C \omega$.
The moment of inertia for a sphere spinning about its center of mass is $\frac{2}{5}MR^2$, derived in the previous lecture. Using $R_{Earth} = 6400$ km and $T = 86400$, which translates into $\omega \approx 7.2722 \times 10^{-5}$ rad/s,

The spin angular momentum of the Earth is then

$$\left(\frac{2}{5}(5.972 \times 10^{24}\,\text{kg})(6400 \times 10^3\,\text{m})^2\right)(7.2722 \times 10^{-5}\,\text{rad/s}) = 7.11 \times 10^{33}\,\text{kg m}^2\ \text{s}^{-1} \tag{11.38}$$

I'm not sure what units to use; the dimension is kg m^2 s^{-1}, which is equivalent to N · m · s and J · s. Presumably one set of units is more common than the others!
... Coming back to this section after having finished this week's lectures, I would guess that the newton-meter-second is the most common unit (in general) for angular momentum. If a torque of 1 Nm acts for 1 second, the angular impulse (the change in angular momentum) is (1 Nm)(1 s) = 1 Nms. In this particular case (above), though, we have the units explicitly in terms of kg, m^2 and rad/s.

Anyhow. The orbital angular momentum around the Sun, relative to the center of the orbit, is

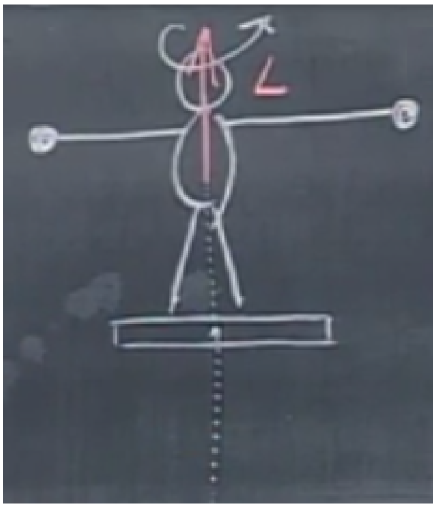
$$L_C = (|\vec{r_C} \times \vec{v}|)m = mr_s \frac{2\pi r_s}{T_{orbit}} = \frac{2\pi m r_s^2}{T_{orbit}} \tag{11.39}$$

where r_s is the radius of the Earth's orbit, about 150 million km, or 1.5×10^{11} m. The mass m is 5.97×10^{24} kg. $T_{orbit} \approx 86400 \times 365$, so This then gives us 2.677×10^{40} kg m^2 s^{-1}.

The ratio of the two, with spin angular momentum on top, is about 2.66×10^{-7}.

11.2.5 Conservation of angular momentum: an experiment

Consider a person, standing on a plate that is free to rotate, like this:



In his hands, he holds two weight of mass $m \approx 1.8$ kg (each). The mass of the person, including the weights plus the turntable itself, is $M \approx 75$ kg. In this counterclockwise rotation, seen from above (clockwise from below, though I find it easier to see this from above), the angular momentum vector will be upwards.

In a case like this, we have a rotation about a center of mass. Angular momentum has a value found simply by $I_C\omega$, and it will be conserved, after the initial push to get the person rotating. When rotating, these weights can be moved either close to his body, or as far out as his arms can reach. Clearly, this does not cause an *external* torque, so angular momentum L_C must be conserved.

However, I_C, the moment of inertia, will change! Since $L = I_C\omega$, and I_C will go down, ω must go up – there is no other way for the conservation to hold.

In a semi-quantitative calculation, we can approximate the professor as a cylinder, with radius $R = 0.2$ m. The cylinder has a mass of 75 kg[1], and so has a moment of inertia of about $\frac{1}{2}MR^2 = 1.5\,\text{kg m}^2$.

Now, if we ignore the weight of the professor's arms, when stretching them out, two additional point masses are added, at arm's length from the axis of rotation. The moment of inertia for each weight is about $mr^2 = 1.5\,\text{kg m}^2$, assuming an arm length of 90 cm.
When his arms are stretched out, we then find the total moment of inertia to be the roughly 1.5 from the professor's body, plus another 3 from the weights! This difference of a factor of 3 (4.5/1.5) when pulling the weight in close causes a difference in ω of a factor of three, so the change in the angular velocity is very apparent!

11.2.6 Conservation of angular momentum (in general, and in stars)

Let's look at conservation of angular momentum in a similar way to how we treated linear momentum.

Say we have a group of objects interacting: stars, interacting gravitationally, point particles of any kind, objects connected together with springs, etc. There can be any kind of internal interactions between these, including collisions, internal friction, explosions/supernovae, and so on.
Because all internal forces will cancel out, there can never be any net internal torque relative to any point Q we choose. In the absence of a net *external* torque, angular momentum will be conserved. In the presence of a net external torque, angular momentum will change according to

$$\frac{d\vec{L_Q}}{dt} = \vec{r_Q} \times \vec{F_{Ext}} = \tau_{Q,\vec{Ext}} \tag{11.40}$$

[1] I think it should be 75 kg minus the 2 times 1.8 kg, but these numbers will clearly not be very accurate either way.

... for the entire system as a whole. The angular moment of any one object *inside* that system is, just as with linear momentum, *not* conserved. Essentially, they can "trade" angular momentum with each other, but the net angular momentum of the entire system cannot change.

Similar to the experiment the professor did, reminiscent of ice skaters, stars can also shrink, and have their moments of inertia go down (since $I \propto R^2$), which means that the star's angular velocity must increase: $L = I\omega$ must be constant in the absence of a net external torque.

In a star, the nuclear fusion going on causes forces that want to expand the star outwards. However, there is also gravity, which does what it can to collapse the star towards its center. In all stars that are actively "burning" fuel, these two forces are balanced out.

What now happens when the fuel runs out, and fusion can't continue on? This won't happen for about 5 billion years for our Sun, but it happens all the time for other stars, considering the amount of stars in the observable universe. There are three possible end products of stars.

The first is that the star becomes a *white dwarf.* They have radii of about 10^4 km, not too far from the radius of the Earth, and a mass of about half our Sun's mass. Our Sun will end up as a white dwarf far in the future.
We can see that the density of such an object must be very high, with half the Sun's mass, but a volume almost comparable with that of the Earth. The density becomes on the order of 1×10^6 g/cm^3.

The second possibility is that the star becomes a *neutron star*, a very interesting type of celestial object. They have radii on the order of 10 kilometers – less than many cities – yet a mass typically in the range 1.4 to 3.2 solar masses! This causes a ridiculous density of some 10^{14} g/cm^3. Neutron stars are so named because they are thought to consist largely of neutrons. In some ways, they are like enormous nuclei.
The Crab pulsar, which we discussed earlier, is a neutron star. In fact, all pulsars are neutron stars. Not all neutron stars are pulsars, however, since they do stop rotating sooner or later, at which point they no longer pulse. Also, pulsars are mostly defined by the fact that they pulse towards the Earth, so the vast majority of such neutron stars may go unnoticed, since their light beams are unlikely to be pointed towards us.

Finally, the third possibility is that the star becomes a *black hole*, an even more bizarre type of celestial object. This can only happen for stars that are more massive than about three solar masses.
Black holes will not be covered in this lecture, but will be in the future.

When a star collapses, due to the lack of outwards pressure, a huge amount of gravitational potential energy is converted to kinetic energy, as the mass of the star falls inward. That energy is ultimately turned into heat and radiation.
In addition, because the moment of inertia is reduced as the mass moves inwards, the rotational period of the star increases, often dramatically. (Especially for neutron stars, see below.)

Our Sun will not become a neutron star, as it is not massive enough, but let's do some calculations on it anyway, just to get a feeling for some numbers. The radius of the Sun is about 700 000 km, while that of a neutron star might be about 10 km.
The mass is about 2×10^{30} kg. In moving all this mass inwards, there is a huge loss in gravitational potential energy, on the order of $\Delta U = 10^{46}$ joules.
This is converted to kinetic energy, and then conserved into heat and radiation, etc.

This energy is on the order of *100 times more* than the Sun releases in its *10 billion year* lifetime – and this explosion releases that energy in a matter of seconds, rather than billions of years! This is what we call a supernova. (There are different types of supernovae; this is one of them.)

"A white dwarf, with a mass of 2.8×10^{30} kg and a radius of about 10 000 km implodes (via gravitational collapse) to becomes a neutron star of the same mass with a radius of about 10 km. The rotational period of the white dwarf was $T_i = 10$ hours. What will the rotational period of the neutron star be? For simplicity, assume that the mass density in both the white dwarf and the neutron star is uniform."

Okay, so let's first convert these numbers a little. The initial radius is $10000 \times 10^3 = 10^7$ m, and the final radius 10^4 m. The initial period is $T_i = 36000$ seconds.

$L = I\omega = I\dfrac{2\pi}{T}$ is conserved, so when I goes down, T must go down by the same factor. $I \propto R^2$, and the change in R is a factor 10^3, so the change in R^2 in a factor of 10^6. The time period becomes $T = 36000 \times 10^{-6} = 0.036$ seconds (27.77 Hz, up from 2.77×10^{-5} Hz!).

11.2.7 More on supernovae, pulsars and neutron stars

The idea behind these notes is not to provide a transcript, and as such, as I do now and then, I recommend simply watching this part of the lecture at this point. There is quite a bit of information, but most of it works better in context in the video, so I see little point in copying it down here almost verbatim!

11.3 Lecture 21: Torque

The lecture begins with a review of the last week. I didn't find anything in particular to take notes of again. After all, being able to review is sort of the point of these notes!

Still, here are a few equations given:

$$\vec{L_Q} = \vec{r_Q} \times \vec{p} \tag{11.41}$$
$$\vec{\tau_Q} = \vec{r_Q} \times \vec{F} \tag{11.42}$$
$$\frac{d\vec{L_Q}}{dt} = \vec{\tau_{Q,Ext}} \tag{11.43}$$

The last equation is often used for a system of objects, thus the "external" qualifier. With no net *external* torque, angular momentum in conserved... relative to that point Q, since that may be the only point with zero torque.

In rotation about some point Q, we can write a variant of Newton's second law, that relates torque with moment of inertia and angular acceleration (rather than relating force with mass and linear acceleration), as well as a relation that relates angular momentum with moment of inertia and angular velocity (rather than relating linear momentum with mass and linear velocity):

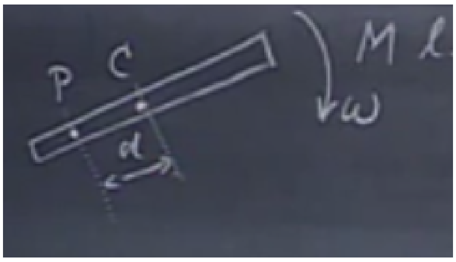
$$|\tau_Q| = I_Q\ \alpha_Q, \text{ where } \alpha = \ddot{\theta},\ \omega_Q = \dot{\theta} \tag{11.44}$$
$$|L_Q| = I_Q\ \omega_Q \tag{11.45}$$

In the special case of a rotation about the center of mass of an object, the angular momentum is instead an intrinsic property of that object, and it *no longer matters* which point Q we choose. In that case

$$|L_{cm}| = I_{cm}\omega_{cm} \tag{11.46}$$

In all other cases, it is crucial to specify the point of origin chosen, since angular momentum depends on not only your reference frame, but also the point of origin.

Let's now look at a case where angular momentum is conserved for exactly one point, but is not conserved for any other.
We have a rod, that we rotate about a point P, which is a distance d from the center of mass C. It has a mass M and a length ℓ, and we rotate it with an angular velocity ω.



The moment of inertia for rotation about the center of axis of a rod of uniform density is $\frac{1}{12}M\ell^2$, so angular momentum relative to point P is

$$|L_P| = I_P\omega = \left(\frac{1}{12}M\ell^2 + Md^2\right)\omega \tag{11.47}$$

where the Md^2 term is added because of the parallel axis theorem.

There will be a force acting on the ruler at point P, as well as one acting on the pin (that it rotates around) *by* the ruler. We can show this by analogy. Consider a massless rod, with two equal masses on each end. We rotate this rod about the center of mass, which is also the center of the rod. There is a centripetal force inwards on each of the masses, of equal magnitude since the masses are equal and the distance from the point of rotation are equal (and the velocities of the masses are then also equal).
However, if we rotate the rod about a point that is closer to one of the masses, then the centripetal force on that mass is smaller, but the centripetal force no the other is larger, since they move in circles of different radius. (As the point of rotations comes closer to one of the masses, it almost doesn't rotate at all, but rather spins about its axis.)

For this reason, there will be a force on the pin (and a reaction force from it) at point P in the ruler. The *torque* relative to point P is zero, however: the force is along the ruler, and the position vector in $\vec{r} \times \vec{F}$ is also along the ruler.

Since $|\vec{\tau_P}| = 0$, angular momentum is conserved in this case. It would not be if we chose any other *stationary* point, however.

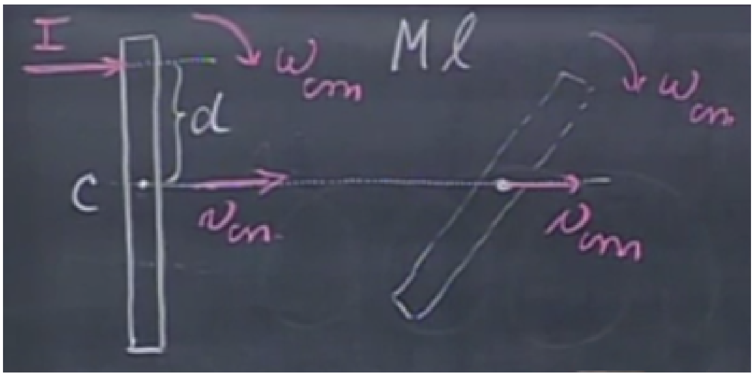
Let's now rotate the same ruler about the center of mass C. The problem is now symmetric, like the case with the two masses mentioned above, and so there is now no net force on the pin/due to the pin. With no net force, $\vec{\tau_Q} = \vec{r_Q} \times \vec{F}$ must be zero for *all* points Q, since $F = 0$. Therefore, in this case, angular momentum is conserved relative to *all* points, as there is no net torque relative to *any* point. The magnitude of the angular momentum for the rotation about the center of mass is then simply $I_C\omega$, which for a rod is

$$|L_{CM}| = \frac{1}{2}M\ell^2\omega \tag{11.48}$$

for all points in space.

11.3.1 Off-center impulse: translation and rotation

Let's now consider a ruler, lying flat on a frictionless table. It has a uniform mass density, and so its center of mass C is located at the geometric center. We give it an impulse I, in other words, a force that acts for a certain amount of time (a very short amount of time in this case, but that is not strictly necessary for this to hold).



What will happen? Clearly, it is going to move towards the right. It will also rotate, assuming you don't aim for the center of mass (i.e. assuming $d \neq 0$).

The object *must* rotate about its center of mass – anything else is impossible. If it were to rotate about a point offset from the center of mass, then the center of mass would have to move in a spiral-ish motion. Due to conservation of (linear) momentum, the center of mass must have a constant velocity (meaning both magnitude and direction are constant!) after the initial push, which is only possible if any rotation is about the center of mass.
(Keep in mind that this is assuming a frictionless table. On a real table, where μ likely differs at different points, the results may also therefore differ: there is then an external force that may vary in unpredictable ways!)

That much is fairly intuitive, in my opinion, but what's interesting that the distance d from the center of mass where the impulse happens does not affect the velocity of the center of mass. For a given amount of momentum given by the impulse, the velocity of the center of mass is constant, regardless of d. I find this nonintuitive, but it is still easy to see mathematically.

$\vec{p} = m\vec{v_{cm}}$ must hold, and since initial momentum is 0, $\vec{p} = \vec{I}$ after the initial push. Nowhere in the equation does d appear – the equation in question is valid for any isolated system, regardless of shape and place where the force is applied, as long as it is rigid.
Since $\vec{p} = \vec{I}$, the above equation also says that

$$\vec{v_{cm}} = \frac{\vec{I}}{m} \tag{11.49}$$

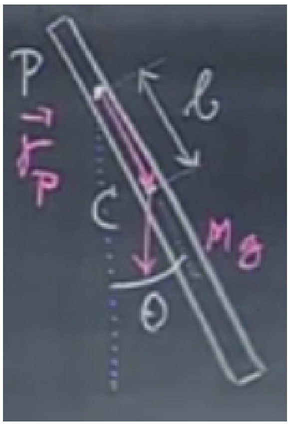
ω, on the other hand, clearly depends on the distance d. If $d = 0$, then $\omega = 0$; that much is clear. It is also clear that ω grows as d grows. The reason is that the amount of torque (relative to the center of mass) provided depends on d and the amount of force provided during the impulse. (The amount of angular impulse, i.e. change in angular momentum, depends on the average torque and the time, $J = \tau\Delta t$, just like how the linear impulse is given by $I = F\Delta t$.)
(ω was calculated in this week's homework: week 9/homework 7, problem 9.)

However, note that if we instead look at the torque relative to a point P on the line of the impulse, i.e. a distance d up from the center of mass, this torque is zero! Zero torque means angular momentum will be conserved, and so angular momentum *relative to point P* is zero not only before, but also after the object starts to rotate. It is therefore conserved, unlike angular momentum relative to the center of mass!

11.3.2 Physical pendulum

Let's have a look at a different ruler. We previously derived an equation describing the motion of a simple pendulum, as a simple harmonic oscillator. We made some unrealistic approximations, however, including a massless string and a point mass hanging from it.
We will now consider a related yet different type of oscillator, called a physical pendulum.

In this case, we have a ruler, though the solution is valid for other shapes as well. We drill a hole through it at point P, and hang it on a small pin. The point P is located a distance b from the center of mass, point C.



The ruler makes an angle θ with the vertical. Using the concept of the center of mass, we can consider gravity acting only at the center of mass; the magnitude is Mg, where M is the total mass of the ruler.

If we choose to use point P as our origin, our lives become much easier. There will be a force at point P, but if we choose it as our origin, we do not have to worry about those forces, since the torque due to those forces will be zero: $\vec{\tau_P} = \vec{0} \times \vec{F}$, since the first term is the position vector from P to P.

There is a torque that matters, relative to point P, however: the torque due to gravity acting on the center of mass. The distance b acts as a lever arm, and the torque is given as the cross product between the distance and the force. The torque can also be found as $I_P\alpha$, as we saw earlier.

$$|\vec{\tau_P}| = \vec{b} \times \vec{F_g} = Mgb\sin\theta = -I_P\alpha \tag{11.50}$$

The torque is always trying to restore things back to equilibrium, which gives us a minus sign, just as how we used $-kx$ for the force when deriving the equation governing a spring oscillator. Note that we have used the Newton's second law equivalent for torques: $F = ma \Rightarrow \tau = I\alpha$.

$\alpha = \ddot{\theta}$ by definition, in a different form of notation. Now, using the small angle approximation that we have used several times earlier, $\sin\theta \approx \theta$. This is fairly valid for small angles; at 5 degrees, the difference is less than 0.15%; at 10 degrees, the difference is about 0.5%.
By using this approximation, and making the substitution for α, we have

$$Mgb\theta = -I_P\ddot{\theta} \tag{11.51}$$

$$I_P\ddot{\theta} + Mgb\theta = 0 \tag{11.52}$$

$$\ddot{\theta} + \frac{Mgb}{I_P}\theta = 0 \tag{11.53}$$

A-ha! This has the exact form of a simple harmonic oscillator! We already know the solutions; the square root of the stuff multiplying θ gives us the angular frequency ω, etc:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi) \tag{11.54}$$

$$\omega = \sqrt{\frac{Mgb}{I_P}} \tag{11.55}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_P}{Mgb}} \tag{11.56}$$

Keep in mind that because $I_P \propto M$, this is in fact independent on the mass, just as with a simple pendulum. We can substitute in the value for I_P, in which case we transform the general result, above, into a result that only holds for a rod.
$I_P = \frac{1}{12}M\ell^2 + Mb^2$ via the parallel axis theorem. Making that substitution,

$$\omega = \sqrt{\frac{gb}{\frac{1}{12}\ell^2 + b^2}} \tag{11.57}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\frac{1}{12}\ell^2 + b^2}{gb}} \tag{11.58}$$

Again, note that these results are now only valid for the case of a rod.

As a side note, we can also write for the angular acceleration $\alpha = \dot{\omega}$. This is a very confusing thing to do, however! The omega in the previous sentence is the *angular velocity*, i.e. how fast the angle is changing with time, analogous to the velocity v for linear motion. This omega is *constantly changing in time*, and has a minimum at θ_{max}, as the pendulum reverses direction, and a maximum at $\theta = 0$.

On the other hand, the ω used in the cosine above, and the only one I have mentioned prior to the paragraph above to avoid confusion, is the *angular frequency*, $\omega = \dfrac{2\pi}{T}$. That ω is a *constant*, and is only related to how many oscillations the pendulum completes per second. This sentence marks the last mention of the angular velocity ω in this section; only the one that represents angular *frequency* will be used from here on.

Let's try to calculate the approximate period, including the uncertainty, of this pendulum when $\ell = 1.00$ m and $b = 0.400 \pm 0.002\,\mathrm{m}$, i.e. the uncertainty is 2 mm or 0.2 cm.

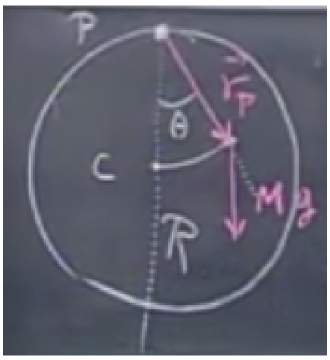
In the ideal case, the number we find by plugging in the numbers is $T = 1.5497\,\mathrm{s}$, using $g = 10\,\mathrm{m/s^2}$.
The largest possible time should be when ℓ and b are both maximized, which gives $T = 1.556$ s. On the other side of things, the smallest possible period is about $T = 1.543$ s.

The uncertainty is about 0.0065 seconds or so; call it 0.01 s.
It turns out the professor used $g = 9.8\,\mathrm{m/s^2}$, which is probably a good idea if you're going to time this in the real world. In either case, he found $T = 1.565$ seconds, which is very close to these numbers.

This is then demonstrated, and the timing indeed works out.

Next, we look at the same type of oscillation, but we use a hula hoop as the pendulum, instead of the ruler. The derivation is almost identical, and except for some variable names, we can still use the general equation we found earlier.

The center of mass of the hoop is at the geometric center, i.e. in the middle of empty space. We again hang it on a pin, at point P, which clearly is at the very top of the hoop.



As before, we consider the force of gravity as acting solely on the center of mass, with a force Mg downwards. Again as before, the position vector $\vec{r_P}$ acts as the lever arm, and the torque relative to point

P is

$$\vec{r_P} \times \vec{F_g} = MgR\sin\theta = -I_P\ddot{\theta} \tag{11.59}$$

As before, the torque must be equal to the negative of $I_P\alpha = I_P\ddot{\theta}$, using Newton's second law for circular motion. If we use the small angle approximation $\sin\theta \approx \theta$ again, and solve for $\ddot{\theta}$:

$$\ddot{\theta} + \frac{MgR}{I_P}\theta = 0 \tag{11.60}$$

Clearly, this is yet another simple harmonic oscillator! The only difference from the one we found for the rod is that we now used R instead of b for the distance to the center of mass; they are identical other than that non-detail.
The moment of inertia about point I_P is the moment inertia about the center of mass I_C, plus MR^2 via the parallel axis theorem. I_C is also MR^2 for a circular object with uniform mass distribution (all mass points are a distance R away (derivation not shown), so $I_P = 2MR^2$. That gives, using the solutions we found previously and using this new I_P,

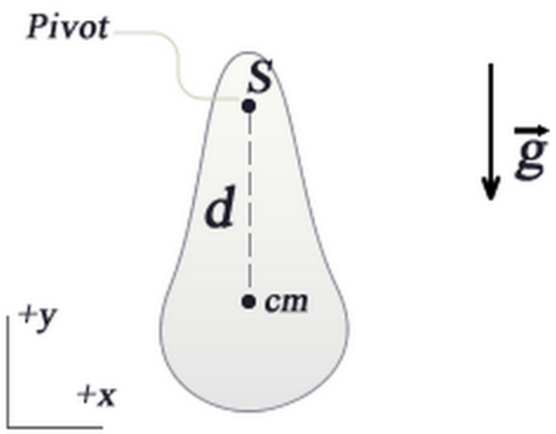
$$\theta(t) = \theta_{max}\cos(\omega t + \phi) \tag{11.61}$$

$$\omega = \sqrt{\frac{g}{2R}} \tag{11.62}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2R}{g}} \tag{11.63}$$

This is the same result as we have found previously for a pendulum with a massless string, if we just call its length $2R$! Quite neat, that they would have the same period.

Time for an interesting lecture question.



"A physical pendulum consists of a body of mass m contained in the xy-plane. The moment of inertia of the object about an axis perpendicular to the plane and passing through the object's center of mass is Icm.

The object oscillates in the xy-plane about the point S a distance d from the center of mass as shown. What is the period of the pendulum for small angle oscillations where $\sin\theta \approx \theta$?"

Since they want the answer in terms of I_{cm} (so that we don't need to actually calculate the moment of inertia) this should be fairly easy. In fact, we just use the general solution with d as the distance between the point and the center of mass,

$$T = 2\pi\sqrt{\frac{I_S}{mgd}} \tag{11.64}$$

I_S can be written as $I_{cm} + md^2$, so

$$T = 2\pi\sqrt{\frac{I_{cm} + md^2}{mgd}} = 2\pi\sqrt{\frac{I_{cm}}{mgd} + \frac{d}{g}} \tag{11.65}$$

There is then a demonstration of the hula hoop, and a simple pendulum (an apple hanging on a lightweight string), to show that their periods are almost synchronized. (Most of the error likely comes from a small difference in length.)

Chapter 12: Week 10

12.1 Lecture 22: Kepler's laws, elliptical orbits, and change of orbits

Let's begin with a quick review of circular orbits, before we move on to the more general (and more realistic) case of elliptical orbits.
In all the following equations, m is the mass of the object that orbits (e.g. a satellite around the Earth, or the Earth itself around the Sun), while M is the object at the center[1] of the orbit. G is the gravitational constant, and R is the (fixed) distance between the two masses. v is the orbital speed (tangential speed), and T the orbital period. With all these variables in place, the following equations hold for circular orbits:

$$T^2 = \frac{4\pi^2 R^3}{GM} \tag{12.1}$$

$$v = \frac{2\pi R}{T} = \sqrt{\frac{MG}{R}} \tag{12.2}$$

$$v_{esc} = \sqrt{2}\ v = \sqrt{\frac{2MG}{R}} \tag{12.3}$$

$$E_{total} = K + U = \frac{1}{2}mv^2 - \frac{mMG}{R} = -\frac{mMG}{2R} = \frac{1}{2}U \tag{12.4}$$

Gravity is the only relevant force, and since gravity is conservative, mechanical energy is conserved. The total mechanical energy is, interestingly enough, always equal to half the gravitational potential energy, which makes it rather easy to find the total energy.
All bound orbits have negative total energy, as can be seen above (with $E_{total} = \frac{1}{2}U$, and $U < 0$ at non-infinite separations). If the total energy is zero, the orbit is unbound and parabolic (the object will never return; this is an escape trajectory), and if it is positive, it is hyperbolic (again, the object will never return).

We can find the orbital period by setting $\frac{mv^2}{R} = \frac{GMm}{R^2}$, where $v = \frac{2\pi R}{T}$, and then solving for T.
The escape velocity can be found by setting $E_{tot} = 0$ and solving for v, since that causes an escape trajectory (as mentioned above).
The total energy is simply found by adding the kinetic energy at some point with the gravitational potential energy at that same point, and then substituting in $v = \sqrt{\frac{MG}{R}}$.

Let's now move on to elliptical orbits. Since a circle is a special case of an ellipse (with a semi-minor axis that equals the semi-major axis, or the eccentricity is 0, or the two foci coincide at the center; all of these must be the case for a circle), the above equations are good approximations for many orbits, since many orbits are very close to being circular (their eccentricity is very low).

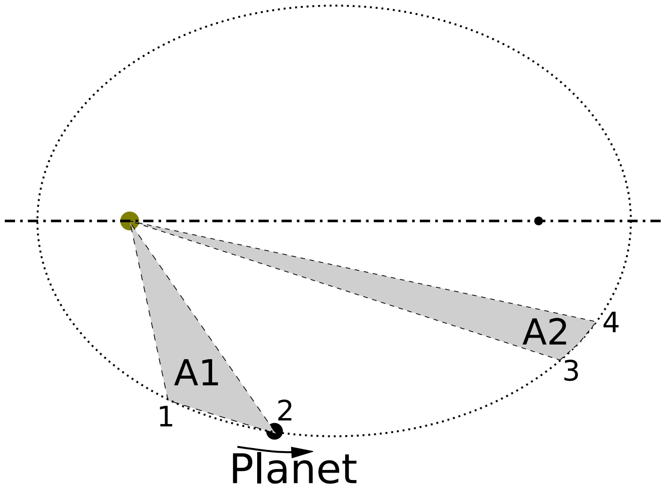
In some cases, however, orbits can be extremely elongated; comets are a common example. Comet Hale-Bopp comes as close as 0.9 AU to the Sun, where 1 AU (astronomical unit) is the mean Earth-Sun distance, it then goes as far as 370 AU away. The eccentricity of the orbit is about 0.995, where > 1 would mean a hyperbolic (unbound) orbit.

12.1.1 Kepler's laws

Kepler's first law states that planetary orbits are ellipses, where the Sun is located at one focus. Note that the Sun is then *not* located at the center, except in the special case of a circular orbit, where both foci are are both located at the center.

[1] For circular orbits; it is not at the center of elliptical orbits!

Kepler's second law is best described with the help of an image.

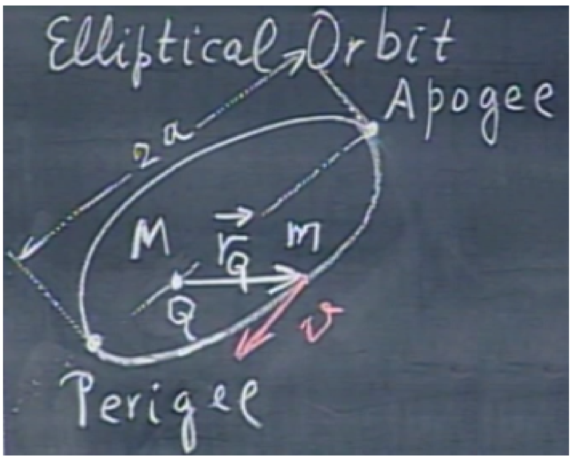


Kepler's second law says that if the area A_1 is the same as the area A_2, the time taken to go between points 1 and 2 is the same as the time taken to go between points 3 and 4.

Kepler's third law says that $T^2 \propto (\text{mean distance})^3$. This is often stated as $T^2 \propto a^3$, where a is the semi-major axis or the orbit, which is equal to the mean distance in some ways of calculating the average, and not equal in other ways. The two statements are usually considered equivalent, however.

12.1.2 Elliptical orbits

Let's now consider the case of elliptical orbits.



Say M here represents the Earth, located an one focus of the ellipse. m is perhaps a satellite or such that orbits the Earth, with velocity $\vec{v}$, with the position vector $\vec{r_Q}$ from Earth.
When the satellite is at the closest point to Earth, we say it is at *perigee*. At the farthest point, it is at *apogee*. The distance between apogee and perigee is $2a$, i.e. the major axis of the orbit.

If M instead represented the Sun, and m orbited the Sun instead (where m could be the Earth, or some other planet), we instead call the closest approach *perihelion*, and the farthest point is at *aphelion*. The distance between the two extremes is still $2a$; only the names change.

We can now re-write a few of our equations:

$$T^2 = \frac{4\pi^2 a^3}{GM} \tag{12.5}$$

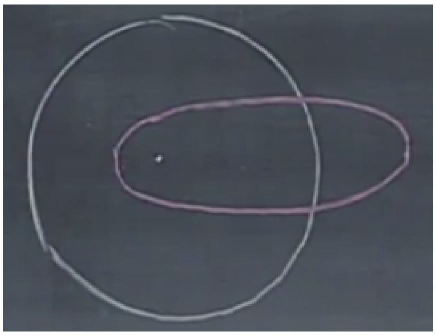
$$v_{esc} = \sqrt{\frac{2MG}{r(t)}} \text{(by setting } E_{tot} = 0) \tag{12.6}$$

$$E_{total} = K + U = \frac{1}{2}mv(t)^2 - \frac{mMG}{r(t)} = -\frac{mMG}{2a} = \frac{1}{2}U \tag{12.7}$$

The total mechanical energy is still a constant (which is not proved here). Not only is it still constant, but it also has the same value as it would for a circular orbit with the same radius as the semi-major axis of the elliptical one. That is, all we do is replace the R by a, and we have the constant energy total.

What is however not constant now is the two energy terms on their own. The kinetic energy now changes, since the orbital speed is no longer constant.
The gravitational potential energy also changes with time, since the distance to the Sun is no longer constant.

The escape velocity now also changes with time, though the expression looks about the same as it did for circular orbits, only that the constant radius R has been replaced by the current distance to the sun, $r(t)$.



These two orbits were drawn to have the same semi-major axis ($R = a$). This means that not only is the total mechanical energy the same for the two, but the time taken to complete one orbit is also exactly the same for either orbit.
What about angular momentum? Is it the same for both orbits, and if not, which one has more?

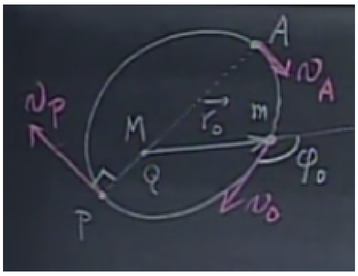
First off, angular momentum is conserved for elliptical orbits. This is clear if you use the same arguments as you do for circular orbits: the only relevant force is the gravitational force; that vector points straight from the planet towards the Sun. The position vector points straight from the Sun towards the planet. Therefore, the torque $\tau_Q = \vec{r_Q} \times \vec{F} = 0$, since the cross product of parallel/anti-parallel vectors is zero. With no external torque, angular momentum must be conserved – *with respect to the center of the orbit* Q, and not in general!

We can write the angular momentum relative to point Q (the center of orbit) as $L_Q = \vec{r} \times \vec{p}$. However, in the case of the elliptical orbit, the magnitude of both these vectors change in time. What doesn't change is the angular momentum relative to this point (see above). Because it is conserved, we can calculate the value at any point of the orbit, and if the value is smaller than it would be for a circular orbit of the same mean distance, that must mean the angular momentum is always smaller. (Or the other way around, if it is bigger.)

The answer is that it is smaller. Using the above information, consider a very elongated orbit, with a semi-major axis the same as the circular orbit we compare it to of course, but with a smaller semi-minor axis. Since angular momentum is constant, we can calculate the angular momentum at a point where it is very far from the Sun. Here, the position vector is large, but the velocity vector is small, and the angle between them is also small. This makes the cross product, proportional to the sine of the angle, small.
Alternatively, consider the case where the two orbits overlap. For the circular orbit, the position vector and velocity vector are always constant, and always at 90 degree angles.

In the case of the elliptical orbit, the position vector is the same (of course, it's the same point), but the velocity vector differs, and makes a smaller angle with the position vector. I find it reasonable that the cross product is then smaller in this case, but I wouldn't call it completely obvious.

Let's now try to find some information about an orbit, given some initial conditions. We are have an object moving at a velocity v_0 near the Earth, which makes an angle φ_0 with the position vector $\vec{r_0}$ from the Earth, at some $t = 0$. Given these details plus the mass M of the Earth, can we find all the details about the orbit, such as: the semi-major axis, the velocity at any given point, the perigee and apogee (closest and furthest distances from the Earth), and so on? The answer is yes, we can.
(Needless to say, we can do this for orbits around the Sun, too; the Earth is merely an example.)



We start out by finding the total mechanical energy, which we can use to find the semi-major axis. One of our equations for elliptical orbits is

$$E_{total} = K + U = \frac{1}{2}mv(t)^2 - \frac{mMG}{r(t)} = -\frac{mMG}{2a} = \frac{1}{2}U \tag{12.8}$$

In this case, we can find the mechanical energy at this instant $t = 0$, since we know everything we need to know in the above equation:

$$\frac{1}{2}mv_0^2 - \frac{mMG}{r_0} = -\frac{mMG}{2a} \tag{12.9}$$

$$a\left(v_0^2 - \frac{2MG}{r_0}\right) = -MG \tag{12.10}$$

$$a = -\frac{MGr_0}{v_0^2 r_0 - 2MG} \tag{12.11}$$

Since a cannot be negative, the second term in the denominator *must* be greater than the first, so that the signs work out. We can rewrite the equation with this in mind:

$$a = \frac{MGr_0}{2MG - v_0^2 r_0} \tag{12.12}$$

This only holds for bound orbits. If $E_{tot} > 0$, a must be negative, which makes no physical sense. The equation is simply invalid at in that case.

We can now apply

$$T^2 = \frac{4\pi^2 a^3}{GM} \tag{12.13}$$

given that we know a, so we also know the orbital period at this point.
The escape velocity also follows easily, since all you need to know there is M and the distance to he center of that mass r.

Next up, we want to find the angular momentum of the orbiting object, relative to the center of the mass M (which we call point Q), as the conservation of angular momentum is helpful for finding some other orbital properties. We can relate the initial angular momentum with the angular momentum at perigee (point P). The distance at perigee is QP, and so we have

$$|L_Q| = mv_0 r_0 \sin\varphi_0 = mv_p(QP) \tag{12.14}$$

The first term is the cross product $(\vec{r_0} \times \vec{v_0})m$, i.e. the angular momentum as we start out. The second term is the cross product $(\vec{r_p} \times \vec{v_p})m$, only we call the distance QP instead of r_p.

Both v_p and QP are unknown, so we need a second equation. We can find another using the conservation of mechanical energy. At point P, we have

$$\frac{1}{2}mv_p^2 - \frac{mMG}{QP} = -\frac{mMG}{2a} \tag{12.15}$$

Now, as an added bonus, when we solve this, we will get two answers, since this equation is quadratic in v_p. One solution will be v_p, and the other will be v_A. We will also find both QP and QA (the perigee and apogee distances). The reason we find both of these values is rather simple: we have said that the angular momentum is $mv_p(QP)$. There are only two places where the position vector and the velocity vector are exactly perpendicular: at perigee, and at apogee. The equations are equally valid at both points, since the equation doesn't know that we said v_p and QP; it might as well have said v_A and QA, and we would have found the same answers.

The solutions I got were extremely ugly; I'm not sure if they can be simplified further using physics knowledge, but Mathematica can't do any better than this.

$$v_p = \frac{aGM - \sqrt{aGM\left(aGM - r_0^2 v_0^2 \sin^2(\varphi_0)\right)}}{ar_0 v_0 \sin(\varphi_0)} \tag{12.16}$$

$$\text{QP} = a + \sqrt{a\left(a - \frac{r_0^2 v_0^2 \sin^2(\varphi_0)}{GM}\right)} \tag{12.17}$$

$$v_A = \frac{aGM + \sqrt{aGM\left(aGM - r_0^2 v_0^2 \sin^2(\varphi_0)\right)}}{ar_0 v_0 \sin(\varphi_0)} \tag{12.18}$$

$$\text{QA} = a - \frac{\sqrt{aGM(aGM - r_0^2 v_0^2 \sin^2(\varphi_0))}}{GM} \tag{12.19}$$

Because the angular momentum $m(\vec{r} \times \vec{v})$ must remain a constant at all times, it must be true that at perigee/apogee when the cross product equals simply mrv, the product of the distance and the velocity must be the same in both cases. That is, $mv_p(QP) = mv_A(QA)$. The mass cancels, of course, so we find that $v_p(QP) = v_A(QA)$ must hold.

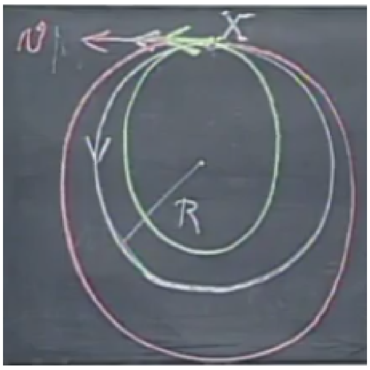
Apogee is by definition farther from the Earth than perigee, so the speed at apogee must be lower.
The difference in speed can be rather enormous, for very elliptical orbits. If apogee is 14 times as far away as perigee is, the speed at perigee will then be 14 times *higher* than the speed at apogee. If this were not the case, angular momentum would not be conserved (relative to the point Q, the only point relative to which it is *ever* conserved).

So in conclusion, by knowing the initial position, velocity (including the direction, i.e. angle between position vector and velocity vector) and the mass of the object we orbit, we can find all the orbital details. This is assuming that the orbital will be elliptical ($a > 0$ when we calculate it, using the first equation). If that is not the case, then there will be no bound orbit, and some of these parameters are meaningless (such as apogee/aphelion, the orbital period, etc). In any case, we can not deal with that using what we have learned *so far*.

12.1.3 Change of orbits

Let's look at a simplified case of a change in orbit. We begin in a fully circular orbit, and then fire a rocket exactly tangentially to the orbit.
Say we begin at the "top" of the orbit (in a diagram), at point X. We fire the rocket a very short amount of time, so that we can consider that it is still at X afterwards. The speed will increase, so the kinetic energy will increase, and the total mechanical energy will increase (we have not moved away, so gravitational potential energy must be the same, assuming our mass has not changed).

Because our velocity is now greater, and we are at a certain distance R from the planet (or star, or whatever), our velocity is no longer the correct velocity for a circular orbit at this distance. Instead, we go into an elliptical orbit, where $a > R$. The total mechanical energy has increased, which means a must increase; total mechanical energy is $-\frac{mMG}{2a}$, and for mechanical energy to increase, that number must become less negative. A larger a does exactly that. Via the relationship in T^2, this also implies that the period of this new orbit is greater than the old period, despite the increase in speed.

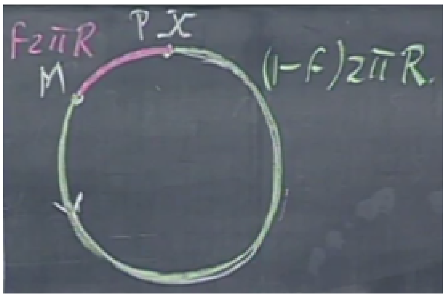


In this graphic, the original orbit (fully circular) is in white. We are orbiting counterclockwise, as seen here.
For the red orbit, we have increased our tangential velocity, and so the orbit grows, and becomes elliptical. $E_{tot} > E_{tot,circular}$, and so the semi-major axis is greater than the original radius, and the period is greater than the original, since $T^2 \propto a^3$.

For the green orbit, we instead fired our rocket exactly opposite to our velocity, so that we slowed down, and lost kinetic energy, and also lost total mechanical energy.
In that case, $a < R$, and the period goes down.
Out of context, it sounds rather crazy: you go around faster by braking! Of course, you also travel a shorter distance per period, so there's nothing particularly crazy about it.

The rest of this lecture will focus on a "practical" (fun, at least) example of this; perhaps we can call it several examples, even.

Two astronauts, Peter and Mary, are in orbit around the Earth. They are both moving at the same speed, and are at the same distance from Earth, sharing a circular orbit. They are not moving along it together, however. Peter is at one place along this circular orbit, point X, while Mary is at a point M, a distance further along. We can specify this separation as a fraction of the circumference $2\pi R$ of the orbit, using f as the fraction. The distance between the two is then $f\ 2\pi R$. The rest of the orbit must then be $(1 - f)2\pi R$, which is the amount of distance Mary must move before she ends up at point X (which she recently passed).



Say the radius of this shared orbit is $R = 7000$ km (7×10^6 m), and $f = 0.05$, which makes the separation between the two about 2200 km. (The actual separation as a straight line is smaller, but we don't really care about that.)
Given these parameters, there is a only one possible circular orbit: the radius of the orbit puts a demand on the velocity and period of the orbit. This velocity is about 7.55 km/s, and the period is about 97 minutes.

If we round things off a bit, it then takes $fT \approx 5$ minutes to travel the distance that exists between Peter and Mary. This means it takes 92 minutes for Mary to get to the point where Peter is currently. (Needless to say, Peter will no longer be there when she gets there.)

Now, here is the problem: Mary forgot her lunch. She radius Peter, who says not to worry; he will throw her a ham sandwich. The question is: how will he do this, so that Mary can catch it? Perhaps the simplest way is to throw the sandwich such that its new orbit brings it to point X, where Mary will be, in exactly the 92 minutes required, so that Mary and the sandwich meets at that point.

Here comes the counter-intuitive part (at least to me): if Peter throws the sandwich forward, it will come back to point X *later* than Mary. The faster he throws it, the longer it will take.
The reason is that by throwing it, he is doing exactly what we did with the rocket earlier on. The sandwich will move into an elliptical orbit, with $a > R$. Since $T \propto a^{3/2}$, this orbit has a longer period than Peter's, and so the sandwich will take a longer time to get into place for Mary's move past point X.

Instead, he needs to throw it *backwards*, and *reduce* its orbital speed. That way, it will move into a smaller orbit, with a smaller period, and come back to point X faster, by having moved a smaller distance.

What we need is that the orbital period of the sandwich T_s is less than Mary's period T_a (a for astronaut), such that $T_s = (1-f)T_a$. T_a is 97 minutes, and the orbit we want should be 92 minutes. $1 - f = 0.95$, and 0.95×97 minutes is about 92 minutes. (Keep in mind that all of these numbers for the period are rounded approximations; the actual period T_s will be exactly right.)

Doing the actual math, we need the sandwich's orbit to be $(1-f)$ times the astronauts' orbit, as mentioned, which using Kepler's third law is

$$\sqrt{\frac{4\pi^2 a^3}{GM}} = (1-f)\sqrt{\frac{4\pi^2 R^3}{GM}} \tag{12.20}$$

$$\sqrt{a^3} = (1-f)\sqrt{R^3} \tag{12.21}$$

$$a = (1-f)^{2/3} R \tag{12.22}$$

So knowing only the radius of their orbit and how far they are apart along that orbit (as measured by a fraction f of the total circumference), we can find what the radius of the sandwich, a, must be.
Knowing a, the semi-major axis of the sandwich's orbit, we can now calculate what velocity it must have, using the equation for total mechanical energy.

$$\frac{1}{2}mv_s^2 - \frac{mMG}{R} = -\frac{mMG}{2a} \tag{12.23}$$

On the right-hand side, we have the current kinetic energy of the sandwich (after the throw), and its current gravitation potential energy. An instant after the throw, we can still consider it to be at point X, so it is still a distance R from the planet. The right-hand side of the equation must always hold for elliptical (and thus also circular) orbits: the total mechanical energy is always $\frac{1}{2}U$.

We can solve this for v_s, now that we know everything else. m cancels, as always, and in the end, we find

$$\frac{1}{2}v_s^2 - \frac{2MG}{R} = -\frac{MG}{a} \tag{12.24}$$

$$v_s^2 = \frac{2MG}{R} - \frac{MG}{a} \tag{12.25}$$

$$v_s^2 = \frac{2MG}{R} - \frac{MG}{a} \tag{12.26}$$

$$v_s = \sqrt{\frac{GM(2a-R)}{aR}} \tag{12.27}$$

In terms of R (since we know a), this is also

$$v_s = \frac{\sqrt{(2(1-f)^{2/3}-1)GMR}}{(1-f)^{1/3}R} \tag{12.28}$$

In terms of numbers, we have $a = 6765$ km – smaller than R, which it must be; $v_s = 7.42$ (though closer to 7.43) km/s.
The speed of the sandwich *relative to Peter* is what matters to hit, though: he doesn't need to throw it at over 7 km/s! He just needs to throw it so that its final velocity v_s is that value, but most will come from his current orbital speed v_a.

The speed he needs to throw it at is $v_s - v_a = 0.13$ km/s. Note how $v_s < v_a$ – it moves in a smaller orbit, but therefore also moves at a lower speed.

So in order for Mary to catch it the *first time she passes point X again*, he must throw the sandwich *backwards* (towards the clockwise direction), slowing its speed and reducing its total energy, even though he can see Mary currently in front of him!

Unfortunately, 130 m/s is a bit much for a person to throw a sandwich. We can find a different solution, where the sandwich moves around the Earth multiple times, and possibly where Mary does too.
Mathematically, we can have Mary pass the point n_a times (n for number of times, a for astronaut), and the sandwich n_s times. Both numbers need to be integers, of course.

We now find

$$a = R\left(\frac{n_a - f}{n_s}\right)^{2/3} \tag{12.29}$$

as the semi-major axis for the sandwich's orbit. If $n_a = n_s = 1$, the equation reduces down to what we had previously.
Not all combinations of integers will work, however.

For $n_a = 1$ and $n_s = 3$, such that the sandwich should make three orbits in the same time Mary completes her one orbit, the result is invalid.
For these numbers, we find $a = 3252$ km (versus $R = 7000$ km).
$2a < R$, which is not allowed; the reason why can be seen in the equation for v_s. If we the a value in, we find an imaginary answer (complex, but the real part is 0).
As a approaches $R/2$, v_s goes to zero. There is the possibility where he throws it at exactly his orbital

speed, in which case it will stand still relative to the Earth, and fall straight down. What if he throws it ever faster? In that case, there is clearly no longer a counterclockwise orbit, which has been our assumption all along.

The final part of the lecture shows a computer simulation of this scenario, including a few cases not mentioned in this text.

12.2 Lecture 23: Doppler effect, binary stars, neutron stars and black holes

The speed of sound, in air, is about 340 m/s, at roughly room temperature. The range is something like 306 m/s near −40 degrees Celcius (which also equals −40 degrees Fahrenheit) and 360 m/s at 50 degrees Celcius (122° F), which hopefully covers most temperatures we encounter.

When a person speaks, his/her vocal cords oscillate at a certain frequency, which causes a pressure wave in the air to move towards you, at the speed of sound. Your eardrums then oscillate at the this same frequency, which your brain can interpret as a certain pitch.

If the sound transmitter is moving away or towards the receiver, the perceived sound frequency will change. The faster the transmitter is approaching you, the higher the perceived frequency, and vice versa. If the transmitter is moving away from you at the speed of sound, you will hear exactly half the original frequency. That is, if we use prime notation for the perceived frequency and f for the original frequency,

$$f' > f \text{ (when sound source is approaching you)} \tag{12.30}$$
$$f' < f \text{ (when sound source is moving away from you)} \tag{12.31}$$

More quantitatively, when a source source is moving towards you, but you are sitting still, the frequency you perceive is

$$f' = f(1 + \frac{v}{v_s}\cos\theta) \tag{12.32}$$

where v_s is the speed of sound, and $v\cos\theta$ is the radial speed, i.e. the speed at which the transmitter is moving towards you. If the transmitter is moving quickly, but at a 90 degree angle to you, you don't hear any shift in the frequency.
If the transmitter is moving *away* from you, the frequency you perceive goes down, which this equation also captures (for negative values of $\cos\theta$).

The professor demonstrates this by using a 4000 Hz tuning fork, moving it back and forth towards and away from the students. If this is done at roughly 1 m/s, the pitch will change with ±0.3% which is ±12 Hz. This difference is clearly audible, including in the recorded video of this lecture.

This effect of frequency change is known as the *Doppler effect*. It is the reason behind the familiar scenario where the pitch of an ambulance/police car/fire truck's siren changes as it travels past you.
When the sound transmitter is moving away from you, the pitch you hear is lower than the pitch that is actually transmitted at the source. If the transmitter is coming towards you, the pitch instead increases.

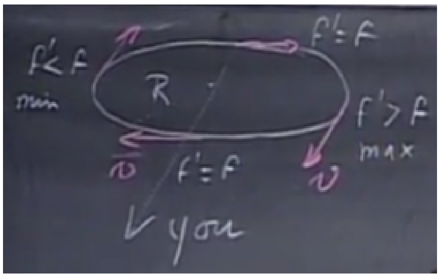
Note that the case of a stationary transmitter and a receiver moving away is *not* equal to the case of a stationary receiver where the transmitter is moving away!

If the transmitter is stationary, and is creating sound waves with a frequency of say 440 Hz (the standard tuning in all modern music is to the A440 note), and the receiver moves away at the speed of sound, then there will be no sound heard at all - the receiver (just barely) outruns the sound waves entirely!

On the other hand, if the *transmitter* is moving away at the speed of sound, while the receiver is stationary, the result is that the perceived frequency is cut in half to 220 Hz.

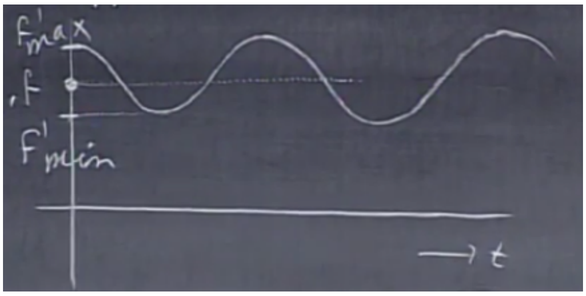
So we still hear the sound, in this case. If you find that nonintuitive, think about it a bit more - it's very clear that while the transmitter moves, the waves will still reach you as they always travel at 340 m/s in your direction. The transmitter may be moving in a direction away from you, but the sound waves don't travel along with the transmitter, but towards you (and it all other directions, too, assuming an omnidirectional speaker).

Let's now consider the case where the transmitter is moving around in a circle. Now, the perceived frequency, assuming the receiver is still stationary, will vary sinusoidally around the base frequency of f that the transmitter is sending out.



When the source is moving straight towards you, f' is at a maximum; the opposite is true when it is moving straight away from you. At the other two extremes, when it is at 90 degree angles, $f' = f$, and so there is no change in frequency at those times. In between, as you might expect, there is a gradual change between these extremes.

If we, as the receiver, plotted the frequency we heard as a function of time, we would get something like this:



If we have this curve, we can calculate an impressive number of things. First, we know f'_{max} and f, which means we can calculate the velocity of the transmitter.
We can also measure the period of one rotation, by measuring the time from one peak to the next (or one valley to the next, etc). Knowing that, we can find the radius R of the circle the transmitter is moving in:

$$\frac{2\pi R}{T} = v_{tr} \tag{12.33}$$

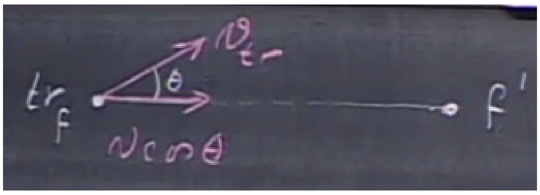
$$R = \frac{T v_{tr}}{2\pi} \tag{12.34}$$

So from simply listening to/measuring the sound over time, we can calculate these three things, rather easily.

How do we intuitively explain the Doppler effect? It is actually quite simple, but is best shown using animated graphics, which I can't really use here. I suggest looking it up online. Wikipedia has some very nice animated figures, for example.
In short, when moving towards a transmitter (or the transmitter is moving towards you), the wave fronts become "compressed", and the wavelength becomes shorter (meaning that for sound, the pitch increases; for light, it becomes *blueshifted*; more on that soon). When the distance between is increasing, the wave fronts become separated from each other, and the wavelength becomes longer (sound: lower pitch; light: "redder" color, or redshift).

12.2.1 The Doppler effect and electromagnetic waves/light

Let's look more at the Doppler effect in light (and other EM radiation).



The equation given in lecture, which is only valid for $v \ll c$, i.e. when the relative velocity between transmitter and receiver is much smaller than the speed of light, is:

$$f' = f(1 + \frac{v}{c}\cos\theta) \tag{12.35}$$

In calculating the Doppler shift due to some relative motion, only the radial component, meaning the part of the velocity that is directly towards or from you, matters. Therefore, we take the cosine of the angle of the velocity vector. Note that the radial component is given as $v\cos\theta$, rather than $v_{tr}\cos\theta$ (tr for transmitter); the index was dropped as it does not matter whether the transmitter or the receiver is moving, or both. (It is even really a meaningless question in relativity; moving relative to what reference frame?)
There is however something known as the transverse Doppler shift, which occurs even at 90 degree angle. I'm not sure when this matters, but it is not mentioned in the lecture, at all, so I suppose it is less important in most cases.

Much of the idea behind special relativity, as the name implies, is that motion is relative. There is no such thing as an absolute reference frame, and therefore, there is only one term of velocity in the equation. It is meaningless to ask whether person A is moving towards person B, or person B is moving towards person A. What we can say is that they are approaching each other.
It does however, of course, matter whether the two are approaching or receding from each other.

In this equation, if $\theta < 90°$, $f' > f$ since the cosine term will contribute to increasing f'. If $\theta > 90°$, the cosine will be negative, and $f' < f$. At 90 degree angles, as mentioned, this equation will tell you that there is zero Doppler shift. And again, this only holds if $v \ll c$, i.e. $\frac{v}{c} \ll 1$.

All electromagnetic radiation, whether it is visible light, gamma rays, microwaves, radio waves etc., has a frequency associated with it. Anything with a frequency of oscillation also has a *period* of oscillation, with the simple relationship that $T = 1/f$.
How far does EM radiation travel in a time T? Well, it travels at the velocity c, the speed of light ($c \approx 3 \times 10^8\,\text{m/s}$, to within 0.07%). That means the distance traveled is $\lambda = cT$, where λ (lowercase Greek letter lambda) is the symbol we use to denote *wavelength*.

$$\lambda = cT = \frac{c}{f} \tag{12.36}$$

For example, if $T = 2 \times 10^{-15}\,\text{s}$, $\lambda = 6 \times 10^7\,\text{m}$, which is the about wavelength of red light.
If instead $T = 1.3 \times 10^{-15}\,\text{s}$, $\lambda = 3.9 \times 10^7\,\text{m}$, which we perceive as blue light.

In optical astronomy, we can only measure the wavelength of light – not frequency or period – so we can rewrite some of our equations to better accommodate this. We can of course calculate frequency, so that

$$f = \frac{c}{\lambda} \tag{12.37}$$
$$f' = \frac{c}{\lambda'} \tag{12.38}$$

With that in mind, we can find

$$\lambda' = \lambda(1 - \frac{v}{c}\cos\theta) \tag{12.39}$$

What was a plus sign in the previous equation of this sort is now a minus sign. Also, keep in mind that this equation also has the restriction that $v \ll c$.

Since frequency is inversely proportional to wavelength, $\lambda' < \lambda$ if the object is approaching you ($\theta < 90°$). This is known as *blueshift*. The name comes from the fact that all radiation becomes more energetic (higher frequency/smaller wavelength), which causes visible light to shift towards the blue end of the spectrum. (EM radiation that is already more energetic than blue/violet light becomes even more energetic, and therefore shifts even further away from the visible spectrum, towards the gamma rays.)

Similarly, if the object is moving away from you, $\theta > 90°$ and $\lambda' > \lambda$. A longer wavelength means less energy, and this is known as *redshift*. Here, all light becomes less energetic and "stretched out", which shifts visible light (and all more energetic light) towards the red end of the spectrum. Light that was already lower energy (microwaves, radio waves) lose further energy and shift further away from the red (and further from visible light altogether).

In case the above is written in a confusing manner: blueshift is called as such because visible light is shifted towards the blue. *All* light, regardless of frequency/wavelength becomes *more* energetic when blueshifted. Conversely, redshift is called as such because visible light is shifted towards the red, and all light, regardless of frequency/wavelength becomes *less* energetic when redshifted.

12.2.2 Emission and absorption spectra

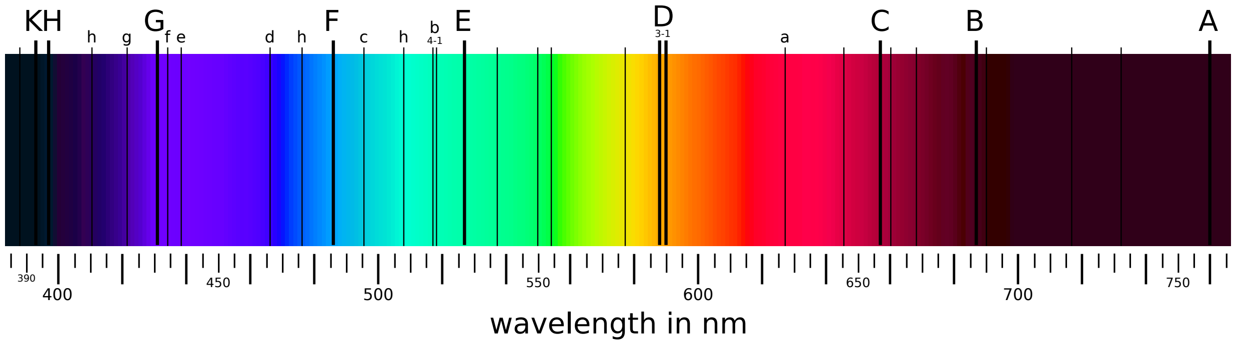
We can look at the light spectra of stars, and look at the intensity of the light as a function of wavelength. When plotting that, we will not see a continuous distribution, as we might expect. Instead, we see a mostly smooth curve, with some sharp spikes downwards. That is, light intensity is sharply reduced for certain wavelengths.
We call these *absorption lines*. Each absorption line is due to some element present in the atmosphere of the star. Through the process of figuring out which elements and isotopes cause which absorption lines here on Earth, we can use the lines to figure out which elements are present in the star.

Here is an example of *emission lines*, in this case of hydrogen:

The red line on the right is known as Hα (hydrogen alpha), and is a very well-known spectral line. Telescopes are often fitted with Hα filters for viewing the Sun.
Emission lines are the *opposite* of absorption lines. In this case, we use a lamp or such that excites hydrogen, which then only produces light in discrete steps, for quantum mechanical reasons.

If you have ever seen street lightning that makes everything appear yellow, those lamps were most likely sodium-vapor lamps. These lamps, especially the low-pressure type, are almost entirely monochromatic, and essentially only emit light at two wavelengths, about 589.0 and 589.6 nm – both of which are yellow. Since vision relies on having light reflect off things and enter our eyes, in the presence of only such light, it is impossible to see colors other than yellow. Objects that completely absorb yellow will become dark. For this reason, the amount of different wavelengths a lamp emits is a common measure of its perceived quality (see color rendering index aka color rendition index, CRI).

As mentioned, absorption lines are the opposite of this. Here is an example of absorption lines:



The letters refer to labels of Fraunhofer lines, a set of spectral lines named after (and identified by) German physicist Joseph van Fraunhofer, in the early 1800s. The sodium lines above can be seen as D1 and D2 in the yellow-orangeish part of the spectrum.

The now-well known element helium was first discovered in Sun, as an absorption line that could not be reproduced in the lab here on the Earth was identified. The element was named helium, after Helios, the Greek Sun god.

What is now interesting, and extremely useful, is that the relative spacing of these lines stays constant. If a star is moving (radially) relative to us, the light will be either redshifted (if it moves away) or blueshifted (if it moves towards us), but since *the entire spectrum* will be shifted, we can still identify what elements are present via their distinct patterns and spacings.
This means that we can not only look at a star and figure out what it is made of, we can also calculate its velocity relative to us!

For example, if $\lambda'/\lambda = 1.000333$, according to the equation we found earlier, the star is moving at $-0.000333c$ (radially), where the minus sign signifies that it moves away from us. That is, $v \cos\theta = -0.000333c = -100$ km/second. Note that since $\lambda' > \lambda$, the light from the object has been redshifted. Also note that we cannot say what v is using this information, only what the radial component is – i.e. how fast it moves towards us, or from us.

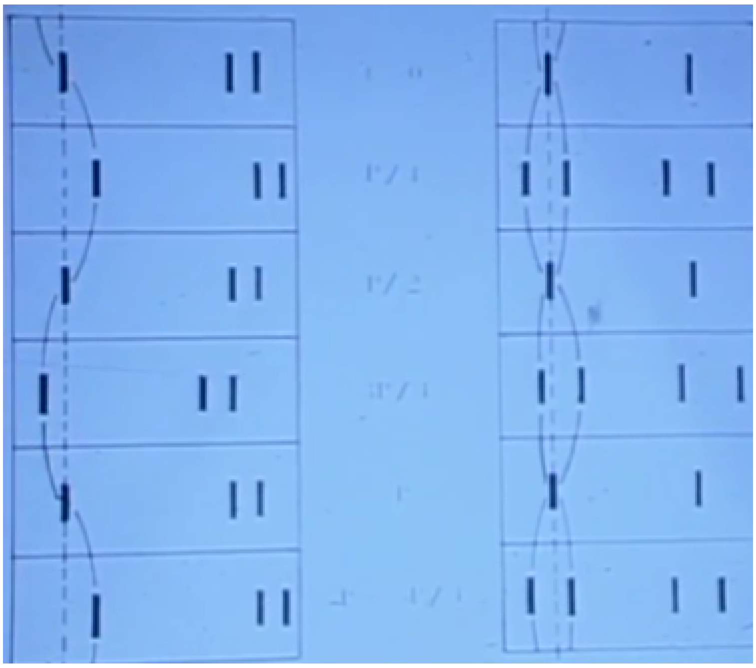
As a very quick aside, this is how police “radar guns”, which can measure the speed of cars, work. They reflect radar waves of a known frequency/wavelength off cars, and measure the wavelength of the returning waves. The radial velocity of the car can then be calculated, using the measured Doppler shift.

Spectra of binary stars

Binary stars, pairs or stars that orbit each other, are extremely common. The lecture states that half of the stars in the sky are binaries. I'm not sure if that refers to visible stars, in which case I would guess nothing has changed, but it *may* (with emphasis, indeed) be that science's view has shifted, and that most stars overall are in fact not binaries, from recent (post-lecture recording) news articles.

When binary stars orbit each other (or rather their shared center of mass), we can measure the Doppler shift they exhibit. While orbiting, they will be going towards us, at right angles, from us, etc. just as with the sound source moving in a circle we looked at earlier.
This is only possible to see if we are in the plane of their orbit, however. If we are looking at the system from “above”, the radial component of the velocity between the stars and us will be constant despite the orbit, as they will not move any closer to us or further from us due to the orbit.

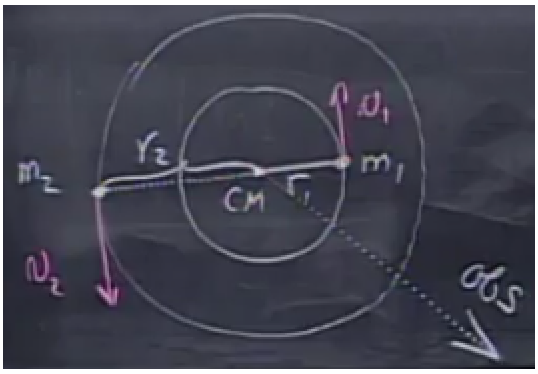
Just as in the case with sound, we can measure and calculate the period of this shift, and therefore calculate their velocities, the orbital period, and the orbital speed.



Above is an illustration on the Doppler shift in the spectrum of a binary star system. On the left, we have the case where we can only see one of the two stars. As time passes, we see the spectral lines shift left and right (as time passes, downwards in the picture) in unison.
On the right, we have the case where we can see both stars. In this simplified case, we only show the same two spectral lines. In this case, when we see the spectral lines of one star redshifting, the other will be blueshifted, and vice versa, in the case that one is moving from you, and the other towards you. This means that one set of lines will move towards the left, as pictured, while the other moves towards the right, doubling the number of spectral lines we observe.

As mentioned previously, based on this data, we can then calculate the orbital radius, velocity in orbit, and the (shared) period of the two stars' orbits.

Let's now consider a binary system. They orbit their common center of mass, and are in different circular orbits, of radii r_1 and r_2, respectively. The stars have masses m_1 and m_2, and velocities v_1 and v_2.



Via the definition of center of mass, $m_1 r_1 = m_2 r_2$.
We, as an observer, are somewhere in the plane of this orbit, but far away from it.

Via Kepler's third law, which we have seen before,

$$T^2 = \frac{4\pi^2(r_1 + r_2)^3}{G(m_1 + m_2)} \tag{12.40}$$

If we measure the Doppler shift of star 1, we can find its period T, its velocity v_1, and its orbital radius r_1. We make a similar measurement for the second star, and find T, v_2 and r_2.
Because we then know r_1 and r_2, we obviously also know $r_1 + r_2$. We also know T. Using this information,

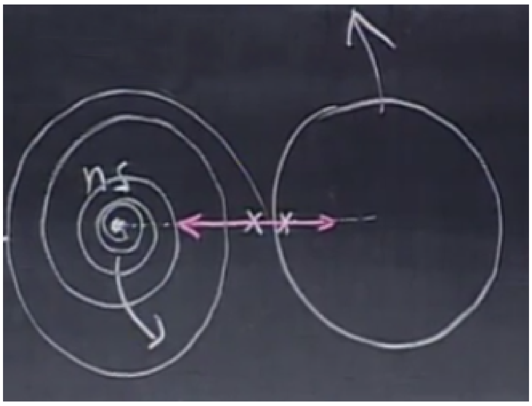
using the equation above, we can find $m_1 + m_2$!
Not only that, but we also know that $m_1 r_1 = m_2 r_2$, so we can find m_1 and m_2 on their own, too, given that we have two equations relating the masses and orbital radii.

Using nothing but two Doppler shift measurements and some calculations, we can find the mass of each star, the radius of their orbit, how fast they move in that orbit, and the time it takes the pair to orbit once. Incredible.
If we are not in the plane of the orbit, however, we must have extra information: we need to know the angle θ we make with the orbital plane. We will only measure the radial components of the stars' velocities, and so only by knowing θ can we make any calculations on their actual velocities in orbit.

12.2.3 X-ray binaries

Let's now have a look at a special case of binary stars. In an X-ray binary system, we have two different types of stars: one large, relatively normal star, not too unlike our Sun.
The other star is a neutron star, or a black hole (or in some cases, a white dwarf). Let's assume it is a neutron star for this discussion.

Consider what happens if the two have the same mass. There will then be a point, right in the middle between them, where the gravitational pull is the same in both directions. We call this the inner Lagrangian point. If instead this point is inside the larger star, matter will fall from that star onto the neutron star, since the gravitational pull for all matter outside that point will have a stronger gravitational force towards the neutron star.



Here, we see the larger star as the ring to the right, with the arrow indicating how it is orbiting. The neutron star is the small dot at the center of the spiral; the spiral is made up by the infalling matter, and is called the *accretion disk*. Matter cannot fall radially inwards towards the neutron star, because of the fact that the two are orbiting each other (or their common center of mass, rather).

The neutron star is also called the accretor, while the larger star is known as the donor.

Consider now a small amount of matter m that is released far from the neutron star. Technically, "far" means infinitely far away, but the answers we find are almost identical for reasonably small distances (starting out at just 1000 km from the neutron star instead of infinitely far away, the impact velocity is 99.5% of what it is if you begin at infinity, so the impact energy is about 99%).

We know that total mechanical energy is conserved, so the velocity of the piece that hits can be found by considering its energy as it hits, and far away, where $U = 0$ and also $K_e = 0$ (if we let it go with zero speed):

$$\frac{1}{2}mv^2 - \frac{mM_{ns}G}{R_{ns}} = 0 \tag{12.41}$$

$$v = \sqrt{\frac{2M_{ns}G}{R_{ns}}} \tag{12.42}$$

The kinetic energy as it hits is clearly $\frac{1}{2}mv^2$, where v is the above impact velocity:

$$K_{impact} = \frac{mM_{ns}G}{R_{ns}} \tag{12.43}$$

For $m = 10$ grams (0.01 kg), $M_{ns} = 1.5$ solar masses ($1.5 \times 2 \times 10^{30}$ kg) and $R_{ns} = 10$ km, we find $K_{impact} = 2 \times 10^{14}$ J. The impact velocity is 2×10^8 m/s – ignoring relativistic effects. Keep in mind that this is for a *10 gram* object! This energy output is comparable to that of the atomic bombs used in world war 2 – all because of a 10 gram object being released from being (relatively) close to a neutron star.

There are hundreds of such systems (that are known) in our galaxy.
The mass transfer rate in such systems is something along the lines of $\frac{dm}{dt} = 10^{14}$ kg/s – which is, of course, a rather insane number. Now, consider how much energy was released from the tiny 10 gram mass falling onto the neutron star! Here, we have 10^{16} times more mass *per second*.
All in all, this gives an energy rate (power) of about 2×10^{30} W, about 5000 times the power output of our Sun.

Because of this enormous energy release, the temperature of the neutron star is about 10 million Kelvin. At such high temperatures, most of the EM radiation emitted is in X-rays.
We humans have body temperatures of about 300 K; at that temperature, we emit infrared radiation – heat. We cannot see this radiation, but we can feel it as heat. An object at 3000 K would glow red-orange due to its temperature. At 3 million K (or degrees Celcius, which are practically the same; the difference is less than 300 K), X-rays begin to matter.

As the matter falls in, it is usually highly ionized, due to the gravitational potential energy released. Highly ionized material has electric charge, which can only reach the neutron star at certain points. The reason is that neutron stars have extremely strong magnetic fields (in particular, one sub-class called magnetars are thought to be the most strongly magnetic objects in the universe), and the movement of charged particles is affected by magnetic fields. They will tend to follow the magnetic field lines, and enter the neutron star near the two magnetic poles.

These two poles then turn into "hot spots", and most matter will fall in a relatively small area – especially considering that the neutron star is very small to begin with.
If the axis of rotation doesn't coincide with these hot spots, we can get the effect where it rotates such that the "jets" created by the infalling matter appear to pulsate at us, creating an X-ray pulsar.
(Consider the case where it rotates around an axis that goes through its north and south (geographic) poles, while the magnetic field is perpendicular to this.)

The timing of these pulses is, as mentioned a few lectures back, extremely precise. However, now that we have a binary system, we can end up with the scenario where the neutron star is coming towards us, and then moving away from us, during an orbit (again assuming we are in the plane of the orbit). The are a bit like a clock; as it is coming towards us, due to Doppler shift, the "ticks" come a little closer together. As it moves away, they are a little further apart. So by timing these pulses, we can measure the Doppler shift of the neutron star, and then as earlier find the orbital radius, period and velocity of the neutron star.

If we combine that X-ray observation with an optical observation of the donor star, we see the Doppler shift in the absorption lines due to its orbit, and so we can calculate from that the donor star's orbital radius, orbital velocity and period.
As before, with this information, we can now also calculate the individual masses of the two stars.

In addition to what we have discussed so far, there may also be a change in activity on a longer time scale, of days rather than milliseconds or seconds.
If we are indeed in the plane of the orbit, then there will be times where the neutron star passes behind the donor – and the donor then absorbs the X-ray emissions from the neutron star. This will cause periods

where the X-ray activity appears to cease, until this X-ray eclipse is over.
This also means that we get a second, independent measurement of the orbital period.

12.2.4 Chandrasekhar limit, black holes

Of all the X-ray binaries measured so far, almost all of the neutron stars have a mass very close to 1.4 solar masses, for good reasons.
Indian-American physicist Chandrasekhar calculated in 1930, at the age of 19, that a white dwarf star could not exist if its mass was greater than about 1.4 solar masses. The reason lies in quantum mechanics (above that mass, the gravitational force wins over the electron degeneracy pressure, so that the star collapses – not that I personally truly know what this means yet). This limit is known as the Chandrasekhar limit; the currently accepted value is about 1.44 solar masses.

So imagine a white dwarf, with a radius about 10 000 km. If we keep adding mass to it, by the time it reaches the Chandrasekhar limit, can collapse down into a neutron star, in a type Ia supernova.

If we instead imagine adding mass to a neutron star, by the time it reaches a mass of approximately 1.5-3 solar masses, it can yet again collapse, in this case into a black hole. This limit is known as the Tolman-Oppenheimer-Volkoff limit (or TOV limit). Similarly to the white dwarf case, a neutron star with a mass under the TOV limit is stable due to the equilibrium between two pressures, this time between the gravitational force and the *neutron* degeneracy pressure.
As a sidenote, it is possible that other types of ultra-dense stars exist, such as quark stars. There is not yet any definite proof one way or the other, but they remain a theoretical possibility.
For now, let us assume that the "next step" from a neutron star is a black hole, and that there exists nothing in between.

What is, then, a black hole?
Classically, a black hole is a point mass – it has no radius in itself. Black hole masses vary by extreme amounts for different types; it is thought that there are types from a few (3-10) solar masses, to ones with a mass of *billions* of solar masses: supermassive black holes. It is also thought that most, if not all galaxies contain a supermassive black hole at their center.

A black hole does have one radius that is useful to talk about: the radius of its *event horizon*. We know how to calculate escape velocities:

$$v = \sqrt{\frac{2MG}{R}} \tag{12.44}$$

As M grows, there is a point that for a certain distance R away, the escape velocity is the speed of light c. Setting the two equal, that radius is

$$\sqrt{\frac{2MG}{R}} = c \tag{12.45}$$

$$R = \frac{2MG}{c^2} \tag{12.46}$$

At all points inside this radius, the escape velocity is greater than the speed of light. In other words, nothing – not even light – can escape, thus the term black hole. It is theoretically possible to escape from all points outside this radius, but it clearly becomes increasingly hard, the closer you come.

This radius is known as the Schwarzchild radius. All black holes have an associated Schwarzchild radius, found using the above formula. We can also use the formula to calculate into how tiny space we would need to compress a mass M for it to become a black hole. Earth's Schwarzchild radius is a bit less than 1 centimeter, which says something about the insane density of black holes (even if they are not truly point masses)!

As a side note, black holes are not magical: there is a common misconception that they are the "vacuum cleaners" of the universe, and that they suck in everything around them. While this is true in a sense – they do have an extremely strong gravitational pull – they do not have any more of a pull than any other object of a similar mass.
If our Sun was magically replaced by a black hole *of the same mass as the Sun*, all orbits in the solar system would remain unchanged. We would still die due to the lack of sunlight, but that's a different story!

Since nothing, including light (of any wavelength) can escape a black hole, how can we still observe them? In fact, *can* we observe them?
The answer is that yes, we can. Matter in the accretion disk, that is still falling in and is still *outside* the event horizon, can be observed with no contradictions. Such matter can be extremely hot, due to the release of gravitational potential energy while falling in, plus frictional forces. It can be and often is hot enough to emit X-rays.

Black holes never pulsate, as they have no surface, so you cannot have the two jets rotating around. This also implies that we cannot measure the Doppler shift of the black hole. We can measure the Doppler shift of the donor star, however. If we can also estimate the donor's mass, we can find the black hole's/accretor's mass from knowing that.

Cygnus X-1 is a famous case. It was discovered in the early 1970s. It is an X-ray binary, with an orbital period of 5.6 days. By looking at the absorption lines, astronomers estimated the donor's mass to be about 30 solar masses. (Note that this is different from what we have discussed, where we need measurements of both to *calculate* the masses.)
The mass of the accretor must then be about 15 solar masses. Given that this is a very compact object (given that it emits X-rays), and it is clearly much more massive than ≈ 3 solar masses, it is concluded that the object most likely is a black hole.

Since then, many other X-ray binaries have been discovered, where the accretor is thought to be a black hole.

12.3 Lecture 24: Rolling motion, gyroscopes

First out this week is rolling motion; specifically, the case where there is no slipping or skipping, which we call *pure roll*.

Say we have a cylinder that is in rolling motion. It rotates with angular velocity ω, while its center of mass is moving in a straight line. Say the radius of the cylinder is R.
Once it has made a complete rotation about its axis, if the distance it has moved relative to the surface below is $2\pi R$, we call that pure roll.
When this is the case, the velocity v_Q of the center point Q, is always the same as the tangential velocity v_c about the circumference. That is, $v_Q = \omega R$ for pure roll. ($v_C = \omega R$ always holds, of course.)

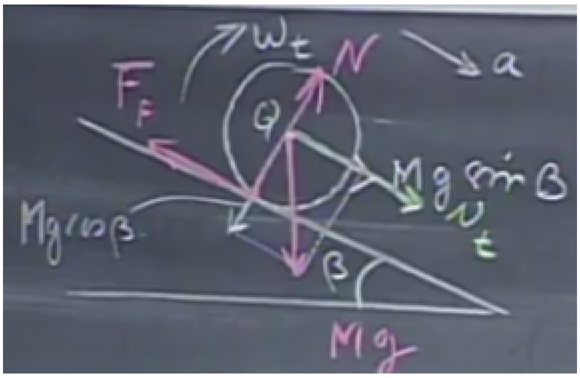
If there is no friction with the surface below, we can imagine both that the cylinder might roll and roll without actually moving anywhere, and the opposite situation where it slides without rolling. In either case, then, we certainly don't have pure roll.
Let's now try to apply this in practice. There are no new physics here; as long as we have pure roll, we can analyze it without too much trouble.

Consider a cylindrical object on an incline of angle β. We want to calculate the acceleration of the cylinder, as it rolls down this slope with pure roll.
The cylinder has a mass M, radius R and a length ℓ.

Given two cylinders, both solid, both having the same mass and length, but differing in radii, which will accelerate faster / reach the bottom of the slope faster?
My semi-intuitive answer was that they both accelerate at the same rate. Considering $F = ma$ of the

center of mass naively (considering only $mg\sin\beta$, the down-slope gravity), neither should win. Considering $\tau = I\alpha$ around the center, the torque is due to friction acting upwards; the larger cylinder has a larger moment of inertia (same mass, larger R), but it also gains a higher torque due to friction creating a torque along that longer R.
In the end, I figured the effects cancel, and α and a are the same.

Let's look at the analytical answer. First, here is the situation will all the forces and such drawn:



For pure roll, we can say that the velocity of the center, point Q, must equal the tangential velocity:

$$v_Q = \omega R \tag{12.47}$$

If we take the time derivative of this, we find

$$a_Q = \alpha R \tag{12.48}$$

$a_Q = a$ is then the linear acceleration of the cylinder down the slope.
Next, we look at torque. The normal force Mg (since there is no acceleration in the y direction, if we set y perpendicular to the slope, $N = F_{gravity}$) and gravity both act through the point Q, so they cannot cause any torque. ($\vec{r}$ in the cross product $\vec{r} \times \vec{F}$ is zero, so the cross product is zero.)

The only force that does cause torque is the frictional force F_f, which acts perpendicularly to the center point Q. Therefore, the torque about point Q is simply $\tau_Q = RF_f$.
The torque must be equal to $I_Q\alpha$; Newton's second law for rotational motion is $\tau = I\alpha$.
Another useful relationship is $a = \alpha R$, which is just the time derivative of $v = \omega R$. Therefore, $\alpha = a/R$.

Next, we can look at Newton's second law of translation, good old $F = ma$. In this case, we have a mass M on the left-hand side, and on the right-hand side, we have $Mg\sin\beta$ acting downhill, and F_f uphill:

$$Ma = Mg\sin\beta - F_f \tag{12.49}$$

From the torque and all that above, we also have

$$RF_f = I_Q\frac{a}{R} \tag{12.50}$$

$$F_f = \frac{I_Q a}{R^2} \tag{12.51}$$

With this, we can eliminate F_f in the first equation and find a:

$$a = g\sin\beta - \frac{I_Q a}{MR^2} \tag{12.52}$$

$$a\left(1 + \frac{I_Q}{MR^2}\right) = g\sin\beta \tag{12.53}$$

$$a = \frac{g\sin beta}{1 + \frac{I_Q}{MR^2}} \tag{12.54}$$

$$a = \frac{MR^2 g\sin\beta}{MR^2 + I_Q} \tag{12.55}$$

Now we just need to enter the moment of inertia of the object, and we're done. This is the fun part. The moment of inertia of a solid cylinder, about the axis of symmetry, is $I_Q = \frac{1}{2}MR^2$. This means that MR^2 in the acceleration cancels!

$$a = \frac{MR^2 g\sin\beta}{MR^2 + \frac{1}{2}MR^2} \tag{12.56}$$

$$a = \frac{g\sin\beta}{1 + \frac{1}{2}} \tag{12.57}$$

$$a = \frac{2g\sin\beta}{3} = \frac{2}{3}g\sin\beta \tag{12.58}$$

A very simple result indeed! It doesn't depend on mass, length or radius in *any way*. This result is valid for all *solid* cylinders (since they have the moment of inertia we used) in *pure roll*.

So the answer is indeed that if we "race" two solid cylinders, neither one wins. We don't need to specify anything further; nothing else than "solid" makes any difference at all.

12.3.1 Pure roll of a hollow cylinder

What if the cylinder is hollow? Either there is a small hole in the center, or it is essentially just a thin edge, or anything in between. In this case, the moment of inertia will be larger for the same mass and radius; in the case where all mass is practically at the edge, the moment of inertia is approximately MR^2, i.e. twice as high. If we substitute that into the equation,

$$a = \frac{MR^2 g\sin\beta}{MR^2 + MR^2} \tag{12.59}$$

$$a = \frac{g\sin\beta}{2} = \frac{1}{2}g\sin\beta \tag{12.60}$$

So the acceleration is now *less*, so it will take longer to reach the end. Any solid cylinder will beat any hollow cylinder, regardless of their masses, lengths or radii.
In the case where the cylinder is hollow, but we can't approximate it either as solid or a thin edge, the moment of inertia is $\frac{1}{2}M(R_i^2 + R_o^2)$ (for inner and outer radius; the math turned too ugly with the full words as indexes). The acceleration becomes

$$a = \frac{MR_o^2 g\sin\beta}{MR_o^2 + \frac{1}{2}M(R_i^2 + R_o^2)} \tag{12.61}$$

$$a = \frac{2R_o^2 g\sin\beta}{3R_o^2 + R_i^2} \tag{12.62}$$

$$a = \frac{2g\sin\beta}{3 + \frac{R_i^2}{R_o^2}} \tag{12.63}$$

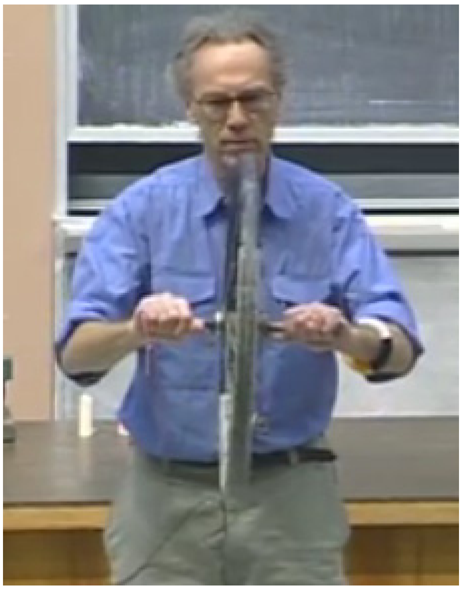
This is then the most general result we can have for a cylinder. Note that when $R_{inner} = R_{inner}$, we get the one-half $g \sin \beta$ we found for the very thin cylinder. For $R_{inner} = 0$, we get the two-thirds $g \sin \beta$ that we found for the solid cylinder. I have not verified this result, but it does produce the correct answers for the two cases mentioned, so I would assume it correctly predicts the behavior in between these cases, i.e. for $R_i = (0, R_o)$, also.
By the way, keep in mind that this result is also, in a way, independent of geometry. Only the ratio matters; a tiny cylinder and a huge cylinder could have the same ratio R_i^2/R_o^2, and would then have the same acceleration.

12.3.2 Gyroscopes and precession

"We now come to the most non-intuitive part of all of 8.01. And arguably, perhaps, the most difficult part in all of physics, and that has to do with gyroscopes."

I believe this lecture might be the hardest one yet to take proper notes of... and for technical reasons, I cannot view the videos on my computer, either, but am forced to do so on my smartphone. Not very ideal. In any case, everything important is in 3D, and so video is a much, much better format than screenshots here either way. I will still try, though.

Say we are somewhere in outer space (to escape any noteworthy gravitational force). We have a bicycle wheel with us, which is mounted such that there an axle sticking out on both sides. That is, we can hold the wheel while it rotates practically freely.



(The reasoning for $\tau = bF$ is touched upon briefly just after the first picture in lecture 25.)

If the professor then were to push *his* right hand forwards, while pulling his left right hand inwards, and apply a torque like that for a short amount of time, clearly the wheel will start spinning, counterclockwise as seen from above, and if we let it go, it will keep spinning like that forever. The torque causes a change in angular momentum, $\Delta L = \tau \Delta t$.

Next, we torque it so that the professor's left hand moves up, and his right hand moves down. This causes a rotation along a different axis, such that it spins counterclockwise as seen from our point of view (the angular velocity and angular momentum vectors are out of the screen). Again, it rotates like that forever along that axis.

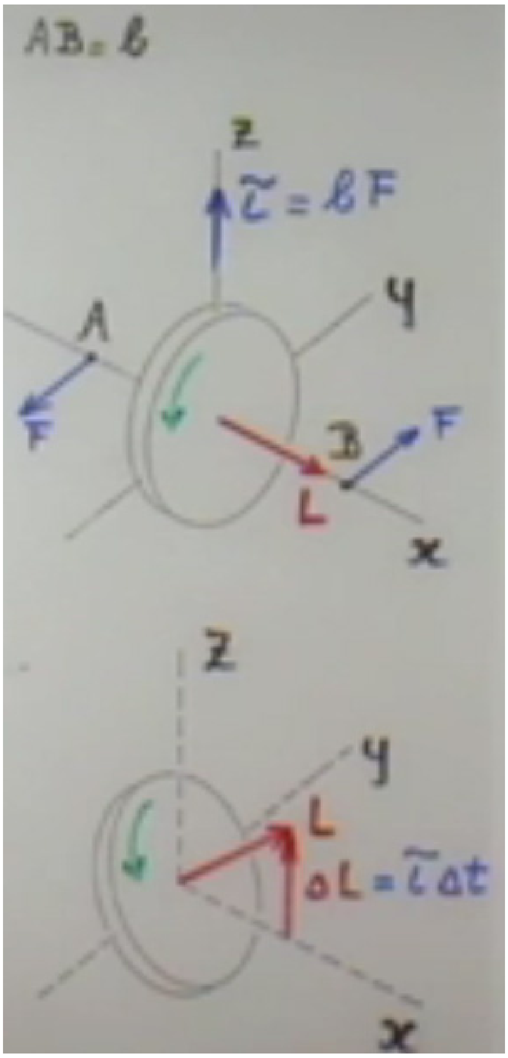
Now... We spin the wheel up (along the axis a bicycle wheel *should* rotate!), such that the angular velocity points to our right. What happens when he torques the wheel now?

The intuitive answer is, of course, that the wheel keeps spinning (it couldn't simply stop due to an unspecified amount of torque during an unspecified time) as a bicycle wheel does, while also rotating about the axis that we torqued it in. Without friction/air drag, both these rotations would continue on forever.

This is not what happens, though. It *cannot* happen, without some external torque applied forever! The reason is that the spin angular momentum is pointing towards our right, as the experiment begins. After the torque, the intuitive answer states that this spin angular momentum would be changing direction constantly! How can a vector change direction and rotate, without any external torque? It cannot! Something else must happen.

What does happen is rather bizarre, and perhaps the most nonintuitive thing in the entire course.

As an important note on notation. any time I use "spin" below, I am talking about the wheel spinning like a bicycle wheel is meant to do.
Any time I use "rotate", I am talking about rotating about a different axis; one that never happens when it is attached to an actual bicycle going in a straight line.



As can be seen from this picture, which describes this exact situation, the torque is upwards. The spin angular momentum will "follow" this torque, and *tilt the wheel* so that the angular spin momentum vector gets closer to the initial position of the torque.
That might not sound so strange, unless you keep the conditions in mind: the professor is doing a forwards/backwards push/pull on each side of the wheel, respectively, and instead of turning in the horizontal plane, it *tilts* to the side! This is something that perhaps must be seen to be believed.

It is, of course, possible to predict what will happen, when we take what we know about physics into account. One very helpful thing to remember is that the spin angular momentum vector will always "chase" the torque vector. In this case, the torque is upwards, while the spin angular momentum is initially towards the right. In this situation, as we have seen, the wheel tilts such that the spin angular

momentum is now pointing slightly upwards, as well.
When we reverse the torque, so the torque is downwards, the wheel tilts in the other direction.

What if the professor were to left his right hand up, and move his left hand down? Well, the spin angular momentum is towards the right to begin with. The torque, found as $\vec{r} \times \vec{F}$, now points either towards the blackboard (into the screen), or towards the audience (out of the screen). The wheel will now rotate as we would have expected it to rotate above. When the force along the moment arm is upwards, the wheel rotates so that the right axle (as we see it) points towards the audience.

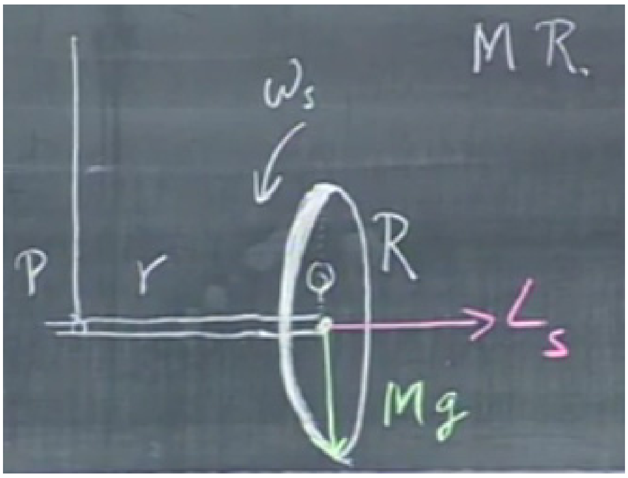
The professor then demonstrates this by sitting a a stool that is free to rotate, while applying torque as mentioned above. The result is that as long as he keeps applying that force, the stool spins. As soon as he stops, the stool stops; if he torques in the opposite direction, the stool rotates in the opposite direction.
This principle can also be applied in space – many satellites have "reaction wheels" that can be used to rotate the satellite, e.g. to keep it pointed in a certain direction.

This slower rotation, that the wheel experiences when exposed to a torque, is called *precession*.

12.3.3 Precession of a bicycle wheel on a string

Next, we change the experiment up a bit. Instead of holding the wheel, we attach a rope to the end of one axis, and one axis only, such that gravity causes a rather strong torque on the wheel, that wants to rotate it downwards (so that it can fall down and simply hang there). That is of course exactly what will happen – as long as the wheel is not spinning.

Here's what the setup looks like:



The thicker part of the wheel is towards the viewer.

The length of the axle is little r, while capital R is the radius of the bicycle wheel. It is attached to the rope at point P.
As the wheel spins, with angular velocity ω_s (s for spin; we will soon see why a subscript is needed), it has spin angular momentum L_s towards the right, as shown.
Gravity acts on the wheel's center of mass, which is approximately at the center of the wheel. Therefore, there is a torque relative to the center point Q, $\tau_Q = (\vec{r} \times \vec{g})M = rMg$, given that there is a right angle between the moment arm (the axle r) and the force.

The direction of this torque is into the blackboard.
Unlike the previous situation, in "outer space", there *is* now an external torque, due to gravity, that will act on the wheel forever. Since that torque vector is directed into the blackboard, what will happen is that, again, the spin angular momentum vector will "chase" that torque. The wheel will keep spinning, but also rotate (precess) *counterclockwise*, as seen from above.

The torque is also changing, since the direction of the axle is constantly changing! It will change in such a way that there is indeed a counterclockwise precession; that is easy to convince oneself of by using the

right-hand rule, especially if you use the entire right arm.
Point the right arm along the initial spin angular momentum vector, i.e. straight towards the right. Curl your fingers along the torque, exactly perpendicular, downwards (since gravity acts exactly downwards). Your right thumb now points "forwards", which is how the wheel will rotate.
A small amount later in time, the spin angular momentum will have chased the torque a bit in that direction, and so you need to rotate your entire right arm a bit forwards. Gravity is still straight down, but the new torque vector is slightly rotated compared to the first, so the spin angular momentum vector will never catch up, and the will will keep precessing (in the absence of losses; in reality, it will of course eventually fall down).

So how in the world can the wheel just stay up like that? Gravity acts on it, so it must fall – $a_{cm} = \dfrac{F_{ext}}{m}$! Well, no – there is a string tension in the problem! The net force upwards/downwards on the system is zero – the string tension equals Mg, so no downwards acceleration of the center of mass is necessary!
So there is no net force on the object. There *is* a net torque, however! This string tension does *not* cancel out the torque due to gravity! The reason is that the tension has no moment arm! It acts through the point P, so it cannot contribute to torque about that point. The wheel's center of mass is located r away from that point, so there is a torque rMg due to gravity, as we saw above.

We can calculate the precession frequency of the wheel. The result is

$$\omega_{pr} = \frac{\tau}{L_s} = \frac{rMg}{I_Q \omega_s} \tag{12.64}$$

The derivation for this is in the book: chapter 22, page 22-4 and forwards.

This is now why we used the subscript on ω previously. ω_s is the angular velocity of the spin, while $\omega_p r$ is the angular frequency of the precession – which is a much, much smaller value. The wheel will spin at several rotations per second, while it will take the wheel about 10 seconds to complete one rotation due to the precession.
Note that this equation is also valid as long as the spin angular momentum is way, way larger than the angular momentum due to the "orbital" motion about point P (the rope). The total angular momentum of the system is L_s plus the component due to the orbital motion, $I_P \omega_{pr}$ (where we have not calculated I_P).
The equation holds while the wheels spins quickly, but it predicts a precession frequency that goes to infinity as the wheel's spin slows down, which clearly doesn't make sense.

This result does make sense, though. If we increase the torque, the precession frequency will also increase. That makes some sense, since the torque is trying to "win" over the spin angular momentum, and force it to change to follow the torque. The stronger the torque, the easier it has to do this, and make the wheel rotate/precess.
On the other hand, the faster the wheel spins, the harder it is to precess. That also makes sense, for the same reason. The same happens as the moment of inertia along the spin axis increases, which also makes it tougher to attempt to change the spin angular momentum.

We can attempt to calculate the period of the precession. Using the above equation, we can use $r = 17$ cm, the length of the axle; $R = 29$ cm, the radius of the bicycle wheel; $f = 5$ Hz (approximate spin frequency of the wheel) so that $\omega_s = 2\pi f_s = 10\pi$ rad/s. With these numbers, we find

$$\omega_{pr} = \frac{rMg}{MR^2 \omega_s} = \frac{rg}{2\pi f_s R^2} \approx 0.631\,\mathrm{rad/s} \tag{12.65}$$

The period is as always $T_{pr} = \frac{2\pi}{\omega_{pr}} \approx 10\,\mathrm{s}$. So the wheel will spin with at about 5 turns per second (300 rpm), and rotate due to precession with about one turn per ten seconds.
This is then demonstrated, in one of the more interesting demonstrations of the course so far.

Finally, the last lecture segment demonstrates a toy gyroscope, and a 3-axis gimbal gyroscope; on this last one, the spinning disk is mounted such that external forces do not cause a torque on the spinning disk, but rather, it is mounted such that the frames it is mounted in will rotate instead. Therefore, the spin angular momentum vector is always pointing in one direction, no matter how you turn the outer part of the gyroscope. This is used in gyrocompasses, and inertial guidance systems, etc.

Chapter 13: Week 11

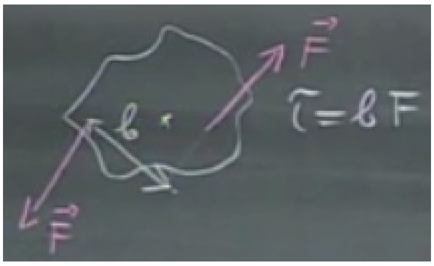
13.1 Lecture 25: Static equilibrium, stability, rope walker

We will begin by introducing the concept of static equilibrium. An object in static equilibrium is one that has no net force *and* and no net torque (relative to any point we choose).
Such an object has zero linear acceleration and zero angular acceleration. In general, we will also assume they are at rest to begin with.
As an example, most objects on your table are likely in static equilibrium.

$$\sum F = 0 \tag{13.1}$$
$$\sum \tau_Q = 0 \text{ (any Q)} \tag{13.2}$$

For the net force to be zero, the net force must be zero along any axis by itself. Therefore, we can also use $\sum F_x = 0$, $\sum F_y = 0$ and $\sum F_z = 0$.

It might be easy to think that if the net force on a rigid object is zero, then it is in static equilibrium. That is far from true, though! If we have two forces of equal magnitude, acting in opposite directions, and they don't act along the same line, then they cause a net torque! The object will rotate, but will *not* have any linear motion, since there is no net force.

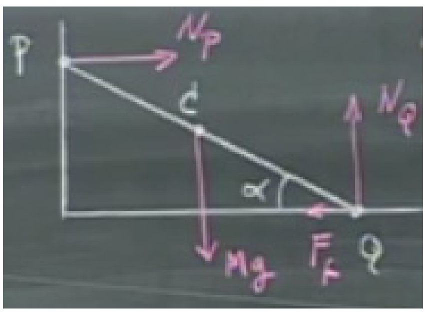


(We can think of each force along causing a torque $(b/2)F\sin\theta$, where $\theta = \pi/2$ so $\sin(\theta) = 1$. They cause a torque in the same direction, so the total torque is bF.)

These two forces form a *couple* (which I believe is a term used mostly in mechanics): together, they cause rotation, but *not* translation.
An example of this are the forces exerted by your hand on a screw driver (or the forces exerted on the screw by the screw driver).

In this lecture, a ladder will be used for the calculations and examples regarding static equilibrium.

We put this ladder against a wall, at an angle α ($\alpha = 0$ meaning it is on the ground, while $\alpha = 90°$ means it is standing straight up, parallel to the wall). Say there is no friction against the wall, but there is static friction μ at point Q, where the ladder touches the ground. We call the ladder's total mass M and its length ℓ.



The center of mass of the ladder is in the middle.
Now... what forces act on this ladder?

First out, we have a gravitational force Mg acting on the center of mass. we have a normal force N_Q where the ladder is in contact with the ground. Because the ladder wants to slide towards the right, there is a frictional force F_f towards the left at point Q.
We said there is no friction at point P, so there can only be a normal force from the wall, towards the right; we call that N_P.

Let's now try to figure out when this static is in static equilibrium, i.e. at what angles α we can leave it, and have it be stable and remain at rest.
Our definition of static equilibrium was that the sum of all forces must be zero, and that the sum of all torques relative to any point must be zero. Let's first look at the forces.

First, in the x direction. We have N_P and F_f, so the two must be the same in magnitude.

$$N_P = F_f \tag{13.3}$$

Next, the y direction. Again, we have two forces, and find

$$N_Q = Mg \tag{13.4}$$

After this, we move on to torque. The torque relative to any point – we can choose freely – must be zero. If we choose point Q, neither F_f nor N_Q can contribute to the torque (since they act through point Q), and so we choose that point to simplify our lives.

First, we have the torque due to N_P. Torque is $\vec{r} \times \vec{F}$; the position vector from point Q has length ℓ (the entire ladder's length), and the angle between the two is α. The cross product is then $\tau_{Q,N_P} = N_P \ell \sin\alpha$. The direction of this torque is into the blackboard, using the right-hand rule. We choose this as our positive direction, so this contributes a positive term to the net torque.

Next, there is a torque due to gravity pulling the ladder downwards. We model the gravity as acting purely at the center of mass, which is a length $\ell/2$ away from point Q. The angle between the two is not α, but $90° - \alpha$. Therefore, the cross product becomes

$$\tau Q, grav = \frac{\ell}{2} Mg \sin(90° - \alpha) = \frac{\ell}{2} Mg \cos\alpha \tag{13.5}$$

The direction is out of the blackboard, and so it contributes with a negative term. The two must be equal in magnitude, so

$$\sum \tau_Q = 0 \Rightarrow N_P \ell \sin\alpha = \frac{\ell}{2} Mg \cos\alpha \tag{13.6}$$

We can solve this for N_P:

$$N_P = \frac{M}{2} g \frac{\cos\alpha}{\sin\alpha} = \frac{M}{2} g \cot\alpha \tag{13.7}$$

This must then be equal in magnitude to the frictional force, as we found earlier, or there will be a net force in the x direction. This must always be smaller than the maximum possible static friction μMg, or the ladder will start to slide.

$$\frac{M}{2} g \cot \alpha \leq \mu M g \tag{13.8}$$

$$\cot \alpha \leq 2\mu \tag{13.9}$$

$$\alpha \leq \arctan \frac{1}{2\mu} \tag{13.10}$$

The larger μ is, the smaller the angle can be without any sliding – that is, the ladder can be closer to the ground, while still being held back by friction.
When μ is very small, it will slide at almost any angle, as we would expect. This is then demonstrated: smaller angles are less stable.

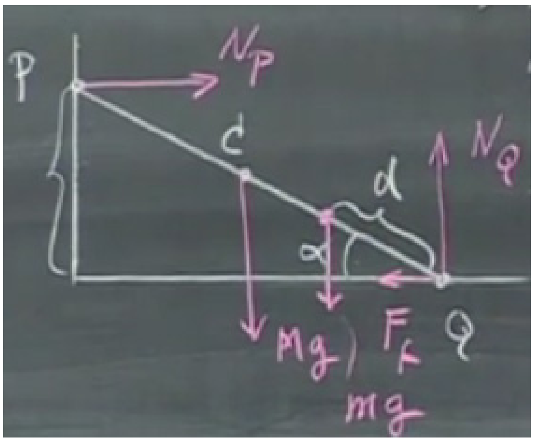
13.1.1 Adding a mass along the ladder

Let's now consider what happens when we actually use this ladder. Suppose we set it up just at the critical point, so that it is just about to slide. What happens if we stand near the bottom of the ladder, closer to point Q and far below the center of mass?
Will the ladder be more stable, less stable, or is there no change?

Let's consider what happens (rather quickly, as we will perform a full analysis soon). N_Q will increase, which also increases the maximum possible frictional force $F_{fmax} = \mu N_Q$. This makes it seem likely that the ladder becomes more stable, as the maximum possible friction is now larger.
Since the actual friction $F_f = N_P$, has N_P increased? I don't see why it would, so the system should become more stable.

What if the person keeps climbing, and moves past point C (the center of mass / center of the ladder), and keeps moving up towards point P? Is it now more stable, less stable, or does it not matter that the person is there?
Here (a bit unlike the previous case) I find it intuitively clear that this is *not* a very safe thing to do. You add extra force downwards near the top of this about-to-fall ladder. This contributes to a torque that wants to rotate this ladder such that you fall down. It should also cause N_P to increase, perhaps so that the required friction is now above the maximum possible. The system becomes less stable, as we will soon see.

We add a person of mass m to the ladder, a distance d away from point Q, measured diagonally along the ladder.



We then re-do the above calculations considering this extra mass.

For the x direction, we still find $N_P = F_f$, as before.
In the y direction, we have an extra downwards force, that N_Q must balance out for the net force to be zero:

$$N_Q = (M + m)g \tag{13.11}$$

This changes the maximum friction possible to

$$F_{fmax} = \mu N_Q = \mu(M+m)g \tag{13.12}$$

which is clearly an increase from the previous case.
For the net torque, the first two terms are unchanged, but we add a third term due to the person of mass m at distance d:

$$\sum \tau_Q = 0 \Rightarrow N_P \ell \sin\alpha = \frac{\ell}{2} Mg\cos\alpha + mgd\cos\alpha \tag{13.13}$$

The angle is calculated in the same way as last time. Again, we solve for N_P:

$$N_P \ell \sin\alpha = \frac{\ell}{2} Mg\cos\alpha + mgd\cos\alpha \tag{13.14}$$

$$N_P \ell \sin\alpha = g\cos\alpha \left(\frac{\ell}{2}M + md\right) \tag{13.15}$$

$$N_P = \frac{g\cos\alpha}{\ell\sin\alpha}\left(\frac{\ell}{2}M + md\right) \tag{13.16}$$

$$N_P = g\cot\alpha\left(\frac{M}{2} + \frac{md}{\ell}\right) \tag{13.17}$$

And again, $N_P = F_f$ in magnitude, since they are the only two forces in the x direction.

Since we have a new term $g\cot\alpha\frac{md}{\ell}$, the frictional force has gone up. However, the maximum possible friction $F_{fmax} = \mu(M+m)g$ has also gone up!
In order to find which matters most, consider the case where the person is moving up the ladder gradually. To begin with, $d = 0$. We then gradually increase it. At $d = 0$, the frictional force has not changed at all, but the *maximum possible* has! Therefore, the system is *more stable* with this extra mass near the bottom. What happens as this mass is moving up along the ladder?

The maximum friction possible is independent of d, so that will always have the same, new value of $F_{fmax} = \mu(M+m)g$. However, as we climb, the actual frictional force $F_f = N_P$ (see above) does go up, due to new term we added.
When a certain point is reached, we are back to the just-about-to-slide situation again. If we climb higher yet, we are past that point, and the ladder will slide.

The condition we care about is that the frictional force is less than the maximum possible (when that is still the case, the ladder won't slide).

$$g\cot\alpha\left(\frac{M}{2} + \frac{md}{\ell}\right) \le \mu(M+m)g \tag{13.18}$$

However, we set the angle at the critical point we found earlier, where $\cot\alpha = 2\mu$. We can substitute that in and solve for the largest d possible for this inequality to hold:

$$2\mu\left(\frac{M}{2}+\frac{md}{\ell}\right) \leq \mu(M+m) \tag{13.19}$$

$$M+\frac{2md}{\ell} \leq M+m \tag{13.20}$$

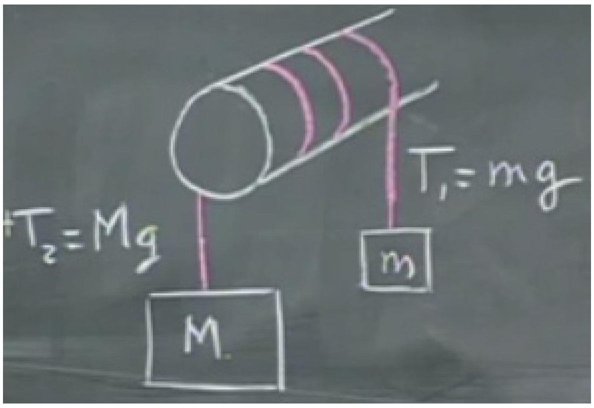
$$2d \leq \ell \tag{13.21}$$

$$d \leq \frac{\ell}{2} \tag{13.22}$$

Quite an awesome result – though perhaps one that could have been guessed, all things considered. Once we walk past (higher than) the center of mass, the situation is less stable that it was before. If we stand exactly *at* the center of mass, we are at the just-about-to-slide stage. If we stand further down than the center of mass, the system is more stable than it was to begin with.

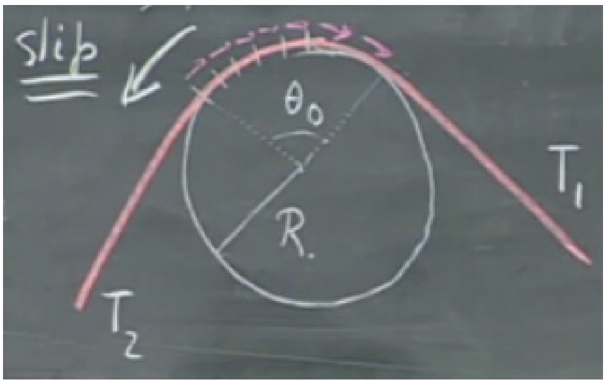
13.2 Rope around a cylinder (capstan equation)

We can often use friction to our advantage. One example of such a case is often used by sailors: by wrapping a rope around a cylinder, we can use friction forces as a "substitute" for string tension. We can have a very large force pulling on one end of the rope, which is countered in part by tension on the other side of the rope, but in larger part due to friction between the rope and the cylinder it is wrapped around.



Here, we have two masses attached to either end of a rope. The middle part of the rope is wrapped several times around a cylinder.
If the system is at rest, $T_1 = Mg$ and $T_2 = mg$. The mass M is much greater than the mass m.

The reason the system can be at rest under these circumstances is that friction between the cylinder and the rope is trying to hold the rope in place.



If we consider the friction at the surface of each tiny length element of the rope, it is clear that the rope wants to slip towards the left (counterclockwise), and therefore, friction is attempting to oppose this motion. T_2 is pulling towards the right, but T_1 *plus* the total frictional force is pulling towards the right. With enough friction, T_1 can be very small compared to T_2 and we can still have static equilibrium.

The result (the derivation is fairly complex and thus not shown) is that the ratio of the two tensions is given by

$$\frac{T_2}{T_1} = e^{\mu\theta_0} \tag{13.23}$$

where e^x is the exponential function, μ is the coefficient of static friction between the rope and the cylinder, and θ_0 is the angle over which the rope is in contact with the cylinder. There is no particular limit on this angle: it could be wrapped 10 turns, which would make $\theta_0 = 20\pi$.
This is known as the *capstan equation*. My dictionary says that a capstan is "a revolving cylinder with a vertical axis used for winding a rope or cable, powered by a motor or pushed around by levers".

Notice that this result is independent of the cylinder's radius; only the angle matters.
The result is also exponential, so it grows extremely quickly. Adding an extra turn can make a tremendous difference in the tension ratio. For example, consider $\mu = 0.3$.

If we wrap it around exactly once, the ratio is $\frac{T_2}{T_1} = 6.59$ (already an impressive number). Two turns makes the ratio 43.38, three turns 285.7... eight turns 3.54×10^6! So the first mass could be one ton, and in theory the tension from a second mass of less than one gram would be enough to counter it. Clearly, we wouldn't need to go to such extremes for this to be very useful. A mere 4 turns is enough for 1 kg to counter more than 1800 kg (or 1 N to counter more than 1800 N; since the ratio is all that matters, we can compare hanging masses in kg and tensions in newton just the same).

Let's go back to a less extreme example. If $\mu = 0.2$ and we wrap the rope 6 turns around, the ratio is about 1881, call it 2000. (If $\mu = 0.202$ instead, it goes above 2000, so it's certainly not far from it!)

With a 10 000 kg mass M hanging on the left side, we could balance that force with a $10\,000\,\text{kg}/2000 =$ 5 kg mass on the right side! Alternatively, we could hold the rope in our hands and have no problem at all balancing this 10 ton mass on the other side.

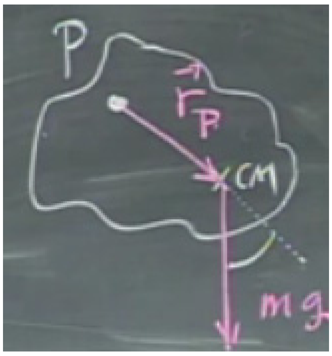
What would happen if we let go just a little, and reduced our force from the approximately 50 newtons $(5\,\text{kg})g$ to just a tiny bit less? Since this ratio is for the no-slip condition, it will start to slip, and the huge mass will win, and move downwards. We can therefore control this large mass, and lower it down gently, with barely any force at all exerted by us.

What about raising the mass up? We now run in to a problem... a very, very big problem. We now want the rope to slip in the opposite direction, so that it comes towards us. This means that all those tiny friction elements between rope and cylinder switch direction, to as always oppose the relative motion between the two surfaces. To lift this object up, we must now overcome that friction *and* the object's weight... so we must provide a force *2000 times larger* than the object's weight in order to move it, which for a 10 ton mass is the same as hanging a 20 000 ton (20 million kg) mass on this side!
We should perhaps stick to lowering things down (and balancing things) using this mechanism.

This (balancing a heavy object) is then demonstrated, with a mass of approximately 30 kg hanging from a rope. After winding the rope a few times around a cylinder, the mass hangs there with almost no counterbalancing force at all. With 12 windings (perhaps even less, as 9, 10 or 11 was not tested) the weight of the remaining rather thin rope itself was enough to balance the 30 kilograms without having to even touch the rope.

13.2.1 More on static equilibrium

Consider an object of some shape (any shape), hanging on a pin somewhere (somewhere that is not through the center of mass, or the point of this will be lost).



Gravity acts on the center of mass, which is not located at point P, so there will clearly be a torque. The object will start to rotate, and practically become a pendulum. Suppose we either wait until friction takes care of that, or we stop it manually. Clearly, at some point, it will reach a static equilibrium and stand completely still.

This can only happen when the center of mass is on a straight, vertical line with the point P from which it is suspended. In any other case, $\vec{r_P} \times \vec{F_{grav}} \neq 0$ and so there is a torque relative to point P, causing it to rotate.
More specifically, the only stable situation occurs when the center of mass is straight *below* the point of suspension. Any time it is straight *above*, any tiny disturbance will of course cause it to turn 180 degrees and then (once stopped) be in static equilibrium like that, instead.

We can then use this information to determine the center of mass very easily (at least in the case of a practically two-dimensional object). We hang it from one location, and draw a line from the pin (that we hang it from) and straight downwards.
We then move the pin to some other location, let it settle, and again draw a line straight downwards.

Since the center of mass must have been straight below the point in both two cases, there will be a unique point where the two lines intersect, where the center of mass is located!
Clearly, we can't find it from a single measurement: we only know that it will be straight below; it could be *anywhere* straight below though (within reason – it can't be below the lowest point of the object, of course)! With two measurements, we can narrow it down to a point on a 2D surface.

13.3 Lecture 26: Elasticity and Young's Modulus

We will now look at elasticity in materials. First, we will look at a similar situation with springs and Hooke's law.

Say we have a regular spring, with length ℓ and spring constant k. We extend it a length $\Delta\ell$ past its natural length, and according to Hooke's law, the spring force is $-k\Delta\ell$ – a force which is trying to return the spring to its natural length. If we pull twice as hard, $\Delta\ell$ will double, and the spring force will also double.

Now, consider instead doubling the natural length of the spring. For the same pulling force, we now get twice the extension $\Delta\ell$. We can consider this as having two identical springs in series, instead of doubling the length of one, as the physics are the same. Each spring will experience the same pulling force, and so each spring individually will get longer by $\Delta\ell$. Therefore, the spring that is twice as long is extended twice as much.

If we instead have two (still identical) parallel springs, i.e. two springs are fastened at some wall, while a block or such is put on the other side and attached to each spring individually, they will have to share the load. Therefore, if we pull with a force F, each spring will respond with a force $-F/2$, so that the net force due to the two springs cancels out our pulling force.
Since they are still identical, with the same k, they must each be extended by only half as much as previously, so that the sum of the force due to both springs equals the pulling force.

If we had three springs in parallel, each would only have to provide a third of the force, and would only extend a third as much.

With these short thought experiments in mind, we have found these three relationships for these springs:

$$\Delta\ell \propto F \tag{13.24}$$
$$\Delta\ell \propto \ell \tag{13.25}$$
$$\Delta\ell \propto \frac{1}{\#\text{ of springs in parallel}} \text{ (for identical springs)} \tag{13.26}$$

Let's now move on to the subject at hand. We replace the springs by rods (say metal rods, for example) with a cross-sectional area A and length ℓ. We apply a force at one end of the rod (while it is fastened at the other end, like the springs).

If we new consider this rod as a spring, pulling on this rod with a force F will again extend it a distance $\Delta\ell$. As long as Hooke's law holds, i.e. that the restoring force is proportional to $\Delta\ell$, we can again say that $\Delta\ell \propto F$.

What if we put two rods together, so that the length doubles? (That is, we put them "in series".)
Again, we get the same result: each rod experience the force F, so each rod extends by $\Delta\ell$, and the extension doubles by doubling the length of the rod. $\Delta\ell \propto \ell$ holds for the rods, too.

Finally, what if we have two next to each other, in parallel? We pull downwards with a force F, that is shared by the two rods. Each rod only needs to counter half of our pull, and so they extend by $\Delta\ell/2$, just like the springs did. With twice the cross-sectional area A of the rods, we get half $\Delta\ell$. Therefore, $\Delta\ell$ is inversely proportional to the cross-sectional area of the rods.

All in all, for the rods, we find

$$\Delta\ell \propto F \tag{13.27}$$
$$\Delta\ell \propto \ell \tag{13.28}$$
$$\Delta\ell \propto \frac{1}{A} \tag{13.29}$$

We can write this as a single proportionality:

$$\Delta\ell \propto \frac{F\ell}{A} \tag{13.30}$$

Reordered,

$$\frac{F}{A} \propto \frac{\Delta\ell}{\ell} \tag{13.31}$$

The proportionality constant Y (or E) is called Young's modulus. The equation becomes

$$\frac{F}{A} = Y\frac{\Delta\ell}{\ell} \tag{13.32}$$

Because the fraction on the right has no dimension, while the left-hand side has dimension force per unit area, which is pressure, Y also has units of pressure (newtons per square meter).

In this equation, $\frac{F}{A}$ is called the *stress* while $\frac{\Delta\ell}{\ell}$ is known as *strain*.

If we compare two rods with different value for Young's modulus, the one with the smaller value is easier to stretch out: it gives a larger strain for a given stress. If the Young's modulus is very high, the rod is very stiff.

As an example, if we hang a mass of 500 kg of a cylindrical steel rod of radius 0.5 cm and length 1 meter, how much will it stretch? We can start by solving the equation for $\Delta\ell$:

$$\frac{F\ell}{AY} = \Delta\ell \tag{13.33}$$

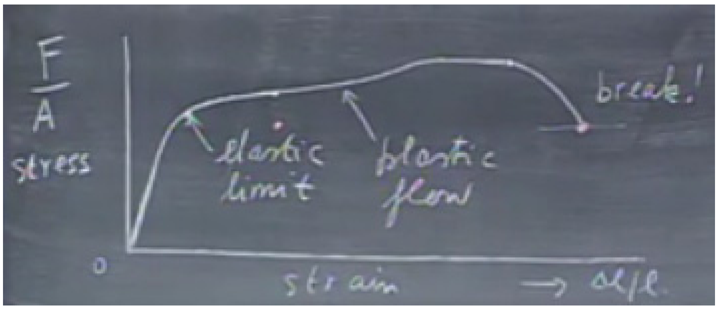
$F = (500\,\text{kg})(10\,\text{m/s}^2) = 5000$ N. $A = \pi R^2 = 7.85 \times 10^{-5}\,\text{m}^2$. Y for steel is given as $20 \times 10^{10}\,\text{N/m}^2$. We find $\Delta\ell = 0.32$ mm.
If we instead use nylon with $Y = 0.36 \times 10^{10}\,\text{N/m}^2$, it would stretch 18 mm.

The stress in this case is about $6.4 \times 10^7\,\text{N/m}^2$ (for either material – it does not depend on Y, only on the force and the cross-sectional area).

If we keep adding more mass, there is clearly a point where the rod will simply break. That breaking point is at the *ultimate tensile strength*, which is given as a pressure. In other words, when $\frac{F}{A}$ becomes too large, the rod breaks.
The ultimate tensile strength of both steel and nylon are such that neither would break given the load we calculated above; even the nylon could tolerate a few times this force. If the bar was made out of aluminium however, which has an ultimate tensile strength of about $7.8 \times 10^7\,\text{N/m}^2$, we would be rather close to the breaking point.

Before the material breaks, our equation will stop working, as the strain will no longer be proportional to the stress. Hooke's law no longer holds. If we overload the material like that, it will not return to its original length again, but will be permanently deformed. This also has an analogue with springs: pull too much on a spring, and the restoring force becomes nonlinear, and the spring will eventually be permanently deformed.

Let's have a look of what a stress vs strain plot may look like.



The first part of the curve is linear: this is where Hooke's law applies. The next part of the curve is nonlinear, and goes to a point known as the *elastic limit.* Even though it is not linear, as long as we do not pass the elastic limit, the material will return to its original length after we return the force.

Once we pass the elastic limit, however, the material will be permanently deformed.
Past this limit, increasing the stress by small amounts will cause very large amounts of strain: That is, the material will be much easier to stretch. If we then remove the force, the material would not return back to where it was. That also implies that if we create a graph like this one by gradually increasing the force and plotting $\frac{\Delta\ell}{\ell}$, if we gradually remove the force, the strain will not follow the curve back to zero, but will instead return to somewhere to the right of the origin.
The work we have done in pulling will go in part to deforming the rod, which takes energy, and in part to heat: the rod will heat up, sometimes quite substantially.

Past the elastic limit, but before the breaking point, there is a completely horizontal part of the curve. This implies that with no change at all in the stress (y axis), the strain will increase anyway, and the rod will stretch without any extra force being applied. We call this *plastic flow*. In this region, the material acts almost like a liquid, flowing towards the force.

Prior to breaking, the stress is lower than it was in the plastic flow region. The reason is that the material can start to pinch, so that it gets thinner at a point. That will cause the cross-sectional area to decrease, and so $\frac{F}{A'}$, where A' is the new cross-sectional area, will be larger than $\frac{F}{A}$ for other points.

There are machines designed to test materials, and create plots like this one. They then increase the value of F very gradually, and measure the strain. In the linear region, and the nonlinear region around the elastic limit, this is straightforward.
Once they start reaching the plastic flow area, however, they decrease the force. If $\Delta\ell$ increases anyway, they decrease it further. It then becomes possible to trace out the entire curve, until the breaking point.

Brittle materials generally have a curve with many of the same characteristics, but they are practically squashed together right-to-left, so that all these regions occur for smaller values of the strain.

Next we have a very long demonstration with a set-up and measuring of these values of strain vs stress, and plotting them on the blackboard. As usual, I don't really take any notes during demonstrations.

As one of many results of the demonstration, we find that in the linear region, the 36 cm copper rod has only expanded with about 0.13%. Any further expansion was not linear, and eventually entered the plastic flow region, where adding 1 kg more (for a total of 5 kg) hanging from the copper string would *double* $\Delta\ell$ – not very linear at all!

A typical breaking point for metals would be at 5% to 10% extension over the original length, or so.

13.3.1 Elasticity and simple harmonic oscillations

In the linear portion, just as with springs, there is a restoring force that is linearly proportional.to the extension distance. Assuming this also holds for compression (which appears to have been silently assumed in the lecture), this forms a simple harmonic oscillator. We can solve the equation regarding the Young's modulus for F:

$$F = \frac{AY}{\ell}\Delta\ell \tag{13.34}$$

Here, we can think of $\frac{AY}{\ell}$ as the spring constant k (the units are indeed N/m), while as we saw earlier in the lecture $\Delta\ell$ is really just a different way of notating the displacement x. Stick a minus sign in there to take care of the direction (it's a restoring force), and we have an equation that is the same as that of a spring oscillator with $k = \frac{AY}{\ell}$, which gives us

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{AY}{\ell m}} \tag{13.35}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell m}{AY}} \tag{13.36}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{AY}{\ell m}} \tag{13.37}$$

For $Y = 11 \times 10^{10}\,\mathrm{N/m^2}$, $m = 3$ kg, $A = 2 \times 10^{-7}\,\mathrm{m^2}$ and $\ell = 0.36$ m, we find a frequency of about 23 Hz.

Chapter 14: Week 12

14.1 Lecture 27: Gases and incompressible liquids

Say we have a vessel containing a fluid, where a fluid is either a liquid *or* a gas[1]. That is, a fluid does *not* refer exclusively to a liquid, unlike colloquial usage of the word.
It has an opening of area A, where we apply a force F.

The pressure at the opening is by definition $P = \dfrac{F}{A}$, measured in pascal (1 Pa = 1 N/m^2).
In the absence of gravity, the pressure everywhere inside this vessel in the same; this is known as *Pascal's principle*.

According to Pascal's principle, a pressure enclosed to an enclosed fluid is transmitted undiminished to every point of the fluid, and to the walls of the container.
Pressure is a scalar, i.e. it has no direction. Force has a direction, of course, though.
The force exerted on the walls of the container must, at all points, be perpendicular to the wall, in a static situation.
If there was a tangential component to any such force, that net force would cause movement in the fluid, and we are no longer in a static situation.

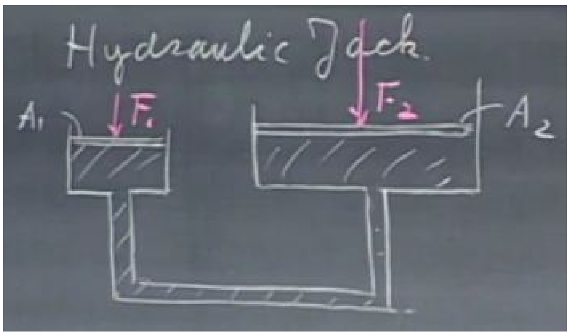
As a result of this, for a small area element ΔA of the container, we can relate the force on that area ΔF with the pressure:

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{14.1}$$

Pascal's principle leads has many interesting consequences, some of which are not very intuitive. First, we shall look at a hydraulic jack.

14.1.1 Hydraulic jack

We have a container, containing a fluid (a practically incompressible liquid); see drawing:



[1]Or more rarely other states of matter; we will only discuss liquids and gases, however.

The left side has a piston of area A_1, and the right area a piston of area A_2.
We apply a downwards force F_1 on the left side, and a downwards force F_2 on the right side.

According to Pascal's principle, the pressure everywhere in this container is the same. Since the pressure below each piston is that force divided by that piston's area,

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2} \tag{14.2}$$

when the liquid is not moving. This is also assuming we can neglect gravity, which we will discuss shortly.

We can then design this system such that $A_2/A_1 = 100$. Rearranging the equation,

$$\frac{A_2}{A_1} = \frac{F_2}{F_1} \tag{14.3}$$

We could then have the situation where $F_2 = 100F_1$, so that we could balance out a very large force with a much smaller one – similarly to a capstan.

Unlike a capstan, we can use this system to lift very heavy weights easily. We could put a mass of 10 kg on the left piston, and a mass of 1000 kg on the right, and the system would be in equilibrium.
This is used, for example, to lift cars. As we expect, if we increase the force F_1 a small amount, that piston will go down, which will force the other to go up, lifting the heavy object using a much smaller force.

So how does this work in regards to energy?
Well, consider we push the left piston down a distance d_1. We displace a volume d_1A_1 of fluid. This fluid has nowhere to go but the right side, where it moves the right piston a distance d_2, displacing a volume $d_2A_2 = d_1A_1$.

Using the above equation,

$$d_2 = d_1\frac{A_1}{A_2} \tag{14.4}$$

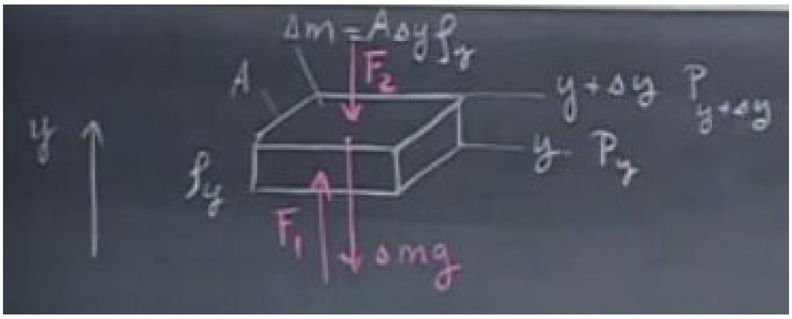
And since $A_2 > A_1$, we see that we must push d_1 down a lot to raise d_2 a little. In other words, the work we do, F_1d_1, is equal to the work done at the right side, F_2d_2, assuming no losses.
With this ratio, you would have to move the left piston a distance $d_1 = 100$ m, to raise the right piston $d_2 = 1$ m – rather unpractical if you want to lift a car, for example.
However, we can design such a jack so that we can move it a short distance by applying a force with a lever, and then lower it down again, and repeat. This way, we only raise the car (or whatever we are trying to lift) a very small distance at a time, perhaps less than a centimeter, but can repeat the process until we reach the height we want.

14.1.2 Pressure due to gravity: hydrostatic pressure

Until now, we ignored the effect gravity would have on such a system; we will now (essentially for the rest of the lecture) discuss pressure in fluids in the presence of gravity.

Consider a liquid inside some container. We will look at a small "slab" of liquid, which is then everywhere surrounded by more of that same liquid.
We look at a piece of area A, height Δy and density ρ_y – i.e. the density may be a function of y.

We have increasing values of y upwards. The call the coordinate at the bottom of the slab y, and the one at the top is then $y + \Delta y$.
The pressures at the two depths are then P_y and $P_{y+\Delta y}$, respectively.



The mass of this "element" of liquid is $\Delta m = A\Delta y \rho_y$, i.e. simply the volume times the density.
Now, what are the forces on this element? First, there is gravity, Δmg, acting downwards. There is then a force F_1 upward, due to the pressure on this element. Keep in mind that the pressure is everywhere perpendicular to a surface – even on imaginary surfaces like this one.
It then also comes in from the top, with force F_2. The forces in the horizontal plane all cancel.

For there to be equilibrium, we apply Newton's second law:

$$F_1 - F_2 - \Delta mg = 0 \tag{14.5}$$

By definition, F_1 is the pressure at that level, times the area, $F_1 = AP_y$. For the same reasons, $F_2 = AP_{y+\Delta y}$.
We can then substitute in the expression we had for Δm, and find

$$AP_y - AP_{y+\Delta y} - A\Delta y \rho_y g = 0 \tag{14.6}$$
$$P_y - P_{y+\Delta y} - \Delta y \rho_y g = 0 \tag{14.7}$$
$$-\Delta y \rho_y g = P_{y+\Delta y} - P_y \tag{14.8}$$
$$-\rho_y g = \frac{P_{y+\Delta y} - P_y}{\Delta y} \tag{14.9}$$

A cancels, and we can then rearrange this a bit, as shown above. Finally, we can take the limit as $\Delta y \to 0$ and we see that what we have is the definition of a derivative,

$$\lim_{\Delta y \to 0} \frac{P_{y+\Delta y} - P_y}{\Delta y} = -\rho_y g \Rightarrow \frac{dP}{dy} = -\rho_y g \tag{14.10}$$

This equation shows us the definition of *hydrostatic pressure*. As the equation tells us, hydrostatic pressure is there because of gravity.

Most liquids are in practice almost completely incompressible, meaning that the density is practically constant, so we can really change ρ_y into ρ above.
Even at an ocean depth of 4 km, at pressures of almost 400 times atmospheric pressure (400 atm is about $4000\,\mathrm{N/cm^2}$) the decrease in volume of water is less than 2%. Gases, on the other, are often very compressible.

Say we have a liquid in a container, and apply a force on the top (like in the case we had in the beginning of the lecture), we could not get a measurable change in the density using any reasonable force we as humans could apply. With machines, of course, we absolutely could compress it.

If we hit an air-filled plastic pillow with a sledgehammer, the air would act as a cushion. If we instead hit a marble floor, the pressure on the sledgehammer (and on the floor) would be way higher than that of the pillow, since the marble floor is almost completely rigid and incompressible, so this "cushioning" effect is gone.

Now consider two metal paint cans. One is completely filled with water (with no air at all inside), while another is filled with air (at atmospheric pressure).
If we hit these two cans with a sledgehammer, there would again be a cushion effect one the one filled with air, while the force (and pressure) on the one filled with water would be much higher.

If we now fire a bullet into these containers instead, what happens? The area where the bullet hits is very small, but the force is clearly very high. With these two effects in combination, the pressure will be extremely high.
Pascal's principle says that the pressure will propagate undiminished in the fluid.
In the one filled with air, there is not much of a problem: the air is glad to change its volume/density to take care of this.
In the one filled with water, however, the pressure is extremely high, and the can may will explode due to the extreme pressure on the sides of the can, as the water won't compress any noticeable amount.

14.1.3 Pascal's law

From now on, we will assume that liquids are completely incompressible.
With that in mind, we can now treat ρ as a a constant, and calculate the pressure change as a function of a change in depth, via separation of variables. Again, considering that $+y$ is upwards, and $y_2 > y_1$ (below),

$$\frac{dP}{dy} = -\rho g \tag{14.11}$$
$$dP = -\rho g\, dy \tag{14.12}$$

We can integrate both sides, from y_1 to y_2 and P_1 (pressure at y_1) to P_2 (pressure at y_2) respectively. g is also constant, so the integrals are just the integrals of the differentials themselves (think of it as $\int_a^b 1\, dx$).

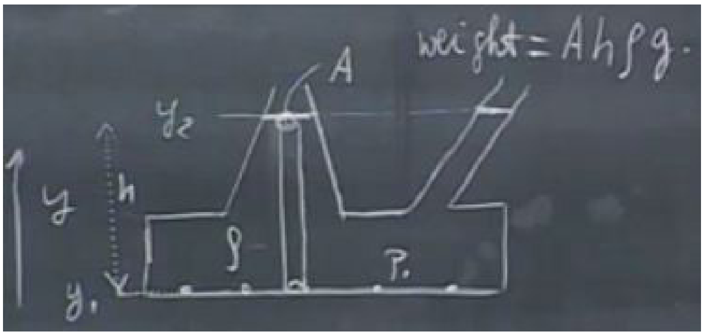
$$\int_{P_1}^{P_2} dP = -\rho g \int_{y_1}^{y_2} dy \tag{14.13}$$
$$P_2 - P_1 = -\rho g(y_2 - y_1) \tag{14.14}$$

Equivalently, we can multiply both sides by -1 and get

$$P_1 - P_2 = \rho g(y_2 - y_1) \tag{14.15}$$

which you may or may not prefer.
This result is known as *Pascal's law.*

Consider a strange-shaped vessel containing a liquid:



According to Pascal's law, the pressure at the bottom must be the same, at all points along the bottom (assuming the liquid is static).
However, consider the point just below the marked cylinder: the water column has a weight $Ah\rho g$ – its

volume Ah, times the density ρ which gives its mass, times g which gives its weight. The pressure at the bottom is this weight divided by the area, i.e ρhg.

However, the pressure just below that column must be exactly the same as the pressure in the corner, where the water column above is much smaller. Not that intuitive.
If we think of it in terms of requiring no net force for a static situation, it does make sense, but from the perspective of weight, it does not.

What is the pressure difference due to gravity for a water column that is 10 meters high?

Well, using $P_2 - P_1 = \Delta P$ and $y_2 - y_1 = \Delta y$ we find, using the previous formulas, $|\Delta P| = \rho g \Delta y$.
ρ for water is $1\,\mathrm{g/cm^3} = 1000\,\mathrm{kg/m^3}$. Using $g = 10\,\mathrm{m/s^2}$ and $\Delta y = 10\,\mathrm{m}$, we find $\Delta P = 10^5$ Pa, which incidentally is very close to 1 atmosphere of pressure (i.e. the pressure the air exerts on us, all the time), which is defined as 101325 Pa; more on that soon.
This is a very useful thing to remember: there is an additional 1 atm of pressure for each 10 meters you go down in water.

14.1.4 Atmospheric pressure and barometers

"We live at the bottom of an ocean of air", as the professor says.

Unlike liquids, the density of air changes noticeable with altitude (clearly: as we go up, sooner or later, the density is almost exactly zero, out in space, and the change is gradual), so we can't do the very simple integration we did earlier with the ρ of air.
We can weigh it, though. Look back to the case of the strange-shaped vessel with water: the pressure at the bottom, below a column of water stretching all the way up, was the same as the weight of that column, divided by the area.

In the same way, if we weigh a column of air stretching up to the edge of the atmosphere, we would measure a weight of approximately 10 N per square centimeter. There are 10000 square centimeters in a square meter, so the pressure is about $10^5\mathrm{N/m^2} = 10^5$ Pa.

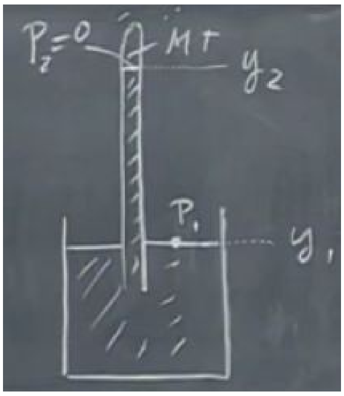
More exactly, atmospheric pressure is, as mentioned earlier, defined to be exactly 101325 Pa. It varies with the weather, altitude, etc., but is often relatively nearby. In my case, living within 25 m of sea level, I find it rare to look at a barometer and see a value outside the range 970-1030 hPa (i.e. 97000-103000 Pa).

If we hold out our hands, we feel a force equivalent to about 150 kilogram-force or kgf[2] pushing downwards. However, there is also a force of almost exactly equal magnitude pushing up on the hand's underside, as well as horizontal forces in many directions. All of these forces are, as mentioned earlier, exactly perpendicular to the hand, if the air is not moving.

We can measure the atmospheric pressure in a rather different way. We emerge a hose completely in a liquid of known density, and block off the the top end. The liquid will stay inside the tube, so that it is filled all the way, up to a certain height. For water, this height is about 10 meters; at that point, the liquid "lets go" and the very top of the tube will contain a vacuum.

Consider now instead a glass tube instead of a hose, and mercury instead of water. Mercury is way denser than water, so the height required will also be way less than the 10 meters required for water.

[2] 1 kgf = 1 kg times $9.806\,65\,\mathrm{m/s^2}$; the unit is used so that we can talk about forces in terms of kilograms, which are more familiar in daily usage than newtons.



(MT means empty, nothing more; I'm not sure why it is written in short.)

P_1 is clearly just the atmospheric pressure, since it is in direct "contact" with the outside air.
A distance $y_2 - y_1 = h$ above, the pressure is $P_2 = 0$, since there is a vacuum.

We have the formula $P_1 - P_2 = \rho g(y_2 - y_1)$, but since $P_2 = 0$ and P_1 is the atmospheric pressure, using the definition of h, we have

$$P_{atm} = \rho g h \tag{14.16}$$

where h then is the height of the column of mercury. This device is known as a *barometer.*
Using $P_{atm} = 101325$ Pa as an example (it can clearly differ), combined with $g = 9.81\,\mathrm{m/s^2}$ and $\rho = 13.6 \times 10^3\,\mathrm{kg/m^3}$, we find that the height of the column will be about 760 mm.
Because mercury barometers were quite common in the past, this pressure is often referred to as 760 mmHg. Other pressures are also measured in mmHg – "millimeters of mercury"; blood pressure is almost always measured in mmHg (the "golden standard" of approximately 120 over 80, for example, is measured in mmHg).

For water, using the same formula, we see that the column would then have to be about 10 meters high (as mentioned earlier), which is impractical, yet possible.

14.1.5 Submarines and hydrostatic pressure

Construction of the world's first submarine is usually credited to Dutchman Cornelius van Drebbel, as early as 1622. He not only built it, but successfully operated it at a depth of 5 meters, where the hydrostatic pressure is about 0.5 atmospheres. Add to that the atmospheric pressure of 1 atmosphere, and you get a total pressure of 1.5 atm at that depth.

Since he had 1 atmosphere of air inside, to breathe, the pressure differential is then 0.5 atmospheres, equivalent to about 50 kPa or 5000 kgf per square meter acting on the outside; very impressive for the time.

The professor mentions that modern submarines have gone down to at least a 900 meter depth, meaning approximately 90 atmospheres of hydrostatic pressure, but for some reason made no mention of the manned descents into the Challenger Deep (once in 1960, once in 2012; the latter was after the lecture was recorded, however) to a depth of about 10920 meters! The pressure down there is over *a thousand* atmospheres, equivalent to over 10^8 Pa, or equivalent to having over ten thousand metric tons of mass on each square meter of the outside, in Earth's gravitational field that is. The fact that this is not only possible, but was done even before the first moon landing, amazes me quite a lot.
These descents are not done in regular submarines, of course, but they are still man-made vessels that can withstand such absurd pressures.

The professor demonstrates what a "small" pressure difference of 0.5-1 atm can do to an object, by sucking the air out of a paint can. There will then be an underpressure inside the can, i.e. the pressure is larger outside than inside.

Long before the pressure difference is 1 atm (i.e. before there is a vacuum inside the can), it has already crumpled up quite a lot. Based on what happens, it's fairly safe to say that this paint can wouldn't survive at a 5 meter depth, if filled with 1 atmosphere of air and then hermetically sealed.

Now, consider what happens when we go scuba diving. Could we snorkel at a 10 meter depth? Far from it, actually!
The air in our lungs would be at 1 atm, connected to the surface via a snorkel (or a simple hose, etc). The pressure on our chests from the outside would be about 2 atm, however, since there is a hydrostatic pressure of 1 atm in addition to the atmospheric pressure of the air at the surface.

Since 1 atm is about 100 kN per square meter, or 10 tons worth of weight, raising your chest to breathe in is absolutely impossible under these conditions. If you can't breathe with a car standing on your chest, how could you breathe with an equivalent hydrostatic force of the same magnitude pushing in on you? (Based on a chest size of about a tenth of a square meter, or so, the force will be about one ton's worth of weight at g.)
So at what depths *could* we snorkel?

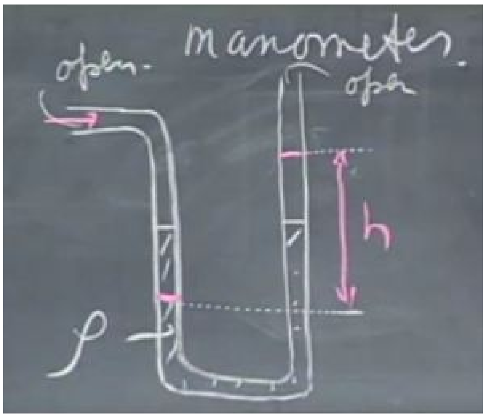
Well, to answer the question, we would need to know approximately what sort of underpressure we can have in our lungs, and still be able to breathe in. It seems reasonable, but tough, that we could perhaps lift a 100 kg weight laying on our chests, using only our lungs. I doubt that it's easy, but it's probably doable, at least for some humans.
Given a chest area of 0.3 times 0.3 meters, or 0.1 square meters, this is equivalent to a pressure of about $\frac{(100\,\mathrm{kg})(10\,\mathrm{m/s^2})}{0.1\,\mathrm{m}^2}$, or about 10000 Pa.
That is, the outside pressure cannot be more than 10000 Pa greater than atmospheric pressure; that is about 0.1 atmospheres.

At what depth is the hydrostatic pressure 0.1 atm? With the rule of 1 atm per 10 meters, this is at about 1 meter, or so. So a roughly calculated answer is than snorkeling at a depth greater than 1 meter is essentially impossible.
We will look at this in more detail now.

By the way, the way divers get around this is to have pressurized breathing gas. That is, the air in their lungs is at about the same pressure as the water outside.

14.1.6 Manometers

A manometer is a very simple device that can be used to measure pressure. We have a U-shaped tube, plastic in this case, which is partially filled with water.



By blowing (or sucking) on one end, we can measure the height difference in the liquid, and calculate the amount of underpressure/overpressure we managed to produce in our lungs.
Call the top height y_2 with pressure P_2, and the bottom y_1 with pressure P_1, we have

$$P_1 - P_2 = \rho h g \tag{14.17}$$

P_2 is at atmospheric pressure, since it is connected to the outside world. Therefore, solving for P_1 and making a substitution,

$$P_1 = 1\,\text{atm} + \rho h g \tag{14.18}$$

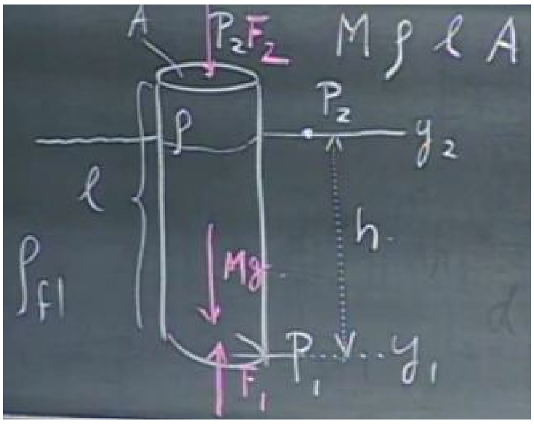
So we can measure the amount of pressure we can generate above or below the atmospheric pressure. We call that overpressure and underpressure, respectively.
If you have ever measured the pressure in a car's tires, that is done by an overpressure gauge.

If we use water in the manometer, the height difference it shows is equal to the depth at which we could snorkel (for a short amount of time, at least), since 1 meter on the manometer means we can generate an overpressure of 0.1 atm, and the hydrostatic pressure at such a depth also is 0.1 atm.
The hard part of snorkeling is breathing *in*, though – out is easy, since your lungs are compressed from the outside. In other to find out out maximum snorkeling depth, we need to measure the maximum *underpressure* we can produce, i.e. when sucking.

The professor then demonstrates this, and then demonstrates an interesting feat: drinking with a "straw" that is much, much longer than the 1 meter he could manage with the manometer. Exactly how it's done is not explained.

14.2 Lecture 28: Hydrostatics, Archimedes' principle, and fluid dynamics

We will now look at how objects float.
Say we have a cylinder of end-cap area A and length ℓ, and therefore volume $A\ell$. It has a uniform density ρ, and therefore a mass $M = A\ell\rho$.
The cylinder is in a liquid of density ρ_{fl}, and a height $h = y_2 - y_1$ of the cylinder is submerged.



There is a downwards force $F_2 = AP_2$ due to the weight of the air above the cylinder, and an upwards force $F_1 = AP_1$ due to the hydrostatic pressure. n addition, there is a gravitational force $Mg = A\ell\rho g$ downwards.

Via Pascal's law, $P_1 - P_2 = \rho_{fl} g h$.

In equilibrium, the forces on the cylinder must be balanced:

$$F_1 - F_2 - Mg = 0 \tag{14.19}$$

If we multiply both sides of the Pascal's law equation by A, we find

$$AP_1 - AP_2 = A\rho_{fl}gh \tag{14.20}$$

The left side here, $AP_1 - AP_2 = F_1 - F_2$, is known as the *buoyant force*. The other side is the *weight of the displaced fluid*: Ah is the volume of the the displaced fluid, times ρ_{fl} gives us the mass, and times g times us the weight.

This is a case of *Archimedes' principle*, which can be stated as: "the buoyant force on an emerged body has the same magnitude as the weight of the fluid which is displaced by the body".

The story is that Archimedes was given the task to find out whether a crown made for his king was made of pure gold. He therefore wanted to measure the density of the crown – but how does one measure the density of something without destroying it? The simple solution would be to measure the volume and the weight, and then calculate the mass and density from there. However, measuring the volume of such an irregularly shaped object with e.g. a meter stick is no easy task!

What he realized (according to the legend, when he noticed the water rise as he stepped into a bath) was that he could measure the volume by submerging the crown in water.
Silver has a lower density than gold, so if part of the crown was silver, for a given mass/weight, it would have to be slightly larger in volume. Measuring the weight is relatively easy, but even then, measuring a fairly small change in volume is still not easy, as the change in water level would be very small (probably less than 1 mm, depending on the container size etc.). Not only that, but there could be other factors causing trouble, such a surface tension, which may well make the difference completely impossible to measure.

What one can do is the following. First, we weigh the crown as per usual, perhaps using a spring, and find a weight $W_1 = V\rho g$, where V is the volume of the crown.
Next, we submerge it in water, and weigh it there. Because of the buoyant force acting on the crown, its weight is less under water. (Its *mass* is of course the same.)

In water, the weight is the original weight, minus the buoyant force $V\rho_{fl}g$, which is the weight of the displaced fluid of volume V. So we have

$$W_1 = V\rho g \tag{14.21}$$
$$W_2 = V\rho g - V\rho_{water}g = W_1 - V\rho_{water}g \tag{14.22}$$

We can solve this to find

$$\rho = \frac{W_1}{W_1 - W_2}\rho_{water} \tag{14.23}$$

and also

$$V = \frac{W_1 - W_2}{g\rho_{water}} \tag{14.24}$$

All of the things needed to find ρ was either known (ρ_{water}) or easily measurable (the weights), with rather high accuracy.

14.2.1 Floating and icebergs

I'm sure most of us have heard the expression "that's just the tip of the iceberg". There's a good reason for that expression, as we will see now.

Consider an iceberg of mass M, volume V_{tot}, density $\rho_{ice} = 0.92\,\mathrm{g/cm^3}$, compared to $\rho_{water} = 1\,\mathrm{g/cm^3}$. Because it is floating, the buoyant force is equal in magnitude to $Mg = V_{tot}\rho_{ice}g$. The buoyant force is given by the weight of the displaced water, $F_b = V_{sub}\rho_{water}g$, where V_{sub} is the volume of the iceberg that is submerged, i.e. under water.

$$V_{tot}\rho_{ice}g = V_{sub}\rho_{water}g \tag{14.25}$$
$$V_{tot}\rho_{ice} = V_{sub}\rho_{water} \tag{14.26}$$

g cancels. We can rearrange the equation:

$$\frac{V_{sub}}{V_{tot}} = \frac{\rho_{ice}}{\rho_{water}} = 0.92 \tag{14.27}$$

So the submerged volume is going to be 92% of the total volume: 9/10 of the iceberg is under water, and we can indeed only see the tip of it from above water.

Going back to our cylinder from the beginning of the lecture, what is the condition for floating? We know already that the buoyant force must equal the weight, and we have already learned that the buoyant force is equal to the weight of the displaced water, $Ah\rho_{fl}g$, where h is the height of the part of the cylinder that is under water.
That must be equal to the weight of the cylinder, $A\ell\rho g$, so after cancelling A and g we have

$$h\rho_{fl} = \ell\rho \tag{14.28}$$

However, $h < \ell$ must be the case: the amount submerged must be less than the total height, or it would be entirely underwater. With that condition, to balance the two sides out to be equal, it must also be the case that

$$\rho_{fl} > \rho \tag{14.29}$$

Very simple, indeed. Things float if their density is smaller than the density of the fluid they are in. Note that the masses or weights don't matter: a small pebble, say a 2 mm radius "rock", will sink in water, because rock has a greater density than water. A 1 km radius iceberg, with a mass of over 10^{12} kg would float in water, however, since the density of ice is smaller than the density of water.

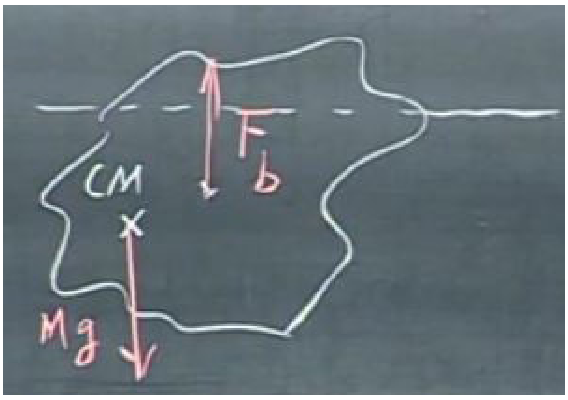
Let's now consider an interesting question. We are in a boat, in some small-ish reservoir of water, like a swimming pool. We have a large/heavy rock in the boat with us. If we throw this rock overboard, so that it sinks, will the water level in the pool go up, go down, or stay the same as it was when the rock was inside the boat?

When it is in the boat, the boat displaces extra water due to the weight of the rock, $W = V\rho_{rock}g$, which causes the water level to rise (compared to the rock not being there at all).
When it is in he water, it displaces water equal to the volume of the rock; the displaced water then has a weight $V\rho_{water}g$.
So which is greater? The rock's weight is $V\rho_{rock}g$, while the weight of the displaced water in case is $V\rho_{water}g$. The former is clearly greater, since the rock 's density is much greater than the density of water.

More water is therefore displaced when the rock is in the boat, and so the water level will go *down* when it is instead in the water.

14.2.2 Stability of immersed objects; balloons

Consider a floating object, which has a center of mass not aligned with its geometrical center. Perhaps it's an iceberg with some rocks in it.



Gravity acts at the center of mass, as usual, but the buoyant force does not! It acts at the center of mass *of the displaced fluid*, which is this case will be to the right of the iceberg's center of mass. Therefore, these two forces create a counterclockwise torque, causing the object to rotate.

Just as we saw previously, with an object on a pin, it will rotate until these forces are vertically aligned, so that is no longer any net torque.
Also as we saw previously, there are two possible cases where this happens: one where the center of mass of the iceberg is *above* the point of application of the buoyant force, which would be a case of unstable equilibrium, or the more stable case where it is *below*.

This is then a very important issue in the construction of ships. If a ship were to have the center of mass above the point where the buoyant force is applied, it could very easily capsize (flip around upside down). The lower the center of mass is, the more stable a ship will be.

Next, let's have a quick look at balloons, specifically ones with light gases inside, such as helium. What is the condition for them to float and rise in air?
Well, the situation is really very similar to an object floating in water.

The balloon has a certain mass M, given by the mass of the gas inside $V\rho_g$ plus the mass of the "rest", i.e. the balloon itself (the rubber, perhaps a string, etc). It then has a weight $W_{balloon} = g(V\rho_g + M_{rest})$. In order to rise, the buoyant force must be greater than this weight. The buoyant force is given by the weight of the displaced fluid – and the fluid is air, here. $W_{air} = V\rho_{air}g$, so the condition is

$$V\rho_{air}g > V\rho_g g + M_{rest}g \tag{14.30}$$

$$V\rho_{air} > V\rho_g + M_{rest} \tag{14.31}$$

It's clear, then, that $\rho_g < \rho_{air}$ must be the case. That is necessary, but not sufficient: it is sufficient only for a massless balloon. Since the rubber has a small mass, the density of the gas must be smaller by a margin wide enough to also carry that mass.

14.2.3 Helium balloon in an accelerated frame

We will now look at a second example involving helium balloons. I will shorten this section compared to the lecture, which should (as always) be watched anyway, especially as this is a demonstration.

In short: in the presence of air, a helium balloon will always move in the direction that opposes gravity. That includes perceived gravity, for example due to a rocket accelerating in outer space.
So say we have a sealed-off "room" somewhere in outer space, where the gravity due to the surrounding stars etc. is completely negligible. We accelerate this system, say "upwards" (as shown in a drawn figure, that is). We will perceive gravity in the opposite direction, which means we will fall down, as will the air inside the room. However, as the air will sink down due to its weight (which was zero prior to the acceleration), we will essentially end up with an atmosphere inside. The pressure will be higher near what has now become the floor, and smaller at the roof. Therefore, the helium balloon will rise towards the roof, in the *same* direction of the acceleration.

So far, this is a bit strange perhaps, but it still appears reasonable, since acceleration creates perceived gravity, which we cannot really tell apart from "regular" gravity.
However, now consider doing this in a room here on Earth, only we accelerate it towards the right, rather than up.
We have a closed compartment filled with air; it indeed needs to be closed, since this effect relies on the pressure difference between the two sides.

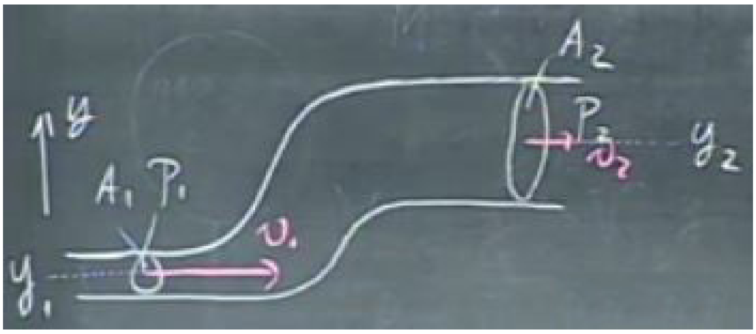
If we hang an apple from a string, we know what will happen: accelerate the box towards the right, and the apple will resist this motion, and appear to lean to the left.
However, what if we have a helium balloon, attached to the floor via a string? Before we push, it will happily float and just sit there, trying to move opposite gravity, but being stopped by the downwards tension in the string. When we accelerate the box towards the right, the air inside this closed compartment "falls" towards the left. Again, a pressure difference is created, such that the pressure at the left side is greater than the pressure on the right side.
This causes a horizontal buoyant force, and the balloon will "float" and move *towards the right*.

That is, unlike what we would expect any object to do, it moves *forward*, along with the acceleration. This can also be replicated in a car, for example. Step on the gas, and the balloon will move forward, while the passengers are pushed back into their seats; slam the brakes, and the balloon will move backward, while everyone moves forward. Not very intuitive.

14.2.4 Bernoulli's equation

We will now show a rather fast derivation of Bernoulli's equation for incompressible fluids.

Say we have a flow between two different heights, with two different areas, pressures and fluid velocities, as follows:



Because the fluid is incompressible, the amount of fluid that passes though A_1 per unit time must be the same as the amount that passes through A_2 per unit time; that is, $A_1 v_1 = A_2 v_2$. For that reason, in this case, $v_1 > v_2$.

In the case where the fluid is static, $v_1 = v_2 = 0$, we could apply Pascal's law: $P_1 - P_2 = \rho g(y_2 - y_1) = \rho g h$, using the usual definition $y_2 - y_1 = h$.
This then implies that the pressure at A_1 is higher than the pressure at A_2, since it is at a lower level, which implies higher pressure.

We know mgh as being a term of gravitational potential energy. However, m divided by volume gives us density ρ; therefore, the above expression is in terms of gravitational potential energy per unit volume, $\frac{mgh}{V} = \frac{m}{V} gh = \rho g h$.
Since we can only equate quantities that have the same dimension, the dimension of pressure must be the same as energy per unit volume.

If we now consider the dynamic case, where the velocities come in to play, we also have kinetic energy to consider. We can then relate the kinetic energy per unit volume, $\frac{1}{2}\frac{m}{V}v^2 = \frac{1}{2}\rho v^2$, gravitational potential energy per unit volume $\rho g y$, and the pressures. The sum of these three terms must then remain a constant, via the conservation of energy.

$$\frac{1}{2}\rho v^2 + \rho g y + P = \text{constant} \tag{14.32}$$

The above is one way of writing *Bernoulli's equation.* Just as Pascal's law, this equation has some very interesting (and strange) properties.

Consider a tube that changes diameter (like above), but where the level y stays constant. We still have an incompressible fluid of density ρ, and two areas A_1 and A_2, with pressures P_1 and P_2, respectively; the fluid has speeds v_1 and v_2 respectively at the two places, where $v_1 > v_2$, because as earlier, $A_1 v_1 = A_2 v_2$ must hold for a fully incompressible fluid.

Since we measure the pressure at the same height y, the total energy equation for both places contain a $+\rho g y$ term, which then cancels. We can then simplify down the result, to arrive at

$$\frac{1}{2}\rho v_1^2 + \rho g y + P_1 = \frac{1}{2}\rho v_2^2 + \rho g y + P_2 \tag{14.33}$$

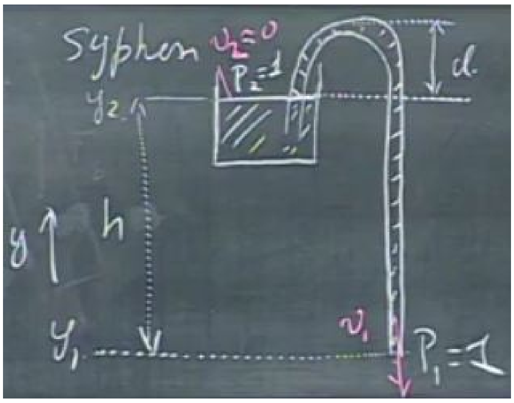
$$P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) = P_2 \tag{14.34}$$

$$\tag{14.35}$$

Because the non-pressure term is positive, it must be the case that $P_2 > P_1$. Very nonintuitive, to me – I would absolutely have guessed that the pressure would be higher at A_1 where not only the velocity is greater, but the liquid seems to be more tightly packed... but that is not the case.

14.2.5 Siphons

Most of us have probably seen a siphon (or syphon) in action. We have a container of water that is at a height, and a hose (with a diameter much smaller than that of the container) going down below the container.



v_2, the velocity of the liquid in the container, can be approximated as zero, if it is much larger than the diameter of the hose.
Both the top of the liquid in the container and the liquid flowing out is directly exposed to the atmosphere, so $P_1 = P_2 = 1$ atm. Therefore, in our conservation of energy equation, we lose the pressure terms. With that in mind, having different y values this time, and $v_2 \approx 0$, we find, also using $y_2 - y_1 = h$,

$$\frac{1}{2}\rho v_1^2 + \rho g y_1 = \rho g y_2 \tag{14.36}$$

$$\rho v_1^2 = 2\rho g(y_2 - y_1) \tag{14.37}$$

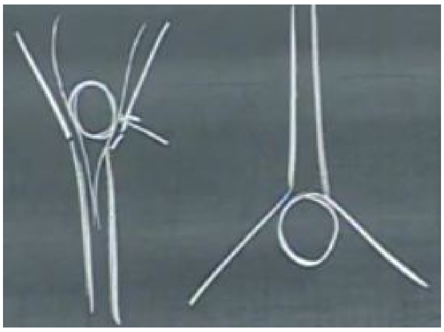
$$v_1 = \sqrt{2gh} \tag{14.38}$$

This is *exactly* the velocity we would find for an object being accelerated down by gravity. When starting at 0 velocity and having fallen a distance h, an object in free fall has the velocity $\sqrt{2gh}$. In other words, the siphoned liquid is acting as if it's in free fall.

That much may be intuitive, but the strange part is once the flow has begun, we can raise the hose up, i.e. increase d, up until the $\approx 10\,\mathrm{m}$ limit discussed earlier (for water), as long as the end of the hose is below the container, and the liquid will keep flowing.
We need to get it started manually, though, but sucking on the free end. Once that's done, the entire container will empty all by itself.

14.2.6 A few quick experiments

Consider a funnel, with a ping-pong ball inside:

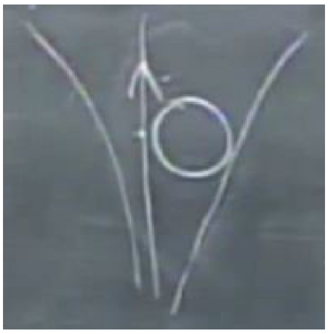


First, we hold it upright. If we try to blow and get the ball to move upwards, what will happen? The opposite of what we might think: the harder to blow, the more the ball is sucked down. According to the Bernoulli principle, the pressure is lower in the thin part, where the velocity is high. Therefore, there is an underpressure there, and the ball is sucked down more than it is blown upwards.

The effect is strong enough that we can do the experiment upside down, and hold it in place (for a short time, at least) merely by blowing out, as the second figure above shows.

Next, we have an air pump, blowing to keep a ping-pong ball floating in mid-air:

It is held up for reasons to do with turbulence, which is more complex than we can discuss here. However, Bernoulli's principle comes into play in another way: the stability. While it's obviously difficult to show this in a still image, the ball wobbles back and forth, but never falls out. Even when the hose is tilted perhaps 10-20 degrees, the system is still stable. This stability is because of Bernoulli's principle.



The air is blowing faster near the center, as it is diverging away (the area is becoming larger, so the velocity goes down, as we saw earlier).
Therefore, the pressure is the lowest near the center, and when the ball moves away, it is being sucked back in by the lower pressure.

Finally, the professor demonstrates what happens if you fill a glass half-way, put a piece of cardboard (or some paper similar to a postcard in thickness, perhaps) over the top, and then turn it upside down. The liquid will tend to stay in place even upside down, with no support, so not only is the paper is held up against gravity, the liquid is as well.
This happens because air pressure is acting to push the paper up, stronger than gravity is pulling the water down.
There are other things at play too, though, including surface tension. I have not been able to find a fully satisfactory explanation of this, even though it seems so simple.

For example, why does this not happen when you simply turn the glass upside down, without the paper? Clearly, the paper isn't increasing the air pressure; if the air pressure can support the liquid via the paper, it must obviously be able to support the liquid itself, too! So why does the water simply run out, as we would expect intuitively, but perhaps not expect considering air pressure?

The explanation appears to be quite a bit beyond this course, in Rayleigh-Taylor instability. If the water surface could be *perfectly* flat, it appears that it would indeed not fall out, though achieving this in practice is clearly either extremely difficult or plain impossible.

Chapter 15: Week 13: Exam review only

15.1 Lecture 29: Exam 3 review

I didn't take any additional notes during this lecture, except on Doppler shift for sound, which I added back to the original lecture on Doppler shift instead.

Chapter 16: Week 14

16.1 Lecture 30: Simple harmonic oscillations of suspended solid bodies

The first half (or so) of this lecture is mostly a review of physical pendula. I will make the notes for that part rather brief, as it's easy to go back to previous chapters of these notes for a review.

We will make frequent use of a formula we derived in lecture 21 (and also derived here in lecture very quickly).

$$T = 2\pi\sqrt{\frac{I_P}{bMg}} \tag{16.1}$$

where b is the distance between the point a pendulum is fixed, and its center of mass. This was derived for a rod, but holds for any geometry, as shown in this lecture. We will use it for rods, hula hoops, solid disks and simple pendula (a mass on a massless string) here. All we need to do is find b and I_P, the moment of inertia about the point where it is fixed (and therefore rotates); when that is done, we can easily calculate the period of that system.

16.1.1 Rod

For a rod, $I_{cm} = (1/12)M\ell^2$. The point P is then located a distance $b = (\ell/2)$ from the center of mass, as we rotate it about its end. Using the parallel axis theorem, $I_P = I_cm + Mb^2 = (1/12)M\ell^2 + M\ell^2/4 = (1/3)M\ell^2$. Using those two equivalences,

$$T_{rod} = 2\pi\sqrt{\frac{(1/3)M\ell^2}{(\ell/2)Mg}} = 2\pi\sqrt{\frac{2\ell}{3g}} \tag{16.2}$$

The lecture has a demonstration about several pendulum types, where each has a period of approximately 1 second. We want to know how long/large each type of pendulum should be to match that, so let's solve this for ℓ also:

$$\frac{3gT_{rod}^2}{8\pi^2} = \ell \tag{16.3}$$

For $T = 1\,\mathrm{s}$, $\ell \approx 37.27$ cm, assuming we rotate it about the exact end.

16.1.2 Simple pendulum

Next, they ask how long a simple ("regular") pendulum should be to have a period of 1 second. We use find the period of such a pendulum using the first equation above, using $b = \ell$ and $I_P = M\ell^2$. Doing so quickly gives you

$$T_{simple} = 2\pi\sqrt{\frac{\ell}{g}} \tag{16.4}$$

Again, let's solve for ℓ.

$$\frac{gT_{simple}^2}{4\pi^2} = \ell \tag{16.5}$$

Here, we find a length of about 24.85 cm.

16.1.3 Ring

We repeat the process for a ring, e.g. a hula hoop, where all the mass can be approximated to be at the circumference. Its center of mass will then be at the geometrical center, i.e. mid-air. Physics doesn't care, and we can use the same formula anyway. Quite amazing, really.
Here, we find a moment of inertia of MR^2 about the center of mass. Since point P is a distance $b = R$ away, we must add to that MR^2 from the parallel axis theorem, and find $I_P = 2MR^2$.

$$T_{ring} = 2\pi\sqrt{\frac{2R}{g}} \tag{16.6}$$

As the professor notes, this is exactly what you would find for a simple pendulum of length $\ell = 2R$, i.e. with the same length as the diameter of the hula hoop. That makes it a bit unnecessary to solve for R, since we could just say that $R = \ell/2$, but I will do so anyway for completeness.

$$\frac{gT_{ring}^2}{8\pi^2} = R \tag{16.7}$$

As expected, we find $R \approx 12.42$ cm, half (ignoring rounding) the length of the simple pendulum.

16.1.4 Solid disk

Finally, we do this for a solid disk (solid except for a negligibly small hole where it's fixed, of course). Here, $I_{cm} = (1/2)MR^2$, and so adding MR^2 for the parallel axis theorem gives us $(3/2)MR^2$.

$$T_{disk} = 2\pi\sqrt{\frac{(3R}{2g}} \tag{16.8}$$

Or, solved for R:

$$\frac{gT_{disk}^2}{6\pi^2} = R \tag{16.9}$$

$R \approx 16.56$ cm, or $d = 33.12$ cm, which is 4/3 times that of the hollow ring.
The rod must also be exactly 50% longer than the simple pendulum.

16.1.5 Lecture question

"Suppose we scale up the radius of our planet (keeping its mass density fixed), and the size and mass of a physical pendulum by a factor of 2. How will the period of oscillation change?"

Hmm... Many things will have to change, g being one. Let's first find the change in Earth's mass, and then after that find the change in g, before we move on to the pendulum.

Even before that, the period of an ideal pendulum (which I chose from the various types as it's simple) is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}} \tag{16.10}$$

For a uniform density, we have $M_{old} = (4/3)\rho\pi R^3$ and $M_{new} = (4/3)\rho\pi(2R)^3$, so the mass goes up by a factor of 8 from the 2^3.
The distance to the center of the Earth also doubles, so g changes not only due to the increased mass:

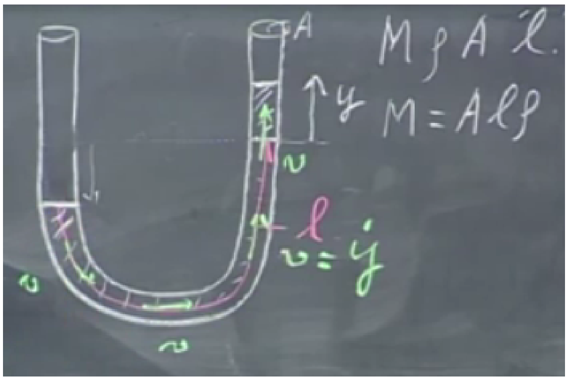
$$\frac{g_{new}}{g_{old}} = \frac{(8GM)/(2R)^2}{(GM)/R^2} = \frac{8R^2}{(2R)^2} = 2 \tag{16.11}$$

So g doubles. Since the period depends on ℓ/g, which becomes $(2\ell)/(2g)$, the period is unchanged.

16.1.6 Oscillating liquid in a U-tube

Let's now move away from the fairly familiar territory above into something new: oscillating liquids.
We have a liquid in a U-shaped tube. The entire mass of the liquid is M, the density ρ and the "length" of the liquid, at the center of the tube, is ℓ.

The tube has a cross-sectional area A everywhere along it.



As shown above, we displace the liquid so that it is at a height y above equilibrium height on one side, and a height y below at the other. We also denote the velocity of the liquid as $v = \dot{y}$, which is the same everywhere, at one instant.
Say we denote the equilibrium point as $U = 0$ for the system, where U is the gravitational potential energy. If we then displace some liquid as shown, say of mass Δm, the increase in gravitational potential energy is simply $\Delta m g y$, using $\Delta U = mgh$ with other variable names. This works because the same amount of liquid that is above equilibrium on the right side must be taken from the left side, and so it is equivalent to simply lifting that liquid up a distance y, regardless of which side this happens on.

There will be frictional losses and such in this system, but if we neglect that for a while, we can derive an approximate period of oscillation by using the conservation of mechanical energy.

In doing so, we find

$$\frac{1}{2}M(\dot{y}) + \Delta m g y = \text{constant} \tag{16.12}$$

If we substitute in $M = A\ell\rho$ and $\Delta m = Ay\rho$,

$$\frac{1}{2}A\ell\rho(\dot{y})^2 + A\rho g y^2 = \text{constant} \tag{16.13}$$

If we take the time derivative, we will have a differential equation in y which has $\ddot{y}$ as the highest derivative.
Let's do that and see:

$$\frac{1}{2}A\ell\rho 2(\dot{y})\ddot{y} + A\rho g 2y\dot{y} = 0 \quad (16.14)$$

$$\ell\ddot{y} + 2gy = 0 \quad (16.15)$$

$$\ddot{y} + \frac{2g}{\ell}y = 0 \quad (16.16)$$

Many things cancel, including $\dot{y}$. We get a result that is clearly a simple harmonic oscillation! However, more in this case than we have seen previously, this result is not that accurate; at least not for the demonstration in lecture. There is a lot of damping, i.e. the amplitude goes down from its maximum quickly, and as that happens, the period is affected.
Anyhow, we know the solution to this differential equation very well by now:

$$y = y_{max}\cos(\omega t + \varphi) \quad (16.17)$$

$$\omega = \sqrt{\frac{2g}{\ell}} \quad (16.18)$$

$$T_{tube} = 2\pi\sqrt{\frac{\ell}{2g}} \quad (16.19)$$

This happens to be the same answers as you would find for a simple pendulum of length $\ell/2$ – nature is funny that way.

Solved for ℓ, since we did that for the rest,

$$\frac{gT_{tube}^2}{2\pi^2} = \ell \quad (16.20)$$

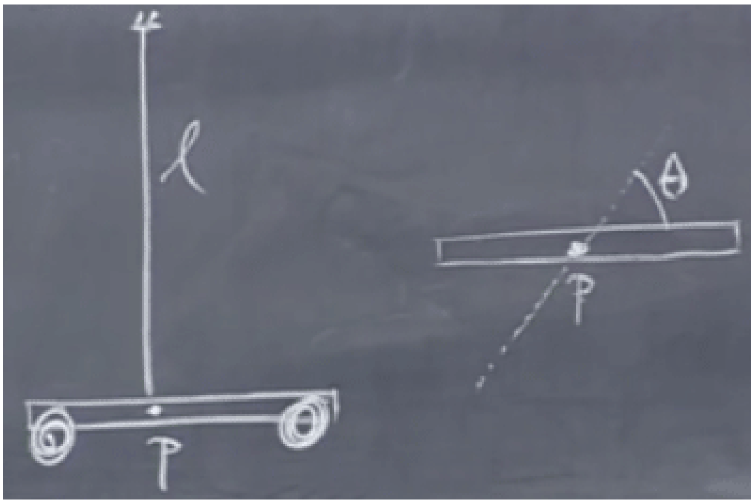
For $T = 1$ s, we need $\ell = 0.497$ m or about 50 cm.

16.1.7 Torsional pendulum

Nature really loves these simple harmonic oscillators. Granted, in many of these derivations we use a small angle approximation, but still.
For this one, we won't need to do that.

In a torsional pendulum, we can have for example an object hanging from a wire, which we rotate, and then let the wire's restoring torque try to get itself back to equilibrium.
Like a simple spring pendulum, a torsional pendulum has a period that is independent of the amplitude, assuming we don't permanently deform the wire, and stay in a region where the restoring torque can be considered linear with regard to the angle we twist the wire.

Consider this system:



(Left: as seen from the front; right: as seen from above, after giving the mass a small twist.)

The torque relative to point P is given by

$$\tau_P = -\kappa\theta \tag{16.21}$$

where κ (Greek letter kappa) is the *torsional spring constant*. Just as with a spring oscillator, we have a minus sign to denote a restoring torque (or force, in that case), a spring constant, and something the torque/force is proportional to: here an angle, but in the case of a regular spring oscillator, a displacement.

Since the product $\kappa\theta$ must be in newton-meters, and θ is in radians, the units of κ must be N m/rad (though radians are dimensionless, and perhaps we could say the units are already in N m, though that would be a bit confusing).

The torque is always equal to $I_P\alpha = I_P\ddot{\theta}$, so

$$-\kappa\theta = I_P\ddot{\theta} \tag{16.22}$$

$$\ddot{\theta} + \frac{\kappa}{I_P}\theta = 0 \tag{16.23}$$

which is a simple harmonic oscillation – without using any small angle approximation. As usual, the solution to this differential equation is

$$\theta = \theta_{max}\cos(\omega t + \varphi) \tag{16.24}$$

$$\omega = \sqrt{\frac{\kappa}{I_P}} \tag{16.25}$$

$$T = 2\pi\sqrt{\frac{I_P}{\kappa}} \tag{16.26}$$

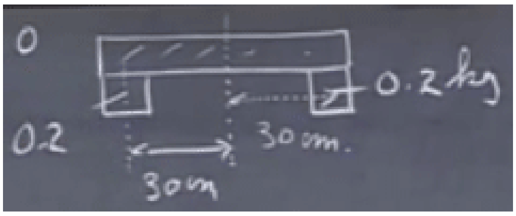
where ω is the angular frequency, a constant, not to be confused with $\dot{\theta}$ which is the angular velocity. The angular velocity varies with time; it is at a maximum at $\theta = 0$, and zero at $\theta = \theta_{max}$.

κ is a function of the cross-sectional area, the length, and of course of the material in question. If the wire is thicker, κ will go up, as we would expect; if you instead make it longer, κ goes down.
Both are intuitive: a thick wire is much harder to turn than a thin one. Also, if you make it longer, it becomes easier to twist – that also makes sense. A very short steel wire/rod is almost impossible to twist 10 degrees while grabbing each end of the wire/rod, but if it's very long, it's not a problem (unless it's also very thick, so that κ is high for that reason).
Exactly how this is calculated is not shown, as it is a apparently more complex than the equivalent calculation for the linear case, using Young's modulus.

In the lecture demonstration, we have a 2.5 meter long piano wire, which is either 25/1000" or 1/25000" thick – the subtitles say the latter, but I'm doubtful. It's a piano wire, which (according to Wikipedia) usually range from about 1/30 to 1/3 inches in diameter. Why would this one be a hundred times thinner? In either case, the professor calculates $\kappa \approx 4 \times 10^{-4}$ Nm/rad for this wire.

If we now calculate the moment of inertia, we can predict the period of the pendulum.
We ignore the moment of inertia of the wire, since it's very thin, and has a near-zero moment of inertia about this rotation axis.

Here's a closeup of the part of the system with a non-negligible moment of inertia:



We can approximate the masses as point particles, each having a moment of inertia of MR^2 about the center axis (point P), so the total moment of inertia is approximately $I_P \approx 2MR^2 = 2(0.2\,\text{kg})(0.3\,\text{m})^2 = 0.036\,\text{kg m}^2$.

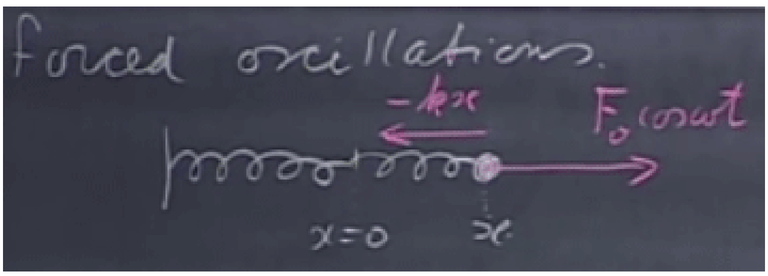
Using the equation we found earlier, we find $T = 59.608\,\text{s} \approx 60\,\text{s}$.

The rest of the lecture is then demonstrations of this concept. The prediction is fairly accurate even for very large angles – multiple rotations, not just some 30 degrees or such. The angular velocity is then very high (at times) for large angular displacements, since it must rotate much longer in the amount of time (since the period is independent of amplitude). As mentioned earlier, this is only true as long as we don't leave the region where Hooke's law is valid, and/or permanently deform the wire.

For half a period at 1 rotation, $T = 28.8$ s is measured. For three full rotations, half a period is measured as 28.5 seconds. For 10(!) full rotations, the period is measured as 29.2 seconds.
There is a reasonable amount of uncertainty in the measurements, as it's hard to define exactly when it stops rotating. Also, our calculations themselves were really approximations (e.g. the moment of inertia about the rotational axis).

16.2 Lecture 31: Pendulums and springs

We have talked considerably about springs in the past, but we will now add a new twist: instead of just setting a spring system off equilibrium and then leave it be, we apply a time-varying force at some fixed frequency that we choose – which does not have to be the same frequency that the system would oscillate at on its own.



An shown, we have a simple system with a mass m connected to a spring. We drive it with some frequency $F_0 \cos(\omega t)$, where F_0 is the amplitude of the applied force.
We apply Newton's second law to the system, with $a = \ddot{x}$:

$$m\ddot{x} = -kx + F_0\cos(\omega t) \tag{16.27}$$
$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t) \tag{16.28}$$

Looks like a simple harmonic oscillator, except that the right-hand side is not zero, which of course changes things considerably.
In the beginning, the behavior can be fairly complex; we call this the transient phase, since it is indeed transient: it goes away after a while.
After the transient phase, we enter the *steady state*. Here, the driver has "won", and the system oscillates at a steady frequency: that of the driver, so $f = \dfrac{\omega}{2\pi}$ for the frequency (unless you prefer the angular frequency ω as-is). The mass will then move as described by $x = A\cos(\omega t)$, once the transient phase is over. The amplitude A of this oscillation is not known, so let's try to find it.

The derivatives of this trial solution are (since we need $\ddot{x}$):

$$x = A\cos(\omega t) \tag{16.29}$$
$$\dot{x} = -A\omega\sin(\omega t) \tag{16.30}$$
$$\ddot{x} = -A\omega^2\cos(\omega t) \tag{16.31}$$

We can then try to substitute x and $\ddot{x}$ above into the differential equation we found earlier.

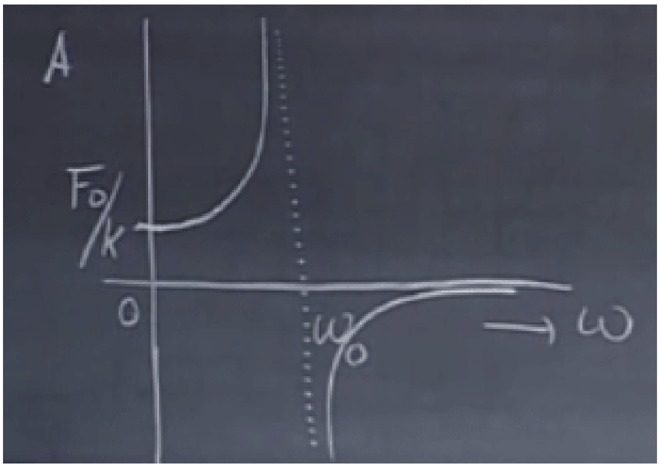
$$-A\omega^2\cos(\omega t) + \frac{k}{m}A\cos(\omega t) = \frac{F_0}{m}\cos(\omega t) \tag{16.32}$$
$$A\left(\frac{k}{m} - \omega^2\right) = \frac{F_0}{m} \tag{16.33}$$
$$A\left(\omega_0^2 - \omega^2\right) = \frac{F_0}{m} \tag{16.34}$$
$$A = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \tag{16.35}$$

Here, we have used $\omega_0 = \sqrt{\dfrac{k}{m}}$, which we call the natural frequency of the system. It is the ω we have seen before in spring systems – the one it oscillates with naturally, if you offset it from equilibrium and then let it be. We add a subscript 0 to denote this, since ω is now the driving frequency.

Let's look at some limiting cases. In the case $\omega \ll \omega_0$; in that case, we find $A = F_0/k$.
If $\omega \gg \omega_0$, the amplitude goes to 0. (That it also becomes negative is something we will discuss shortly.)

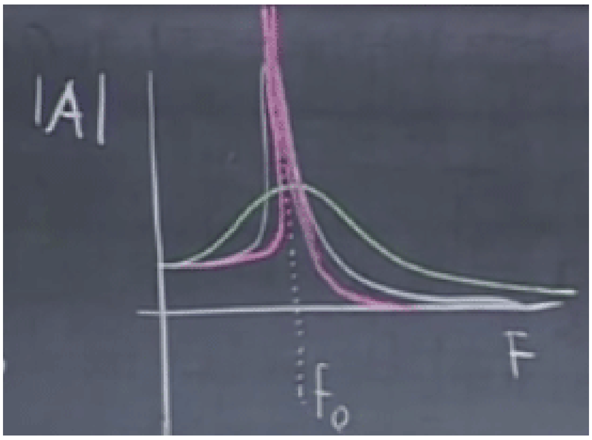
If $\omega \to \omega_0$, i.e. we drive it at the natural frequency, the denominator goes to zero, and the amplitude goes to infinity. It doesn't go to infinity it practice, of course, but the amplitude does tend to become very large. We call this *resonance*.
Frictional forces etc. limit the actual amplitude in practice.

If we plot the amplitude A versus the driving (angular) frequency ω, we find something like this:



The negative values mean that the object is now 180 degrees out of phase with the driver, which we don't go into any detail about in this lecture.

If we take a more realistic case (where the amplitude stays finite), and also plot the absolute value of the amplitude to get rid of the discontinuity of the phase shift, we get something like this:



The less damping there is, the narrower the resonance peak is (shown in pink). With lots of damping, the peak becomes just a small bump (in green).

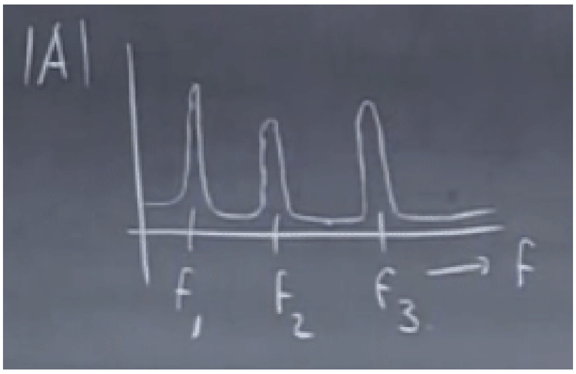
If we have a more complex system with more than 1 mass, all joined together with springs, we find the same number of resonance frequencies as there are masses, so a plot of amplitude vs driving frequency would have multiple peaks.
We can take this to the extreme, and consider a practically infinite number of such oscillators, in for example a violin string. We can think of each atom being a mass, connected to the others via a "spring" (in reality via electromagnetic forces).

In the case we have looked at earlier, we have a case of longitudinal oscillations: the objects move in the same direction as the spring, so to speak.
Oscillations/waves can also be transverse; electromagnetic waves are transverse, for example. Water waves are not good as an example, as they are a combination; in a fully transverse water wave, each molecule of water would simply move up and down, as the wave passes from side to side.

Sound is a longitudinal wave; at least in air, there appears to be some discussion about whether transverse waves in other media can be considered sound or not.

16.2.1 Harmonics

Let's look now at the example of shaking a string. As mentioned earlier, there will be many, many resonance frequencies. Consider the first few:

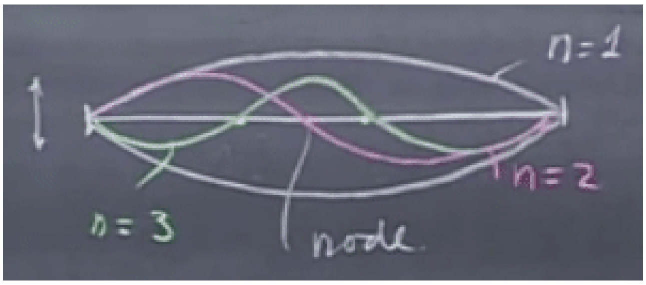


We start off by shaking the string up and down at a low frequency, which we slowly increase. At one point, we will hit the first resonance frequency, also known as the first harmonic, or the fundamental. We denote this frequency as f_1.
At that point, the amplitude will be much greater than it was just before, and we will have a *standing wave*. Each point of the curve will bob up and down, but that is all that happens. The movement is the greatest at the center, and zero at the two ends where the string is held.
Points where the string is standing still are called *nodes* (while points of maximum amplitude are called *antinodes*).

If we keep increasing the frequency, we will eventually find the second resonance frequency, or second harmonic, f_2. There will now be a node at the center of the string, while there will be two antinodes, evenly spaced. The two antinodes will be 180 degrees out of phase, so when one is at its highest point, the other is at its lowest.

Increase the frequency further, we find the third harmonic, f_3. This adds another node, so there are now two nodes and three antinodes.
Animated graphics are extremely useful here, so I suggest looking some up. Wikipedia has one on the page "Node (physics)" and several more in the article "Vibrating string", which shows the first five harmonics.

Here is a still image from the lecture, which is about the best I can do in these notes (in the lecture, Prof. Lewin also did a demonstration):



The first harmonic, $n = 1$ (in white) just has the string being either all high, all low, or in transition between the two states.
The second harmonic, $n = 2$ (in pink) has a node at the center, with the antinodes out of phase.
The third harmonic, $n = 3$ (in green) has two nodes, with the leftmost and rightmost antinodes in phase with each other, and out of phase with the middle antinode.

These frequencies follow a simple relationship of $f_n = nf_1$, i.e. they are integer multiples of the fundamental frequency. If $f_1 = 100$ Hz, then $f_2 = 200Hz$, $f_3 = 300Hz$ etc.
In musical instruments, the second harmonic is therefore one octave higher than the first. The third harmonic is not one octave higher than the second harmonic, though, but rather an interval of a perfect fifth above.[1] (An octave doubles the frequency, which would be 400 Hz, so the fourth harmonic in an

[1] In just intonation, a perfect fifth has a frequency ratio of 3:2, i.e. 1.5 times; in equal temperament, which most instruments use in modern times, a perfect fifth is exactly $2^{7/12} \approx 1.4983$ times the frequency of a given note, but this is now really becoming music theory.

octave above the second.)

The frequency of the fundamental/first harmonic depends on the string's mass, length and tension (or, if you prefer, the mass per unit length, length and tension).

Many musical instruments are of course stringed instruments. In the case of a piano, the strings vary in all three attributes.
In a violin, there are four strings, of essentially equal length. The thickness (and therefore mass per unit length) varies. Tuning is set by adjusting the tension in each string individually.
A higher tension causes a higher pitch, while a *shorter* string causes a higher pitch (for a given tension and mass).
When playing a violin, the player changes the pitch by shortening the strings. When you fret a note (i.e. hold a string down against the violin's neck), the effective length that vibrates is shortened, and so the pitch goes up.

In the case we looked at earlier, with the driven spring and the long shaken string, the driver alone decided the frequency. How does that work in the case of e.g. a violin?
When we use a bow on a violin, that rubbing motion can be thought to consist of many, many different frequencies at which the string is "driven". The string picks out its resonance frequencies, and so the frequencies we hear are mostly the different harmonics, i.e. integer multiples of the string's fundamental frequency.

The ratio of the amplitudes of the different harmonics is what gives an instrument its *timbre*. Consider a theoretical violin string that only oscillated at a single frequency f_1. The sound it would make is a pure sine wave – which sounds like a very boring "beep" and nothing at all like a violin.
For a middle A note, both a violin and a piano produce a 440 Hz sound with several harmonics, but they sound very different, as the harmonic content of the two are very different.
Adding up only the odd harmonics – 1, 3, 5 etc – will produce a square wave, which adding up all harmonics gives a sawtooth wave. These terms are mostly used in synthesis of sound, but are also useful for describing the sound of real-world instruments. Violins have a lot of harmonic content, and are much closer to a sawtooth-shaped waveform than to a square-shaped one in timbre.

16.2.2 Woodwind instruments

Consider an overly simple woodwind instrument: a closed box of length L, filled with air, and a loudspeaker at the end that can generate different frequencies. We can find several such resonant frequencies, however, this time it's not the material itself that resonates, but rather the air inside it.
The air acts a bit like a spring if you excite it at the right frequencies. The harmonics are easily calculated as

$$f_n = \frac{nv}{2L} \tag{16.36}$$

where n is the harmonic number, v is the speed of sound in air (about 340 m/s) and L is the length of the box.

This instrument isn't very practical though; it's not only closed, so that the sound will barely be audible outside, but it is driven by a loudspeaker. We can take care of that by opening up either one side, or both.
What is now interesting (and rather strange, in my opinion) is that if we open up both sides of this box, and again put the loudspeaker at one end, we can still find the exact same resonance frequencies as we did before!
This would be referred to as an open-open instrument. Flutes are open-open (more or less). We can also open up only one side, to create a closed-open instrument, such as a clarinet. In this case, the formula above doesn't apply; there is no additional detail to how or why it is different, though.

In reality, the loudspeaker is of course replaced by the player (or the player's mouth, rather).

16.2.3 Other resonances

Next, there is some talk about how everything has a resonance frequency: from car keys to frying pans and refrigerators, people and wine glasses.
The wine glass is demonstrated: by moving a clean, wet finger around the top of a wine glass, you can generate a fairly loud sound. What happens is just as with the violin string, our rubbing causes a ton of different frequencies to be generated, and the glass "picks out" its resonance frequency/frequencies and hums along at those.

Resonances can also be destructive. By playing back a tone that corresponds to the glass' fundamental resonance frequency at a loud volume, we can make the amplitude of oscillation so great that the glass breaks. This is demonstrated using a strobe light setup to allow us to see the deformations in the glass (which are about 470 Hz, which of course is far too fast to see otherwise).

Another well-known and oft-cited example is the Tacoma Narrows Bridge, first opened on July 1, 1940. It collapsed barely more than 4 months later, on a day of strong wind. However, while this is often presented as a resonance phenomena, it appears opinion has changed, and it is now consider to be due to aerodynamic flutter rather than forced oscillation. This is the same phenomena that causes a paper to oscillate when held steadily in constant airflow. Some also present it as a combination of multiple phenomena, and I certainly don't have the expertise to say who is correct, so I'll leave it at that.

Finally, the professor demonstrates what happens when the speed of sound in a medium changes – or when the medium itself changes, rather, by filling his lungs with helium while speaking. Sound travels about 2.7 times faster in helium than in air, and so the pitch created by our vocal chords goes way up if your lungs are filled with helium rather than air.
Since helium is, well, helium, which doesn't contain the $\approx 20\%$ oxygen we need to live, this experiment is dangerous if done incorrectly (or for anything but a short period of time).

16.3 Lecture 32: Thermal expansion

We begin the lecture by introducing the concept of thermometric properties. A thermometric property is a property of an object that depends on the object's temperature. A typical one, that we will look at in this lecture, is that many objects expand when heated, and contract when cooled.
If we heat up a gas in a closed container, the pressure in the container goes up, which is a thermometric property. If we heat an electric conductor, the electric resistance will tend to go up.
(That is why regular light bulbs often break when turned on; the current through them is at a maximum the first split second before it starts to reach its operating temperature (which happens extremely quickly). After that, the resistance goes up, and so the current goes down, to its steady state level.)

If we heat a metal bar, it will expand. Cool it, and it will shrink. If we bring a hot and a cold iron bar together, there will be heat transfer between the objects until they are in thermal equilibrium, i.e. when their temperatures are equal. Until then, the two bars will both change in size as their temperature changes.

We can construct a simple thermometer this way. We have a bar of some length L of a known material, at a known temperature. We put the bar in melting ice, and measure a length L_1; we then put it in boiling water, and find a length L_2. We can then define a temperature scale such that, for example, L_1 means the temperature is 0 degrees, and L_2 means 100 degrees. This is basically how the Celsius scale works.
This scale is (or was) often called *centigrade* (from Latin's centum, meaning hundred and gradus meaning step), though that name was formally obsoleted in 1948, and the scale is now known as the Celsius scale, after Anders Celsius, the Swedish astronomer who came up with it.[2]

Another common temperature scale is the Fahrenheit scale, invented by German scientist Daniel Gabriel Fahrenheit. He used brine, a mixture of salt and ice, as the zero degree definition, and human body

[2]Apparently, his original scale was the opposite: ice melted at 100 degrees, while water boiled at 0. Carl von Linné, also know as Carl Linnaeus, reversed the scale soon after Celsius' death.

temperature as 100... roughly speaking, as 98.6 F is the most commonly quoted number for human body temperature these days, and 100 F is defined as having a fever.

Conversion between the two scale is relatively straightforward. 0 degrees C is 32 degrees F, while 100 degrees C is 212 degrees F. To convert, we use

$$T_F = \frac{9}{5}T_C + 32 \tag{16.37}$$

$$T_C = \frac{5}{9}(T_F - 32) \tag{16.38}$$

The two scales "cross over" at -40 degrees, so $-40\,°\mathrm{F} = -40\,°\mathrm{C}$.

The third temperature scale (or perhaps rather unit) that is fairly common, especially very common in science, is the kelvin, which is an absolute temperature scale. Because it is absolute (see below), we do not talk about "degrees" kelvin, but just kelvin (just as we don't talk about degrees pascal). The coldest temperature with any physical meaning, absolute zero, is by definition 0 K. At this temperature, it is often said that all motion stops (which is not entirely true, due to to the world of quantum mechanics), and so colder temperatures are not very meaningful. This might be expanded upon as early as next week, when Heisenberg's uncertainty principle is introduced.

The kelvin scale is closely related to the Celsius scale: it is offset by exactly 273.15 degrees C, so that 0 K $= -273.15\,°\mathrm{C}$. Therefore, water boils at 373.15 K (at 1 atm of pressure).

16.3.1 Thermal expansion

We can now get to the focus of this lecture: thermal expansion.
Say we start with a rod of length L. We heat it up by ΔT degrees C (or kelvin), and it gets longer by an amount ΔL. We can approximate this amount in a simple way:

$$\Delta L = \alpha L \Delta T \tag{16.39}$$

where α is known as the *linear expansion coefficient*, with units of 1 over degrees Celsius (1/°C).

As the values of α are often small, we can write them in terms of 10^{-6}/°C, equivalent to ppm/°C or ppm/K.
α for a few common materials is

α	ppm/°C
Copper	17
Brass	19
Pyrex	3.3
Invar	0.9
Steel	12

Invar is an alloy often used for its unusually low thermal expansion coefficient. There are several variations; the one usually called invar is 64% iron and 36% nickel. It was invented by Swiss scientist Charles Édouard Guillaume in 1896; the invention won him the Nobel prize in physics in 1920.
Having a material with a low thermal expansion coefficient is important for many precision instruments, for example.

Consider a steel railroad track, 1000 meters long. In many climates, it might need to be usable at -15 degrees C, a cold winter's day, and also at +35, a hot summer's day, so $\Delta T = 50$ degrees. Using the formula above, and the value of α of steel, we find $\Delta L = 0.6$ meters. If the rail is continuous and can't expand in the "forward" direction, it will start to bulge either sideways or upwards, whichever is easier.

How is this taken care of? The rail needs to be able to expand, or it will deform and become unusable. One solution is very simple: the railroad has gaps in it. We need gaps of up to 60 cm per km, so for example 5 cm per 80 meters gives the rail space to expand. With large enough wheels, this causes a "clunk" when riding over it, but nothing more.

The professor then demonstrates the expansion of a brass bar, by using an "amplifier" device to turn the small (millimeter-scale) expansion into something more clearly visible (a large change in the angle of an indicator).

16.3.2 Bimetals

Bimetals are a very useful type of material. We take two metals with different linear expansion coefficients, and put them together (perhaps using welding):

When we heat this system, what happens? The copper must get longer, but the change in the invar's length is much less. Since we join them such that one can not expand without affecting the other, it will bend upwards, so that the invar is on the inside of an arc, and the copper is along the (longer!) outside. The difference in the length change of the two materials is

$$\Delta L_{Cu} - \Delta L_{invar} = (\alpha_{Cu} - \alpha_{invar}) L \Delta T \tag{16.40}$$

For a 10 cm rod composed of copper and invar, the difference in expanded length is about 0.16 mm. However, the difference in height between the two sides will be about 3-4 mm[3], despite the small difference in length.

We can use bimetals for example in thermostats, so that a bimetal being sufficiently cold makes contact in an electric circuit, to turn a heater on. Once the bimetal is warm enough, it expands and "bends away", so that the contact is broken, and the heater turns off.

They can also be used for safety devices. Gas stoves sometimes use a "pilot light", basically a small flame, that is used to light the main burners. If the pilot light is off, but the gas is on, the room will fill up with a flammable and explosive gas, which can of course cause horrible accidents. One of several ways to prevent this is to use a bimetal, such that the gas supply is only on as long as the pilot light is burning. When it goes out, the bimetal cools down, and in some way automatically turns off the gas supply.

We can build thermometers of bimetals. We could have a construction like this:

[3]The professor says about 4 mm; a scientific paper on bimetal thermostats has a formula that gives about 3 mm; one or both are probably estimates, though.

The outer end is attached to some casing and cannot move. The pink arrow is some form of indicator, attached at the center.
When the bimetal is heated, it will try to curl up even tighter than it already is, and the arrow moves clockwise. When the bimetal is cooled, the arrow moves towards the left. All we need, then, is to calculate how much it will turn, and then add a temperature scale around this.

Here is such a thermometer:

16.3.3 Volumetric expansion

Let's now consider how much the *volume* of an object increases when heated.
For simplicity, we use a cube, of side L. We increase the temperature by ΔT.

The old volume is $V = L^3$, and the new volume $V + \Delta V = (L + \Delta L)^3$. Let's try to approximate this for a small increase in L.

$$\Delta V = (L + \Delta L)^3 - V \tag{16.41}$$

$$\Delta V = L^3 \left(1 + \frac{\Delta L}{L}\right)^3 - L^3 \tag{16.42}$$

Here, we simply factor out L^3, and then also substitute $V = L^3$.
Next, we use the first-order term of the Taylor expansion of $(1 + x)^n \approx 1 + nx$, where $x = \Delta L/L$ and $n = 3$:

$$\Delta V = L^3(1 + 3\frac{\Delta L}{L}) - L^3 \tag{16.43}$$

$$\Delta V = 3\frac{L^3 \Delta L}{L} \tag{16.44}$$

$$\Delta V = 3L^2(L\alpha\Delta T) \tag{16.45}$$

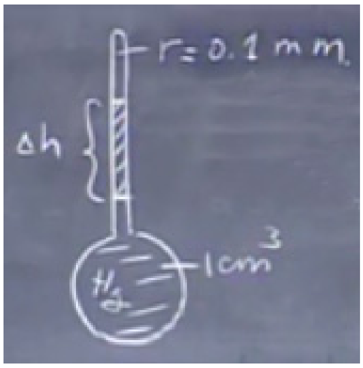
$$\Delta V = 3\alpha L^3 \Delta T \tag{16.46}$$

$$\Delta V = 3\alpha V \Delta T \tag{16.47}$$

In (16.45) we simply substitute $\Delta L = L\alpha\Delta T$.
We find, then, that the result depends on some value 3α. We usually write this as $\beta = 3\alpha$, and call it the *cubic expansion coefficient* (or volumetric expansion coefficient).

The reason we can use the linear term only is that the next term in the Taylor series contains α^2, which is extremely small. (For $L = 1$ m, $\Delta T = 100$ K and $\alpha = 10^{-5}/°\mathrm{C}$, the quadratic term is 3×10^{-6}, versus 1.003 for the two terms we do have.)The higher-order terms are much smaller yet.

We can now look at how a(nother) simple thermometer can work, e.g. a mercury thermometer, though other fluids are used these days, for safety reasons.
Mercury has a β of about $18 \times 10^{-5}/°\mathrm{C}$, while Pyrex has a value of roughly 1×10^{-5} (3 times the value of α we had earlier). A Pyrex container with mercury inside will then barely expand when heated, but the mercury inside certainly will – about 18 times as much. If we make a reservoir of mercury at the bottom, and then a very narrow column upwards, we get this:



When the mercury/liquid expands, it has nowhere to go except up the display tube. We calculate how much the column will grow per degree, and create the scale accordingly. All that remains is then to fill it up to the correct level, and nature takes care of the rest.

Consider a tiny radius of 0.1 mm, as shown, for the display tube. Ignoring the expansion of the Pyrex (which we probably shouldn't do if we actually built this), if our 1 cubic centimeter of mercury/liquid goes up in temperature by 10 degrees C, it expands by $V\beta\Delta T = 0.0018\,\mathrm{cm}^3$.
A very small increase, but hold on. A height h in the tube can hold a volume $\pi r^2 h$, so we find $h = \dfrac{0.0018\,\mathrm{cm}^3}{\pi(0.01\,\mathrm{cm})^2} = 5.73$ cm! That gives us almost 6 mm per degree, which is quite a bit more than most such thermometers I've seen; very easily readable.

16.3.4 Expansion of water

Water is a peculiar substance. We have so far only talked about substances that expand when heated, but water behaves rather differently at some temperatures.
If we take room-temperature water and cool it down slightly, it shrinks, as expected. However, once we reach 4 degrees C, cooling it *further* down to 0 will cause the water to expand!
Put in other words, the density of water is at a *maximum* when it is at 4 degrees C. This also implies that in this region between 0 and 4 degrees C, $\beta < 0$, so it changes sign at 4 degrees C.

This causes several important phenomena. For one, the 4 degree water sinks to the bottom, while ice tends to float; therefore, the bottom of lakes and such tend to remain liquid all year round, so that fish can survive below the ice.

For most materials, the solid of a material tends to sink in its liquid (i.e. the solid tends to have a higher density, so that a given amount, measured by mass, is more compact), but water is an exception.

Chapter 17: Week 15

17.1 Lecture 33: Ideal gas law

While liquids are almost entirely incompressible, as we have seen, gases are not. In a liquid, the molecules are still moving around (as opposed to a solid), but are quite closely packed, at least compared to a gas. In a gas, there is a fairly large distances between molecules, unless the pressure is very high. Therefore, we can compress gases rather easily, until the molecules become about as closely packed as in a liquid. If we keep compressing a gas at that point, it may undergo a phase change, usually to a liquid, but this depends on the compound. Carbon dioxide is perhaps the most well-known compound to only exist in gas and solid phases at atmospheric pressures; the liquid phase only exists at higher pressures (higher than about 5.1 atmospheres), so it either *deposits* (goes from gas directly to solid) or *sublimes* (also known as sublimates), meaning it goes from solid directly to gas.
Air at 1 atmosphere has a density 1/1000 times that of water; that says something about the relative distances between molecules involved.

Here are some definitions we'll soon use, from a lecture supplement sheet:

Symbol	Meaning	Unit/value
P	Pressure	Pascal (N/m^2)
V	Volume	m^3
T	Temperature	Kelvin (K)
N	Number of molecules	
n	Number of moles (see below)	mol
$\mathrm{N_A}$	Avogadro's constant	$6.022 \times 10^{23}\mathrm{mol}^{-1}$
R	Universal gas constant	8.31 J/(K mol)
k	Boltzmann constant	$R/N_A = 1.38 \times 10^{-23}$ J/K
Z	number of protons in a nucleus	
N	number of neutrons in a nucleus	
A	atomic number, $A = Z + N$	

We will use the unit of a *mole*, which is a unit not unlike terms like a dozen, only way larger. 1 mol is defined as the number of atoms in 12 grams of carbon-12; 1 mol of water means approximately 6.022×10^{23} molecules of water, for example. The unit can be used for anything. 1 mol of eggs is a lot; something like 10^{12} eggs were produced in 2002, so 1 mol of eggs would, at that rate, take 602 214 129 000 years to produce! Nevertheless, it is not much more than a number – only that it has a unit attached to it. I think I'll stick to dozens as far as eggs go.

Note that moles are about a number of something, but not *necessarily* number of *atoms*. One mole of helium molecules is the same as one mole of helium atoms, since helium doesn't tend to group into molecules at all.
On the other hand, one mole of oxygen gas (O_2) contains 2 moles of oxygen atoms. Unless specified otherwise, one mole will here refer to the molecular count, so that 1 mole of carbon dioxide and 1 mole of helium has the same number of molecules, but *not* the same number of individual atoms.

17.1.1 Ideal gas law

The *ideal gas law* states that

$$PV = nRT \tag{17.1}$$

using the definitions we introduced above. Both sides of this equation have the dimension of energy, i.e. units of joules using the MKS units. PV has units of (N/m^2)(m^3) = N m = J, while nRT has units of (mol)(J/(K mol))(K) = J.

Using the Boltzmann constant $k = R/N_A \approx 1.38 \times 10^{-23}\,\text{J/K}$ that we listed in the table above, we can also write the ideal gas law as

$$PV = NkT \tag{17.2}$$

where N is now a dimensionless number relating the number of molecules (not in moles, but the actual number), k is the Boltzmann constant as in the table above, and the rest of the variables remain as they were.

Before we use the ideal gas law, we'll have a quick look at atomic number and related things.
An atom has Z protons (that define which element it is), N neutrons (which define the isotope) and, if it is electrically neutral (i.e. not an ion), also Z electrons to balance out the change. (As we learn in 8.02 if not in high school, the proton and the electron have exactly the same magnitude of change, only opposite signs.)
The *atomic mass number* A is then simply $A = Z + N$, and defines how many protons plus neutrons there are in the nucleus.
Protons and neutrons have very close to the same mass (they differ by about 0.14%), while is this context, electrons have almost zero mass (an electron only has 0.05% of a proton's mass) that we can often neglect.

Let's look at carbon as an example. Carbon has 6 protons (6 protons defines the element, so anything else wouldn't be carbon). Carbon-12 also has 6 neutrons, so $A = 12$, which is also what we specify in its name.
Other forms of carbon have differing number of neutrons; known isotopes range from carbon-8 (2 neutrons) to carbon-22 (16 neutrons), though most of these are highly unstable. Only carbon-12 and carbon-13 are stable; carbon-14 has a half-life of 5730 years and is commonly used for radiometric dating of organic things.

As shown in the definition above, 1 mol of carbon-12 has a mass of 12 grams exactly. 1 mol of carbon-14 has a mass of approximately 14 grams (the approximate mass of 1 mol, in grams, of any atom is simply the number of nucleons), though because of the small difference in mass between protons and neutrons, the actual mass is closer to 14.00324 g.

Another example would be that of oxygen gas; it has a *molar mass* of about 32 g/mol. Each oxygen atom has 8 protons and 8 neutrons (some oxygen atoms are oxygen-17 and oxygen-18, but the vast majority are oxygen-16, so the average atomic mass number is about 16). Each O_2 molecule consists of two oxygen atoms, so we find $2 \times (8 + 8) = 32$ g/mol.

Since the mass of a proton and a neutron is almost equal, we can to a reasonable approximation write the mass of a molecule as $m_{molecule} = A \times 1.67 \times 10^{-27}\,\text{kg}$, where the mass of a proton is $m_p \approx 1.672\,621 \times 10^{-27}\,\text{kg}$.

17.1.2 Ideal gas law example

The ideal gas law is an approximation, but one that holds reasonably well for most gases. Therefore, we don't need to specify what the gas is to use it.

Say we have a gas at 1 atmosphere, so $P \approx 1.03 \times 10^5\,\text{Pa}$. We also have $n = 1\,\text{mol}$ of the substance. We do this at room temperature, so $T = 293\,\text{K}$.

$PV = nRT$, and we know everything except V. (R is a constant, so we know that, too.)
We solve for V, and find

$$V = \frac{nRT}{P} = \frac{(1\,\text{mol})(8.31\,\text{J/(K mol)})(293\,\text{K})}{1.03 \times 10^5\,\text{Pa}} \approx 0.0236\,\text{m}^3 \approx 23.6\,\text{L} \tag{17.3}$$

So this is (approximately) true whether the gas is helium, oxygen, nitrogen etc., as long as there is 1 atm of pressure. Of course, this only holds as long as the substance in question would actually *be* a gas as this

temperature and pressure. If we try to use water at 1 atmosphere and room temperature, then our results will be nonsense; we still find almost 24 liters, but the correct answer is about 18 mL, so this "estimate" is over 1000 times too high. (It also doesn't hold very well for water vapor either, because water molecules are fairly attracted to each other, which makes the ideal gas law not hold.)
We will soon look at phase diagrams, which will help us figure out whether a substance will be a gas, liquid or solid (or a mixture of two or three of these) at a given temperature–pressure combination.

As the name implies, the law is exactly true for *ideal* gases (by definition: an ideal gas is one that obeys this law). Many real gases are close to ideal under common circumstances, though. 1 mole of oxygen at atmospheric pressure and room temperature is within 0.1% of what the ideal gas law predicts (the true value is smaller than the approximation). At 20 atmospheres, the result is about 2% off, still with the correct result being smaller than the approximation.

17.1.3 Ideal gas law with different molar mass gases

Consider the case when we have two gases where the molar masses are very different, but we have the same number of moles of each gas. Both are at room temperature, and they are in identical containers. n, T and V are the same, and via the ideal as law, that means P is also the same. The masses of the molecules are very different however, and since we have the same amount, the total mass of one gas must also be much greater than the mass of the other.

The molecules in the gas are flying around in all directions, with different speeds. We consider an average speed $\bar{v}$ for simplicity.
Say a molecule of mass m hits the container wall with speed $\bar{v}$. It bounces back in an elastic collision, which implies a momentum change of $2m\bar{v}$ in magnitude – its forward momentum is replaced with backwards momentum of the same magnitude.

That is just the momentum change of one molecule, though. We want the rate of momentum transfer over time.

If we consider a cube of side L, it takes a molecule a time $t = \dfrac{2L}{v}$ to come back to a wall after bouncing off it, in the simple case where it moves in one dimension only. Therefore, the rate of momentum transfer (per second) is $\dfrac{2mv}{t} = \dfrac{2mv}{\frac{2L}{v}} = \frac{mv^2}{L}$. (Thanks to Grove for this derivation.)

The rate of momentum transfer for the entire system is therefore proportional to mv^2. Rate of momentum transfer is force, and force is proportional to pressure. It certainly looks as if $m \propto P$ – which the ideal gas law clearly says is not the case!
The only way this works out, and how it actually does work, is if mv^2 is constant for a given temperature, i.e. it is independent of m! In other words, the speed of the particle is such that it balances out its mass; the smaller the mass, the larger the speed, and vice versa.

For example, comparing helium and oxygen gas, we can write that

$$m_{He}\overline{v_{He}}^2 = m_{O_2}\overline{v_{O_2}}^2 \tag{17.4}$$

$$\overline{v_{He}} = \sqrt{\frac{m_{O_2}}{m_{He}}}\,\overline{v_{O_2}} \tag{17.5}$$

Oxygen molecules have an average speed of about 480 m/s at room temperature. Since the ratio of masses here is about 8, helium molecules move, on average, about $\sqrt{8} \approx 2.82$ times faster than oxygen molecules, which is about 1350 m/s.
If we mix the two cases, the only way the ideal gas law can hold is if these speeds still stay true, so that the lighter molecules move faster.

17.1.4 Ideal gas law example #2

"A closed container with a volume of $8000\,\mathrm{cm}^3$ is filled with Xenon gas. The gas temperature is 273 K (the container is placed in ice water) and the pressure is 2.0 atm.

How many moles of Xenon are in the container?"

Well, we use $PV = nRT$. We want n, so

$$n = \frac{PV}{RT} = \frac{(2 \times 1.03 \times 10^5\,\mathrm{Pa})(0.008\,\mathrm{m}^3)}{(8.31\,\mathrm{J/(K\ mol)})(273\,\mathrm{K})} \approx 0.726\,\mathrm{mol} \tag{17.6}$$

"The container is now submerged in boiling water until the gas inside the container is at 373 K. You may assume that the increase in volume of the container is negligible.

What is the pressure of the xenon?"

We re-rearrange the equation to give

$$P = \frac{nRT}{V} \tag{17.7}$$

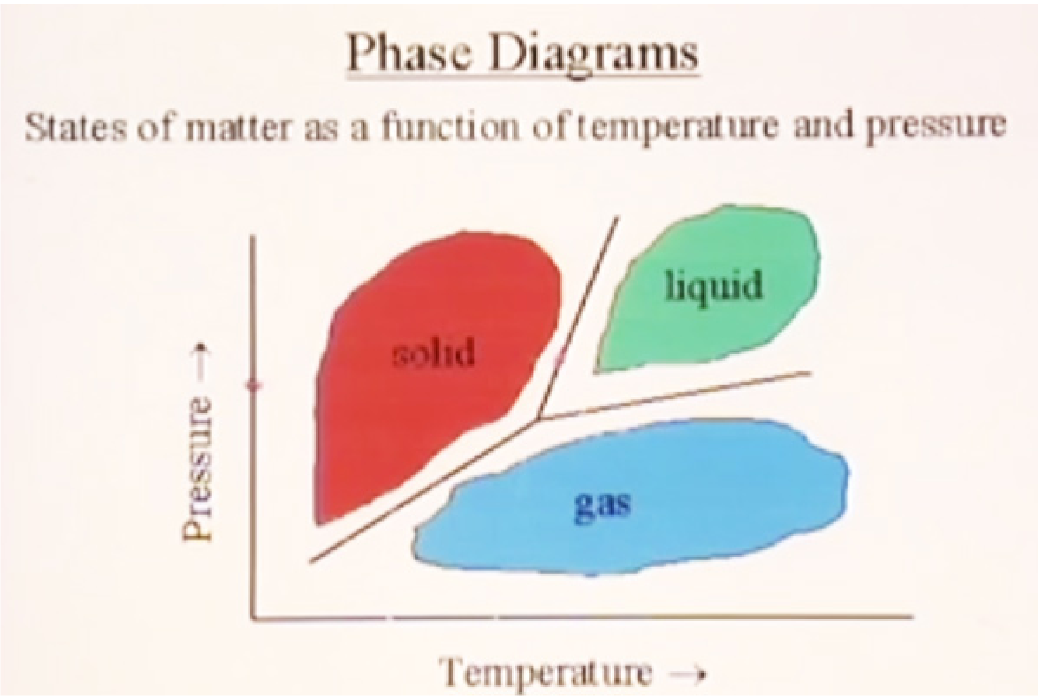
Plugging in the given numbers, plus the n we found above,

$$P = \frac{nRT}{V} = \frac{(0.726\,\mathrm{mol})(8.31\,\mathrm{J/(K\ mol)})(373\,\mathrm{K})}{0.008\,\mathrm{m}^3} \approx 281\,300\,\mathrm{Pa} = 2.7\,\mathrm{atm} \tag{17.8}$$

17.1.5 Phase diagrams

We will now look at phase diagrams. The *phase* of a substance is basically whether it is a solid, liquid or a gas; however, phase and state of matter are not the same thing. For example, while all water ice is solid (its state of matter), there are many different phases. Ice Ih is by far the most common water ice; Ice Ic essentially the only other form naturally occurring on Earth. Over a dozen other forms of water ice have been created in labs, by varying the temperatures and pressures.
I will mostly (perhaps entirely) use phase and state of matter interchangeably in the rest of these notes.

Here's a simple, general phase diagram:



Consider starting at a low pressure, with the temperature at about the center, say around the "g" in gas. Clearly, the substance is a gas at this point. We increase the pressure, while keeping the temperature constant. The volume decreases, and the pressure increases, with PV being kept constant (which is called Boyle's law; it states that $P \propto \frac{1}{V}$ or $PV =$ constant if temperature/amount of gas is held constant). The ideal gas law holds until we reach the dividing line into the liquid. At this point, *some* of the gas will turn into a liquid, and there will be an equilibrium between the two.
If we try to push down harder, the pressure *will not increase* until *all* the gas has been converted into a liquid. Only at that point will pressing harder again increase the pressure; while there is still gas present, pressing down harder will only convert more of the gas into liquid.

Suppose we instead do this at a lower temperature; we can then see that there comes a point in this example phase diagram where we go directly from gas to solid (this is called *deposition*; the reverse process, solid to gas, is called *sublimation*). So we increase the pressure and decrease the volume, until the solid starts forming. Once again, we can't increase the pressure further until all gas molecules are part of the solid.

Let's now look at the case of constant pressure instead of constant temperature.
We start out in the solid phase, at about the vertical center – at the tiny red mark in the y axis, about next to the "e" in pressure.
Say we start out with water ice; or even iron. We heat it, keeping the pressure constant, meaning we move horizontally towards the right. We eventually hit the dividing line between solid and liquid, i.e. the substance will begin to melt.
Once that happens, the temperature will stop increasing until *all* of the solid has melted into liquid, similarly to what we saw with pressure above. Once all of it has become a liquid, we can increase the temperature further.
If we do so, it will eventually boil, i.e. go from a liquid to a gaseous form. Yet again, we can no longer increase the temperature at this point, until all of the liquid has become a gas (water vapor). This might be the only one of these that we are familiar with: when boiling water, it doesn't matter if the platter is just barely hot enough than necessary, or *much* hotter than necessary. In either case, the liquid water will not become any hotter than 100 degrees C (unless the pressure is greater than 1 atm), no matter how violent the boiling is.

As a side note: water vapor is completely invisible (it as as transparent as clean air). Any time we think we see water vapor, for example when boiling water in the kitchen, what we actually see is tiny condensed liquid droplets. The water vapor condenses back into liquid as it comes in contact with the much colder surrounding air.
For this reason, you may be able to see that just above the surface of boiling water, there is invisible water vapor (i.e. it looks as if there's only air there), and only *above* that is the visible steam showing up, since it hasn't had time to cool down yet when just above the surface of the boiling water.

17.1.6 Pressure and phase in a CO2 fire extinguisher

Carbon dioxide fire extinguishers are fairly common. They work by displacing oxygen, so that a fire can't be sustained. This has two important meanings, by the way: one, it can be dangerous to use on/near people or in closed spaces, as you may suffocate; two, burning materials that contain enough oxygen by themselves may well keep on burning.

So is there gas or liquid (or even a solid?) inside such a fire extinguisher?
Prof. Lewin calculated the volume of one such extinguisher to be $7.1 \times 10^{-3}\,\text{m}^3$. It is at room temperature, so $T = 293\,\text{K}$.

To find the pressure, which helps us find the phase, we now need to know n, the number of moles of CO2 inside. By reading on the label, we can find that the difference between a full extinguisher and an empty one is about 10 pounds, or 4500 grams.

Carbon has an atomic weight of about 12, and oxygen one of about 16; that gives us $A = 12 + 2 \times 16 = 44$. With a molar mass of 44 g/mol, we can simply find the number of moles as

$$n = \frac{4500\,\mathrm{g}}{44\,\mathrm{g/mol}} = 102.28\,\mathrm{mol} \approx 100\,\mathrm{mol} \tag{17.9}$$

We now have all we need to know to use the ideal gas law. We aren't sure if it will hold, but let's try. Plugging the values in,

$$P = \frac{nRT}{V} = \frac{(100\,\mathrm{mol})(8.31\,\mathrm{J/(K\ mol)})(293\,\mathrm{K})}{7.1 \times 10^{-3}\,\mathrm{m}^3} = 3.43 \times 10^7\,\mathrm{Pa} \approx 340\,\mathrm{atm} \tag{17.10}$$

This pretty much rules out the possibility of there being gas inside, for two reasons! First, it seems doubtful that the container could withstand such a tremendous pressure. Second, if we look at a phase diagram, we would likely find that CO2 becomes a liquid (or a solid) at a way lower pressure than 340 atm at room temperature. And indeed, looking at one, we find that at room temperature, the phase transition to liquid happens at something like 60-70 atm.

The answer is that the extinguisher contains a pressure of about 60 atm, according to a fire department called by the professor. Looking at a phase diagram of CO2, this makes it clear that there is either 100% liquid, or a combination of liquid and gas inside. When we open the valve, some of the liquid will turn into gas, but the pressure will not change until all the liquid is gone. Up until that point, they must exist together, which can only happen at certain combinations of temperature and pressure. For a *fixed* temperature (say 20 C), however, there is only one pressure at which this can happen; that pressure must then be constant inside until the liquid "runs out", having been converted into gas.

This also means we can fit a lot more CO2 into a canister than we could otherwise. Remember how the original calculations said the pressure would have to be over 300 atm if it were pure gas; with this part-liquid mix, we can fit that same amount at about 60 atm instead, which doesn't require as strong a canister. This is put to the test in a lecture question:

"The density of liquid carbon dioxide is about $0.8\,\mathrm{g/cm}^3$. What volume fraction inside the fire extinguisher (when it is full) is occupied by liquid CO2 ? (The density of CO2 gas is negligible compared to the density of liquid CO2)

Hint: Review what is given in the lecture: Volume of the extinguisher is $7.1 \times 10^{-3}\,\mathrm{m}^3$, total mass of CO2 = 4.5 kg, temperature= 293 K , pressure at that temperature has to be $60 \times 10^5\,\mathrm{Pa}$."

Hmm. Well, if 100% was liquid, its volume would be

$$V = \frac{M}{\rho} = \frac{4500\,\mathrm{g}}{0.8\,\mathrm{g/cm}^3} = 5625\,\mathrm{cm}^3 \tag{17.11}$$

... which is less than the full volume of $7100\,\mathrm{cm}^3$. In fact, it is about 80%... a little more than 75%, which is one of the answer options. We were told that the density of CO2 gas in negligible, so this should in fact be the answer, and it is. Not a very rigorous process, but they did tell us to neglect the gaseous portion.

17.1.7 Isothermal atmosphere

We have earlier looked at hydrostatic pressure, and found the relationship

$$\frac{dP}{dy} = -\rho g \tag{17.12}$$

between pressure, density and depth. Because we can treat both g and ρ as constants (depth differences are small enough, and liquids are practically incompressible, respectively), this gives us a very simple linear relationship

$$P_2 - P_1 = -\rho g(y_2 - y_1) \tag{17.13}$$

where $y_2 > y_1$ (positive upwards), and therefore $P_1 > P_2$. If you ever lose track of the minus signs and that, you just need to keep in mind that pressure must *increase* at greater depths, and you can't go wrong. With that in mind, I probably prefer to think of this as

$$|\Delta P| = \rho g|\Delta y| \tag{17.14}$$

which is hard to get wrong, using the above (rather obvious) trick.

The reason that was easy to do is that we could treat ρ as constant, with little loss of accuracy. In reality, ρ is a function of pressure. This is a smaller detail for liquids, but a crucial one for gases, which we'll look at now. We can no longer treat ρ as a constant.

We will now look how the pressure changes in altitude in our atmosphere. We will assume that the temperature everywhere is 0 degrees C everywhere in the atmosphere; that is not true, but the full calculation is still a bit too complex. We call this simplification an isothermal atmosphere. (In general, the iso- prefix is used in physics for things that are the same in one way or another; from the Greek word isos, meaning equal.)

Density is mass per unit volume, so if we have N molecules inside a certain volume V, each of mass m, we can say that

$$\rho = \frac{Nm}{V} \tag{17.15}$$

where ρ is the average density inside that volume.

Using the ideal gas law,

$$PV = NkT \tag{17.16}$$

$$\frac{P}{kT} = \frac{N}{V} \tag{17.17}$$

So we substitute that, and find

$$\rho = \frac{Pm}{kT} \tag{17.18}$$

We can then substitute that into the differential form we had earlier,

$$\frac{dP}{dy} = -\rho g = -\frac{Pm}{kT}g \tag{17.19}$$

Rearranged,

$$\frac{dP}{P} = -\frac{mg}{kT}dy \tag{17.20}$$

As it was before, this is a separable differential equation. m is a constant, k is a constant, and we said we consider T and g constants. The right-hand side is an easy integral, and the left-hand side isn't much harder. We integrate from 0 (sea level) and P_0 to h and P_h:

$$\int_{P_0}^{P_h} \frac{dP}{P} = -\frac{mg}{kT}\int_0^h dy \tag{17.21}$$

$$\ln P_h - \ln P_0 = -\frac{mg}{kT}h \tag{17.22}$$

$$\ln \frac{P_h}{P_0} = -\frac{mg}{kT}h \tag{17.23}$$

Exponentiating both sides:

$$\frac{P_h}{P_0} = e^{-\frac{mg}{kT}h} \tag{17.24}$$

We can still simplify this a bit. We have a constant involved; we can find its value. If we flip it upside down, the constant I'm talking about is

$$H_0 = \frac{kT}{mg} \tag{17.25}$$

This is called the *scale height*; it has the dimension of length, because its reciprocal (in the exponential above) must be 1 over length, so that it cancels out with h; you need a dimensionless number in exponentials. We know $k = R/N_A \approx 1.38 \times 10^{-23}\,\mathrm{J/K}$, $T = 273\,\mathrm{K}$ (0 degrees C, as we chose earlier), and $g \approx 9.8\,\mathrm{m/s^2}$. What about m, the mass of an air molecule?

Well, the professor chose to use 29 atomic mass units, with this reasoning: air is about 20% oxygen (32 amu per molecule) and about 80% nitrogen (28 amu per molecule), and some spare change (argon, CO2 etc) that we ignore. Since there is more nitrogen, we choose a number closer to 28 than 32 amu, and so we end up with 29 amu. (The actual value appears to be 28.964, so this approximation is very good.)
Each amu represents a mass of about $1.66 \times 10^{-27}\,\mathrm{kg}$, so all in all, we find $H_0 \approx 8000$ meters. Rewriting our exponential, we now have

$$P_h = P_0 e^{-h/H_0} \tag{17.26}$$

Using $H_0 = 8$ km and $P_0 = 1$ atm, we can then for example find that the air pressure at 3 km above sea level is about $(1\,\mathrm{atm})e^{-3/8}$, or about 0.7 atmospheres. At 8 km, it is only $1/e$ times 1 atm, which is about 0.37 atmospheres.

This not only has implications for human life (the air is basically too thin to support human life, though some do climb Mount Everest with no supplemental oxygen), but also for basic things like boiling water.
The boiling point is defined as the temperature where the liquid's *vapor pressure* (which we have not really learned about in this course) equals the (atmospheric) pressure of the air surrounding the liquid.
The vapor pressure of a substance, e.g. water, is a constant for a given temperature. At 100 degrees, it is about 101.3 kPa (1 atm) – that is to say, it boils at 100 degrees at 1 atmosphere. However, if the atmospheric pressure is *lower*, the water will boil at a lower temperature. For example, water would boil at 80 degrees C if the atmospheric pressure were 47.3 kPa (about half an atmosphere); in other words, the vapor pressure of water is 47.3 kPa at 80 degrees C.
This can cause some trouble for cooking at higher altitudes – in extreme cases, it will be hard to prepare certain types of food as they will need to boil for very long.

Even more interestingly, the vapor pressure of water at 22 degrees C is about 2.6 kPa – which means that if we place water in a near-vacuum, it will boil at or even below room temperature. This is demonstrated in the lecture.
To see this on a phase diagram, locate the (likely vertical) line of constant temperature of about 20 degrees, and then find the point along that line of 1 atmosphere pressure. That's where we start out; you will

undoubtedly find that the water should be in its liquid phase. Follow that line of constant temperature downwards, and you will eventually reach the line where gas and liquid coexist – which is where it starts to boil.

17.1.8 More lecture experiments

A second lecture experiment is to add a very small amount of liquid water to a paint can (of the same type that imploded in a previous lecture, when we sucked out the air from inside it), boil it, and then seal the can. It is now filled with almost 100% water vapor (a tiny fraction liquid water may remain), at 100 degrees C, and 1 atmosphere of pressure. We seal the can, and let it cool.

The amount of water, in moles, must of course be a constant. However, now that the temperature goes down, so does the volume of the water vapor gas; we can see this by looking at the vapor pressure for water at various temperatures. (Perhaps also by using the ideal gas law, but it doesn't hold very well for water vapor.)
Indeed, as it cools back down to room temperature, the vapor pressure is about 1/45 atm, so there is practically a vacuum inside (as far as the thin walls are concerned, at least), and the can will implode.

Next, we cool a regular air-filled balloon in liquid nitrogen. We change the temperature of the air inside from 293 K to about 77 K (about -196 C, the boiling point of liquid nitrogen). What will happen to the balloon?
We can see, using the ideal gas law, that it must shrink; if we hold P approximately constant, that is clear.

$$V_1 = \frac{nRT_1}{P} \tag{17.27}$$

$$V_2 = \frac{nRT_2}{P} \tag{17.28}$$

$$\frac{V_2}{V_1} = \frac{T_2}{T_1} \tag{17.29}$$

This gives us $V_2/V_1 \approx 0.26$. The radius changes less, though; $R \propto V^{1/3}$, so the radius should shrink by something like 60%. However, in practice, the balloon shrinks down to almost nothing – the volume goes down to perhaps 1-2% of the original, or something of that order.

What did we miss? Well, we used the ideal gas law – but we won't have gases when we're done! Remember that air is 80% nitrogen, and we dip it into liquid nitrogen. The nitrogen inside the balloon may turn into liquid, since we cool it to approximately[1] its boiling point.
The boiling point of oxygen at 1 atm is about 90.1 K, so we cool that down below its boiling point, too, so the oxygen should certainly become a liquid. We know, of course, that liquids have a *way* higher density than gases, so it should not come as a huge surprise (when you consider the phase changes) that the volume is very small.

17.2 Lecture 34:Heisenberg's uncertainty principle

17.2.1 Off-topic intro

This is a course on classical mechanics. It is extremely useful in many everyday situations, whether we talk about toy gyroscopes, elastic collisions in billiards or in all kinds of mechanical engineering. However, it it doesn't always hold true. At large velocities or in very strong gravitational fields, the laws of physics we've learned about here gradually become more and more incorrect. The velocities involved are on the order of 1/10 of the speed of light and greater (or a bit less, depending on the accuracy required); here, we need Einstein's theory of special relativity to find correct answers. For example, using what we have learned in this course, we can calculate the kinetic energy of a 1 kg mass moving at the speed of light

[1]The heat transfer may not be ideal, etc., so it may not go all the way down to 77 K.

as $\frac{1}{2}(1\,\text{kg})c^2 \approx 4.5 \times 10^{16}\,\text{J}$, but this is a meaningless result. It would take an infinite amount of energy to accelerate a mass to that speed, and so via the work-energy theorem it would have an infinite kinetic energy.

The actual kinetic energy is found as

$$K_e = m\gamma c^2 - mc^2 = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \tag{17.30}$$

Note that as $v \to c$, $K_e \to \infty$, which is not the case in Newtonian mechanics. If you try to plug in $v > c$, you get an imaginary result with no physical meaning.

If we calculate the Taylor expansion of the above function centered at $v = 0$ and only keep the lowest-order term, $\frac{1}{2}mv^2$ is the only term that remains. That is, classical physics agrees with special relativity, only that the latter can handle the more extreme cases.
If we plot these two functions between $v = 0$ and $v = 10^7$ m/s, they are almost impossible to tell apart. The plot starts to visually diverge at about 3×10^7 m/s, where Einstein's version gives 4.534×10^{14} J and the classical variant only 4.500×10^{14} J.

The term

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{17.31}$$

is called the Lorentz factor, and appears in many special relativity equations, regarding kinetic energy, momentum, time dilation, length contraction and probably other things I'm not yet aware of. It is a useful thing to know even if you only know Newtonian mechanics: the closer γ is to 1, the more accurate the physics we've learned so far will be. For everyday speeds below 200 km/h, γ is still 1 to within well over 10 decimal points. At 1000 km/s (yes, per second!), it is about 1.00000556, so calculating momentum as mv instead of $mv\gamma$ will yield an error of much less than 1%. At 10^8 m/s, 1/3 the speed of light, γ is *still* only 1.06, though this is about where it starts to shoot off towards infinity and begins to truly matter.

Now for a much quicker look on gravity.
As an example, the orbit of Mercury cannot be predicted accurately using Newton's law of universal gravitation; its orbit precesses, at a rate of about 1.55 degrees per century relative to the Earth. Einstein's theory of general relativity explains this anomaly, and is currently the best theory of gravity we have. As mentioned, though, classical mechanics is still "correct enough" for the vast majority of applications. General relativity is heavily used in astrophysics, where it describes many things that Newtonian mechanics does not (and describes the rest of them more accurately), one of which is gravitational lensing: light bending in gravitational fields.

I had to add a subsection label to this as it became a bit too strongly off topic – I was going to write a few short sentences on the limits of classical mechanics and simply introduce relativity and quantum mechanics, but got a bit carried away!

17.2.2 The smaller world

After that off-topic introduction, let's now look at the second case where Newtonian mechanics stops working: the world of the very small, i.e. the atomic and sub-atomic scale.
An atom is about 10^{-10} meters, including the surrounding electron cloud. The nucleus itself is much, much smaller yet, and yet it contains virtually 100% of the atom's mass. The distance between the nucleus and the electrons is on the order of 100 000 times greater than the size of the nucleus.
The nucleus consists of protons (positively charged) and neutrons (with no electric change, thus the name) bound together by the strong nuclear force, which is much stronger than the electromagnetic force, but

has an extremely short range, on the order of a nucleus and less. The strong nuclear force is the reason why the nucleus can be held together despite the electromagnetic repulsion of the equal charges; without it, all nuclei would simply fall apart. (Neutrons and protons themselves couldn't exist without it either, as the quarks that make *them* up are also held together by the strong force.)

Atoms are basically all vacuum, as we can see above (the nucleus is almost zero size compared to the atom's total size) – we're used to thinking of "empty" spaces being filled with air, but since air *consists* of atoms, well, there isn't much of anything that fits inside an atom![2] So why can we not walk through walls, if we all consist of mostly vacuum? The professor mentions that it's not easy to answer and that we cannot answer it using classical physics, which I find surprising.
All (neutral) atoms have electrons in "shells" surrounding the nucleus. All electrons are negatively charged, and like charges repel; therefore, atoms repel each other. Your hands are being repelled by the wall even when you're standing ten meters from the wall, but with a force small enough that you cannot notice it. When you are "touching" the wall, the repulsive force is very large, and the closer you try to come, the stronger the repulsive force. That is, you never really touch a wall – just you get really, really close, and the normal force the wall exerts on you really comes from this electromagnetic repulsion of the electron shells.

As far as I know, the above explanation is fairly correct, and if not else a useful way to think of it. I'm not sure of the details of it, however. There must be a reason the professor didn't use it, and especially mentioned that it cannot be explained by classical physics.

In 1913, Danish physicist Niels Bohr postulated that electrons move around the nucleus in well-defined energy levels, which are all distinctly separated from each other, and that electrons cannot exist in between these allowed energy levels. This is then the reason why the electron does not simply crash into the nucleus, as it would seem like it should, considering the electromagnetic attraction between the two.

This concept of quantization was groundbreaking. It also implies that planetary orbits should be quantized; you couldn't orbit at an arbitrary distance, so that you couldn't move an object in orbit in or out just a tiny bit.
it also implies that we can't bounce a tennis ball on the ground and have it reach any level; instead, there would be a set of allowed heights it could reach.

However, the quantization here is on a very, very small scale. Small enough that we could never measure the effect on the scale of planetary orbits or tennis balls; the effect is *much* too tiny for that. It is for exactly this reason that quantum mechanics has no meaning when it comes to the motion of tennis balls and planets.
Some quantum phenomena are absolutely observable in daily life, however: magnetism has its origin in quantum physics, for example. We will also soon perform an experiment where we could say that quantum physics is observable using the naked eye.

The professor stresses this point of quantized electron energy levels, as it is a very important one.

When we heat a substance, the electrons can gain energy, and therefore jump to higher energy levels. Later on, they lose that energy, and fall back down. As they do, they emit photons: they need to lose that extra energy somehow (since the higher energy levels have, well, higher energy than their previous states). The professor makes an analogy with the work you do while lifting a vase in a gravitational field. You do work, but that energy is not lost; if you drop the vase, the stored gravitational potential energy is "released" by being converted into kinetic energy.

17.2.3 Photon energy and momentum

The energy of a photon can be written as

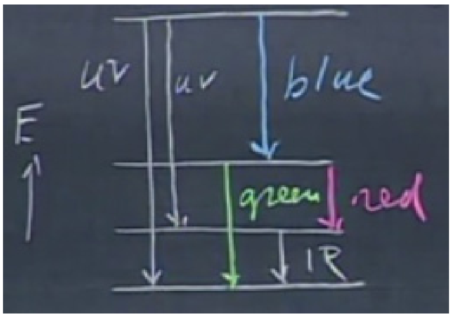
[2] In quantum physics, there *are* "things" inside this vacuum: empty space is not truly empty, but contains random energy fluctuations and short-lived virtual particles, in what's called a quantum foam. I won't (and can't!) go into more detail, though.

$$E = hf = \frac{hc}{\lambda} \tag{17.32}$$

where h is the Planck constant, $h \approx 6.6 \times 10^{-34}$ J s, perhaps the most important constant in all of quantum physics. There is also the related constant $\hbar$, "h-bar", which is $\hbar = \dfrac{h}{2\pi} \approx 1.05 \times 10^{-34}$ J s.
f is the photon's frequency in hertz, and λ the wavelength in meters.

This definition makes it clear that the greater the photon's energy, the shorter the wavelength, and vice versa.
This also means that when an electron jumps from a high to a low energy state, and the energy difference between the two levels is very high, a short-wavelength photon will be generated, since it must contain all the energy of the jump. (A single energy jump always releases exactly one photon; an electron can fall down multiple levels in *steps* however, in which case one photon is released per step. These photons will then have a smaller individual energy than if the entire jump were to be done in one step.)

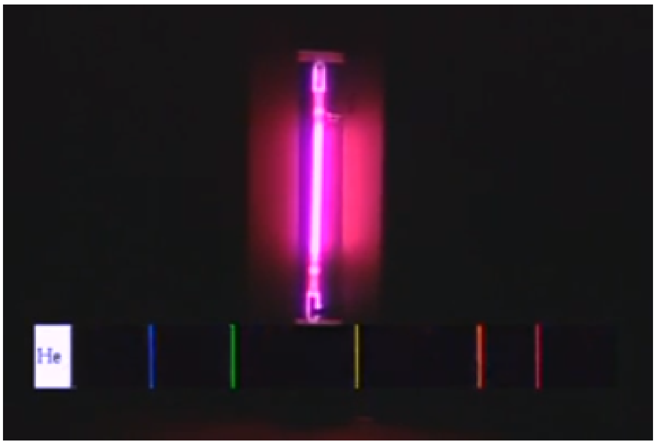
Here's a diagram illustrating these energy jumps.

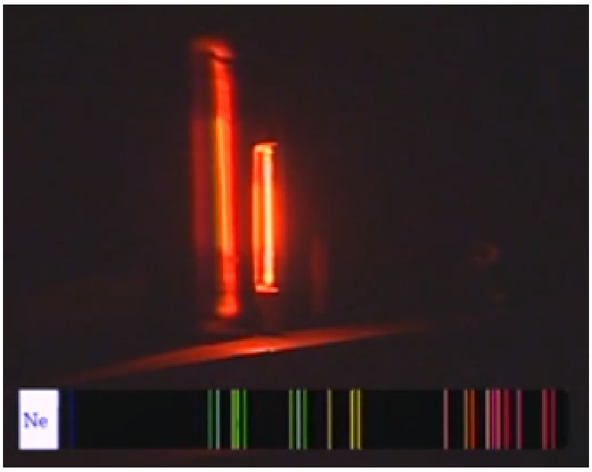


The lowest-drawn line is the lowest allowed energy level, with energy increasing towards the higher levels. A jump from the highest energy level to the lowest one may generate a photon with too much energy (too short a wavelength) to be visible – ultraviolet (or even beyond that), while some of the smaller jumps may correspond to visible wavelengths: blue for the more energetic visible ones, green for the mid-range ones, red for the lowest-energy visible jumps. Finally, even smaller jumps may generate invisible frequencies again, such as infrared or beyond.

So when a material emits photons in this manner, we expect to see these very discrete photon wavelengths, and nothing at all in between. Indeed, we can test this. By using a diffraction grating (a concept later introduced in 8.02/8.02x), we can split the light into its constituent colors, in concept not unlike a prism, so that we see the colors laid out in a nice horizontal line, looking very much like pictures of emission and absorption spectra that we have seen earlier, e.g. like this:

This is what we would expect to see from a pure-hydrogen source emitting light.





Above are two "simulations" of the gratings given to the students in lecture, as they could not capture the effect on video. The image on the left is from a helium light source, while that on the right is from a neon light source.

Surprisingly, light also carries momentum! We know that in classical mechanics, $p = mv$, which clearly cannot hold for a photon (if they indeed have nonzero momentum), without some fancy tweaking; the mass of a photon is exactly zero, so $mv = 0$.
What we do instead is to use Einstein's mass-energy equivalence, the famous $E = mc^2$:

$$m = \frac{E}{c^2} \Rightarrow p = \frac{E}{c^2}v \Rightarrow p = \frac{E}{c} \tag{17.33}$$

... since $v = c$ for a photon.

We can also find this more directly from the full, less-famous version of the equation:

$$E^2 = (m_0c^2)^2 + (pc)^2 \tag{17.34}$$

where m_0 is the rest mass (which I denoted by m above). Since $m_0 = 0$, the equation simplifies down considerably to

$$E = pc \tag{17.35}$$

which is what we found before.

This momentum also gives rise to interesting properties such as radiation pressure/light pressure: shining a light onto something causes a pressure (and therefore a force) due to this momentum transfer! This is used in practice to create spacecraft with "solar sails" that use the momentum transfer from reflected light to accelerate. The effect is too tiny to be noticed in daily life, though, considering that visible light has a momentum on the order of perhaps 10^{-27} kg m/s per photon. Despite that they come in large numbers, the radiation pressure of for example a regular lamp is negligibly small compared to just about any other force we experience.

17.2.4 Wavelength of a particle

Before quantum mechanics, physicists were divided into two camps: those who thought light to consist of particles, and those who thought it was made of waves.
Newton believed that light was made of particles, while Dutch physicist Huygens believed they were waves.
In 1801, British scientist Thomas Young showed fairly conclusively that light is a wave, by performing the famous double-slit experiment.
You shine a monochromatic light onto two very thin slits in some material (that otherwise blocks light), and project that onto some surface a distance behind.

What you see is an interference pattern: there is bright light at the center, then darkness a bit further out, then light again even further out, etc. This can be "easily" explained (and is discussed in detail in 8.02/8.02x, and likely even more in detail in 8.03) in terms of light being a wave, as the peaks of the wave cancel out with the valleys when the two arrive in phase, causing darkness; likewise, when two peaks arrive in phase, they add instead of cancel, and the result is a bright area.
The two forms of interference are called destructive and constructive interference, respectively.

So it looked like Huygens was right; light is a wave. However, later on, experiments were made that showed quite conclusively that light is made up of particles, perhaps most notably the photoelectric effect observed by Einstein (which won him his only Nobel prize) that required light to arrive in quantized "packets" of energy, rather than in a continuous wave. (Einstein didn't discover the effect itself, however.)

Quantum mechanics says the answer to this disagreement is that light acts as *both*, depending on the situation; this concept is known as wave-particle duality.

One of the truly strange things about quantum mechanics is that we can consider *matter particles* to be waves, as well. We now know that finding the wavelength of a photon of some given energy is easy, but what about the wavelength of an electron, or of a baseball?
Louis de Broglie suggested that matter can act as waves in this manner. He also specified that the wavelength of such a particle would be

$$\lambda = \frac{h}{p} \tag{17.36}$$

where p is simply the momentum mv. We can derive this result ourselves using what we know of the momentum of light.

We know that $E = pc$ and $E = hf = \dfrac{hc}{\lambda}$; if we put these together, we find

$$pc = \frac{hc}{\lambda} \Rightarrow p = \frac{h}{\lambda} \tag{17.37}$$

or, equivalently, $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$. This wavelength is called the *de Broglie wavelength*, after him. This derivation assumes that the result is equally valid for matter particles as it is for light, though, which is by no means obvious.

Note that if the momentum is higher, the wavelength is shorter. An electron moving at 10^7 m/s, and with a rest mass on the order of 10^{-30} kg (9.1×10^{-31} kg) has a classical momentum of about 9.1×10^{-24} kg m/s, which translates into a wavelength of about $\lambda = 7.25 \times 10^{-11}$ m – about 73 picometers, several times larger than at atomic nucleus.

A daily life-sized object such a baseball, with a mass of 145 grams, moving at 130 km/h (36.11 m/s) has a momentum of about 5.24 kg m/s, which gives it a de Broglie wavelength of about 1.3×10^{-34} meters – which is of course ridiculously small. It's far, far below what we could ever measure directly (at a billion billion times smaller than a proton), so this wavelength is meaningless in the macroscopic world. These kinds of quantum effects simply aren't observable at this scale.

17.2.5 Heisenberg's uncertainty principle

In 1926, Austrian physicist Erwin Schrödinger formulated the Schödinger equation, which is at the heart of quantum mechanics; it is a wave equation, which describes how a quantum system evolves over time. It unifies these wave and particle behaviors into one set of rules.

We talked earlier about the double-slit experiment and interference of waves. Amazingly, we can do this experiment with particles, too, and get the same end result! This is one of the many extremely nonintuitive

results we can find in quantum mechanics.
It seems bizarre that e.g. two electrons can be shot through two slits, and then combine and disappear. We really need to think in terms of waves for this to make any sense at all; if the electrons are instead two waves moving through the slits, it does make sense that they can cancel each other out at certain locations.

Let's now look at another bizarre effect in the quantum world. In classical physics, we can measure the momentum and position of an object with any precision that we need, as long as we have the equipment and cleverness. The object has a certain mass, and we can measure its position and momentum at the same time with no problems.

In quantum mechanics, this is not possible. We can measure the position to an arbitrary accuracy, and the momentum to an arbitrary accuracy, but not at the same time! The more exactly we know one, the less exactly we know the other, in this one measurement. This is known as Heisenberg's uncertainty principle.

"The very concept of exact position of an object and its exact momentum, together, have no meaning in nature."

One way of writing this down mathematically is

$$\Delta p \Delta x \gtrapprox \frac{\hbar}{2} \tag{17.38}$$

where again $\hbar = \dfrac{h}{2\pi} \approx 10^{-34}$ joule-seconds.
The right-hand side is often written as a factor 2 larger, and I'm not sure which we actually should use. The principle is also often stated in terms of standard deviations. I think we'll have to accept that this is approximate, and take a proper quantum mechanics course for more detail.

Roughly speaking, then, if we know the position to an accuracy Δx, the momentum is non-determined by an amount

$$\Delta p \gtrapprox \frac{\hbar}{2\Delta x} \tag{17.39}$$

The lecture uses twice this ($\hbar$ rather than $\hbar/2$), but there was a caption suggesting that the above values are the ones that *should* have been used in the lecture, i.e. a post-recording correction, so I chose to use their corrected information instead of the one actually shown in the lecture video itself.

The professor uses a story from a book, trying to describe quantum mechanics in an intuitive way (more or less). In this story, we set $\hbar = 1$, instead of about 10^{-34}. This essentially has the effect of scaling up these quantum effects to a level where we can observe them.
So in this world where $\hbar = 1$, a character in the book takes a billiard ball and puts it in a triangle (which is used to align the balls at the start of the game; it can fit exactly 15 such balls in the case of pool).

Assuming the ball stays inside the triangle (which may not be a safe assumption in this crazy quantum world; see quantum tunneling), we have constrained its position to $\Delta x \approx 0.3$ meters. We *know* that it must be somewhere inside the triangle. Via Heisenberg's uncertainty principle, this implies that the ball's momentum is ill-defined to about 1/0.3 kg m/s (using $\hbar = 1$, and using $\Delta x \Delta p \gtrapprox \hbar$, as in the lecture), so about 3 kg m/s. If the ball has a mass of 1 kg, the ball's velocity is undetermined to about 3 m/s – $\Delta p = m\Delta v$. It's moving around like crazy, simply because we constrained its position.

Professor Lewin reads a passage from the book ("the professor" in what follows refers to a character in the book):

"Look here", the professor said. "I'm going to put definite limits on the position of this ball by putting it inside a wooden triangle."

As soon as the ball was placed in the enclosure, the whole inside of the triangle became filled up with the glittering of ivory.
"You see", said the professor, "I defined the position of the ball to the extent of the dimensions of the triangle. This results in considerable uncertainty in the velocity, and the ball is moving rapidly inside the boundary.
"Can't you stop it?", asks Mr. Tompkins.
"No, it is physically impossible. Any body in an enclosed space possess a certain motion. We physicists call it zero point motion. For example, the motion of electrons in any atom."

So with $\hbar = 1$, the bizarre consequences of the quantum world become more apparent to us, though hardly much easier to grasp.
What happens when we perform this experiment in the real world? Well, we perform the same math, but with $\hbar$ being 10^{-34} times smaller than above. The effects scales linearly with $\hbar$, so the uncertainty in the ball's velocity is now on the order of 3×10^{-34} m/s – a value so tiny that we could never measure it. In one billion years, the ball would move at most 1/100 of the diameter of a proton. Such distances and velocities are of course completely meaningless to us, and so we again see that these effects are irrelevant in the macroscopic world "of baseballs and billiards and pots and pans".

For this reason, it is no problem for us to talk about a billiard ball being exactly at a certain position, and having exactly zero speed. The error is so far beyond measurable that we could never show whether it actually exists or not, so it is completely safe to ignore it and keep working as usual.

Let's now look at an atom, and more specifically the electrons "orbiting" it. Say an atom is about 10^{-10} meters. An electron is then confined to $\Delta x \approx 10^{-10}$ m. Using the uncertainty principle, we can find that

$$m\Delta v\Delta x \overset{>}{\approx} \hbar \tag{17.40}$$

$$\Delta v \overset{>}{\approx} \frac{\hbar}{m\Delta x} \tag{17.41}$$

(Again, this is using the lecture's possibly incorrect equations and not the ones they later added as corrections via overlaid text, though I'm not sure if either form can be used for accurate calculations.)
Using $\Delta x = 10^{-10}$ m and $m \approx 10^{-30}$, we find $\Delta v \overset{>}{\approx} 10^6$ m/s, a third of a percent of the speed of light. So the electron moves simply because it is confined.

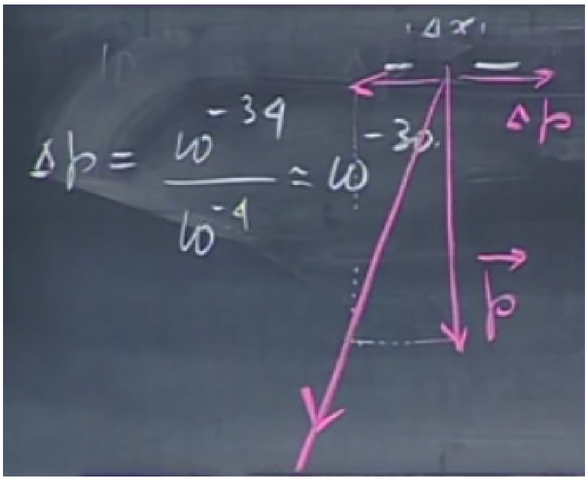
17.2.6 The single-slit experiment

The professor then makes a demonstration that can be explained in terms of the uncertainty principle. We shine a laser beam, monochromatic light (i.e. it consists of only one wavelength, as opposed to e.g. white light) onto a thin vertical slit in a material that otherwise blocks light. A distance L away (where L is several meters), we have a wall, that this light pattern is projected upon.

During the experiment, we then shrink the (vertical) slit's width. In doing so, less light will manage to pass through – that much is clearly unavoidable – and the laser dot projected on the wall will have its sides "chopped off", just as we would predict.
However, in doing this, we are constraining Δx. The thinner the slit is, the better we know the x position of the photons that pass through (if we consider them as photons rather than waves, that is). Because of this, via the uncertainty principle, we lose accuracy in our knowledge of the momentum in the x direction (the direction we are constraining the photons in). We are not constraining the beam in the y directions, so nothing out of the ordinary will happen in the vertical direction.
Horizontally, however, the light begins to *spread out*. The *thinner* the slit becomes, the *wider* the light projection becomes! Simply because we reduce Δx and constrain the light's position, the x-component of the momentum is starting to become more and more ill-defined, and some of the light spreads out accordingly.

We can work this out semi-quantitatively. The momentum of each photon is about 10^{-27} kg m/s, according to the professor. This indeed corresponds to the wavelength of red light; the laser in the lecture is green, but everything here is an order-of-magnitude approximation, so there's little point in being more precise. Say we start with the slit at 1/10 mm, that is, it can pass though a slit of width $\Delta x \approx 10^{-4}$ meters, constraining its position. That makes the x-component of the momentum ill-defined to about $h/\Delta x \approx 10^{-34}/10^{-4} = 10^{-30}$ meters. (I'm not sure if $\hbar$ was/should be used here, but it seems it was not; consider this an order-of-magnitude type result either way.)

The total momentum is then the sum of the light's original momentum and this Δp, so the path changes path according to this vector addition:



We then expect some of the light to shoot off at an angle, but only in the x direction; the light is not constrained at all in the y direction (the slit's height is much greater than the light beam's diameter).

The angle θ between the original vector and the new one can now be easily calculated. Using trigonometry, $\tan\theta = \Delta p/p$. For small angles, $\tan\theta \approx \sin\theta \approx \theta$, we can consider this simply as $\theta = \Delta p/p$. (The professor did this implicitly.)

For the values we have, with $\Delta x = 1/10$ mm leading to $\Delta p \approx 10^{-30}$ kg m/s, we find $\theta \approx 10^{-3}$ radians. Using the definition of a radian, we can multiply this by the distance L to the screen to get the approximate size at the screen. With $L = 10$ meters, we find $\theta L \approx 1$ cm (in each direction from the center, so a total width of about 2 cm).

However, if we make the slit 10 times smaller, Δp grows by 10 times, which causes θ to grow by 10 times, and therefore θL also grows by 10 times. The "uncertainty" is now 10 cm in each direction of the center, so the light has spread out way more. This is then demonstrated in the lecture – which is clearly something that must be *seen*!

The professor then makes it clear that this can be explained without the uncertainty principle – and was explained to high accuracy even in the 19th century; this demonstration is however entirely consistent with the uncertainty principle.
(This experiment is also discussed in further detail in 8.02/8.02x, in the context of interference of light waves and diffraction; there, we also explain the dark bands that appear, that I briefly mentioned in regards to the *double* slit experiment, where they are more prominent and appear in two ways, rather than one way as seen here.)

Modern quantum physics can make some incredibly accurate predictions (I read that in some cases, we can measure quantum phenomena to an accuracy a million times better than that of some astronomical phenomena), however, we cannot predict exactly where each photon is going to end up. Quantum mechanics is, by its very nature, a probability-based theory. We can calculate how the pattern will look after a whole lot of photons have hit the screen, as they are more likely to end up in some places. However, we cannot predict anything at all about what an individual photon will do; as far as we can tell, nature has an intrinsic randomness built into it.

It has been argued that perhaps this behavior is not random; perhaps it would be predictable, only that there are some variables we are not aware of, and that a more complete theory *would* be able to predict all behavior (given all necessary initial conditions). Certain types of these theories seem to have been disproven, and as of yet, there is no proof of these so-called "hidden variables" existing, though there is still discussion and research being done in this area, to the best of my understanding. That is, quantum mechanics still appears to be indeterministic: given all possible information about every particle in the universe, you still cannot predict exactly what will happen next, only come up with accurate probabilities.

One more crazy detail in regards to this: it's important to realize that this experiment cannot be adequately explained by photons (particles) interfering with each other. The exact same pattern will build up over time *regardless* of the rate you shoot the photons through. Even if you shoot one photon, wait 5 minutes, shoot the next, etc. the same pattern will emerge over time. The photon must in some way interfere with... itself? This only really makes any kind of sense if we consider waves.

Also, the same experiment can be and has been done with particles, ranging from electrons up to multi-atom structures such as "buckyballs" (each of which consists of 60 carbon atoms), and the results are exactly as predicted by quantum mechanics. Photons, electrons or buckyballs – nature doesn't really care and treats them the same way, it would seem!

17.2.7 Some notes on the uncertainty principle

(This last section is not at all from the lecture, and is (just as the intro section to this lecture) all written by me. That is, you shouldn't take anything in here as hard fact, as I have not studied quantum mechanics to any greater extent than what this course teaches, plus some popular science which often is just as misleading as it is informative.)

A common question (and misconception) is that this uncertainty is a technical limit in our measuring equipment; it is not. It is a physical limit built into nature. I hardly have the expertise (or even basic knowledge) of quantum mechanics to know this myself yet, but many describe the entire notion of perfectly knowing both momentum and position as *meaningless* in quantum mechanics, including that quote in the lecture earlier.

Another common explanation for this uncertainty (one that I've liked myself) is that in order to for example measure the position of an electron, we need to probe it somehow, perhaps using light (sending photons to interact with it). A photon of long wavelength has little energy and thus little momentum, and won't disturb the electron a lot; we get a fairly accurate measurement of its momentum, but since the light wavelength is large, we don't know *exactly* where it is; we only have a fuzzy idea about the position!

If we want to measure the position accurately, we instead need to use light of a shorter wavelength (smaller Δx), i.e. greater energy (and greater momentum, $p = E/c$ for light). This means we will know where the electron is (was) very accurately, but because we transferred a large amount of momentum to it, we can now not know its momentum exactly.

The above explanation (Heisenberg's microscope) is not technically accurate, though. It is a metaphor, rather than an explanation. In reality, the uncertainty predicted by the uncertainty principle is greater than that in the above experiment.

My current understanding (again, without having actually studied any quantum mechanics – this is still unlikely to be 100% correct!) is that the uncertainty arises from the wave nature of matter. That is, the electron doesn't really *have* a perfectly defined position until it is probed; prior to the probing, the electron's position is only a probability distribution. It may be likely to be confined in a small volume, but it is still *possible* for it to be outside it – even infinitely far away, only with a probability moving closer to 0 the further away you go.

17.3 Lecture 35: Professor Lewin's early days at MIT

I did not take any notes for this lecture. It is as always absolutely worth watching, but it feels pointless to write it down – professor Lewin is telling *his* story, and I should leave that to *him* to do so! If you want it in text form, his book "For the Love of Physics" talks about these topics and several others!

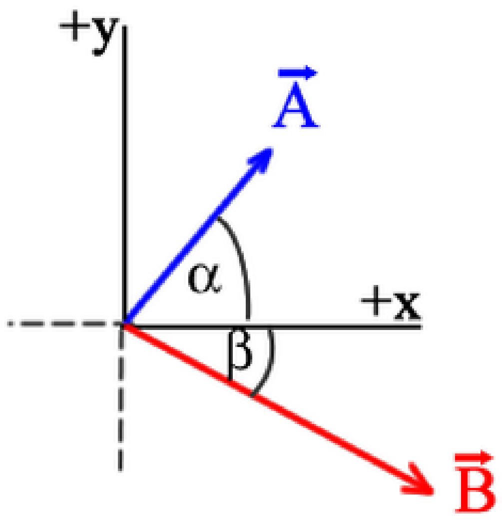
Part III

Homework problems

Chapter 1: Week 1: Homework 1

1.1 Problem 1: Decomposing vectors

"Consider two vectors in the xy-plane as shown.



Vector $\vec{A}$, in the first quadrant, has a magnitude $|\vec{A}| = 2.0$ and is at an angle $\alpha = 40°$ with respect to the positive x-axis. Vector $\vec{B}$, in the fourth quadrant, has a magnitude $|\vec{B}| = 1.5$ and is at an angle $\beta = 20.0°$ with respect to the positive x-axis.

Find the x and y components of the vectors $\vec{A}$ and $\vec{B}$."

Well, this ought to be fairly simple. First, let's consider what sort of values we expect. α is in the first quadrant that is, angled upwards and to the right. That means $A_x > 0$ and $A_y > 0$.
We could use the formulas listed in the first part of these notes, or simply re-derive them from the basic trig definitions. I prefer the latter route, since it can be done in a few seconds once you're comfortable with it, and it means you can't remember them the wrong way. (Unless you misremember everything central to trigonometry!)

A_x is the adjacent side to α, while $\vec{A}$ is the hypotenuse, and A_y is the opposite. Using trig definitions, we find

$$\sin\alpha = \frac{A_y}{|\vec{A}|} \tag{1.1}$$

$$A_y = |\vec{A}|\sin\alpha \tag{1.2}$$

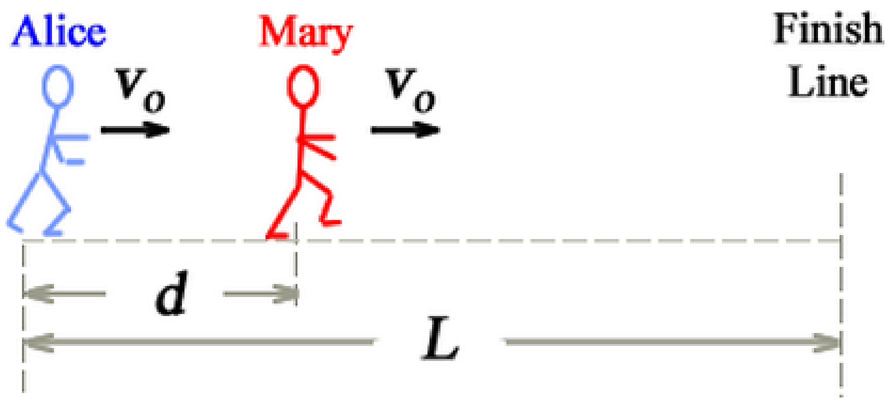
$$\cos\alpha = \frac{A_x}{|\vec{A}|} \tag{1.3}$$

$$A_x = |\vec{A}|\cos\alpha \tag{1.4}$$

The same relations hold for β and $\vec{B}$ as well, of course, so we just need to enter these answers into the form (and convert the degree values to radians inside the trig functions), and we're done!

1.2 Problem 2: Catching up

“During a track event two runners, Mary, and Alice, round the last turn and head into the final stretch with Mary a distance $d = 3.0$ m in front of Alice. They are both running with the same speed of $v_0 = 7.0$ m/s. When the finish line is a distance $L = 45.0$ m away from Alice, Alice accelerates at $a_A = 1.5$ m/s^2 until she catches up to Mary. Alice then continues at a constant speed until she reaches the finish line.



(a) How long (in s) did it take Alice to catch up with Mary?”

First up, we need to choose a reference frame to work in. I considered choosing Mary’s reference frame, so that she is seen as stationary, but that would probably just cause problems in some aspects of the problem. I’ll therefore choose the simple one, where the track and the finish line are stationary.

Since there are five sub-question, let’s set up some equations. We know the initial position of each runner (we’ll call Alice’s position 0, and therefore Mary’s initial position is d), initial velocity, and acceleration.

For Alice:

$$x_A(t) = v_0 t + \frac{3}{4}t^2 \tag{1.5}$$

$$v_A(t) = v_0 + 1.5t \tag{1.6}$$

(Alice’s acceleration is due to the $1.5\,\text{m/s}^2$ times the one-half present in the formula.)

And for Mary:

$$x_M(t) = d + v_0 t \tag{1.7}$$

$$v_M(t) = v_0 \tag{1.8}$$

So, let’s restate the question: “(a) How long (in s) did it take Alice to catch up with Mary?”

We set their position equations equal, and solve for t:

$$\frac{3}{4}t^2 = d \tag{1.9}$$

$$t^2 = \frac{4}{3}d \tag{1.10}$$

$$t = +\sqrt{\frac{4d}{3}} = 2\,\text{s} \tag{1.11}$$

“(b) How far (in m) did Alice still have to run when she just caught up to Mary?”

It took 2 seconds exactly, so Mary must have moved 14 meters (2 seconds, 7 m/s) in that time, to position $d + 14\,\text{m}$.
The distance remaining is $L - d - 14\,\text{m} = 28\,\text{m}$.

“(c) How long (in s) did Alice take to reach the finish line after she just caught up to Mary?”

Keep in mind that she stopped accelerating when she passed, so her velocity is now a constant again. She started out at 7 m/s and accelerated at $1.5\,\mathrm{m/s^2}$ for 2 seconds, so her velocity is now $10\,\mathrm{m/s}$. The answer is the distance remaining divided by her velocity, so the answer is

$$\text{time taken} = \frac{L - d - 14\,\mathrm{m}}{10\,\mathrm{m/s}} = 2.8\,\mathrm{s} \tag{1.12}$$

"Mary starts to accelerate when Alice just catches up to her, and accelerates all the way to the finish line and crosses the line exactly when Alice does. Assume Mary's acceleration is constant.
(d) What is Mary's acceleration (in $\mathrm{m/s^2}$)?"

To sum up where we're at: Mary is still running at 7 m/s, with 28 meters left to go. Alice is running at 10 m/s, also with 28 meters left to go (meaning she will get there in 2.8 seconds).
Mary must now accelerate such that $x_M(t) = L$ at $t = 2.8\,\mathrm{s}$ (we reset the current time to $t = 0$ for simplicity).

Mary's new position equation is

$$x_M(t) = (L - 28\,\mathrm{m}) + v_0 t + \frac{1}{2}at^2 \tag{1.13}$$

We need that it to be equal to L at $t = 2.8\,\mathrm{s}$ as mentioned, so we substitute the values for $t = 2.8\,\mathrm{s}$ and $v_0 = 7\,\mathrm{m/s}$ in, set it equal, and solve for a:

$$(L - 28\,\mathrm{m}) + (7\,\mathrm{m/s})(2.8\,\mathrm{s}) + \frac{1}{2}\ a\ (2.8\,\mathrm{s})^2 = L \tag{1.14}$$

$$-28\,\mathrm{m} + 19.6\,\mathrm{m} + 3.92\,\mathrm{s}^2 \cdot a = 0 \tag{1.15}$$

$$3.92\,\mathrm{s}^2 \cdot a = 28\,\mathrm{m} - 19.6\,\mathrm{m} \tag{1.16}$$

$$a = \frac{28\,\mathrm{m} - 19.6\,\mathrm{m}}{3.92\,\mathrm{s}^2} \approx 2.14\,\mathrm{m/s^2} \tag{1.17}$$

"(e) What is Mary's velocity at the finish line (in m/s)?"

That would be given by her old velocity, 7 m/s, plus the acceleration multiplied by the time accelerated (2.8 seconds):

$$v_{M_{final}} = 7\,\mathrm{m/s} + 2.14\,\mathrm{m/s^2} \cdot 2.8\,\mathrm{s} \approx 13\,\mathrm{m/s} \tag{1.18}$$

1.3 Problem 3: Speeding ticket

This problem was removed from the grading, i.e. assigned 0 points, after there had been some trouble with it. The problem is supposed to be fixed now, however, so I will attempt it.

"A motorist traveling with constant speed of $v_c = 18.0$ m/s passes a school-crossing corner, where the speed limit is 10 m/s. Just as the motorist passes, a police officer on a motorcycle stopped at the corner starts off in pursuit. The officer accelerates from rest at $a_m = 3.00\,\mathrm{m/s^2}$ until reaching a speed of 30.0 m/s. The officer then slows down at a constant rate until coming alongside the car at x = 270.0 m.
Consider a coordinate system with origin at the school-crossing corner, $x = 0$, and the +x-axis in the direction of the car's motion.

(a) How long does it take for the motorcycle to catch up with the car (in s)?"

Okay. The car moves at a constant velocity, so that part is easy. Now, as for the offices, he takes 10 seconds to accelerate to his top speed. During that time, his new position is

$$x_m(t=10) = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}(3\,\mathrm{m/s^2})(10\,\mathrm{s})^2 \tag{1.19}$$
$$= 150\,\mathrm{m}\ \text{(answers question c)} \tag{1.20}$$

As mentioned above, this also answers part (c): "(c) How far (in m) is the motorcycle from the corner when switching from speeding up to slowing down?"

In fact, I think the next question that should be answered is (d), not (a), so let's see.

"(d) How far (in m) is the motorcycle from the car when switching from speeding up to slowing down?"

We don't really need to write down the position equation here, as it's a bit too simple: $x = 18.0\,\mathrm{m/s} \cdot 10\,\mathrm{s} = 180\,\mathrm{m}$. Since the motorcycle is 150 m from the corner, the answer here must be 30 m.

So, we are now at: motorist at $x = 180$ m at 18.0 m/s, cop at $x = 150$ m at 30.0 m/s. The cop must slow down with constant acceleration so that he hits $x = 270$ m at the same time as the motorist. At the motorists' speed, that happens at

$$t = 10\,\mathrm{s} + \frac{270\,\mathrm{m} - 180\,\mathrm{m}}{18.0\,\mathrm{m/s}} = 15\,\mathrm{s} \tag{1.21}$$

... where 10 seconds is the time that has already passed. Alternatively, we could simply take the 270 meters divided by the velocity to find the same result. This answers question (a). Back to (e):

All in all, we set up a new equation for the cop. $x_0 = 150\,\mathrm{m}$, $v_0 = 30.0\,\mathrm{m/s}$, and a is our unknown. The position equation must equal 270 m at $t = 5\,\mathrm{s}$ – we reset t to start over from the instant where the above parameters are true:

$$150\,\mathrm{m} + (30.0\,\mathrm{m/s})(5\,\mathrm{s}) + \frac{1}{2}a(5\,\mathrm{s})^2 = 270\,\mathrm{m} \tag{1.22}$$
$$\frac{1}{2}a(5\,\mathrm{s})^2 = -30\,\mathrm{m} \tag{1.23}$$
$$a(25\,\mathrm{s}^2) = -60\,\mathrm{m} \tag{1.24}$$
$$a = \frac{-60\,\mathrm{m}}{25\,\mathrm{s}^2} = -2.4\,\mathrm{m/s^2} \tag{1.25}$$

1.4 Problem 4: Position, velocity and acceleration

"An object is moving along a straight line parallel to the x-axis. Its position as a function of time is given by:
$x(t) = 30\,\mathrm{m} - (21\,\mathrm{m/s})t + (3\,\mathrm{m/s^2})t^2$
where the position x is in meters and the time t is in seconds.

(a) What is the object's velocity at $t = 0$ s, 2 s, and 5 s?"

We take the derivative of the above equation, and end up with

$$v(t) = -21\,\mathrm{m/s} + (6\,\mathrm{m/s^2})t \tag{1.26}$$

All we need to do now is plug in the values for t.

"(b) What is the object's acceleration at $t = 0$ s, 2 s, and 5 s?"

We take the derivative of $v(t)$ above:

$$a(t) = 6\,\mathrm{m/s^2} \tag{1.27}$$

We don't even need to plug in values here - the answer is $6\,\mathrm{m/s^2}$ for all three cases.

"(c) At what time T is the object's velocity zero?"

We set $v(t) = 0$ and solve for t:

$$-21\,\mathrm{m/s} + (6\,\mathrm{m/s^2})t = 0 \tag{1.28}$$
$$t = \frac{21\,\mathrm{m/s}}{(6\,\mathrm{m/s^2})} \tag{1.29}$$
$$t = 3.5\,\mathrm{s} \tag{1.30}$$

"What is the object's position when its velocity is zero?"

We plug $t = 3.5\,\mathrm{s}$ into $x(t)$ and we're done. Note that the answer (in my case, at least) is negative.

"(d) What is the average velocity between $t_1 = 1.0\,\mathrm{s}$ and $t_2 = 3.5\,\mathrm{s}$?"

With a constant acceleration, as here, we can calculate the average simply by averaging between the velocities at t_1 and t_2.

$$v_{t_1 t_2 avg} = \frac{v(1.0\,\mathrm{s}) + v(3.5\,\mathrm{s})}{2} = \frac{-15\,\mathrm{m/s} + 0\,\mathrm{m/s}}{2} = -7.5\,\mathrm{m/s} \tag{1.31}$$

"(e) What is the object's average velocity between $t_1 = 0\,\mathrm{s}$ and $t_2 = 7.0\,\mathrm{s}$?"

Same deal as above.

$$v_{t_1 t_2 avg} = \frac{v(0\,\mathrm{s}) + v(7\,\mathrm{s})}{2} = \frac{-21\,\mathrm{m/s} + 21\,\mathrm{m/s}}{2} = 0\,\mathrm{m/s} \tag{1.32}$$

"(e) What is the object's average speed between $t_1 = 0\,\mathrm{s}$ and $t_2 = 7.0\,\mathrm{s}$?"

Aha! Keep in mind that speed and velocity are *not* the same in physics! Here, because the object has reversed, the average speed will be greater than the average velocity. How do we calculate the speed, though?

Well, we know that the object stops at $t = 3.5\,\mathrm{s}$, from (c) above. We can also show very simply that it reverses direction at that point. We can find the average speed as the total distance traveled, divided by the 7 seconds.
The first part of the distance is $|x(3.5\,\mathrm{s}) - x(0\,\mathrm{s})|$, and the second part is $|x(7\,\mathrm{s}) - x(3.5\,\mathrm{s})|$.

$$\text{avg speed} = \frac{|x(3.5\,\mathrm{s}) - x(0\,\mathrm{s})| + |x(7\,\mathrm{s}) - x(3.5\,\mathrm{s})|}{7\,\mathrm{s}} = 10.5\,\mathrm{m/s} \tag{1.33}$$

That wasn't very pretty, but it worked.

"(g) At what time t_3 does the object reverse its direction?"

That was answered in passing above: at $t_3 = 3.5\,\mathrm{s}$.

1.5 Problem 5: One-dimensional kinematics

"Two stones are released from rest at a certain height, but at different times. When answering the following questions, ignore air drag.

(a) Will the difference between their speeds increase, decrease, or stay the same?"

The two will have the same acceleration ($-g \approx -9.8\,\mathrm{m/s^2}$), of course, but they are released at different times. The one released first will have fallen for a longer time, all the way until it hits the ground.
The acceleration causes a linear change in the velocity, adding $-9.8\,\mathrm{m/s}$ every second, to both stones.
If one is released at $t = 0$ and the other at $t = 1\,\mathrm{s}$, at the instant where the second stone is released, one stands still and one moves at $-9.8\,\mathrm{m/s^2}$. Ten seconds later, one moves with $-107.8\,\mathrm{m/s^2}$ and the other at $-98\,\mathrm{m/s^2}$.

So indeed, the difference is constant, as we expected from the third sentence above.

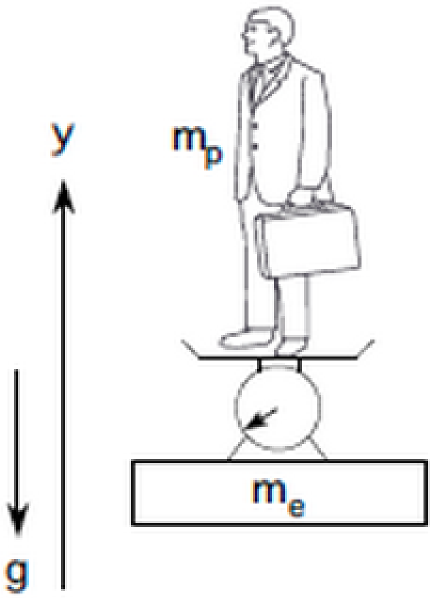
"(b) Will their separation distance increase, decrease, or stay the same?"

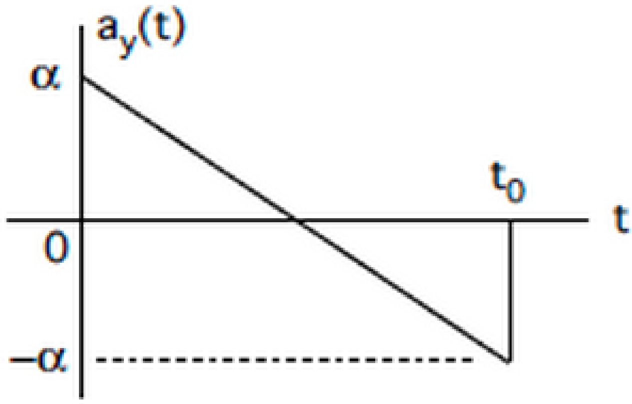
Now this is a bit interesting. When something falls in a gravitational field, its velocity increases all the time (as long as we ignore air drag, which creates an upper limit to the velocity). In the above example, if the separation begins at $4.9\,\mathrm{m}$ ($x(t) = \dfrac{-9.8\,\mathrm{m/s^2}}{2}t^2$), but after another 10 seconds, the separation is much greater. Thus the answer is that it increases.

"(c) Will the time interval between the instants at which they hit the ground be smaller than, equal to, or larger than the time interval between the instants of their release?"

This is something I expect could get a bit tricky if we get into the equations, but it's obvious that the answer *must* be "equal". Both have the position function $x(t) = \dfrac{-9.8\,\mathrm{m/s^2}}{2}t^2$, so they must fall in the same trajectory, taking the same time to fall. Therefore, if we release them at the same time, they must hit the ground at the same time. If we release them one second apart, they must hit the ground one second apart, etc.

1.6 Problem 6: Elevator





"A person of mass m_p stands on a scale in an elevator of mass m_e. The scale reads the magnitude of the force F exerted on it from above in a downward direction. Starting at rest at $t = 0$ the elevator moves upward, coming to rest again at time $t = t_0$. The downward acceleration of gravity is g . The acceleration of the elevator during this period is shown graphically above and is given analytically by
$a_y(t) = \alpha - \dfrac{2\alpha}{t_0}t$
a) Find the maximum speed of the elevator. Express your answer in terms of α and t_0.
b) Find the total distance traveled by the elevator."

Uh, okay. Honestly, I'm a bit confused – it doesn't appear as if the man, his mass, the scale, the mass of the elevator or g matter whatsoever! These questions don't tend to include information to make them appear harder than they are, though.
I suppose I'll get started by ignoring them and see what happens.

The elevator is stationary to begin with. That means we can say not only $y_0 = 0$, but also $v_0 = 0$. However, as nice as it would be, we cannot use $v(t) = a_y t$, since the acceleration is not constant. We must integrate the acceleration. Note, however, that the acceleration starts at α, and progresses all the way to $-\alpha$, and that the graph is symmetric around the middle. The integral of the entire interval is zero. We want the maximum speed, which should happen just as it reverses, so at $t_0/2$.

$$\begin{aligned}
s_{max} &= \int_0^{t_0/2} a_y(t)dt = \int_0^{t_0/2} \left(\alpha - \frac{2\alpha}{t_0}t\right) dt = \left[\alpha t - \frac{\alpha}{t_0}t^2\right]_{t=0}^{t=t_0/2} && (1.34)\\
&= \left(\alpha\frac{t_0}{2} - \frac{\alpha}{t_0}\frac{t_0^2}{4}\right) - (0-0) && (1.35)\\
&= \alpha\left(\frac{t_0}{2} - \frac{t_0}{4}\right) && (1.36)\\
&= \frac{\alpha t_0}{4} && (1.37)
\end{aligned}$$

That indeed works.

"b) Find the total distance traveled by the elevator, $y(t_0)$."

Let's begin by finding the velocity function (as an indefinite integral).

$$\begin{aligned}
v(t) &= \int \left(\alpha - \frac{2\alpha}{t_0}t\right) dt = \alpha t - \frac{2\alpha}{t_0}\frac{t^2}{2} && (1.38)\\
&= \alpha\left(t - \frac{t^2}{t_0}\right) && (1.39)
\end{aligned}$$

We can forget about the $+C$. The constant here would be v_0, and we know that to be zero.

We now know the velocity at any point between $t = 0$ and $t = t_0$, but that doesn't help us much yet – it's not constant, so we need to integrate again, to find the position function.

$$\begin{aligned}
y(t) &= \int \alpha\left(t - \frac{t^2}{t_0}\right) dt && (1.40)\\
&= \alpha\left(\frac{t^2}{2} - \frac{t^3}{3t_0}\right) && (1.41)\\
& && (1.42)
\end{aligned}$$

We substitute $t = t_0$:

$$\begin{aligned}
y(t_0) &= \alpha\left(\frac{t_0^2}{2} - \frac{t_0^3}{3t_0}\right) && (1.43)\\
&= \alpha t_0^2\left(\frac{1}{2} - \frac{1}{3}\right) && (1.44)\\
&= \frac{\alpha t_0^2}{6} && (1.45)
\end{aligned}$$

... and we're done, indeed by ignoring the majority of the information given. Strange problem.

1.7 Problem 7: Position, velocity, and acceleration in 3D

"A particle is moving in three dimensions. Its position vector, $\vec{r}$, is given by
$\vec{r}(t) = (6 - 2t)\hat{x} + (3 + 4t - 6t^2)\hat{y} - (1 + 3t - 2t^2)\hat{z}$

Distances are in meters, and the time, t, in seconds.
(a) What are the components of the velocity vector (in m/s) $\vec{v}$ at $t = +3$?"

As usual, we need to take the derivative. We calculate the derivative of each dimension on its own, and sum up the results.

$$\vec{v} = -2\hat{x} + (4 - 12t)\hat{y} - (3 - 4t)\hat{z} \tag{1.46}$$
$$v_x = -2 \tag{1.47}$$
$$v_y = 4 - 12 \cdot 3 = -32 \tag{1.48}$$
$$v_z = -3 + 4 \cdot 3 = 9 \tag{1.49}$$

"(b) What is the speed (in m/s) at $t = +3$?"

Speed at an instant is simply the magnitude of the velocity (as they point out when they also ask for $|\vec{v}|$).

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(-2)^2 + (-32)^2 + 9^2} \approx 33.3\,\text{m/s} \tag{1.50}$$

"(c) What are the components of the acceleration vector $\vec{a}$ (in m/s^2) at $t = +3$?"

We calculate the derivative of the velocity, i.e. equation (1.46).

$$\vec{a} = -12\hat{y} + 4\hat{z} \tag{1.51}$$

"(d) What is the magnitude of the acceleration vector $\vec{a}$ (in m/s^2) at $t = +3$?"

We do what we did for the velocity vector.

$$|\vec{a}| = \sqrt{(-12)^2 + 4^2} \approx 12.65\,\text{m/s}^2 \tag{1.52}$$

1.8 Problem 8: Vertical collision

"Mary wants to throw a can straight up into the air and then hit it with a second can. She wants the collision to occur at height $h = 5.0$ m above the throw point. In addition, she knows that she needs $t_1 = 4.0\,\text{s}$ between successive throws. Assuming that she throws both cans with the same speed. Take g to be $9.81\,\text{m/s}^2$.

(a) How long it takes (in s) after the first can has been thrown into the air for the two cans to collide?
(b) Find the initial speed of the cans (in m/s)."

The grammar in the question text might need a bit of a double-check (I spot two errors, and one slightly confusing statement)! Re-stated, I read it as something like this:
She throws the two cans up in into the air, with the same unknown velocity v_0, 4 seconds apart (one at $t = 0$, one at $t = 4.0\,\text{s}$). How long after $t = 0$ do the cans collide, assuming it happens at $h = 5.0\,\text{m}$?

In these equations, increasing y is defined as upwards. Therefore, gravitational acceleration is $-g$. I choose to define $y_0 = 0$, since that simplifies things. (Rather, choosing anything else would needlessly complicate things.)

$$y_1(t) = v_0 t - \frac{1}{2} g t^2 \tag{1.53}$$

$$y_2(t) = v_0 t' - \frac{1}{2} g t'^2 \tag{1.54}$$

$$\tag{1.55}$$

... where $t' = t - 4.0\,\mathrm{s}$. Let's set $y_1 = y_2$ and see what happens.

$$v_0 t - \frac{1}{2} g t^2 = v_0 t' - \frac{1}{2} g t'^2 \tag{1.56}$$

$$v_0 t - \frac{1}{2} g t^2 = v_0 (t-4) - \frac{1}{2} g (t-4)^2 \tag{1.57}$$

$$v_0 t - \frac{1}{2} g t^2 = v_0 t - 4v_0 - \frac{1}{2} g (t^2 - 8t + 16) \tag{1.58}$$

$$t^2 = \frac{-4v_0}{-\frac{1}{2} g} + (t^2 - 8t + 16) \tag{1.59}$$

We simplify this into a linear equation, and solve it.

$$0 = \frac{8v_0}{g} + -8t + 16 \tag{1.60}$$

$$t = \frac{v_0}{g} + 2 \tag{1.61}$$

Hmm, we're a bit stuck here, since v_0 is an unknown. Let's try to add another equation to the mix and see if we can solve for that. The *next* sub-question is about finding v_0.

We do have another piece of information, that we have not used: $y_1(t) = y_2(t) = h$, for the time t where they collide. We have not used h yet.

$$v_0 t - \frac{1}{2} g t^2 = h \tag{1.62}$$

$$v_0 t = h + \frac{1}{2} g t^2 \tag{1.63}$$

$$v_0 = \frac{h}{t} + \frac{1}{2} g t \tag{1.64}$$

Combine the two equations and finally solve for t:

$$t = \frac{1}{g} \left(\frac{h}{t} + \frac{1}{2} g t \right) + 2 \tag{1.65}$$

$$t = \frac{h}{gt} + \frac{1}{2} t + 2 \tag{1.66}$$

$$\frac{t}{2} = \frac{h}{gt} + 2 \tag{1.67}$$

Multiply both sides by $2t$ and we finally have a quadratic to solve:

$$t^2 - 4t - \frac{2h}{g} = 0 \tag{1.68}$$

$$t = \frac{4 \pm \sqrt{(-4)^2 + \frac{8h}{g}}}{2} = \frac{4 \pm 4.48}{2} = 4.24\,\mathrm{s} \tag{1.69}$$

... neglecting the other solution, which is negative and therefore not applicable. Our equations are only valid from $t = 0$.

"(b) Find the initial speed of the cans (in m/s)."

We have that in equation (1.64), so we plug in the values:

$$v_0 = \frac{5\,\mathrm{m}}{4.24\,\mathrm{s}} + \frac{1}{2}(9.81\,\mathrm{m/s^2})(4.24\,\mathrm{s}) \approx 22.0\,\mathrm{m/s} \tag{1.70}$$

1.9 Problem 9: Vector operations

Since these are essentially just multiplication, I didn't bother to take notes for them. Cross products can be a bit painful when using components, but as far as I know, using calculators is allowed for this course. I wouldn't recommend doing that unless you feel certain that you could also do it *without* a calculator, however.

1.10 Problem 10: Perpendicular vectors

"The vectors $\vec{A} = 1\hat{x} - 2\hat{y}$ and $\vec{B} = -4\hat{x} + a\hat{y} - 2\hat{z}$ are perpendicular to each other. What is the value of a?"

Honestly, I looked up a small hint for this one: that the dot product of two perpendicular vectors has a special property. That was enough to see the solution: the dot product will always be zero. We can set the dot product equal to zero and solve for a. Simple!

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.71}$$

$$(1)(-4) + (-2)(a) + (0)(-2) = 0 \tag{1.72}$$

$$-4 - 2a = 0 \tag{1.73}$$

$$a = -2 \tag{1.74}$$

Dead simple once you realize how to solve it.

That's it for this week!

Chapter 2: Week 2: Homework 2

2.1 Problem 1: Roundtrip by plane

An airplane makes a roundtrip between point A and point B (starting at A). The purpose of this problem is to figure out if the roundtrip will take longer without wind or with wind. Let the distance between A and B be d and the speed of the plane relative to air be v.

"(a) First, assume there is no wind. How long does it take to finish a round trip between A and B (starting at A)? Express your answer as a function of d and v as needed."

The distance is d, the velocity v, and since there is no wind, both parts of the trip take equal time, so we simply double the time taken for the one-way trip:

$$t = 2\frac{d}{v} \tag{2.1}$$

"(b) Now, assume that the wind is blowing in the direction from A to B. How long does it take to finish a round trip between A and B (starting at A), with the wind? Let the distance between A and B be d, the speed of the plane relative to air be v, and the speed of the wind be w (where $w < v$). Express your answer as a function of d, v, or w as needed."

The wind blows with the velocity for one part of the trip (net velocity is $v + w$), and against the other (net velocity $v - w$). We add up the two, and that's really it:

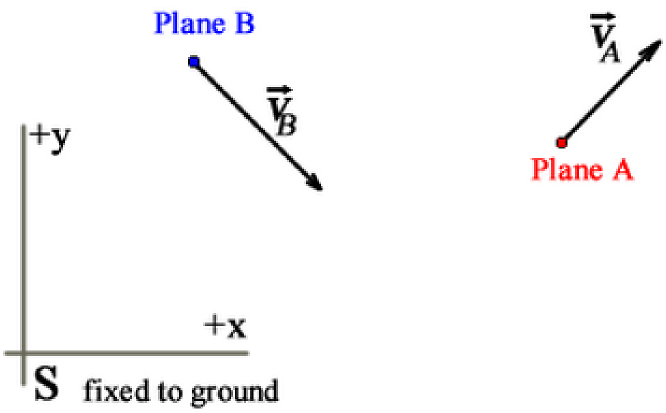
$$t = \frac{d}{v+w} + \frac{d}{v-w} \tag{2.2}$$

"Reflect on your answers: What would happen if the wind speed became so high that $w = v$? How would your answer change if the wind were blowing in the direction from B to A?"

In the first case, a plane flying against the wind would stand still, relative to the Earth.
If the direction of the wind were reversed from above, the round-trip time should be the same: we would simply change the order of the two fractions, but addition is commutative, so the sum is the same.

2.2 Problem 2: Passing planes in flight

"The velocities of airplanes A and B are measured with respect to a frame of reference S fixed to the ground as shown. Airplane A is traveling northeast (45° measured counterclockwise from the x axis) with a speed of $v_A = 140\,\mathrm{m/s}$ and airplane B is traveling southeast (45° measured clockwise from the x axis) with a speed of $v_B = 240\,\mathrm{m/s}$.

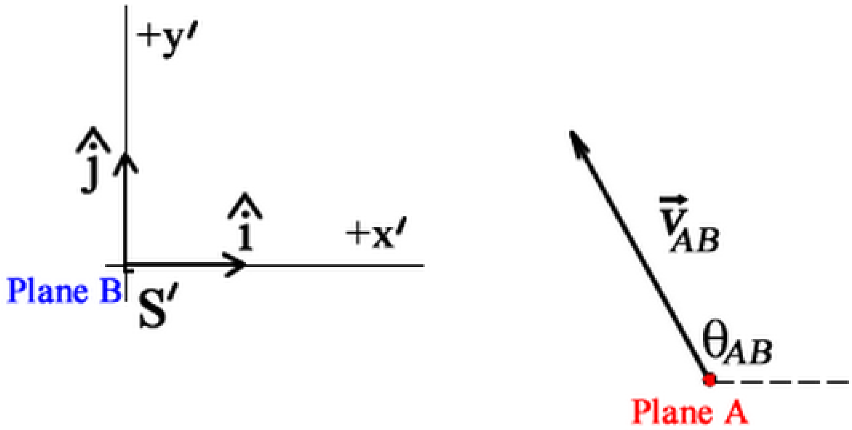


"(a) What are the components of the velocity of each airplane $\vec{v_A}$ and $\vec{v_B}$ in the xy-coordinate system of the stationary frame S?"

Because the system is fixed as shown, this is simply a vector decomposition problem.

$$v_{Ax} = v_A \cos(45°) \tag{2.3}$$
$$v_{Ay} = v_A \sin(45°) \tag{2.4}$$
$$v_{Bx} = v_B \cos(-45°) \tag{2.5}$$
$$v_{By} = v_B \sin(-45°) \tag{2.6}$$

"(b) Consider the frame of reference S' fixed to airplane B. Find $\vec{v_{AB}}$, the velocity of aircraft A as seen from an observer flying in aircraft B?"



Let's see. First, a warning: I want to point out that my method here might be overkill... and very ugly. The reason is that this is the first exercise I do these transformations in, and I want to learn how to do them in general, not just in this one case, so I went the full, ugly way.

I will first calculate the transformation from S to S', so that we understand the reference frame we work in. Because the plane is moving relative to S in two dimensions, we need to use a Galilean transformation on both axes separately, using vector decomposition.

The transformation for a single axis x in one-dimensional motion is $x' = x - vt$, so we apply that concept to both axes using the decomposed velocity vector v_B, which represents the velocity relative to the ground (reference frame S):

$$x' = x - v_{Bx}t = x - v_B \cos(-45°)t \tag{2.7}$$
$$y' = y - v_{By}t = y - v_B \sin(-45°)t \tag{2.8}$$
$$\tag{2.9}$$

Let's now attempt to define v_A in terms of x' and y', in other words, v_{AB} as notated in the problem (v_A as seen by plane B).
In terms of the components in reference frame S, v_A is

$$v_A = v_{Ax}\hat{x} + v_{Ay}\hat{y} = \frac{d(A_x)}{dt}\hat{x} + \frac{d(A_y)}{dt}\hat{y} \tag{2.10}$$

We know the velocities, we know that the acceleration is 0, and so we can find the position equations simply by multiplying the velocity by t, assuming $x_0 = 0$ and $y_0 = 0$. (That should be irrelevant is this context, since we are taking the derivative of them, so that constants disappear.)

$$v_A = \frac{d(v_A \cos(45^\circ) t)}{dt}\hat{x} + \frac{d(v_A \sin(45^\circ) t)}{dt}\hat{y} \tag{2.11}$$

We then apply the transformation to S' by subtracting $v_{Bx}t$ and $v_{By}t$ respectively, and then calculate the derivative and simplify:

$$\begin{aligned}
v_{AB} &= \frac{d(v_A \cos(45^\circ)t - v_B \cos(-45^\circ)t)}{dt}\hat{x} + \frac{d(v_A \sin(45^\circ)t - v_B \sin(-45^\circ)t)}{dt}\hat{y} && (2.12)\\
v_{AB} &= (v_A \cos(45^\circ) - v_B \cos(-45^\circ))\hat{x} + (v_A \sin(45^\circ) - v_B \sin(-45^\circ))\hat{y} && (2.13)\\
v_{AB} &= (v_{Ax} - v_{Bx})\hat{x} + (v_{Ay} - v_{By})\hat{y} && (2.14)
\end{aligned}$$

Well, that's one long derivation for something we could have guessed! We need some numerical values for the components, or the answer becomes way too ugly (too many terms and too many parenthesis to keep track of).

$$\begin{aligned}
v_{Ax} &= 140\cos(45^\circ) \approx 98.995\,\text{m/s} && (2.15)\\
v_{Ay} &= 140\sin(45^\circ) \approx 98.995\,\text{m/s} && (2.16)\\
v_{Bx} &= 240\cos(-45^\circ) \approx 169.706\,\text{m/s} && (2.17)\\
v_{By} &= 240\sin(-45^\circ) \approx -169.706\,\text{m/s} && (2.18)
\end{aligned}$$

With those in mind, we can finally answer the questions, for the magnitude:

$$|v_{AB}| = \sqrt{(98.995 - 169.706)^2 + (98.995 - (-169.706))^2} \approx 277.85\,\text{m/s} \tag{2.19}$$

and:
"Express the direction of $v_{\vec{AB}}$ as an angle θ_{AB} measured counterclockwise from the x' axis (in degrees)."

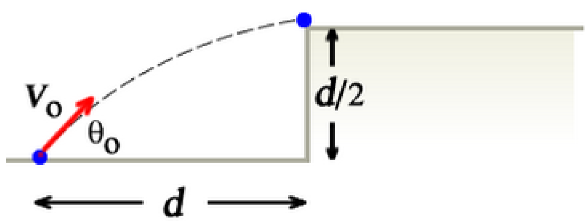
$$\theta_{AB} = \arctan(v_{ABx}, v_{ABy}) = \pi + \arctan\frac{268.701\,\text{m/s}}{-70.711\,\text{m/s}} \approx 104.74^\circ \tag{2.20}$$

I used the two-argument arctangent function here, often known as "atan2" (sometimes with arguments reversed, i.e atan2(y, x)). It it, as shown, equivalent to $\pi + \arctan\frac{y}{x}$ in this case ($y \geq 0$, $x < 0$).

In either case, if we dare trust the graphic, it's obvious that the angle is a bit over 90 degrees, and so the answer makes sense, while an answer of e.g. 14.74 degrees would clearly not (since the question asked for the angle from the x' axis, not the y' axis).

2.3 Problem 3: Throwing a projectile

"A person is a playing a game that requires throwing an object onto a ledge. The ledge is a distance d and a height $d/2$ above the release point. You may neglect air resistance. You may use g for the magnitude of the gravitational acceleration (i.e. $g = 9.81\,\text{m/s}^2$)."



"(a) At what angle θ must the person throw the object and with what magnitude of the velocity v_0 if the object is to be exactly at the top of its flight when it reaches the ledge? Express your answer for the speed in terms of the given quantities d and g, as needed. For the angle, enter the numerical answer in degrees."

I first tried to solve this by starting from the kinematics equations for x and v, but that turned out a bit painful (d is unknown, θ is unknown, t in unknown), so I went back and decided to use the equations we derived in lecture instead. Here they are:

$$t_p = \frac{v_0 \sin\alpha}{g} \tag{2.21}$$
$$h = \frac{(v_0 \sin\alpha)^2}{2g} \tag{2.22}$$
$$t_s = \frac{2v_0 \sin\alpha}{g} \tag{2.23}$$
$$\mathrm{OS} = \frac{v_0^2 \sin 2\alpha}{g} \tag{2.24}$$

t_p is the time to reach the apex (or p for peak); h is the maximum height reached by the projectile; t_s is the time until it comes back down to the y coordinate it was launched from (this one is certainly not useful here), and OS is the total horizontal distance traveled.
The angle α is called θ in this problem, but they are of different names for the same thing.

Which should we use? h is useful, since we know that the peak should be at $d/2$. OS is also useful, since we know that the horizontal distance should be $2d$ (not d! OS is the distance where it would land on the ground again, but we want it to travel exactly halfway there, to the ledge).

$$\frac{v_0^2 \sin^2\theta}{2g} = \frac{d}{2} \tag{2.25}$$
$$\frac{v_0^2 \sin 2\theta}{g} = 2d \tag{2.26}$$

Let's try to solve these for θ. Let's start with the top one:

$$\sin^2\theta = \frac{gd}{v_0^2} \tag{2.27}$$
$$\sin\theta = \sqrt{\frac{gd}{v_0^2}} \tag{2.28}$$
$$\theta = \arcsin\frac{\sqrt{gd}}{v_0} \tag{2.29}$$

Then there's this one:

$$\frac{v_0^2 \sin 2\theta}{g} = 2d \tag{2.30}$$
$$\sin 2\theta = \frac{2gd}{v_0^2} \tag{2.31}$$
$$2\theta = \arcsin\frac{2gd}{v_0^2} \tag{2.32}$$
$$\theta = \frac{1}{2}\arcsin\frac{2gd}{v_0^2} \tag{2.33}$$

These two equations both contain d and v_0, so we should be able to solve for them:

$$\arcsin\frac{\sqrt{gd}}{v_0} = \frac{1}{2}\arcsin\frac{2gd}{v_0^2} \tag{2.34}$$

I'm pretty sure I've missed *something* along the way, because this looks much more complex than I would have assumed this problem to be from the beginning... Either way, I solved this with computer algebra software (which is allowed, but I prefer to do everything myself to make sure that I know how to!), and found

$$v_0 = \sqrt{2dg} \tag{2.35}$$

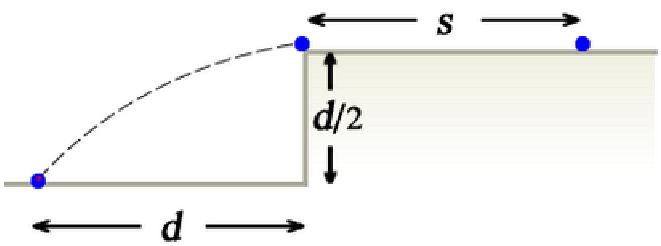
... which is correct. We can then find the angle θ,which we above stated was equal to either of the two expressions we equated above... so we stick in this value for v_0 and see what it turns out to equal, after simplification:

$$\arcsin\frac{\sqrt{gd}}{\sqrt{2dg}} \tag{2.36}$$

$$\arcsin\frac{1}{\sqrt{2}} = 45° \tag{2.37}$$

Wohoo, I didn't even need to use a regular calculator for that one.
But wait, there's more!

"Once the object reaches the ledge it slows down with a constant deceleration and comes to a stop after sliding a distance s.



(b) What is the magnitude of the horizontal component of the acceleration? Express your answer in terms of the given quantities s, d, and g."

The horizontal velocity is constant at $v_0 \cos\theta$ until it starts gliding. We know both v_0 and θ, so let's call this v_{0x} (we will also set $t = 0$ for simplicity):

$$v_{0x} = v_0 \cos\theta = \sqrt{2dg}\cos(45°) = \frac{\sqrt{2dg}}{\sqrt{2}} = \sqrt{dg} \tag{2.38}$$

The rest should be easy. We use $x_0 = 0$, $v_0 = v_{0x}$ and a_x as an unknown, starting at $t = 0$:

$$v_{0x}t + \frac{1}{2}a_x t^2 = s \tag{2.39}$$

$$t\sqrt{dg} + \frac{1}{2}a_x t^2 = s \tag{2.40}$$

Unfortunately, we still have a t in there. We can eliminate that, since we know the initial velocity, so we can set up

$$v_{0x} + a_x t = 0 \tag{2.41}$$
$$t = -\frac{\sqrt{dg}}{a_x} \tag{2.42}$$

That gives us the final equation we need, after solving it for a_x:

$$\left(-\frac{\sqrt{dg}}{a_x}\right)\sqrt{dg} + \frac{1}{2}\frac{\sqrt{dg}^2}{a_x} = s \tag{2.43}$$
$$\left(-\frac{dg}{a_x}\right) + \frac{1}{2}\frac{dg}{a_x} = s \tag{2.44}$$
$$-dg + \frac{1}{2}dg = sa_x \tag{2.45}$$
$$-\frac{1}{2}dg = sa_x \tag{2.46}$$
$$-\frac{dg}{2s} = a_x \tag{2.47}$$
$$|a_x| = \frac{dg}{2s} \tag{2.48}$$

Finally! I look forward to reading the staff's solution... I expect it to be about a third of mine in sheer length!

2.4 Problem 4: Falling apple and arrow

"An archer stands a horizontal distance $d = 50$ m away from a tree sees an apple hanging from the tree at $h = 8$ m above the ground. The archer chooses an arrow and prepares to shoot. The arrow is initially 1.5 m above the ground. Just as the archer shoots the arrow with a speed of 70 m/s, the apple breaks off and falls straight down. A person of height 2.0 m is standing directly underneath the apple. The arrow pierced the apple. Ignore air resistance, and use $g = 9.81\,\mathrm{m/s^2}$ for the acceleration of gravity.

(a) What angle did the archer aim the arrow? Enter your answer in degrees."

We really need to sketch this to make sure we don't screw something up re: the coordinate system, etc. Unfortunately, I draw on paper, and can't really show the picture here. (In theory I could photograph it, but I don't have a scanner. Besides, it's ugly!)

Let's first calculate the trajectory of the apple. I choose a coordinate system centered on the ground below the archer, such that $y = 0$ is below the arrow, which starts at $y_{0p} = 1.5\,\mathrm{m}$ and $x_0 = 0$. Since both apple and arrow unfortunately begin with an a, I choose p for projectile as the subscript for the arrow, and a for the apple.

We then have, for the apple:

$$x_{0a} = 50\,\mathrm{m}\ \text{(constant)} \tag{2.49}$$
$$y_{0a} = 8\,\mathrm{m} \tag{2.50}$$
$$v_{y0a} = 0 \tag{2.51}$$
$$a_{ya} = -g \tag{2.52}$$

So the only relevant kinematics equation for the apple is

$$y_a(t) = 8\,\mathrm{m} - \frac{1}{2}gt^2 \tag{2.53}$$

Next, the arrow. Here, we need to do some decomposition, since it moves in both x and y.

The arrow will fall at the exact same acceleration as the apple. Because they start falling the same instant, this means that they will always "fall together", despite the fact that the arrow has initial velocity upwards. What this means in practice is that because he hit the apple, and it started falling at the same time as the arrow, he *must* have aimed *exactly* at the apple when he fired.
This holds true regardless of the arrow's velocity, as long as it gets to the apple before it hits the ground.

Therefore, we can find the angle θ via basic trigonometry, instead of struggling with multiple unknowns in ugly equations!
We draw a triangle in our sketch, with the adjacent side being the horizontal 50 m to the apple, and the opposite side being the vertical 6.5 m to the apple *from the arrow's initial position* of 1.5 m. Using trigonometry, we see that

$$\tan\theta = \frac{6.5\,\mathrm{m}}{50\,\mathrm{m}} \tag{2.54}$$

$$\theta = \arctan\frac{6.5\,\mathrm{m}}{50\,\mathrm{m}} \approx 7.407^\circ \tag{2.55}$$

That answers part (a), and simplifies things greatly for the next part. We can now calculate the arrow's trajectory without any unknowns.

$$x_{0p} = 0 \tag{2.56}$$

$$y_{0p} = 1.5\,\mathrm{m} \tag{2.57}$$

$$v_{0p} = 70\,\mathrm{m/s} \tag{2.58}$$

$$v_{0px} = (70\,\mathrm{m/s})\cos\theta \approx 69.416\,\mathrm{m/s} \tag{2.59}$$

$$v_{0py} = (70\,\mathrm{m/s})\sin\theta \approx 9.024\,\mathrm{m/s} \tag{2.60}$$

$$a_{yp} = -g \tag{2.61}$$

We don't really care at which velocity it hits the apple, so we have two relevant equations:

$$x_p(t) = v_{0px}t = (69.416\,\mathrm{m/s})t \tag{2.62}$$

$$y_p(t) = 1.5\,\mathrm{m} + (9.024\,\mathrm{m/s})t - \frac{1}{2}gt^2 \tag{2.63}$$

The arrow hits the apple when their y location is equal, and $x = 50\,\mathrm{m}$ (the apple falls straight down, with x always being 50 m).

We can find t from the x equation for the arrow:

$$50\,\mathrm{m} = (69.416\,\mathrm{m/s})t \tag{2.64}$$

$$t = \frac{50\,\mathrm{m}}{69.416\,\mathrm{m/s}} \approx 0.7203\,\mathrm{s} \tag{2.65}$$

Since this is the only time where the arrow is at $x = 50\,\mathrm{m}$, it must be where it hit the apple.

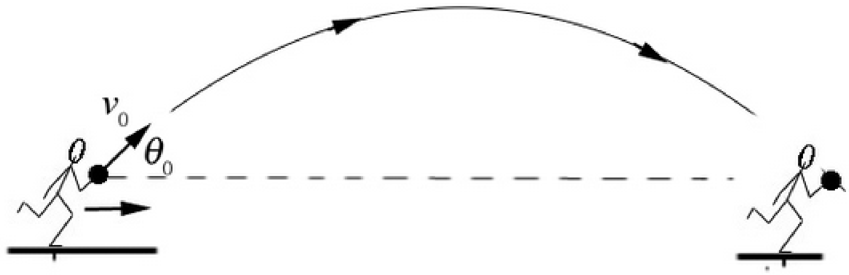
"(b) How high above the person's head did the arrow hit the apple?"

We can find the y coordinate easily, since we wrote a kinematic equation for that:

$$y_a(t) = 8\,\mathrm{m} - \frac{1}{2}gt^2 = 8\,\mathrm{m} - 2.544\,\mathrm{m} = 5.466\,\mathrm{m} \tag{2.66}$$

The answer is then that minus two meters, since the question want the distance between the person's head and the apple.

2.5 Problem 5: Catch



"A person initially at rest throws a ball upward at an angle $\theta_0 = 70°$ with an initial speed $v_0 = 15\,\text{m/s}$. He tries to catch up to the ball by accelerating with a constant acceleration a for a time interval of 1.01 s and then continues to run at a constant speed for the rest of the trip. He catches the ball at exactly the same height he threw it. Let $g = 9.81\,\text{m/s}^2$ be the gravitational constant. What was the person's acceleration a (in m/s^2)?"

OK, since we know the initial velocity and angle right off the bat, let's calculate the ball's initial velocity components:

$$v_{0x} = (15\,\text{m/s})\cos(70°) \approx 5.13\,\text{m/s} \tag{2.67}$$
$$v_{0y} = (15\,\text{m/s})\sin(70°) \approx 14.095\,\text{m/s} \tag{2.68}$$

We define $x_0 = 0$ to be the x position where he starts out, and y_0 to be the height the ball is as he throws it. The ball's trajectory is described by

$$x(t) = (5.13\,\text{m/s})t \tag{2.69}$$
$$y(t) = (14.095\,\text{m/s})t - \frac{1}{2}gt^2 \tag{2.70}$$

He catches the ball as $y(t) = 0$ again, so let's solve for that time:

$$(14.095\,\text{m/s})t - \frac{9.8\,\text{m/s}^2}{2}t^2 = 0 \tag{2.71}$$

Using the quadratic formula $t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a = -\frac{9.8\,\text{m/s}^2}{2}$, $b = 14.095\,\text{m/s}$ and $c = 0$ yields

$$t = \frac{-14.095\,\text{m/s} \pm \sqrt{(14.095\,\text{m/s})^2 - 0}}{-9.8\,\text{m/s}^2} \tag{2.72}$$
$$= -\frac{14.095\,\text{m/s} \pm 14.095\,\text{m/s}}{-9.8\,\text{m/s}^2} \tag{2.73}$$
$$= 2.876\,07\,\text{s} \tag{2.74}$$

(ignoring the other solution, which is clearly zero, as it should be).

Thus he runs at constant speed for $2.876\,07\,\text{s} - 1.01\,\text{s} = 1.866\,\text{s}$.

$t = 0$ to $t = 1.01\,\text{s}$: constant acceleration at unknown acceleration a.
$t = 1.01\,\text{s}$ to $t = 2.876\,07\,\text{s}$: constant speed (also unknown).
$t = 2.876\,07\,\text{s}$: catches the ball.

Using the $x(t)$ equation for the ball, he catches it after having moved 14.75 m, which is then also the total distance he must move in the time periods above.

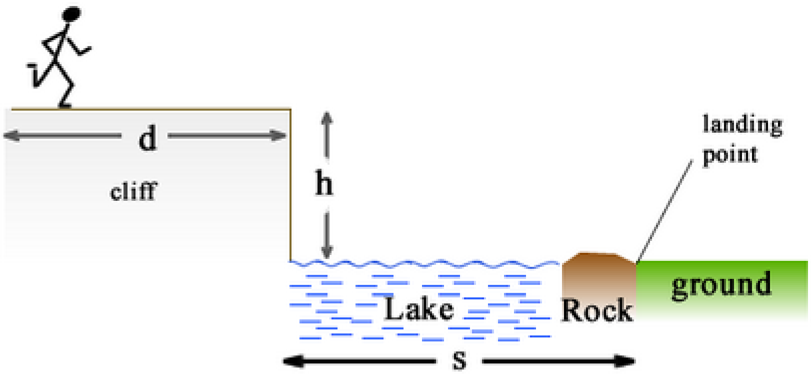
This causes a bit of a problem: we don't know his position when he changes to constant velocity, nor do we know the velocity. We still have enough information, though:

$$a \cdot (1.01\,\mathrm{s}) = \text{the constant velocity, after having accelerated} \tag{2.75}$$
$$(a \cdot 1.01\,\mathrm{s}) \cdot 1.866\,\mathrm{s} = \text{distance covered at constant speed} \tag{2.76}$$
$$\frac{1}{2}a(1.01\,\mathrm{s})^2 = \text{distance covered while accelerating} \tag{2.77}$$

We add the distances covered with the total distance covered, and solve for a:

$$(\frac{1}{2}a(1.01\,\mathrm{s})^2) + (a \cdot 1.01\,\mathrm{s}) \cdot 1.866\,\mathrm{s} = 14.75\,\mathrm{m} \tag{2.78}$$
$$(\frac{1}{2}a(1.02\,\mathrm{s}^2) + (a \cdot 1.88\,\mathrm{s}^2) = 14.75\,\mathrm{m} \tag{2.79}$$
$$a(0.51\,\mathrm{s}^2 + 1.88\,\mathrm{s}^2) = 14.75\,\mathrm{m} \tag{2.80}$$
$$a \approx 6.17\,\mathrm{m/s}^2 \tag{2.81}$$

2.6 Problem 6: Jumping off a cliff



"A person, standing on a vertical cliff a height h above a lake, wants to jump into the lake but notices a rock just at the surface level with its furthest edge a distance s from the bottom of the cliff. The person realizes that with a running start it will be possible to just clear the rock, so the person steps back from the edge a distance d and starting from rest, runs at a constant acceleration a and then leaves the cliff horizontally. The person just clears the rock. Find s in terms of the given quantities d, a, h, and the gravitational acceleration g. You may neglect all air resistance."

Well, there is a typo in the problem. The person doesn't want to "jump into the lake", but rather wants to jump *past* the lake! Either way, this problem is similar to the previous problem: the person can only accelerate (with constant acceleration, apparently) for part of the journey, and will travel the rest at constant velocity (in x).

Let's first look at the second part of the motion: he needs to fall h meters and move s meters forward in exactly the same time; the fall is at 0 initial speed and constant acceleration $-g$, while the horizontal motion is at an unknown initial speed and no acceleration, i.e. constant velocity.
The fall time t_f can be calculated easily. Let's say $y = 0$ is on the ground, so that he starts out at $y_0 = h$:

$$h - \frac{1}{2}gt_f^2 = 0 \tag{2.82}$$
$$t_f = \sqrt{\frac{2h}{g}} \tag{2.83}$$

A familiar result. This time is then exactly the time he must spend to travel the distance s, which means we can calculate the (average, but it is constant, so that's good) velocity:

$$v_x = \frac{s}{t_f} = \frac{s\sqrt{g}}{\sqrt{2h}} \tag{2.84}$$

He must then reach this velocity v_x at constant acceleration, while running the distance d. Let's call the time taken t_r, with r for run.

$$\frac{1}{2}at_r^2 = d \tag{2.85}$$

Not only that, but he must reach the velocity v_x in the same time t_r:

$$at_r = \frac{s\sqrt{g}}{\sqrt{2h}} \tag{2.86}$$

This is all we need – we can now solve for s using these equations.

$$s = at_r\frac{\sqrt{2h}}{\sqrt{g}} \tag{2.87}$$

t_r is something I made up though, so we need to solve for it in the other equation:

$$t_r = \sqrt{\frac{2d}{a}} \tag{2.88}$$

Combining the two and simplifying:

$$s = a\sqrt{\frac{2d}{a}}\frac{\sqrt{2h}}{\sqrt{g}} \tag{2.89}$$

$$s = 2\sqrt{ad}\frac{\sqrt{h}}{\sqrt{g}} \tag{2.90}$$

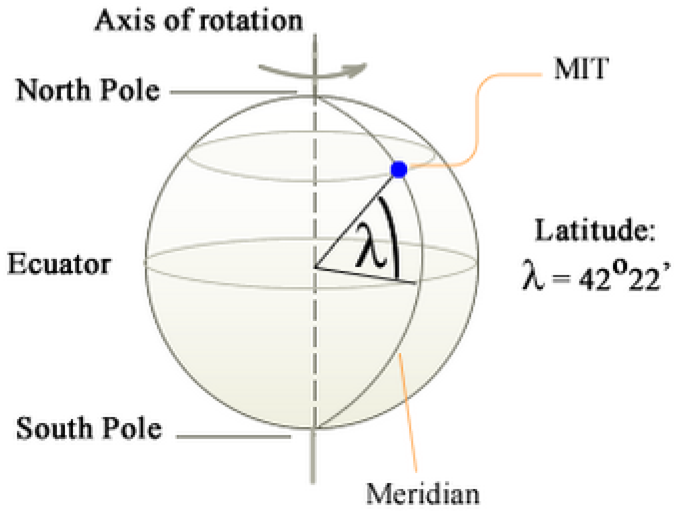
$$s = 2\frac{\sqrt{adh}}{\sqrt{g}} \tag{2.91}$$

... and we have our answer.

2.7 Problem 7: Earth rotation and centripetal acceleration

"The Earth is spinning about its axis with a period of 23 hours 56 min and 4 sec (a sidereal day). The equatorial radius of the Earth is 6.38×10^6 m. The latitude of MIT (Located in Cambridge, Massachusetts) is $42°44'$.

Note: The latitude of a point on Earth, in this case MIT, is the angle from the Equator to that point measured along the meridian of that point. In the figure below the latitude of MIT is indicated with the angle λ. (A meridian is a half of a circle that passes through the north and south poles).



a) Find the velocity of a person at MIT as they undergo circular motion about the Earth's axis of rotation. Enter your answer in m/s."

Let's start by converting the period T to seconds, and the latitude to degrees (with decimals, instead of arcminutes):

$$T = 23 \cdot 3600 + 56 \cdot 60 + 4 = 86\,164\,\mathrm{s} \tag{2.92}$$

$$\lambda = 42°22' = 42° + \frac{22}{60}° = 42.3\overline{666}° \tag{2.93}$$

With the measurements in useful units, let's now calculate the "effective radius", so to speak. (Clearly, a person at the middle of the North Pole will have a near-zero velocity and centripetal acceleration.)

$$r = R_{earth}\cos(42.3\overline{666}°) \approx 4.714 \times 10^6\,\mathrm{m} \tag{2.94}$$

The circumference at MIT's latitude is then

$$C = 2\pi r \approx 2.962 \times 10^7\,\mathrm{m} \tag{2.95}$$

So the speed is

$$v = \frac{C}{T} = \frac{2.962 \times 10^7\,\mathrm{m}}{86\,164\,\mathrm{s}} \approx 343.76\,\mathrm{m/s} \tag{2.96}$$

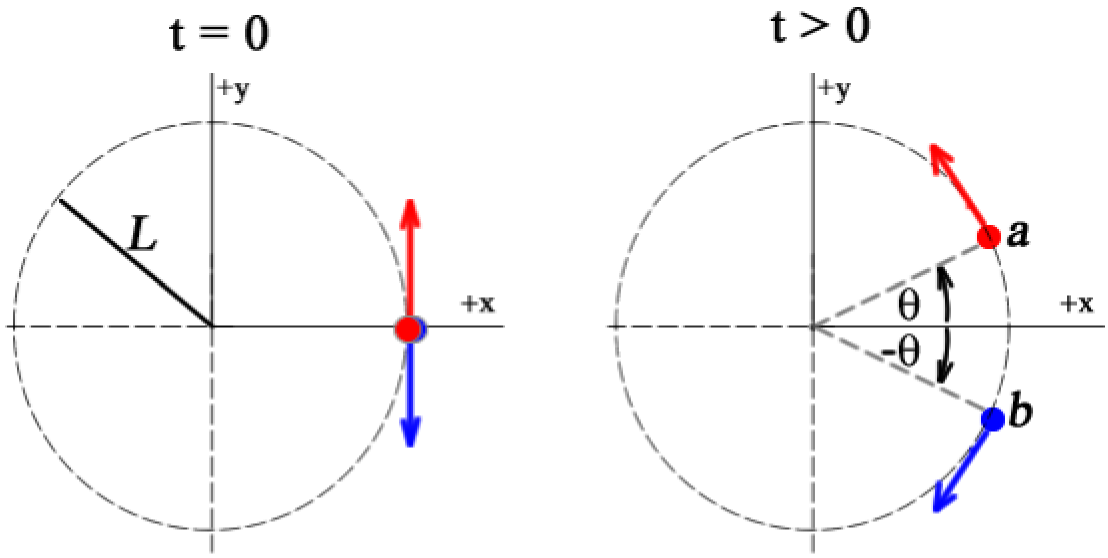
"b) Find the person's centripetal acceleration. Enter your answer in $\mathrm{m/s^2}$."

We can use $|a_c| = v^2/r$ very simply:

$$|a_c| = \frac{v^2}{r} = \frac{(343.76\,\mathrm{m/s})^2}{4.714 \times 10^6\,\mathrm{m}} \approx 0.0250\,\mathrm{m/s^2} \tag{2.97}$$

2.8 Problem 8: Relative velocity on a rotating disk

"Particles a and b move in opposite directions with angular velocity ω around a circle of radius L. At $t = 0$ they are both passing through the +x axis (left figure). The angular position of particle a, θ (> 0), is measured from the positive x-axis as shown in the right figure below. The angular position of particle b is $-\theta$.



Find the x and y components of the velocity vector $\vec{v_{ab}}$, the velocity of particle a relative to particle b. Express your answer in terms of ω, L and θ as needed."

This took me a while, but once I realized how to do it, it was fairly easy.
The book discusses a VERY similar problem, and the exact answer is actually in the book. I didn't realize that until I had solved it, though! I had a bit of difficulty grasping a few things in their analysis, until I managed to solve it on my own.

Anyway, I'll recap my version here, even though it's similar to the book's.

First, let's define $\vec{r_a}$ and $\vec{r_b}$ as vectors from the exact center of the circle, to the respective particle they are named after. By finding the difference $\vec{r_a} - \vec{r_b}$, we find a vector pointing from b to a, with the correct magnitude (the distance between the two). Let's call this vector R_{ab}:

$$R_{ab} = \vec{r_a} - \vec{r_b} \tag{2.98}$$

What is this vector, in terms of components?
We use the unit circle definitions of the trig functions to find that. For example, for the a particle, θ is positive, as is both x and y (at this time). Its x position can be found as $L\cos\theta$, and y position as $L\sin\theta$, by some basic trigonometry. (Draw it and try to calculate the components if you don't see it.)
The same applies to $\vec{r_b}$, except θ is negative. That makes the y coordinate negative, as $\sin(-y) = -\sin(y)$, though $\cos(-x) = \cos(x)$ so nothing changes there. All in all we have

$$\vec{r_a} = L\cos\theta\hat{x} + L\sin\theta\hat{y} \tag{2.99}$$
$$\vec{r_b} = L\cos\theta\hat{x} - L\sin\theta\hat{y} \tag{2.100}$$

the vector R_{ab} is then found as the difference, as shown above:

$$\vec{R_{ab}} = L\sin\theta\hat{y} - (-L\sin\theta\hat{y}) = 2L\sin\theta\hat{y} \tag{2.101}$$

The x part cancels out. If you sketch the problem (or look at the provided sketch) and imagine the particles' motion, this should be intuitive.

The only step remaining is to find the velocity vector instead of the position vector. This is simply done by taking the time derivative of the above vector.

$$\vec{V_{ab}} = \frac{d}{dt}\vec{R_{ab}} = 2L\frac{d\theta}{dt}\cos\theta\hat{y} \tag{2.102}$$

We get a $\frac{d\theta}{dt}$ outside because of the chain rule. By convention, we write that simply as ω, so all in all we have:

$$\vec{V_{ab}} = 2L\omega\cos\theta\hat{y} \tag{2.103}$$

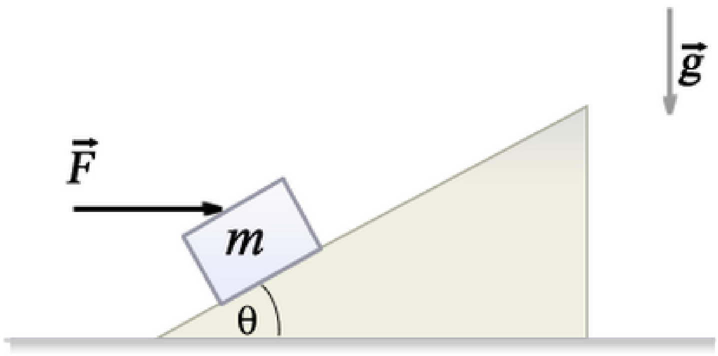
$$V_{abx} = 0 \tag{2.104}$$

$$V_{aby} = 2L\omega\cos\theta \tag{2.105}$$

And we are done!

Chapter 3: Week 3: Homework 3

3.1 Problem 1: A block on a frictionless ramp



"A block of mass m = 4 kg is pressed with a horizontal force F against a frictionless ramp of angle $\theta = 38°$.

Assuming the block is at rest on the ramp, answer the following:
(a) What is the magnitude of the normal force exerted by the incline surface on the block?
(b) What is the magnitude of the force F exerted on the block?"

We should start out by drawing a free-body diagram, but even before we do that, we need to decide on a coordinate system. I'm not sure if it'd be best to choose one where y is perpendicular to the incline, or where it is parallel with $\vec{g}$. In either case, there are things we need to decompose; I'll therefore choose one where $+x$ is in the uphill direction, and $+y$ is perpendicular to the incline, so "diagonally upwards" in this case.

The forces we need to worry about are $\vec{F}$, $m\vec{g}$, and the normal force $\vec{N}$. Let's consider them together in the x direction. The object is at rest, so via Newton's second law, the net force is zero.

$$F \cos\theta = mgsin\theta \tag{3.1}$$

The normal force is perpendicular to the incline, and so the x component is zero. Meanwhile, the x component of the force F applied must balance out the gravitational force $mg\sin\theta$ in the x direction.

It's always a good idea to test the extremes and ensure the correct trig functions are used. If $\theta = 0$, we expect there to be no gravitational force at all in the x direction, and indeed $\sin(0) = 0$. Using the same argument, if $\theta = 90$, the gravitational force should be exclusively in the x direction, and again, it will be. As for $F\cos\theta$, the opposite is true, as it should be.

So far, so good. Next, let's look at the y direction. Newton's second law, again:

$$N = mg\cos\theta + F_y \tag{3.2}$$
$$N = mg\cos\theta + F\sin\theta \tag{3.3}$$

The normal force must balance out the gravitational force (angled) downwards, plus the y component of the force F, which is also in the $-y$ direction (that becomes easier to see if θ is increased).

We now have two equations, and two unknowns (F and N). Let's write the equations with the numbers substituted, and solve:

$$F\cos(38°) = (4\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\sin(38°) \tag{3.4}$$
$$N = (4\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\cos(38°) + F\sin(38°) \tag{3.5}$$

The second equation alone gives us

$$N = (4\,\text{kg})(9.8\,\text{m/s}^2)\cos(38^\circ) + F\sin(38^\circ) \approx 30.89\,\text{N} + 0.6156F \tag{3.6}$$

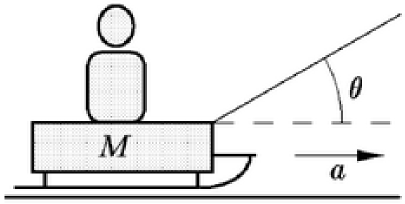
And the first alone tells us

$$F = \frac{(4\,\text{kg})(9.8\,\text{m/s}^2)\sin(38^\circ)}{\cos(38^\circ)} = 39.2\,\text{N}\tan(38^\circ) = 30.63\,\text{N} \tag{3.7}$$

So the answer for (b) is 30.63 N, while the first one is 49.75 N.

3.2 Problem 2: Towing a sled

"A mother tows her daughter on a sled on level ice. The friction between the sled and the ice is negligible, and the tow rope makes an angle of θ to the horizontal. The combined mass of the sled and the child is M. The sled has an acceleration in the horizontal direction of magnitude a.



(a) Calculate the tension, T, in the rope. Express your answer in terms of M, a, g, and θ.
(b) Calculate the magnitude of the normal force, N, exerted by the ice on the sled. Express your answer in terms of M, a, g, and θ."

First, let's identify the forces involved. There's the gravitational force mg downwards, the normal force N straight upwards, and the force from the rope, which will need some decomposition. Because of this, we can't simply state that $|N| = Mg$; the rope is pulling upwards a bit, too.

Since there's no incline involved (for the sled itself), I choose a simple coordinate system where $+x$ is to the right, and $+y$ is straight upwards. The gravitational force is $-Mg$, purely in the y direction, and the acceleration is $a > 0$.
We can also express that in terms of a force $F_x = Ma$, but let's be careful: this is not the force that the mother exerts; that force is at an angle θ, so $F_x = F_{mother}\cos\theta$.

Writing Newton's second law for each of the two axes independently:

$$F_x = Ma \tag{3.8}$$
$$N + F_y = Mg \tag{3.9}$$

We know that $F_x = F_{mother}\cos\theta$, so we can solve for F_{mother} in terms of the acceleration and mass:

$$F_{mother} = \frac{F_x}{\cos\theta} = \frac{Ma}{\cos\theta} \tag{3.10}$$

How does this force relate to the tension T in the rope, that we want to find out? It's actually not specified, but I assume we are to take the rope to be massless and of fixed length, as previously; especially as no mass is shown for the rope (nor is its length, by the way). Because of that, we can ignore the gravitational force on the rope.
So, long story short, $T = F_{mother}$, and we've already found the answer!

Next up, (b): $F_y = T\sin\theta$, so the third law equation becomes

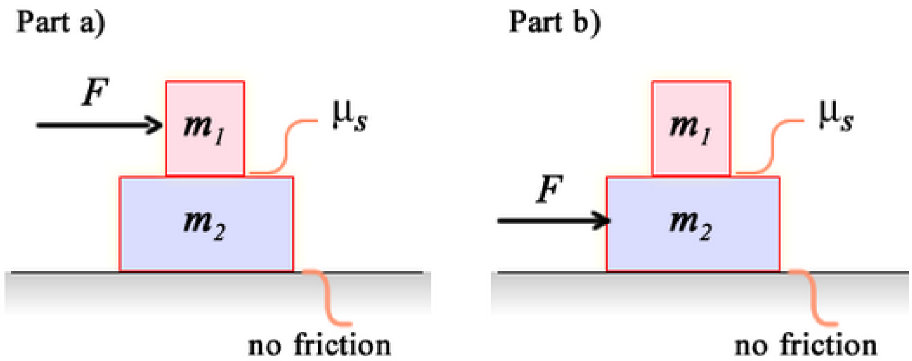
$$N = Mg - T\sin\theta \tag{3.11}$$

If we substitute in the value for $T = F_{mother}$, we find

$$N = Mg - \frac{Ma\sin\theta}{\cos\theta} = M(g - a\tan\theta) \tag{3.12}$$

... and we are done.

3.3 Problem 3: Stacked blocks



"Consider two blocks that are resting one on top of the other. The lower block has mass $m_2 = 4.3\,\text{kg}$ and is resting on a frictionless table. The upper block has mass $m_1 = 1.2\,\text{kg}$. Suppose the coefficient of static friction between the two blocks is given by $\mu_s = 0.6$.

Part a) A force of magnitude F is applied as shown in the left figure above. What is the maximum force for which the upper block can be pushed horizontally so that the two blocks move together without slipping?"

As usual, let's start by looking at the forces involved. In the vertical direction, we have gravitational forces gm_1 and gm_2 acting on each of the blocks, respectively.
Block m_1 (or block 1) pushes downwards on block m_2 (or block 2) with that same force gm_1, and via Newton's third law, we find the reaction force (the normal force, in this case) from block 2 to block 1.

The total forces on block 1 are the gravitational force downwards, and the normal force upwards, from block 2 to 1. Net force: zero – as it must be, since it is at rest.

As for block 2, the downward forces are as mentioned above gm_1 from the upper block, and gm_2 from gravity on the block itself. This is cancelled out by a normal force from the ground on the block, of magnitude $g(m_1 + m_2)$[1]. Again, the net force is zero, at it must be.

With the normal force on block 1, we know that the maximum frictional force that will oppose motion in mass m_1 is $\mu_s N = \mu_s g m_1$. As for block 2, there is no friction to the ground, so we need not worry about the maximum frictional force there.

If we write a second law equation for mass m_1 on its own, and one for the entire system, both exclusively in the x direction:

$$F - F_{Fmax} = m_1 a \text{ (top block)} \tag{3.13}$$
$$F = a(m_1 + m_2) \text{ (entire system)} \tag{3.14}$$

[1]Calculating like this may be a bad idea. I'll try to re-think for next time, and always consider one block at a time.

The acceleration a as seen from an external reference frame is equal for both, since the condition is that they move together. We can solve the second equation for a, and stick it into the first, and then solve for F:

$$F - F_{Fmax} = m_1 \frac{F}{m_1 + m_2} \tag{3.15}$$
$$F - F\frac{m_1}{m_1 + m_2} = F_{Fmax} \tag{3.16}$$
$$F\left(1 - \frac{m_1}{m_1 + m_2}\right) = F_{Fmax} \tag{3.17}$$
$$F = \frac{F_{Fmax}}{1 - \frac{m_1}{m_1+m_2}} = \frac{\mu_s g m_1}{1 - \frac{m_1}{m_1+m_2}} \approx 9.03\,\mathrm{N} \tag{3.18}$$

"Part b) A force of magnitude F as shown in the right figure above. What is the maximum force for which the lower block can be pushed horizontally so that the two blocks move together without slipping?"

Okay, so we need to reverse the situation a bit. Except for the second law equations and such from above which clearly change, what else changes? The vertical forces don't; the maximum frictional force also doesn't, as it's based on the normal force, which is unchanged.
So, the force is now on m_2.

It seems like all we need to do is write a new pair of second law equations, again in the x direction only. One equation remains unchanged, the one for the entire system. However, F no longer acts on m_1!
Instead, it holds on via the frictional force, and can only accelerate together as long as that is "strong" enough.

If we push the lower block towards the right with too much force, what will happen? The upper block will glide "backwards", relative to the lower block. That means that the frictional force is now in the forward direction! Indeed, it's the only force acting on m_1 (horizontally), so we find

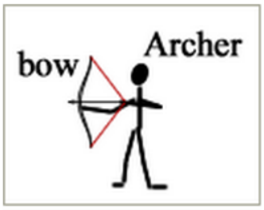
$$F_{Fmax} = m_1 a \text{ (top block)} \tag{3.19}$$
$$F = a(m_1 + m_2) \text{ (entire system)} \tag{3.20}$$

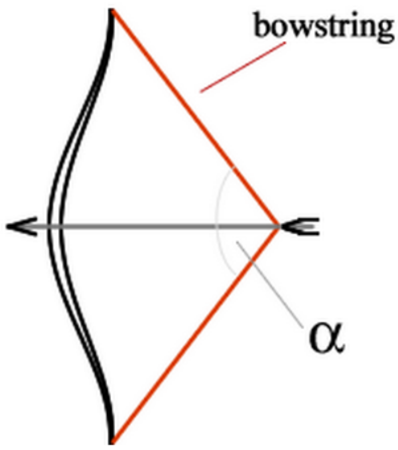
Solving the first equation for a and substituting into the second:

$$F = \frac{F_{Fmax}}{m_1}(m_1 + m_2) \tag{3.21}$$
$$= \frac{\mu_s g m_1}{m_1}(m_1 + m_2) \tag{3.22}$$
$$= \mu_s g(m_1 + m_2) \tag{3.23}$$

That's the second and final answer!

3.4 Problem 4: Tension in string





"An archer is preparing to shoot an arrow. He grabs the center of the bowstring and pulls straight back with a force of magnitude $F = 118$ N. The upper and lower halves of the string form an angle $\alpha = 124°$ with respect to each other. Assume that the bowstring is massless.

(a) What is the magnitude of the tension in the upper half of the bowstring?"

Because the string is massless, we ignore the pull of gravity. That makes this the first problem of this week not to feature g at all!

Instead, because the string is at rest when he's done (I assume that's what they mean: he pulls the string backs, and then hold it in place such that $\alpha = 124°$), the tension must balance his force out exactly, so that $a = 0$.

Let's choose a coordinate system where the archer pulls the string in the $+x$ direction (towards the right), and $+y$ is straight upwards. We call his force $F = (118\,\mathrm{N})\hat{x}$. Any y components in the tension must cancel each other out, and the x components will cancel out with F.
Let's call the two tensions T_U for upper, and T_L for lower; each then have x and y components.

Because the archer pulls the rope towards the right, the tension points "upwards to the left" and 'downward to the left" in the upper and lower part of the string, respectively.

Decomposing the tension vectors, we find

$$T_{Lx} = T_L \cos(-\alpha/2) \tag{3.24}$$
$$T_{Ly} = T_L \sin(-\alpha/2) \tag{3.25}$$
$$T_{Ux} = T_U \cos(\alpha/2) \tag{3.26}$$
$$T_{Uy} = T_U \sin(\alpha/2) \tag{3.27}$$

Writing Newton's second law for the archer's force ($+x$) and the tension forces ($-x$):

$$F = T_{Lx} + T_{Ux} \tag{3.28}$$
$$F = T_L \cos(-\alpha/2) + T_U \cos(\alpha/2) \tag{3.29}$$

One equations, two unknowns. We can also write a second law equation for the vertical components of the tension, which will cancel:

$$T_{Ly} = -T_{Uy} \tag{3.30}$$
$$T_L \sin(-\alpha/2) = -T_U \sin(\alpha/2) \tag{3.31}$$
$$T_L \sin(\alpha/2) = T_U \sin(\alpha/2) \tag{3.32}$$
$$T_L = T_U \tag{3.33}$$

The second-to-last step is because $\sin(-x) = -sin(x)$, so the minus signs cancel, and so we find that the tension is equal (in magnitude) for both parts of the string. With that in mind, we can write the other equation in terms of T_L alone, and solve for it. Also, $\cos(-x) = \cos(x)$, so we can get rid of the duplicate cosine terms:

$$F = T_L \cos(-\alpha/2) + T_L \cos(\alpha/2) \tag{3.34}$$
$$F = T_L(2\cos(\alpha/2)) \tag{3.35}$$
$$T_L = T_U = \frac{F}{2\cos(\alpha/2)} \approx 125.67\,\mathrm{N} \tag{3.36}$$

And we are done!

3.5 Problem 5: Measurement of friction coefficient

"In Lecture 8 Video Segment 5, Prof. Lewin does two different experiments to calculate the coefficient of static friction of an inclined plane.

Experiment 1 took a measurement of the critical angle θ_c at which the block began to slide down the plane. Prof. Lewin measured the angle $\theta_c = 20° \pm 2°$.

Experiment 2 took a measurement of the critical mass m_2 which caused the block to begin to slide up the plane. Prof. Lewin measured the angle $\theta = 20°$, the mass $m_1 = 361 \pm 1\,\mathrm{g}$, and the mass $m_2 = 270 \pm 25\,\mathrm{g}$.

For each of the following questions use only the uncertainties given above. Enter your answer to 3 or 4 significant figures to make sure it is within the grader's tolerance. take the value of g to be $9.81\,\mathrm{m/s^2}$.

(a) What is the upper bound of the coefficient of static friction calculated from the data in Experiment 1?
(b) What is the lower bound of the coefficient of static friction calculated from the data in Experiment 1?"

Okay, so this first part should be easy. We found in lecture that $\mu_s = \tan\alpha$, only we call the angle θ_c here. The bounds are then found by taking the tangent of 18° and 22°, and we're done: that gives us two values of μ_s, one larger than the other; the larger is obviously the upper bound, then.

$$\mu_{s_{max}} = \tan(22°) \approx 0.4040 \tag{3.37}$$
$$\mu_{s_{min}} = \tan(18°) \approx 0.3249 \tag{3.38}$$

We then move on to the second part, which is no doubt more work:

"(c) What is the upper bound of the coefficient of static friction calculated from the data in Experiment 2?
(d) What is the lower bound of the coefficient of static friction calculated from the data in Experiment 2?"

Okay. We know from the video that it's just about to go uphill. The condition that holds at that point is

$$m_2 g = m_1 g \sin\alpha + F_{Fmax} \tag{3.39}$$
$$m_2 g = m_1 g \sin\alpha + \mu_s m_1 g \cos\alpha \tag{3.40}$$

F_{Fmax} is given by $\mu_s N$, where N is the normal force, $m_1 g \cos\alpha$. All of these equations are found in the video, and derived in that lecture (which I took notes of), so I won't repeat that. Let's try to solve this equation for μ_s.

$$m_2 g = m_1 g \sin\alpha + \mu_s m_1 g \cos\alpha \tag{3.41}$$
$$\frac{m_2}{m_1} = \sin\alpha + \mu_s \cos\alpha \tag{3.42}$$
$$\frac{m_2}{m_1} - \sin\alpha = \mu_s \cos\alpha \tag{3.43}$$
$$\frac{\frac{m_2}{m_1} - \sin\alpha}{\cos\alpha} = \mu_s \tag{3.44}$$
$$\frac{m_2}{m_1 \cos\alpha} - \tan\alpha = \mu_s \tag{3.45}$$

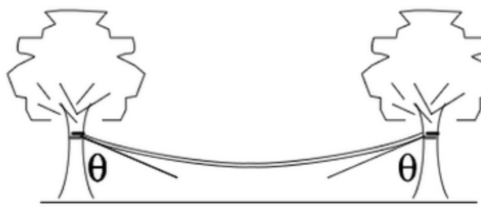
Ah, not too bad. Now, for this part of the problem, $\theta = 20°$ exactly, with no uncertainty given, while the masses have an uncertainty. It should be easy to find the upper bound: maximize m_2, and minimize m_1. For the lower bound, we do the opposite. Easy!

$$\mu_{s_{max}} = \frac{295\,\mathrm{g}}{(360\,\mathrm{g})\cos(20°)} - \tan(20°) = 0.5081 \tag{3.46}$$
$$\mu_{s_{min}} = \frac{245\,\mathrm{g}}{(362\,\mathrm{g})\cos(20°)} - \tan(20°) = 0.3563 \tag{3.47}$$
$$\tag{3.48}$$

Both answers (or all four, I suppose) are marked as correct. Excellent!
The ranges don't quite agree with each other, but there is indeed a range where the friction coefficient could be equal: 0.3563 - 0.4040 is allowed by both experiments.

3.6 Problem 6: Rope between trees

"Suppose a rope of mass m hangs between two trees. The ends of the rope are at the same height and they make an angle θ with the trees.



(a) What is the tension at the ends of the rope where it is connected to the trees? Express your answer in terms of m, g, and θ."

First off, note that θ is measured with respect to the *vertical*, not the horizontal! If $\theta \approx 90°$ then the rope is almost horizontal! In the figure above, I would estimate it around $\theta \approx 70° - 80°$ or something.

Now... For this first part, we can assume that all the mass of the rope is located at the exact middle, and we will find the same result for the tension at the trees (but clearly we can't use this method in part b).

We can therefore apply the same method Prof. Lewin used in a problem solving video. I don't recall it exactly, and I will try to re-derive it instead of re-watching, since my goal is to learn, not just to get a green checkmark! First off, he simplified the rope by drawing it as two straight lines, meeting at a large angle in the middle. If we then draw a dotted line horizontally between the points where the rope attaches, we get an obtuse triangle. I'll call the two side angles α (they are clearly equal), and the bottom, obtuse angle β. From the diagram, it's clear that

$$\theta + \alpha = 90° \tag{3.49}$$
$$\alpha = 90° - \theta \tag{3.50}$$

With that in mind, plus the fact that $2\alpha + \beta = 180°$, we find $\beta = 180 - 2(90 - \theta) = 2\theta$.

Okay, so having all that done, let's look at some forces. At the exact center, there is a downwards force mg due to gravity, which must be exactly balanced out. Only the y component of the string tension could possibly counter this, so $T_y = mg$ (Newton's second law, relating only magnitudes) must hold, or there would be acceleration.

What is T_y, then? By some vector decomposition, it must be

$$T_y = T \sin\alpha \tag{3.51}$$

... since T_y is opposite the angle, while T is the hypotenuse. We put this into the second law equation:

$$T \sin\alpha = mg \tag{3.52}$$
$$T = \frac{mg}{\sin\alpha} \tag{3.53}$$
$$T = \frac{mg}{\sin(90° - \theta)} \tag{3.54}$$
$$T = \frac{mg}{\cos\theta} \tag{3.55}$$

What exactly is T, though? It's not the answer for either question – we are not quite done yet. Instead, this is the total tension (or total force) the trees need to carry. Since there is symmetry in the problem, each tree carries *half* this weight, which makes the answer for (a)

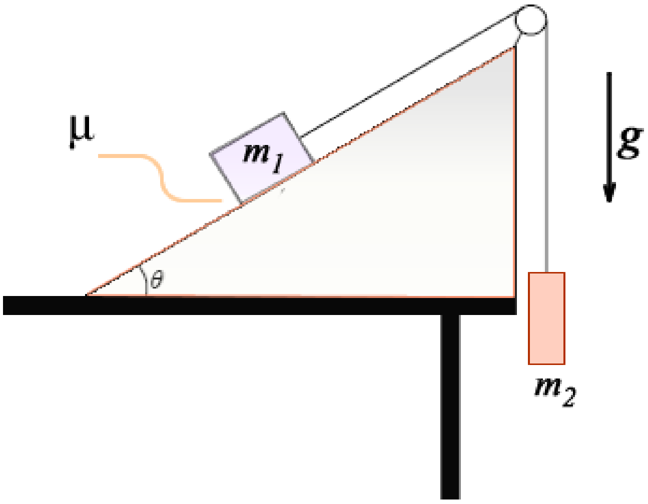
$$T_{tree} = \frac{mg}{2\cos\theta} \tag{3.56}$$

To find the next part, we need to use some symmetry.At the *exact* center, which is mostly a theoretical concept, there is a point that has no weight. It has an infinitely small size, and it doesn't need to "carry" any other part of the rope, either. Therefore, the vertical component of the tension is zero, and only the horizontal component of the tension remains. Therefore, we take the horizontal component of the above: $T_x = T\cos\alpha = T\sin\theta$

$$T_{middle} = \frac{mg}{2\cos\theta}\sin\theta = \frac{mg}{2}\tan\theta \tag{3.57}$$

3.7 Problem 7: Blocks and ramp with friction

"A block of mass $m_1 = 28\,\text{kg}$ rests on a wedge of angle $\theta = 47°$ which is itself attached to a table (the wedge does not move in this problem). An inextensible string is attached to m_1, passes over a frictionless pulley at the top of the wedge, and is then attached to another block of mass $m_2 = 3\,\text{kg}$. The coefficient of kinetic friction between block 1 and the plane is $\mu = 0.8$. The string and wedge are long enough to ensure neither block hits the pulley or the table in this problem, and you may assume that block 1 never reaches the table. Take g to be $9.81\,\text{m/s}^2$.



The system is released from rest as shown above, at $t = 0$.
(a) Find the magnitude of the acceleration of block 1 when it is released (in $\mathrm{m/s^2}$)."

Since m_1 is much greater than m_2, plus the fact that they only give us the *kinetic* friction coefficient, along with "and you may assume that block 1 never reaches the table", I think it's quite safe to assume the system will accelerate "counterclockwise", so that m_1 slides down towards the table.

If we draw up a free-body diagram, we find the following forces acting on block m_1, assuming a coordinate system where $+x$ is downhill and $+y$ is perpendicular to the surface (diagonally upwards to the left):

- $m_1 g \cos\theta$ acting in the $-y$ direction
- $N = m_1 g \cos\theta$ acting in the $+y$ direction, to cancel out the gravitational force
- $m_1 g \sin\theta$ acting in the $+x$ direction
- $F_f = \mu N = \mu m_1 g \cos\theta$ acting in the $-x$ direction
- T (unknown magnitude) acting in the $-x$ direction

As for the mass m_2, there are only two forces:

- $m_2 g$ acting downwards (which we call $-y$ in another coordinate system)
- T acting upwards, to counteract gravity (partially, not entirely)

In both cases, the net force must equal the object's mass times the acceleration, which will be the same for both due to the inextensible string that connects them. We can write two Newton's second law equations, and find

$$m_1 a = m_1 g \sin\theta - T - \mu m_1 g \cos\theta \tag{3.58}$$
$$m_2 a = T - m_2 g \tag{3.59}$$

We can solve the second equation for T and substitute it into the first to find the acceleration:

$$m_1 a = m_1 g \sin\theta - (m_2 a + m_2 g) - \mu m_1 g \cos\theta \tag{3.60}$$
$$m_1 a + m_2 a = m_1 g \sin\theta - m_2 g - \mu m_1 g \cos\theta \tag{3.61}$$
$$a(m_1 + m_2) = m_1 g \sin\theta - m_2 g - \mu m_1 g \cos\theta \tag{3.62}$$
$$a = \frac{m_1 g \sin\theta - m_2 g - \mu m_1 g \cos\theta}{m_1 + m_2} \approx 0.697\,\mathrm{m/s^2} \tag{3.63}$$

That answers part (a).

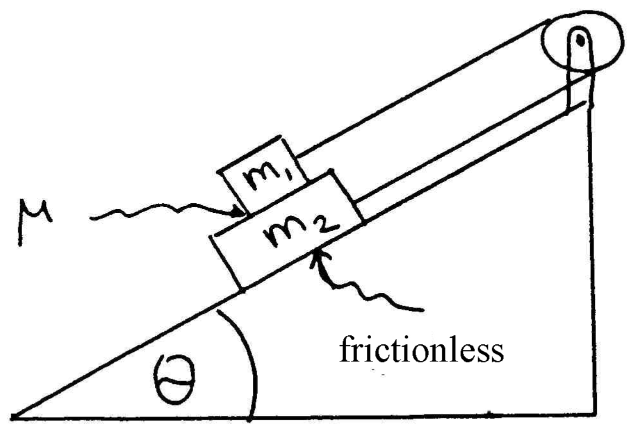
"(b) How many cm down the plane will block 1 have traveled when 0.47 s has elapsed?"

We use the basic kinematics equation, with $x_0 = 0$ and $v_0 = 0$:

$$\frac{1}{2}at^2 = \frac{0.697\,\mathrm{m/s^2}}{2}(0.47\,\mathrm{s}^2) \approx 0.0769\,\mathrm{m} \approx 7.7\,\mathrm{cm} \tag{3.64}$$

3.8 Problem 8: Friction between blocks on a ramp

"Two blocks with masses m_1 and m_2 such that $m_1 \ll m_2$ are connected by a massless inextensible string and a massless pulley as shown in the figure below. The pulley is rigidly connected to the top of a wedge with angle θ. The coefficient of friction between the blocks is μ. The surface between the lower block and the wedge is frictionless. The goal of this problem is to find the magnitude of the acceleration of each block.



What are the magnitudes of the acceleration of the two blocks? Express your answer in terms of g, μ, m_1, m_2, and θ."

Since m_2 is much greater than m_1, m_2 will slide downhill and m_1 uphill... until they slide off each other, that is. The only other possibility is that $a = 0$ and that the system is in equilibrium, because the friction is great enough. I will assume the answer is not zero, though!

Drawing a free-body diagram (a must for most of these questions, but especially this one), we find a lot of forces.
As usual, I chose a coordinate system with x parallel to the incline, and y perpendicular. $+x$ is downhill, for no reason in particular.

On block m_1, there is friction, gravity/normal force (gravity in 2 dimensions) and tension. On block m_2, there is also gravity in two dimensions and a normal force, but we don't need to pay much attention to the y forces, since there is no friction on the ramp. We know that the normal force will cancel gravity, but that's about it for its usefulness. In addition to those, there's tension and a third law reaction force for the friction.

Let's try to write Newton's second law equations in the x direction. I will add up downhill forces, subtract uphill forces, and set it all equal to the mass times acceleration:

$$m_1 g \sin\theta + \mu m_1 g \cos\theta - T = -m_1 a \tag{3.65}$$
$$m_2 g \sin\theta - T - \mu m_1 g \cos\theta = m_2 a \tag{3.66}$$

Not very pretty, is it? I will admit, it took me a few tries to get it right; I first forgot about the third law reaction force for the friction (there's a frictional force uphill on the second block!).
As for directions, the first equation has $-m_1 a$ since the acceleration is positive downhill, but the motion

will surely be uphill. The second equation has it without the minus sign, since that block will indeed move downhill.

Let's try to solve this by addition; that is, add the left sides to a new left side, and the two right sides to a new right side. The friction should cancel, so finding a should be less painful than by substitution.

$$
\begin{aligned}
m_1 g\sin\theta + \mu m_1 g\cos\theta - T + m_2 g\sin\theta - T - \mu m_1 g\cos\theta &= -m_1 a + m_2 a && (3.67)\\
m_1 g\sin\theta - 2T + m_2 g\sin\theta &= -m_1 a + m_2 a && (3.68)\\
g\sin\theta(m_1+m_2) - 2T &= a(m_2 - m_1) && (3.69)\\
a &= \frac{g\sin\theta(m_1+m_2)-2T}{m_2-m_1} && (3.70)
\end{aligned}
$$

Unfortunately, that doesn't quite get us all the way; we don't know T! Let's solve for it from, say, the second equation (either should work, and they're equally complex, so I just picked one). I suppose we'll do substitution after all:

$$
\begin{aligned}
T &= m_2 g\sin\theta - \mu m_1 g\cos\theta - m_2 a && (3.71)\\
T &= g(m_2\sin\theta - \mu m_1\cos\theta) - m_2 a && (3.72)
\end{aligned}
$$

Combining the two, we get... this monstrosity, which we need to solve for a again:

$$
\begin{aligned}
a &= \frac{g\sin\theta(m_1+m_2) - 2(gm_2\sin\theta - g\mu m_1\cos\theta - m_2 a)}{m_2-m_1} && (3.73)\\
a &= \frac{g\sin\theta(m_1+m_2) - 2gm_2\sin\theta + 2g\mu m_1\cos\theta + 2m_2 a}{m_2-m_1} && (3.74)\\
a &= \frac{g\sin\theta(m_1+m_2) - 2gm_2\sin\theta + 2g\mu m_1\cos\theta}{m_2-m_1} + \frac{2m_2 a}{m_2-m_1} && (3.75)\\
a\left(1 - \frac{2m_2}{m_2-m_1}\right) &= \frac{g\sin\theta(m_1+m_2) - 2gm_2\sin\theta + 2g\mu m_1\cos\theta}{m_2-m_1} && (3.76)\\
a &= \frac{g\sin\theta(m_1+m_2) - 2gm_2\sin\theta + 2g\mu m_1\cos\theta}{m_2-m_1} \cdot \frac{1}{1-\frac{2m_2}{m_2-m_1}} && (3.77)
\end{aligned}
$$

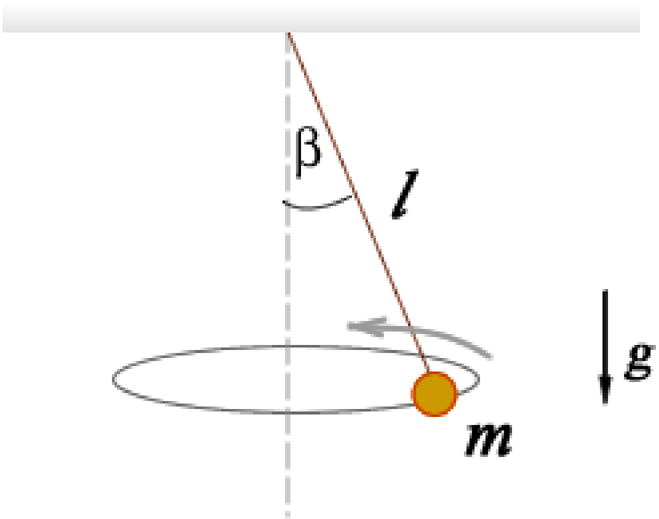
Goodness, I could use Mathematica to simplify that, but it is accepted as correct!

For the sake of readability, here's a simplified version:

$$
a = \frac{g(\sin\theta(m_2-m_1)) - 2gm_1\mu\cos\theta}{m_1+m_2} \tag{3.78}
$$

3.9 Problem 9: Conical pendulum

"A conical pendulum is constructed from a rope of length ℓ and negligible mass, which is suspended from a fixed pivot attached to the ceiling. A small ball of mass m is attached to the lower end of the rope. The ball moves in a circle with constant speed in the horizontal plane, while the rope makes an angle β with respect to the vertical, as shown in the diagram.



(a) Find the tension F_T in the rope. Express your answer in terms of g, m, ℓ, and β.
(b) Find the period of the motion (how long does it take the ball to make one circle in the horizontal plane). Express your answer in terms of g, m, ℓ, and β."

Okay, so let's see. The mass moves in a circle at constant speed: uniform circular motion. We don't know ω or T, though, as that's what we are looking for. We do know the angle and the rope's length, so we should be able to calculate the radius of the (horizontal) circle traced out by the mass itself, however.

In fact, if we forget about the third dimension, we have a very simple right triangle formed by the rope and the axes. We can see that

$$\sin\beta = \frac{r}{\ell} \tag{3.79}$$
$$r = \ell\sin\beta \tag{3.80}$$

I will use cylindrical coordinates for this problem; that is, $\hat{r}$ is radially outwards, $\hat{\theta}$ is tangential to the traced out circle (positive counterclockwise, as the motion is), and $\hat{z}$ is upwards.

There is a centripetal acceleration

$$a_c = \omega^2(-\vec{r}) \tag{3.81}$$
$$= \omega^2\ell\sin\beta(-\hat{r}) \tag{3.82}$$
$$= \frac{4\pi^2}{T^2}\ell\sin\beta(-\hat{r}) \tag{3.83}$$

towards the center of the traced circle, caused by a centripetal force m times the above.

What other forces are there? Well, there's certainly gravity, $-mg$ if we call upwards $+z$. There's the tension in the string, F_T (T is used for the period) which consists of z and r components. Let's decompose the tension.

$$F_{Tz} = F_T\cos\beta \tag{3.84}$$
$$F_{Tr} = F_T\sin\beta \tag{3.85}$$

The centripetal force is purely in the $-\hat{r}$ direction, so we don't need to decompose that. Neither do we need to decompose gravity, which is purely in the $-\hat{z}$ direction.

The net force on the mass must be the centripetal force, or there wouldn't be *uniform* circular motion. The z component of the tension must cancel out gravity, too, or the mass wouldn't move in a horizontal plane, as it does.
Time for Newton's second law. Let's just gather a list of the forces first, so there's no confusion while writing the equations. In the r axis, we have the centripetal force $F_r = a_c m$ inward, and the string tension also inward. In other words, the string tension *provides* (or *is*, essentially) the centripetal force, and thus the cause of the centripetal acceleration.
In the z axis, there is gravity downwards, and a tension component upwards, which must cancel out to yield zero net force.
Lastly, in addition to $F_r = a_c m$, we can say that $a_c = \omega^2 r$, and we derived an expression involving the period earlier, so we find, for the r and z axes respectively,

$$a_c m = F_T \sin\beta \Rightarrow \frac{4\pi^2}{T^2}\ell m \sin(\beta) = F_T \sin\beta \tag{3.86}$$
$$mg = F_T \cos\beta \tag{3.87}$$

And we now at the point where we have two equations with two unknowns. I'll try to solve them manually. Solving the second for F_T is easy:

$$F_T = \frac{mg}{\cos\beta} \tag{3.88}$$

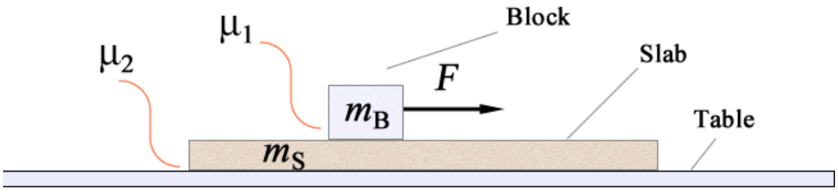
A-ha, nice! It's already in terms of g, m and β, so that's the finished answer for part (a)! Now, let's substitute that into the other one and solve for the period T, which was surprisingly easy:

$$\frac{4\pi^2}{T^2}\ell m \sin(\beta) = \frac{mg}{\cos\beta}\sin\beta \tag{3.89}$$
$$\frac{4\pi^2}{T^2}\ell m = \frac{mg}{\cos\beta} \tag{3.90}$$
$$\frac{2\pi\sqrt{\ell cos\beta}}{\sqrt{g}} = T \tag{3.91}$$

Tension cannot be negative, so we ignore the second solution.

3.10 Problem 10: Stacked blocks 2

"A block of mass $m_B = 15\,\text{kg}$ is on top of a long slab of mass $m_S = 9\,\text{kg}$, and the slab is on top of a horizontal table as shown. A horizontal force of magnitude $F = 294\,\text{N}$ is applied on the block. As a result the block moves relative to the slab and the slab moves relative to the table. There is friction between all surfaces. The coefficient of kinetic friction between the block and the slab is $\mu_1 = 0.7$, and the coefficient of kinetic friction between the slab and the table is $\mu_2 = 0.1$. Take g to be $9.81\,\text{m/s}^2$, and enter your answer to 3 significant figures.



(a) What is the magnitude of the block's acceleration?
(b) What is the magnitude of the slab's acceleration?"

I guessed they saved the best for last! I was expecting to see something like this (friction in two places) on the exam, but not quite on the homework.

Ah well, let's get to it. The free-body diagram comes first, as always. Lots of forces; let's start with the vertical ones, since the frictional forces depend on the normal forces.
The block and a slab each have a gravitational force downwards, and a normal force upwards; I'll denote these by N_B for the normal force *on* the block (by the slab), and N_S for the normal force on the slab (by the table):

$$N_B = m_B g \tag{3.92}$$
$$N_S = g(m_B + m_S) \tag{3.93}$$

This then gives us the frictional forces F_{F1} (friction that limits the block's movement) and F_{F2} (friction that limits the slab's movement), named after the friction coefficients in the problem description:

$$F_{F1} = \mu_1 m_B g \tag{3.94}$$
$$F_{F2} = \mu_2 g(m_B + m_S) \tag{3.95}$$

What is the direction of these forces? Some books appear to write this badly, but here's a quote from (Serway & Jewett, 2010, p. 123):

> Sometimes, an incorrect statement about the friction force between an object and a surface is made – "the friction force on an object is opposite to its motion or impending motion" – rather than the correct phrasing, "the friction force on an object is opposite to its motion or impending motion *relative to the surface*."

So in other words, since the slab moves to the right relative to the table, the friction force there is to the left.
The block should also move right relative to the slab (how could the slab possibly accelerate *faster*?), so that frictional force should also be to the left.

Do we now have all the forces? We have covered the y axis with gravitational forces and normal forces, and friction on all surfaces. Left are the third law reaction forces due to friction.

Because there is a frictional force F_{F1} by the slab (middle) on the block (top), there must be a force of equal magnitude in the opposite direction on the slab, so we have a rightwards force F_{F1} on the slab that we must not forget about.
There is also a leftwards frictional force on the slab from the table, so there is a reaction force there too, but since it's on the table, which we take to be immovable, we can ignore that force.

All in all we have, ignoring vertical forces, on the block: the external force F to the right, friction F_{F1} to the left.
On the slab, we have a reaction force F_{F1} to the *right*, and "regular" friction with the table F_{F2} towards the left.

Let's also not forget that they don't accelerate together; the forces add up to some $m_B a_B$ and $m_S a_S$, but we can't solve for a combined a.

We can finally start writing second law equations, and substituting in the actual values. I will take $+x$ to be towards the right. First the block, then the slab:

$$F - F_{F1} = m_B a_B \Rightarrow F - \mu_1 m_B g = m_B a_B \tag{3.96}$$
$$F_{F1} - F_{F2} = m_S a_S \Rightarrow \mu_1 m_B g - \mu_2 g(m_B + m_S) = m_S a_S \tag{3.97}$$

Two equations, two unknowns (the accelerations), how unusual! However, they don't depend on each other at all, so this should be simple! Let's solve them one at a time:

$$F - \mu_1 m_B g = m_B a_B \tag{3.98}$$

$$a_B = \frac{F - \mu_1 m_B g}{m_B} \approx 12.733\,\mathrm{m/s^2} \tag{3.99}$$

$$\mu_1 m_B g - \mu_2 g(m_B + m_S) = m_S a_S \tag{3.100}$$

$$a_S = \frac{\mu_1 m_B g - \mu_2 g(m_B + m_S)}{m_S} \approx 8.829\,\mathrm{m/s^2} \tag{3.101}$$

Nice!

Chapter 4: Week 4: No homework

There was no homework this week, due to the midterm exam.

Chapter 5: Week 5: Homework 4

5.1 Problem 1: Oil drop

"We release an oil drop of radius r in air. The density of the oil is $670\,\mathrm{kg/m^3}$. C_1 and C_2 for 1 atmosphere air at 20° C are $2.90 \times 10^{-4}\,(\mathrm{kg/m})/\mathrm{s}$ and $0.82\,\mathrm{kg/m^3}$, respectively.

How small should the oil drop be so that the drag force is dominated by the linear term in the speed (in lectures we called this Regime I). In this regime, the terminal velocity is $(mg)/(C_1 r)$. [m is the mass of the drop].

$r \ll ...$"

Well, there clearly isn't a truly "correct" answer here (estimates will vary), but we can use the definitions we have seen previously, which are that $v \gg v_{crit}$ means regime II, and $v \ll v_{crit}$ means regime I. The critical velocity v_{crit} is when the force from each term is equivalent, which is at

$$C_1 r v_{crit} = C_2 r^2 v_{crit}^2 \tag{5.1}$$

$$C_1 = C_2 r v_{crit} \tag{5.2}$$

$$v_{crit} = \frac{C_1}{C_2 r} \approx \frac{3.54 \times 10^{-4}}{r}\ \mathrm{m/s} \tag{5.3}$$

The condition is then that the velocity is much, much smaller than this. Let's set up the terminal velocity in regime I (since the condition is that we must be way inside regime I, and the terminal velocity is the maximum one possible) one the left hand side of an equality, with the critical velocity on the other:

$$\frac{mg}{C_1 r} \ll \frac{C_1}{C_2 r} \tag{5.4}$$

$$\frac{4}{3}\pi\rho r^3 \frac{g}{C_1 r} \ll \frac{C_1}{C_2 r} \tag{5.5}$$

$$r^3 \ll \frac{3C_1^2}{4\pi\rho g C_2} \tag{5.6}$$

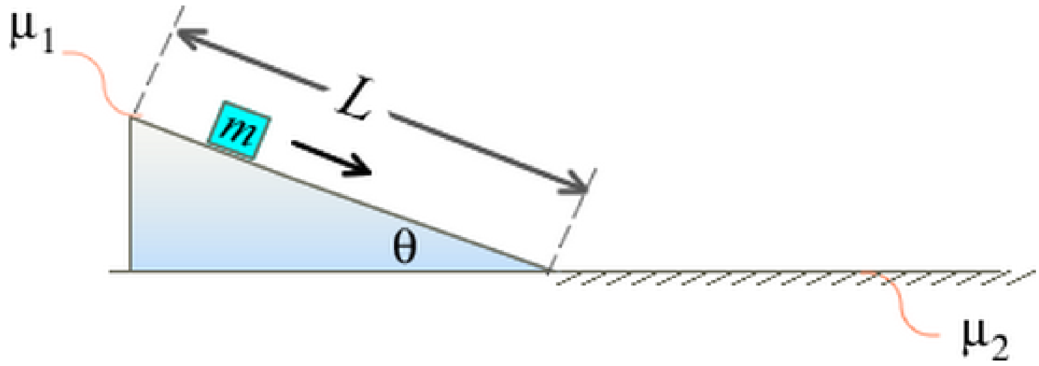
$$r \ll \left(\frac{3C_1^2}{4\pi\rho g C_2}\right)^{1/3} \tag{5.7}$$

$$r \ll 1.54 \times 10^{-4}\,\mathrm{m} \tag{5.8}$$

That's very small! 0.154 mm, though the condition isn't just smaller than that, but *much, much* smaller.

5.2 Problem 2: Rough surfaces

"An block of mass m, starting from rest, slides down an inclined plane of length L and angle θ with respect to the horizontal. The coefficient of kinetic friction between the block and the inclined surface is μ_1. At the bottom of the incline, the block slides along a horizontal and rough surface with a coefficient of kinetic friction μ_2. The goal of this problem is to find out how far the block slides along the rough surface.



(a) What is the work done by the friction force on the block while it is sliding down the inclined plane?
(b) What is the work done by the gravitational force on the block while it is sliding down the inclined plane?
(c) What is the kinetic energy of the block just at the bottom of the inclined plane?
(d) After leaving the incline, the block slides along the rough surface until it comes to rest. How far has it traveled?
Express your answers in terms of g, m, L, θ, μ_1 and μ_2."

Now that we've learned about the conservation of mechanical energy, this problem should be easier to solve than it would be with basic kinematics and friction equations. The work done by gravity should be very easy to find: the work done by gravity is the change in potential energy, which is mgh if we define h to be the height at which the block starts out, and $y = 0$ to be at the ground, so that $U = 0$ there.

We thus need to find h. The illustration makes it look a bit as if the block starts a bit down the ramp, but I assume it travels the distance L, or this would be hard to solve indeed! Via trigonometry, $\sin\theta = h/L$ so $h = L\sin\theta$. That gives us, for the work done by gravity,

$$W_g = mgL\sin\theta \tag{5.9}$$

... which answers part (b).
Next, we must find the work done by frictional forces as the block slides down. The magnitude of that force is

$$|F_f| = \mu_1 N = \mu_1 mg\cos\theta \tag{5.10}$$

We decompose the normal force, since gravity is straight downwards, while the block is on an incline.
Since the force is constant, and work is force times distance, we can find the work easily as $W_f = |F_f|L$. However, let's keep track of the signs here! The frictional force is always opposing the motion relative to the surfaces, so it is "backwards" (to the left) while the block only moves to the right. Therefore, the work is negative:

$$W_f = -(|F_f|L) = -\mu_1 mgL\cos\theta \tag{5.11}$$

... which answers part (a).
Next up is then the kinetic energy of the block as it has just reached the bottom (or end) of the incline. The kinetic energy started out at zero, and must now be at a maximum (since the potential energy is $U = 0$ at the bottom, by our definition). Without friction, it would be equal to the work gravity has done, but we must now add the work done by friction (subtract, in a way, since it is negative, but I prefer "add" to avoid confusion; subtracting a negative would give a larger value, which is clearly incorrect!).

$$K = W_f + W_g = mgL\sin\theta - \mu_1 mgL\cos\theta \tag{5.12}$$
$$= mgL(\sin\theta - \mu_1\cos\theta) \tag{5.13}$$

The work-energy theorem at work... no pun intended.

Finally, part (d): how long does the block slide on the rough surface? It has a certain amount of kinetic energy, above; friction uses up a constant amount per unit length traveled, since it is constant at $\mu_2 N = \mu_2 mg$ (since the surfaces are now horizontal).

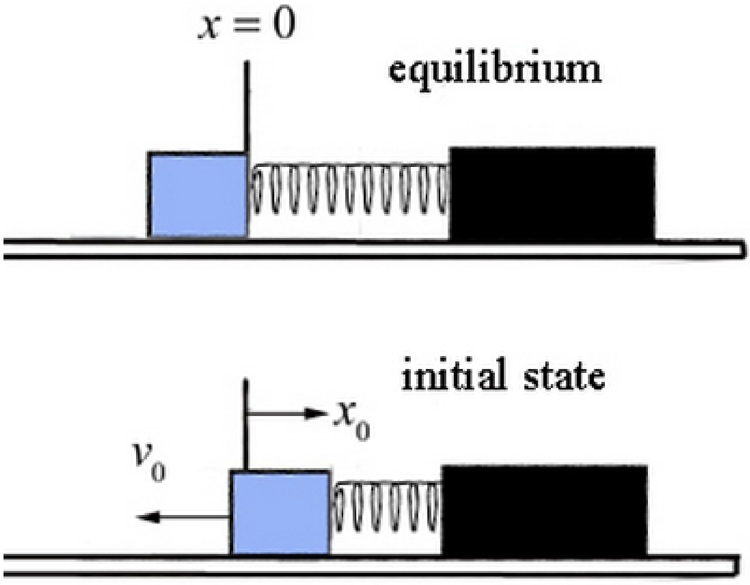
Using d for the distance traveled, the work done by friction is then $\mu_2 mgd$ ($W = Fd$). That work equals the initial kinetic energy, so we set them equal and solve for d:

$$\mu_2 mgd = mgL(\sin\theta - \mu_1\cos\theta) \tag{5.14}$$
$$d = \frac{L(\sin\theta - \mu_1\cos\theta)}{\mu_2} \tag{5.15}$$

That's all!

5.3 Problem 3: Oscillating block

"Consider an ideal spring that has an unstretched length $\ell_0 = 3.1\,\text{m}$. Assume the spring has a constant $k = 36\,\text{N/m}$. Suppose the spring is attached to a mass $m = 7\,\text{kg}$ that lies on a horizontal frictionless surface. The spring-mass system is compressed a distance of $x_0 = 1.8\,\text{m}$ from equilibrium and then released with an initial speed $v_0 = 3\,\text{m/s}$ toward the equilibrium position.



(a) What is the period of oscillation T for this system?
(b) What is the position of the block as a function of time. Express your answer in terms of t.
(c) How long will it take for the mass to first return to the equilibrium position?
(d) How long will it take for the spring to first become completely extended?"

Since the spring is ideal, Hooke's law holds, and we can use the equations we found in lecture, by solving a differential equation for this simple harmonic oscillator. The equation we found was

$$x(t) = A\cos(\omega t + \varphi) \tag{5.16}$$

where A is the amplitude in meters, ω the angular frequency in radians/second, and φ the phase angle in radians. A and φ are found from the initial conditions, while ω can be found as

$$\omega = \sqrt{\frac{k}{m}} \tag{5.17}$$

The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{7}{36}} \approx 2.77\,\mathrm{s} \tag{5.18}$$

To find the position as a function of time, we need to find the amplitude and the phase, by using the initial conditions. At $t = 0$, $x(0) = x_0 = 1.8$ meters, as given in the problem. We substitute those values into the $x(t)$ equation:

$$x_0 = A\cos(\varphi) \tag{5.19}$$

That only gets us so far, since there are two unknowns, A and φ. We can find a second equation in taking the time derivative of $x(t)$ to find $v(t)$, though, since we know the initial velocity.

$$v(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t + \varphi) \tag{5.20}$$

At $t = 0$, this should be equal to -3 (if x_0 is positive, then $+\hat{x}$ is towards the right, but v_0 is towards the left). Combined with the equation for $x(t)$, we have these two equations:

$$x_0 = A\cos(\varphi) \tag{5.21}$$
$$-v_0 = -A\omega\sin(\varphi) \tag{5.22}$$

$$-\frac{x_0}{v_0} = -\frac{\cos(\varphi)}{\omega\sin(\varphi)} \tag{5.23}$$
$$\omega\frac{x_0}{v_0} = \frac{1}{\tan\varphi} \tag{5.24}$$
$$\arctan\frac{v_0}{\omega x_0} = \varphi \approx 0.6338 \text{ rad} \approx 36.31^\circ \tag{5.25}$$

Solving for A should now be dead simple, using the equation $x_0 = A\cos(\varphi)$:

$$1.8 = 0.80578A \tag{5.26}$$
$$A = 2.23\,\mathrm{m} \tag{5.27}$$

ω, using the formula above, is about 2.2678 rad/s, so all in all, the formula for $x(t)$ is

$$x(t) = 2.23\cos(2.2678t + 0.6338) \tag{5.28}$$

Evaluated at $t = 0$, this equals 1.7969 m, and the problem states $x_0 = 1.8\,\mathrm{m}$ – close enough; it's clearly due to rounding errors.

"(c) How long will it take for the mass to first return to the equilibrium position?"

That happens when $x(t) = 0$, so we set it up and solve for t:

$$2.33\cos(2.2678t + 0.6338) = 0 \tag{5.29}$$
$$2.2678t + 0.6338 = \frac{\pi}{2} \text{ (by taking the arccosine of both sides)} \tag{5.30}$$
$$t = \frac{\pi/2 - 0.6338}{2.2678} \approx 0.413\,\mathrm{s} \tag{5.31}$$

"(d) How long will it take for the spring to first become completely extended?"

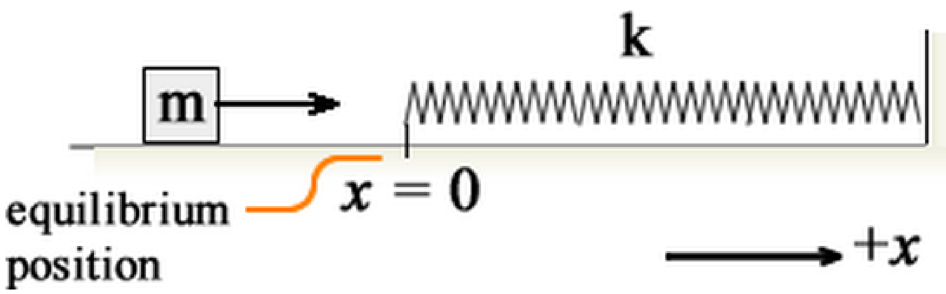
I assume that by "completely extended", they mean when it is as long as it will ever become – since it is at its natural length at $x = 0$, which is what we found above. Since the initial velocity is in the "extending direction", this should happen the first time $v = 0$, so let's set the derivative, which we found earlier, equal to zero:

$$-A\omega\sin(\omega t + \varphi) = 0 \tag{5.32}$$
$$-2.33 \cdot 2.2678\sin(2.2678t + 0.6338) = 0 \tag{5.33}$$
$$\sin(2.2678t + 0.6338) = 0 \tag{5.34}$$
$$2.2678t + 0.6338 = \pi \text{ (by taking the arcsine of both sides)} \tag{5.35}$$
$$t = \frac{\pi - 0.6338}{2.2678} \approx 1.106\,\mathrm{s} \tag{5.36}$$

I chose π instead of 0 for the arcsine because choosing 0 yields a negative time, which is clearly incorrect. Honestly, I'm not completely happy with this solution, but it worked, at least.

5.4 Problem 4: Spring block with friction

"A block of mass $m = 4\,\mathrm{kg}$ slides along a horizontal table when it encounters the free end of a horizontal spring of spring constant $k = 16\,\mathrm{N/m}$. The spring is initially on its equilibrium state, defined when its free end is at $x = 0$ in the figure. Right before the collision, the block is moving with a speed $v_i = 4\,\mathrm{m/s}$. There is friction between the block and the surface. The coefficient of friction is given by $\mu = 0.83$. How far did the spring compress when the block first momentarily comes to rest? Take $g = 10\,\mathrm{m/s^2}$."



This problem can be conceptualized similarly to problem 2, i.e. conservation of energy. The block has an initial kinetic energy of $K = \frac{1}{2}mv_i^2 = 32$ joule; by definition, that kinetic energy must go down to 0 when $v = 0$, which is of course when it first comes to a halt. Part of the kinetic energy will be eaten up by friction (turned into heat, mostly), and part will be transferred into the spring and stored there as potential energy.

The kinetic friction force is $\mu N = \mu mg$, which is constant regardless of position or velocity; the direction is opposite the motion, so to the left here, $-\hat{x}$. The spring's force is $-kx\,\hat{x}$, also to the left.

The work done by the forces together equals the sum of the forces times the distance x the block travels; this work then equals the initial kinetic energy of the block. After having set the two equal, we can solve for x, which is how far the spring has compressed (and how far the block has traveled, after the "collision" with the spring). We can either set the sum of them equal to zero, or set the two work quantities equal, which is the same thing. I chose the latter:

$$\frac{1}{2}mv_i^2 = x\left(\mu mg + kx\right) \tag{5.37}$$

Ah, but here's a snag: kx, the force from the spring, is not constant! It is 0 at the start, kx only at the end of the motion, and somewhere in between for the rest of the time. *However*, it is linear, which is good news for us! That means we can find the average force simply as $\frac{kx}{2}$, and keep going, with no calculus:

$$\frac{1}{2}mv_i^2 = x\left(\mu mg + 0.5kx\right) \tag{5.38}$$

$$\frac{1}{2}mv_i^2 = x\mu mg + 0.5kx^2 \tag{5.39}$$

$$\frac{mv_i^2}{k} = 2x\frac{\mu mg}{k} + x^2 \tag{5.40}$$

$$x^2 + \frac{2\mu mg}{k}x - \frac{mv_i^2}{k} = 0 \tag{5.41}$$

$$\tag{5.42}$$

Using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$x = -\frac{\mu mg}{k} \pm \frac{\sqrt{\left(\frac{2\mu mg}{k}\right)^2 + \frac{4mv_i^2}{k}}}{2} \tag{5.43}$$

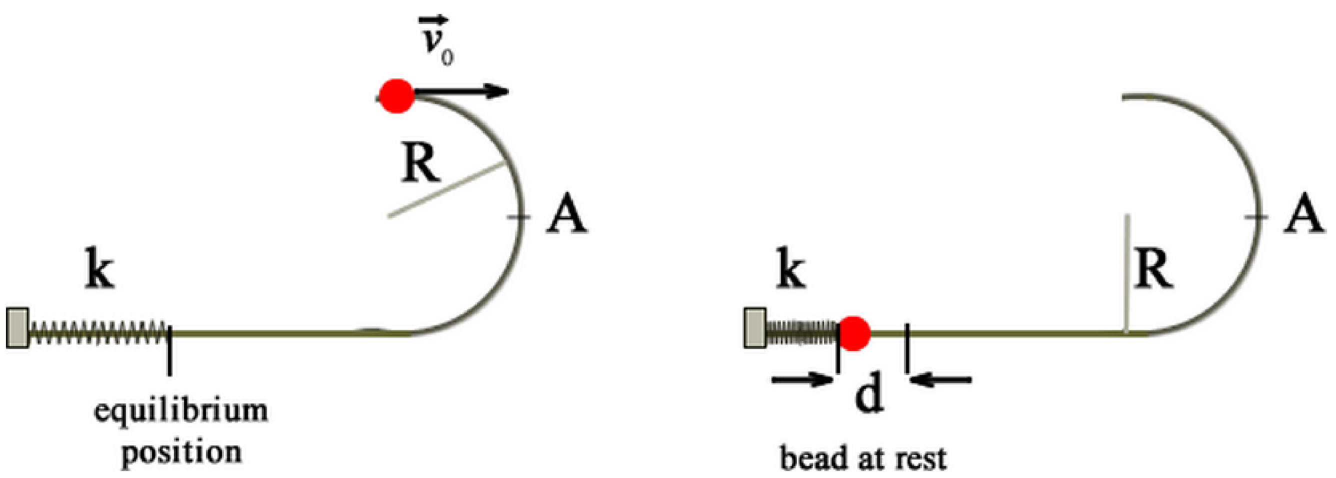
If we stick some values into that mess, we find

$$x = -2.075 \pm 2.88195 \tag{5.44}$$

Since the answer is clearly positive as defined in the problem, it must be $x = -2.075 + 2.88195 = 0.80695 \approx 0.807\,\text{m}$.

5.5 Problem 5: Half loop

"A small bead of mass m is constrained to move along a frictionless track as shown. The track consists of a semicircular portion of radius R followed by a straight part. At the end of the straight portion there is a horizontal spring of spring constant k attached to a fixed support. At the top of the circular portion of the track, the bead is pushed with an unknown speed v_0. The bead comes momentarily to rest after compressing the spring a distance d. The magnitude of the acceleration due to the gravitational force is g.



What is the magnitude of the normal force exerted by the track on the bead at the point A, a height R above the base of the track? Express your answer in terms of m, k, R, d, and g but *not* in terms of v_0."

Okay, let's see. There is no friction, so we should be able to rely on conservation of energy to find the initial velocity from the spring's compression. It is compressed a distance d, with a spring constant k. Now, unfortunately, I don't know how to calculate the stored potential energy in a spring; it's a common formula, easy to find – but I would prefer to figure it out myself! Looking up a formula doesn't teach you much, but deriving it yourself can be very helpful indeed, especially if you've never seen it before.

So, let's take a sidestep for a moment.

5.5.1 Potential energy stored in a spring

Spring forces are conservative, so the amount of work done in compressing a spring should equal the amount of potential energy stored in it. We need to exert a force $F(t) = kx(t)$ to compress a spring, where $x(t)$ is the amount we have compressed it so far. The total work done, and the total energy stored, must therefore be the integral of this:

$$U_{spring} = \int_0^d F(t)\,dx = \int_0^d kx\,dx = k\left[\frac{x^2}{2}\right]_0^d = k\frac{d^2}{2} = \frac{1}{2}kd^2 \tag{5.45}$$

Neat! It looks a lot like the equation for kinetic energy (and many other equations in physics, for that matter).

5.5.2 Back to the problem

Now that we know how much energy is stored in the spring when the bead comes to a temporary halt, before being "shot out" again, we can find v_0, in case we need it later. The energy stored in the spring must come exclusively from the bead's kinetic energy (some of which come from gravity). If we define gravitational potential energy as 0 at the bottom, then it must be be $2mgR$ at the top of the loop.

The spring starts out with no stored energy, while the bead starts out with its kinetic energy $K_E = \frac{1}{2}mv_0^2$ and its gravitational potential energy $2mgR$. Since there is no friction or other resistive forces, the sum of all these must be conserved.

The speed at point A can be found by finding the bead's kinetic energy at that point, which is the sum of its initial kinetic energy and potential energy, minus the energy used up working against gravity, mgR, to reach point A:

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_0^2 + 2mgR - mgR \tag{5.46}$$

$$mv_A^2 = mv_0^2 + 2mgR \tag{5.47}$$

$$v_A = \sqrt{v_0^2 + 2gR} \tag{5.48}$$

We can find v_0. When the bead has compressed the spring fully, all of the initial kinetic energy plus all of the gravitational potential energy is now stored in the spring, so we can equate them:

$$\frac{1}{2}mv_0^2 + 2mgR = \frac{1}{2}kd^2 \tag{5.49}$$

$$mv_0^2 + 4mgR = kd^2 \tag{5.50}$$

$$v_0^2 = \frac{kd^2 - 4mgR}{m} \tag{5.51}$$

$$v_0 = \sqrt{\frac{kd^2 - 4mgR}{m}} \tag{5.52}$$

v_0^2 is what we need to find v_A, however:

$$v_A = \sqrt{\frac{kd^2 - 4mgR}{m} + 2gR} \tag{5.53}$$

Almost there! Now that we know the speed at A, we can apply the formula for centripetal acceleration, $|a_c| = \frac{v^2}{r}$, and then multiply by the mass m to find the centripetal force.

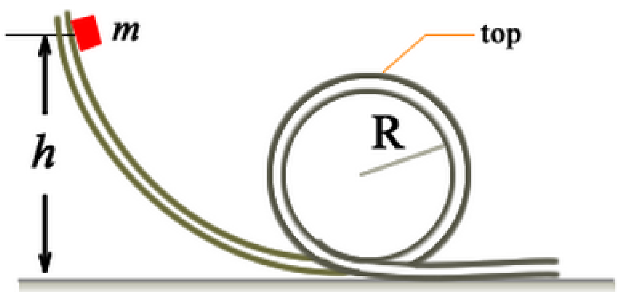
The normal force from the track is the only possible source for this centripetal force, which is necessary for the bead to move along the (semi)circular track. Therefore, we find the centripetal force:

$$N = m\frac{v_A^2}{R} = \frac{m}{R}\left(\frac{kd^2 - 4mgR}{m} + 2gR\right) \tag{5.54}$$

and that solves the problem!

5.6 Problem 6: Full loop

"An object of mass m is released from rest at a height h above the surface of a table. The object slides along the inside of the loop-the-loop track consisting of a ramp and a circular loop of radius R shown in the figure. Assume that the track is frictionless.



When the object is at the top of the loop it barely loses contact with the track. What height h was the object released from? Express your answer in terms of some or all of the given variables m, g, and R."

Well... Unless I'm missing something, I remember the answer from lecture! I'll still try to re-derive it, though, to make sure I fully understand the problem. If I do, this shouldn't take long.

Okay, so the track is frictionless, and we can use conservation of energy to simplify things. Since the object is released from rest, its initial potential energy is mgh, assuming $U = 0$ at $y = 0$; since that is my choice to make, I decide it shall be so.

When entering the loop, the potential energy is zero, and the object's speed is at a maximum, as is the kinetic energy. It then travels up $2R$ against gravity, which causes it to lose kinetic energy again.

Let's first find the condition for the object not falling down at the middle of the loop. $|a_c| > g$ must be the case, or the object will not move in a circle. This puts a constraint on v_{top}, the speed at the top:

$$a_{c,top} = \frac{v_{top}^2}{R} \geq g \tag{5.55}$$

Next, we need to figure out what v_{top} is, as a function of the initial height h. At that height, it will have a potential energy of $mg2R$, which is smaller than the mgh it begins with (or it will never reach that point).

$$\frac{1}{2}mv_{top}^2 = mgh - 2mgR \tag{5.56}$$
$$v_{top}^2 = 2gh - 4gR \tag{5.57}$$
$$v_{top} = \sqrt{2g(h-2R)} \tag{5.58}$$

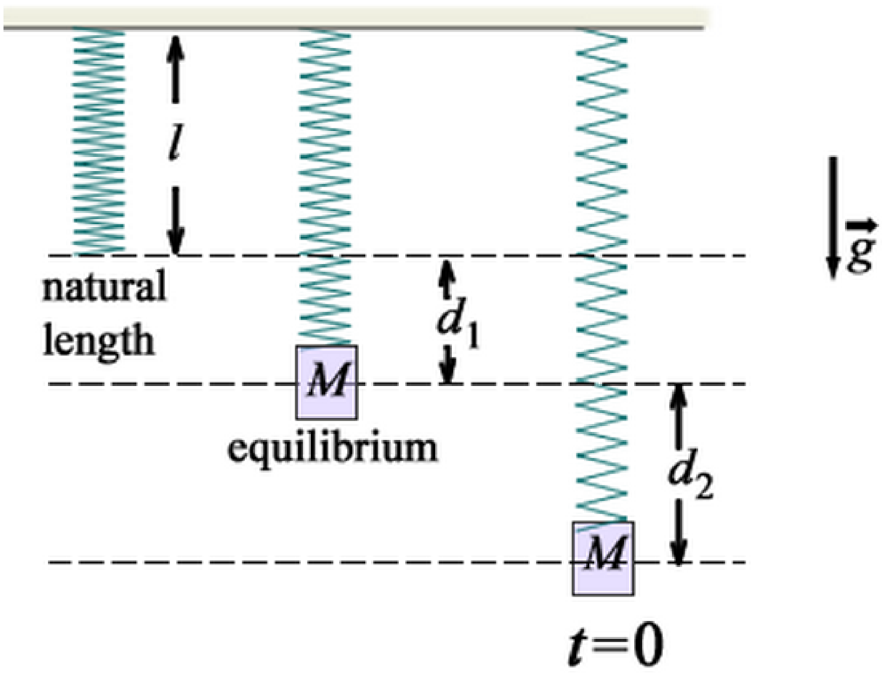
Now we just need to put the two together, and solve for h.

$$\frac{2gh - 4gR}{R} \geq g \tag{5.59}$$
$$2gh \geq 5Rg \tag{5.60}$$
$$h \geq \frac{5}{2}R \tag{5.61}$$

Since the question is when it "just barely" loses contact, the answer is $h = \frac{5}{2}R$.

5.7 Problem 7: Vertical spring

"A spring of negligible mass, spring constant $k = 99\,\mathrm{N/m}$, and natural length $\ell = 1.3\,\mathrm{m}$ is hanging vertically. This is shown in the left figure below where the spring is neither stretched nor compressed. In the central figure, a block of mass $M = 2\,\mathrm{kg}$ is attached to the free end. When equilibrium is reached (the block is at rest), the length of the spring has increased by d_1 with respect to ℓ. We now lower the block by an additional $d_2 = 0.4\,\mathrm{m}$ as shown in the right figure below. At $t = 0$ we release it (zero speed) and the block starts to oscillate. Take $g = 9.81\,\mathrm{m/s^2}$.



(a) Find d_1.
(b) What is the frequency (Hz) of the oscillations?
(c) What is the length of the spring when the block reaches its highest point during the oscillations?
(d) What is maximum speed of the block?"

I'll start off by finding d_1, not only because it's the first question, but because it should be independent of everything else.
For this problem, I choose a coordinate system of one axis, y, which is positive downwards, and has its

origin at the spring's natural length. In other words, $y = +d_1$ when the system is at equilibrium with the mass.

Since it is in equilibrium, with the spring force upwards, and gravity downwards, with no acceleration:

$$d_1 k = mg \tag{5.62}$$
$$d_1 = \frac{mg}{k} \approx 0.19818 \text{ m} \tag{5.63}$$

Now then, onto the rest of the problem. I will use the same coordinate system, by the way.

In a horizontal oscillator (as in lecture), there is only one horizontal force, which is that of the spring. I know (from a quick and dirty test) that the period is the same for this vertical oscillator, but how can we show that to be the case, now that gravity is present along the oscillating axis? If this were an exam question, I would *not* have wasted a try on that assumption!

We can actually show that this system is equivalent to the horizontal one.
We've just shown that the "new" equilibrium position is at $y = d_1$. However, we can re-define y instead, so that $y = 0$ at that point. Why? Because the block will oscillate around that point, moving equal amounts up as down from the new zero point, which is not the case for the old one. In other words, we will get a symmetrical problem if we change the zero point, so we do just that.

The spring force is upwards, in magnitude $k(d_1 + y)$ in this case, now. At $y = 0$, it should be kd_1, and for greater values of y (further down), it should be greater, so that looks about right. Gravity is mg, always downwards. Putting this all together, $a = \ddot{y}$ being positive downwards, we set $m\ddot{y}$ equal to the net force, adding the downwards force (gravity) and subtracting the upwards force (spring force):

$$m\ddot{y} = mg - k(d_1 + y) \tag{5.64}$$

However, note that since $d_1 = \frac{mg}{k}$, $mg = kd_1$, we can replace mg by kd_1:

$$m\ddot{y} = kd_1 - k(d_1 + y) \tag{5.65}$$
$$m\ddot{y} = -ky \tag{5.66}$$
$$\ddot{y} = -\frac{k}{m}y \tag{5.67}$$
$$\ddot{y} + \frac{k}{m}y = 0 \tag{5.68}$$

A-ha! This is clearly the exact same differential equation we had earlier in lecture, only we call the axis y instead of x, so we can safely use the same solutions! That it,

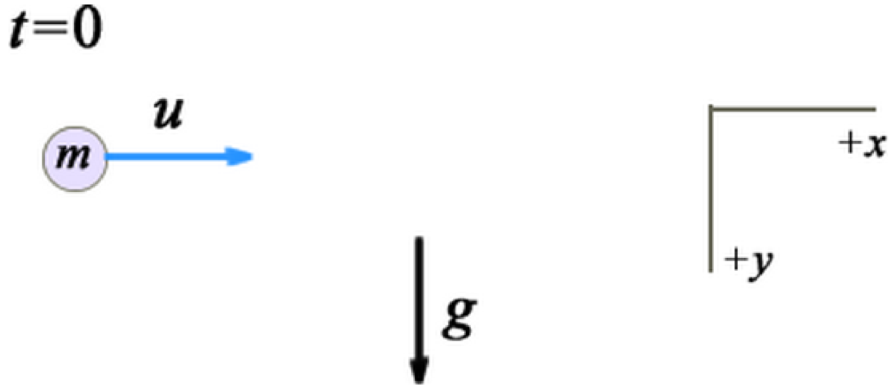
$$\omega = \sqrt{\frac{k}{m}} \tag{5.69}$$
$$T = 2\pi\sqrt{\frac{m}{k}} \tag{5.70}$$
$$y = A\cos(\omega t + \varphi) \tag{5.71}$$
$$\dot{y} = -A\omega\sin(\omega t + \varphi) \tag{5.72}$$
$$f = \frac{\omega}{2\pi} \tag{5.73}$$

We have already solved (a), so let's calculate the frequency for part (b). Using the above formulas, we find $\omega \approx 7.03562$ rad/s, so $f \approx 1.12$ Hz.

Next, the spring's length when the block reaches its highest point. The amplitude of the oscillation is d_2, the amount we extended it from the (new, with the mass) equilibrium point, so the answer is the spring's original length plus d_1, which is the new equilibrium point, minus the amplitude d_2. All in all, $\ell_{top} = \ell + d_1 - d_2 \approx 1.098\,\text{m}$.

Finally, the maximum speed of the block. The velocity is given by $\dot{y}(t)$ above, which is clearly maximized when the sine function is 1. We don't care when that happens, only that the speed at that point is the magnitude of the function's value when the sine term is 1, i.e. $A\omega = d_2\omega \approx 2.814\,\text{m/s}$, and that's it for this question!

5.8 Problem 8: Drag force at low speeds



"At low speeds (especially in liquids rather than gases), the drag force is proportional to the speed rather than its square, i.e., $\vec{F} = -C_1 r\vec{v}$, where C_1 is a constant. At time $t = 0$, a small ball of mass m is projected into a liquid so that it initially has a horizontal velocity of u in the +x direction as shown. The initial speed in the vertical direction (y) is zero. The gravitational acceleration is g. Consider the Cartesian coordinate system shown in the figure (+x to the right and +y downwards).

Express the answer of the following questions in terms of some or all of the variables C_1, r, m, g, v_x, v_y, u and t:

(a) What is component of the acceleration in the x direction as a function of the component of the velocity in the x direction v_x? express your answer in terms of v_x, C_1, r, g, m and u as needed.
(b) What is the acceleration in the y direction as a function of the component of the velocity in the y direction v_y? express your answer in terms of v_y, C_1, r, g, m and u as needed.
(c) Using your result from part (a), find an expression for the horizontal component of the ball's velocity as a function of time t? Express your answer in terms of C_1, r, g, m, u and t as needed (enter e^(-z) for $\exp(-z)$).
(d) Using your result from part (b), find an expression for the vertical component of the ball's velocity as a function of time t? Express your answer in terms of C_1, r, g, m, u and t as needed: (Enter e^(-z) for $\exp(-z)$).
(e) How long does it take for the vertical speed to reach 99% of its maximum value? express your answer in terms of C_1, r, g, m and u as needed.
(f) What value does the horizontal component of the ball's velocity approach as t becomes infinitely large? express your answer in terms of C_1, r, g, m and u as needed.
(g) What value does the vertical component of the ball's velocity approach as t becomes infinitely large? express your answer in terms of C_1, r, g, m and u as needed."

Wow! Okay, let's get started, I suppose. I will most likely use Mathematica for parts of this.

Anyway. The initial velocity in the x direction is u, a given, while that in the y direction is 0. In the x direction, there is only the resistive force, acting towards the left, opposing the motion.

In the y direction, there is gravity pulling the mass downwards, and a resistive force *upwards*, slowing the "fall".

Newton's second law in the x direction:

$$ma_x = m\ddot{x} = -F_{res_x} = -C_1 r v_x \tag{5.74}$$

Any in the y direction, with downwards defined as positive in the problem:

$$ma_y = m\ddot{y} = mg - F_{res_y} = mg - C_1 r v_y \tag{5.75}$$

Well, at least they ask for stuff in one direction at a time, so we don't need to worry too much about having *two* differential equations. Still, rearranged, they are

$$\ddot{x} + \frac{C_1 r}{m} v_x = 0 \tag{5.76}$$

$$\ddot{y} + \frac{C_1 r}{m} v_y - g = 0 \tag{5.77}$$

Oh! This actually answers parts (a) and (b), I almost didn't notice. Just move everything except the $\ddot{x}$ or $\ddot{y}$ terms to the right-hand side, and those are the answers.

Next, they ask us for the velocity, as a function of *time*. I think this means solving the differential equation, and then taking the derivative once (since the solution gives us the position).

Note that the differential equations involve $\ddot{x}$ and $\dot{x}$, or $\ddot{y}$ and $\dot{y}$, respectively, only we call the latter two v_x and v_y, respectively. Therefore, these equations are *not* identical to ones we've seen previously – and they shouldn't be, as there clearly will be no oscillating motion in this case!

As far as I can tell, we have two second-order linear differential equations with constant coefficients. Honestly, I stuck this into Mathematica with $x(0) = x_0$ and $\dot{x}(0) = u$ as boundary conditions, differentiated the answer, and got

$$\dot{x} = u \exp\left(-\frac{C_1 r t}{m}\right) \tag{5.78}$$

... which is marked as correct, and certainly looks plausible. I expected an exponential as a solution, and a decaying exponential that starts out at the initial velocity surely is a reasonable solution.

Next, onto the y direction. Same deal, solve differential equation with boundary conditions ($y(0) = y_0$, $\dot{y}(0) = 0$), differentiate, simplify:

$$\dot{y} = \frac{mg}{C_1 r}\left(1 - \exp\left(-\frac{C_1 r t}{m}\right)\right) \tag{5.79}$$

With the one minus the exponential there, we get an equation that grows up to $\frac{mg}{C_1 r}$ after a certain time, dictated by the term $\frac{C_1 r}{m}$.

For part (e), we can set the multiplying term, $1 - \exp(...)$ equal to 0.99 and solve for t to find the answer:

$$1 - \exp\left(-\frac{C_1 r t}{m}\right) = 0.99 \tag{5.80}$$

$$-\frac{C_1 r t}{m} = \ln(0.01) \tag{5.81}$$

$$\frac{C_1 r t}{m} = \ln(100) \tag{5.82}$$

The last step is because $\ln(0.01) = \ln(1/100) = \ln(1) - \ln(100) = -\ln(100)$, and the minus signs then cancel from both sides. Moving on...

$$t = \frac{m \ln(100)}{C_1 r} \tag{5.83}$$

And finally, the limits of the velocities as t becomes infinitely large. We don't need no calculus for this one, just common sense. The x velocity has a force opposing the motion, proportional to the speed. It will reduce the motion until there is none, and then disappear; so $v_x = 0$ as $t \to \infty$.

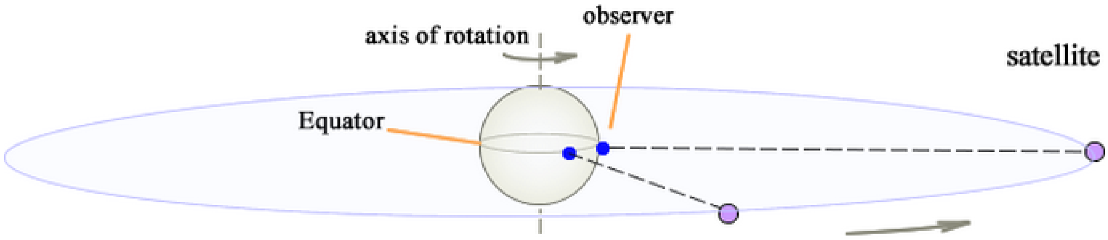
As for v_y, the "multiplying term" above goes to 1 as $t \to \infty$, so the velocity goes to $\dfrac{mg}{C_1 r}$.

That's it for this week!

Chapter 6: Week 6: Homework 5

6.1 Problem 1: Geosynchronous orbit

A satellite with a mass of $m_s = 3 \times 10^3$ kg is in a planet's equatorial plane in a circular "synchronous" orbit. This means that an observer at the equator will see the satellite being stationary overhead (see figure below). The planet has mass $m_p = 5.16 \times 10^{25}$ kg and a day of length $T = 0.7$ earth days (1 earth day = 24 hours).



(a) How far from the center (in m) of the planet is the satellite?
(b) What is the escape velocity (in km/sec) of any object that is at the same distance from the center of the planet that you calculated in (a)?"

The day's length is $0.7 \cdot 24$ hours = 16.8 hours, or 60480 seconds. This must then be the orbital period of the satellite, since it is supposed to remain over the same point at all times.
I don't recall the exact formulas we learned from lecture (and if I did, I likely wouldn't a year from now), but I do remember that the total mechanical energy is exactly $\frac{1}{2}U$. The mechanical energy is then the sum of the current kinetic energy, and the gravitational potential energy:

$$\frac{1}{2}\left(-\frac{Gm_pm_s}{r}\right) = \frac{1}{2}m_sv_{orb}^2 - \frac{Gm_pm_s}{r} \tag{6.1}$$
$$\frac{Gm_pm_s}{r} = m_sv_{orb}^2 \tag{6.2}$$
$$\frac{1}{r} = \frac{v_{orb}^2}{Gm_p} \tag{6.3}$$
$$r = \frac{Gm_p}{v_{orb}^2} \tag{6.4}$$

We can then write v_{orb}, the tangential velocity of the satellite, in terms of r and T:

$$v_{orb} = \frac{2\pi r}{T} \tag{6.5}$$
$$v_{orb}^2 = \frac{4\pi^2r^2}{T^2} \tag{6.6}$$

Substitute into r (by multiplying by the reciprocal, instead of having a 3-layer fraction):

$$r = Gm_p \cdot \frac{T^2}{4\pi^2r^2} \tag{6.7}$$
$$r^3 = \frac{Gm_pT^2}{4\pi^2} \tag{6.8}$$
$$r = \left(\frac{Gm_pT^2}{4\pi^2}\right)^{1/3} \tag{6.9}$$

Next, part (b): what is the escape velocity at this distance r from the planet?
I could re-derive the expression for the escape velocity as well, which wasn't that hard, but I recall that $v_{esc} = \sqrt{2} \times v_{orb}$, and we already have an expression for v_{orb}. Multiplying v_{orb} by $\sqrt{2}$ and then simplifying:

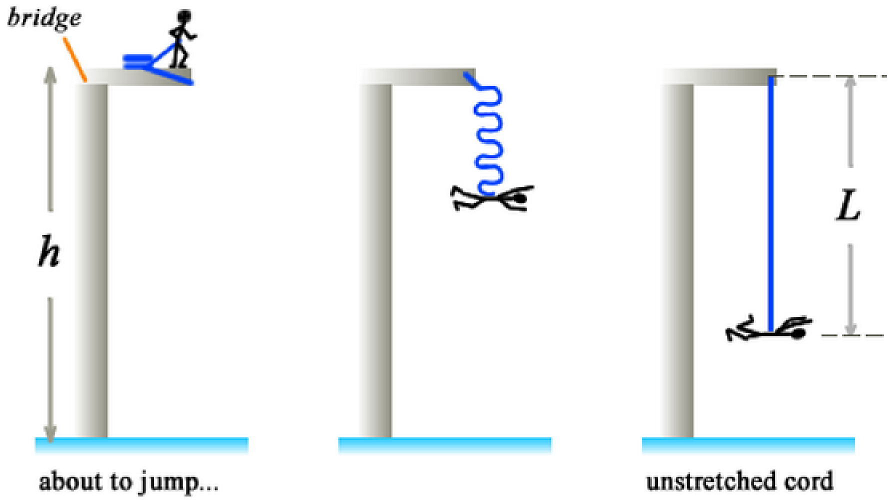
$$v_{orb} = \frac{2\pi}{T}\left(\frac{Gm_pT^2}{4\pi^2}\right)^{1/3} \tag{6.10}$$

$$v_{esc} = \sqrt{2}\left(\frac{2\pi G m_p}{T}\right)^{1/3} \tag{6.11}$$

However, they want the answer in *km/sec*, so we need to divide that by 1000.

6.2 Problem 2: Bungee jumper

"A bungee jumper jumps (with no initial speed) from a tall bridge attached to a light elastic cord (bungee cord) of unstretched length L. The cord first straightens and then extends as the jumper falls. This prevents her from hitting the water! Suppose that the bungee cord behaves like a spring with spring constant $k = 90\,\text{N/m}$. The bridge is $h = 100\,\text{m}$ high and the jumper's mass is $m = 65\,\text{kg}$. Use $g = 10\,\text{m/s}^2$.



(a) What is the maximum allowed length L of the unstretched bungee cord (in m) to keep the jumper alive? (Assume that the spring constant doesn't depend on L).
(b) Before jumping, our jumper verified the spring constant of the cord. She lowered herself very slowly from the bridge to the full extent of the cord and when she is at rest she measured the distance to the water surface. What was the measured distance (in m)?"

Hitting the water at, say, 0.1 m/s will surely not be lethal, but I assume the condition is that she doesn't touch the water whatsoever, or we can't find an exact answer to the question.

I will use a coordinate system where y increases downwards, and is centered on the bridge; thus the water is at $y = h$.
Also, I will use conservation of energy to solve this problem. My first solution was to find the total energy at $y = L$, after a period of free fall, and then the total energy at $y = h$, solving for L that way. I realized later, reading the forums, that this is unnecessarily complex, so my much simpler solution is below.

The kinetic energy is zero both just as you jump (since it is done with zero speed) and as you almost reach the water: the velocity vector reverses at that point, so $v = 0$ at the lowest point (which is $y = h$).

The change in gravitational potential energy is mgh, and all of that goes into the spring. (That's the only possibility other than kinetic energy, which we already ruled out).

The energy stored in the spring is given by $\frac{1}{2}kx^2$, where x in this case is $h - L$, the distance the cord is stretched beyond its natural length of L. (It is the distance to the water, from the natural length.)

We set the two equal, and solve for L:

$$mgh = \frac{1}{2}k(h-L)^2 \tag{6.12}$$
$$2mgh = k(h^2 - 2hL + L^2) \tag{6.13}$$
$$0 = h^2 - 2hL + L^2 - \frac{2mgh}{k} \tag{6.14}$$
$$0 = L^2 - (2h)L - \left(\frac{2mgh}{k} - h^2\right) \tag{6.15}$$

We use the quadratic formula:

$$L = h \pm \frac{1}{2}\sqrt{(-2h)^2 + 4\left(\frac{2mgh}{k} - h^2\right)} \tag{6.16}$$
$$L = h - \frac{1}{2}\sqrt{\frac{8mgh}{k}} \tag{6.17}$$
$$L = h - \sqrt{\frac{2mgh}{k}} \approx 61.9942\,\mathrm{m} \tag{6.18}$$

The plus-solution gives $L > h$, so that is clearly not the solution we want, so I got rid of that one between steps 1 and 2.

Next, part (b).
Same as last week: the spring's natural length is L, but at equilibrium, it is stretched a bit further due to the downwards force mg balancing out with the upwards force kx (where x how far it has stretched beyond its natural length L). We simply set them equal:

$$kx = mg \tag{6.19}$$
$$x = \frac{mg}{k} \tag{6.20}$$

So the equilibrium point is at $L + \frac{mg}{k} \approx 69.22\,\mathrm{m}$. The distance left down to the water is then $h - 69.22\,\mathrm{m} \approx 30.78\,\mathrm{m}$.

Full disclosure: my initial solution, which *was* marked as correct, was actually invalid. The reason I tried the energy approach later despite the green checkmark was because the equation I got was way too complex for it to make sense – but that was due to a bit of a miss on my side: I used both g and the value 10 instead of g, and tried to simplify... 10 and g didn't cancel, of course, so it turned out very complex... until I realized, used g everywhere, and it was only slightly more complex than the answer above.

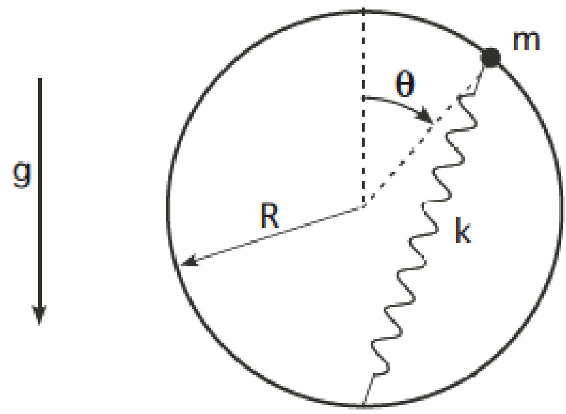
Anyway, my process there was to treat it as a spring oscillator, like last week's problem 7. The problem with that is, I realized, that this cord only acts as a spring when *stretched*, not otherwise. I'm not 100% sure why that affects the answer even when we *only* consider the way down, but the answer was about 0.7 meters greater. (Close enough to be considered correct!)
The larger the mass is, the further apart the two solutions become. The symbolic solution I got there was

$$L = h - \frac{\sqrt{2gmhk - g^2m^2}}{k} \text{ (invalid!)} \tag{6.21}$$

6.3 Problem 3: Loop, spring and bead

"A bead of mass m slides without friction on a vertical hoop of radius R. The bead moves under the combined action of gravity and a spring, with spring constant k, attached to the bottom of the hoop.

Assume that the equilibrium (relaxed) length of the spring is R. The bead is released from rest at $\theta = 0$ with a non-zero but negligible speed to the right.



(a) What is the speed v of the bead when $\theta = 90°$? Express your answer in terms of m, R, k, and g.
(b) What is the magnitude of the force the hoop exerts on the bead when $\theta = 90°$? Express your answer in terms of m, R, k, and g."

Alright, let's start by identifying the forces on the bead. Gravity and spring forces are quite obvious, but is there anything else? Yes, there is: a normal force by the hoop itself – which they ask for in part (b).

The centripetal force required for this motion is still $m\frac{v^2}{R}$ at all times, but v is not a constant in this problem (since both gravity and the spring will change the bead's speed), so the centripetal force will vary, too.

Since there is no friction, and gravitational forces and spring forces are both conservative, let's try conservation of energy.

The initial energy is all either gravitational potential energy or and spring potential energy. Let's set $U_g = 0$ at the center of the circle; in that case, the initial gravitational potential energy is mgR, and the final, at $\theta = 90°$ is 0 by our definition.
There is no initial kinetic energy, since the initial speed was negligible.
What about the spring? It is stretched a distance R beyond its natural length (total length $2R$, natural length R) so it stores a potential energy $U_s = \frac{1}{2}kR^2$ at the top.

$$E = mgR + \frac{1}{2}kR^2 \tag{6.22}$$

At $\theta = 90°$, all gravitational potential energy, and part of the spring's, will have turned into kinetic energy in the bead.

Here, the kinetic energy is $\frac{1}{2}mv^2$. The spring's stored energy is related to how far it is stretched beyond R; how far is that, at this point?
If we draw this up, with a θ as a right angle, and we draw a triangle with the spring length as the hypotenuse, the left and top sides of the triangle are both R in length, so the hypotenuse (the spring's current length) is $x = \sqrt{2R^2} = \sqrt{2} \times R$. It is then stretched $d = R\sqrt{2} - R = R(\sqrt{2} - 1)$ beyond its natural length. That gives it a potential energy of $U_s = \frac{1}{2}kR^2(2 - 2\sqrt{2} + 1)$.

Adding it all up, and setting it equal to E above, which is the total energy at all times:

$$\frac{1}{2}mv^2 + \frac{1}{2}kR^2(2 - 2\sqrt{2} + 1) = mgR + \frac{1}{2}kR^2 \tag{6.23}$$
$$mv^2 = 2mgR + kR^2 - kR^2(2 - 2\sqrt{2} + 1) \tag{6.24}$$
$$mv^2 = 2mgR + kR^2(1 - (2 - 2\sqrt{2} + 1)) \tag{6.25}$$
$$mv^2 = 2mgR + kR^2(-2 + 2\sqrt{2}) \tag{6.26}$$
$$v = \sqrt{\frac{2mgR + kR^2(-2 + 2\sqrt{2})}{m}} \tag{6.27}$$

Next, we need to find the magnitude of the normal force from the hoop on the bead.

The *radial* force (inwards) must always add up to the centripetal force, so we can decompose the forces and set that equal to $\frac{mv^2}{R}$.

Gravity at $\theta = 90°$ is clearly purely tangential; there's no left-or-right force due to gravity. In other words, we can ignore gravity for this part.

The spring force, on the other hand, clearly has components both tangential (up/down) and radial (left/right) at this point.
The total spring force is proportional to its extension past R (its natural length), which we found earlier, so

$$F_{spr} = k(\sqrt{2R^2} - R) = k\sqrt{2}R - kR = kR(\sqrt{2} - 1) \tag{6.28}$$

The above is the total spring force; we only want the radial component, which is $1/\sqrt{2}$ times that, or $F_{spr,rad} = kR(1 - 1/\sqrt{2})$.

The normal force is then the centripetal force $\frac{mv^2}{R}$, minus the force in that direction that the spring provides. (That is, the hoop must provide all the necessary force that the spring isn't.)

$$N + kR(1 - \frac{1}{\sqrt{2}}) = \frac{mv^2}{R} \tag{6.29}$$
$$N + kR(1 - \frac{1}{\sqrt{2}}) = 2mg + kR(2\sqrt{2} - 2) \tag{6.30}$$
$$N = 2mg + kR\left(2\sqrt{2} - 2 - 1 + \frac{1}{\sqrt{2}}\right) \tag{6.31}$$
$$N = 2mg + kR\left(\frac{5}{\sqrt{2}} - 3\right) \tag{6.32}$$

That's it!

6.4 Problem 4: Moon

"A planet has a single moon that is solely influenced by the gravitational interaction between the two bodies. We will assume that the moon is moving in a circular orbit around the planet and that the moon travels with a constant speed in that orbit. The mass of the planet is $m_p = 3.03 \times 10^{25}$ kg. The mass of the moon is $m_m = 9.65 \times 10^{22}$ kg. The radius of the orbit is $R = 2.75 \times 10^8$ m.

What is the period of the moon's orbit around the planet in earth days (1 earth day = 24 hours)."

The moon is about 300 times more massive than the planet; I will assume that makes it valid to use the formulas we've already used (that are not valid if the masses are close to each other; more on that and center on mass very soon – in the next problem).

As with the previous problem regarding orbit, I will use $E = \frac{1}{2}U$ here – it's easy to remember, so why not?

$$K_E + U = \frac{1}{2}U \tag{6.33}$$
$$K_E + \frac{1}{2}U = 0 \tag{6.34}$$
$$\frac{1}{2}m_m v_{orbit}^2 - \frac{1}{2}\frac{Gm_p m_m}{R} = 0 \tag{6.35}$$
$$v_{orbit}^2 - \frac{Gm_p}{R} = 0 \tag{6.36}$$
$$v_{orbit} = \sqrt{\frac{Gm_p}{R}} \tag{6.37}$$

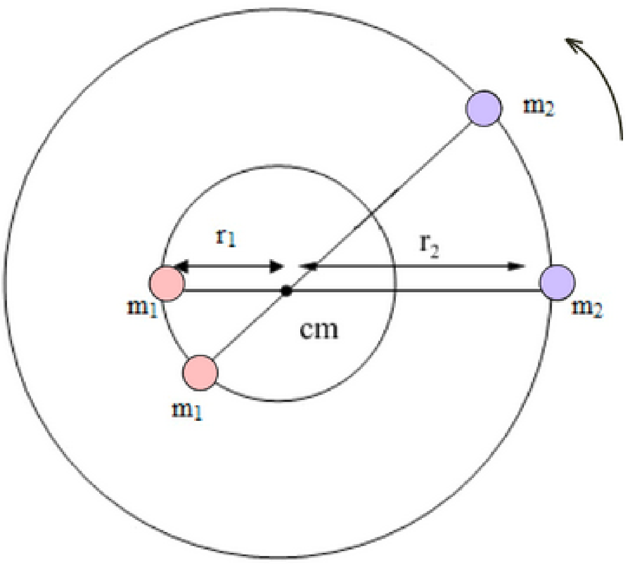
The period is then simply the distance divided by the velocity:

$$T = \frac{2\pi R}{v_{orbit}} = 2\pi R\sqrt{\frac{R}{Gm_p}} \tag{6.38}$$
$$T = 2\pi\sqrt{\frac{R^3}{Gm_p}} = 2\pi\frac{R^{3/2}}{\sqrt{Gm_p}} \approx 637\,374\,\text{s} \tag{6.39}$$

Finally, we just need to divide this by one "Earth day" of 86400 seconds, so the answer is 637374/86400 ≈ 7.38 days.

6.5 Problem 5: Double star system

"Consider a double star system under the influence of the gravitational force between the stars. Star 1 has mass $m_1 = 2.22 \times 10^{31}$ kg and Star 2 has mass $m_2 = 1.64 \times 10^{31}$ kg. Assume that each star undergoes uniform circular motion about the center of mass of the system (cm). In the figure below r_1 is the distance between Star 1 and cm, and r_2 is the distance between Star 2 and cm."



Ah, this week's possibly-scary problem. The concept of *center of mass* should make it easy, though, especially since the period is the same for both stars.

The center of mass of a system is a point around which *both* stars orbit. (In our solar system, the center of mass is inside the Sun, since it's such a dominant mass, but it's not at the Sun's center – so the Sun actually makes a tiny orbit around the center of mass).

Apparently, in the case of two bodies, $m_1 r_1 = m_2 r_2$ will hold. Combined with $s = r_1 + r_2$ where s is a given, we already have two equations and two unknowns. Too easy.

We can solve the second equation to give $r_1 = s - r_2$ and substitute into the first, to give one equation with one unknown:

$$m_1(s - r_2) = m_2 r_2 \tag{6.40}$$
$$m_1 r_2 + m_2 r_2 = m_1 s \tag{6.41}$$
$$r_2(m_1 + m_2) = m_1 s \tag{6.42}$$
$$r_2 = \frac{m_1 s}{m_1 + m_2} \tag{6.43}$$

We can then find r_2 easily, and $r_1 = s - r_2$ as mentioned, so that too is easy. For the given values,

$$r_1 = 1.411 \times 10^{18}\,\mathrm{m} \tag{6.44}$$
$$r_2 = 1.909 \times 10^{18}\,\mathrm{m} \tag{6.45}$$

Now, we just need to find the period. If the bodies orbit as shown, the gravitational attraction between them is always towards the center of mass. We can find ω this way, by equating the centripetal force $m|a_c| = m\omega^2 r$ with the gravitational force on one of the masses:

$$m_1 \omega^2 r_1 = \frac{G m_1 m_2}{s^2} \tag{6.46}$$
$$\omega^2 = \frac{G m_1 m_2}{m_1 r_1 s^2} \tag{6.47}$$
$$\omega = \sqrt{\frac{G m_2}{r_1 s^2}} \tag{6.48}$$

Finally, $T = \dfrac{2\pi}{\omega}$:

$$T = 2\pi\sqrt{\frac{r_1 s^2}{G m_2}} \approx 7.505 \times 10^{17}\,\mathrm{s} \approx 23.8\,\text{billion years} \tag{6.49}$$

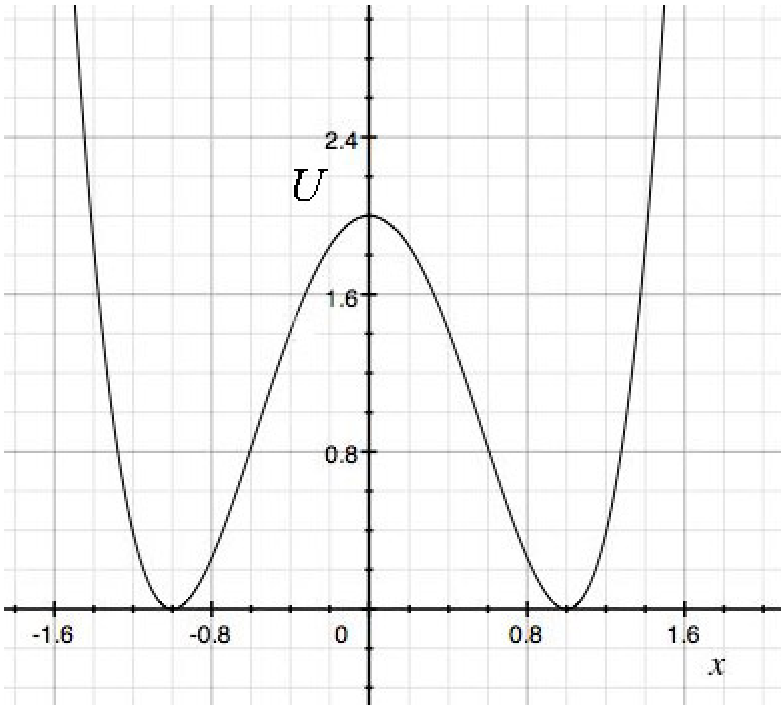
This is, incredibly enough, correct. The staff admitted in a forum post that the value for the distances was way, way larger than what is realistic (by 6 orders of magnitude), and so the period grew to about 10^9 times larger than expected!

6.6 Problem 6: Potential energy diagram

"A body of mass $m = 1$ kg is moving along the x-axis. Its potential energy is given by the function

$$U(x) = 2(x^2 - 1)^2$$

Note: The units were dropped for the numbers in the equation above. You should note that 2 would carry units of $\mathrm{J} \cdot \mathrm{m}^{-4}$ and 1 would carry units of m^2.



a) What is the x component of the force associated with the potential energy given by $U(x)$? Give an expression in terms of x.
b) At what positive value of x ($x > 0$) in m, does the potential have a stable equilibrium point?
c) Suppose the body starts with zero speed at $x = 1.5$ m. What is its speed (in m/s) at $x = 0$ m and at $x = -1$ m?"

Well, (b) is easy from the graph – it is at $x = 1$. But let's avoid getting ahead of ourselves.

The important thing to remember here is that $\frac{dU}{dx} = -F_x$. So far part (a), we need to find the derivative of $U(x)$, and then remember to negate the answer. Using the chain rule,

$$\frac{dU}{dx} = 4(x^2 - 1)(2x) = 4(2x^3 - 2x) = 8x^3 - 8x \tag{6.50}$$

$$F_x = -\frac{dU}{dx} = 8x - 8x^3 = 8(x - x^3) \tag{6.51}$$

For a more rigorous solution of part (b), we can find where $F_x = 0$, and only look at the cases where $x > 0$, which is the condition given:

$$8(x - x^3) = 0 \tag{6.52}$$

$$x^3 = x \tag{6.53}$$

$$x^2 = 1 \tag{6.54}$$

$$x = \pm\sqrt{1} \tag{6.55}$$

For $x > 0$, the only solution is $x = 1$. As a last step, we can confirm whether this is a stable equilibrium point, or an unstable one. It's clear from the graph that it's stable (if there is a small amount of force on the body, it will tend to roll back down from the "hills", rather than roll away, as it would from one of the peaks).

Mathematically, the condition here is that the second derivative of U is positive; that makes the curve "concave upward", i.e. looks like a U shape, so that things tend to stay inside. If $\frac{d^2U}{dx^2} < 0$, the opposite is true, and we are at a peak.

We calculate the second derivative, and stick $x = 1$ in there:

$$24x^2 - 8 \overset{?}{>} 0 \tag{6.56}$$
$$16 > 0 \tag{6.57}$$

The second derivative is positive, and so this is indeed a *stable* equilibrium point. If we try this at $x = 0$, we find -8, less than zero, and indeed, that is an unstable equilibrium point according to the graph.

Next, part (c), which asked

"c) Suppose the body starts with zero speed at $x = 1.5$ m. What is its speed (in m/s) at $x = 0$ m and at $x = -1$ m?"

Okay, so what does this imply? It starts at rest (zero kinetic energy), and we can easily calculate $U(1.5)$. We can then easily calculate $U(0)$, subtract the two, and we know the change in kinetic energy, and can solve for v.

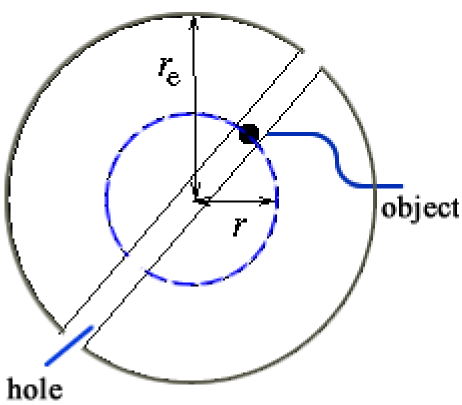
$$K_E(0) = U(1.5) - U(0) \tag{6.58}$$
$$\frac{1}{2}mv^2 = 2(1.5^2 - 1)^2 - 2(-1)^2 \tag{6.59}$$
$$\frac{1}{2}mv^2 = 3.125 - 2 = 1.125 \tag{6.60}$$
$$v = \sqrt{2.25} = 1.5\,\mathrm{m/s} \tag{6.61}$$

For $x = -1$, we simply do the same thing, but use $U(-1)$ instead of $U(0)$.

$$K_E(0) = U(1.5) - U(-1) \tag{6.62}$$
$$\frac{1}{2}mv^2 = 3.125 \tag{6.63}$$
$$v = \sqrt{6.25} = 2.5\,\mathrm{m/s} \tag{6.64}$$

6.7 Problem 7: Earth drilling

"A hole is drilled with smooth sides straight through the center of the earth to the other side of the earth. The air is removed from this tube (and the tube doesn't fill up with water, liquid rock or iron from the core). An object is dropped into one end of the tube and just reaches the opposite end. You can assume the earth is of uniform mass density. You can neglect the amount of mass drilled out and the rotation of the earth.



(a) The gravitational force on an object of mass m located inside the earth a distance $r < r_e$ from the center (r_e is the radius of the earth) is due only to the mass of the earth that lies within a solid sphere of radius r. What is the magnitude of the gravitational force as a function of the distance r from the center of the earth? Express your answer in terms of the gravitational of the r, m, g, and r_e.

Note: you do not need the mass of the earth m_e or the universal gravitation constant G to answer this question but you will need to find an expression relating m_e and G to g and r_e."

Ah, I actually solved this on the forum last week or so, using Gauss's law. I will try to do it this way instead, here, though.

At the surface,

$$F_g = mg = \frac{Gmm_e}{r_e^2} \tag{6.65}$$

The fraction of Earth's mass inside this smaller radius r is just the ratio of the volume of r to the volume of r_e. I will call this mass m_r, so

$$m_r = \frac{4/3\pi r^3}{4/3\pi r_e^3} m_e \tag{6.66}$$

$$m_r = \frac{r^3}{r_e^3} m_e \tag{6.67}$$

$$F_i = \frac{Gm}{r^2}\frac{r^3}{r_e^3} m_e \tag{6.68}$$

$$F_i = \frac{Gmr}{r_e^3} m_e \tag{6.69}$$

Almost there... we need to get rid of that G, and write it in terms of g instead. We have an equation for $F_g = mg$ in terms of G and so on above, so we solve that one for G, and substitute in in here:

$$G = \frac{gr_e^2}{m_e} \tag{6.70}$$

$$F_i = \frac{mr}{r_e^3} m_e \left(\frac{gr_e^2}{m_e}\right) \tag{6.71}$$

$$F_i = \frac{mgr}{r_e} \tag{6.72}$$

Got it!

Next, part (b): "How long would it take for this object to reach the other side of the earth? Express your answer in terms of the gravitational constant at the surface of earth g, m, and r_e as needed."

Okay, so the force experienced by the mass, at all times, is the force shown above. We can find the acceleration simply by dividing out m. If the acceleration were constant, we could use a simple kinematics equation here... but it's not constant. The velocity will not be constant, either, so we can't simply find a value for the velocity and calculate the time from knowing distance and velocity.

However...! The force is in the form $F = kr$, where $k = \frac{mg}{r_e}$ is a *constant*, in newtons per meter. In other words, this looks like a spring problem, in a way. Not exactly, perhaps, but close enough: consider a spring of near-zero natural length, attached at the center of the Earth. It will always have an inwards force, which is proportional to r, the distance you've stretched it beyond its original zero length.

Once you've passed the center, it will still be an inwards force, that is now trying to make you stop and reverse. One full oscillation of this system will then bring you all the way to the other side, and then back, in a symmetric motion. Therefore, the answer is half the period.

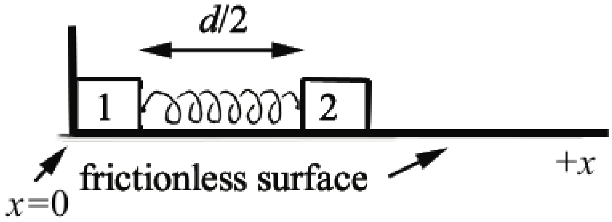
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/r_e}} = 2\pi\sqrt{\frac{r_e}{g}} \tag{6.73}$$

Half this is then simply

$$\frac{T}{2} = \pi\sqrt{\frac{r_e}{g}} \tag{6.74}$$

Chapter 7: Week 7: Homework 6

7.1 Problem 1: Two blocks and a spring



"A system is composed of two non-identical blocks connected by a spring. The blocks slide on a frictionless plane. The unstretched length of the spring is d. Initially block 2 is held so that the spring is compressed to $d/2$ and block 1 is forced against a stop as shown in the figure above. Block 2 is released.

Which of the following statements is true? (Note: more than one statement may be true.)

(a) When the position of block 2 is $x_2 > d$, the center of mass of the system is accelerating to the right.
(b) When the position of block 2 is $x_2 > d$, the center of mass of the system is moving at a constant speed to the right.
(c) When the position of block 2 is $x_2 > d$, the center of mass of the system is at rest.
(d) When the position of block 2 is $x_2 < d$, the center of mass of the system is accelerating to the right.
(e) When the position of block 2 is $x_2 < d$, the center of mass of the system is moving at a constant speed to the right.
(f) When the position of block 2 is $x_2 < d$, the center of mass is at rest."

All right, let's see. The spring is compressed, so as we start this experiment, block 2 will accelerate towards the right. The blocks are "non-identical", so we can't say anything qualitative about the center of mass, other than that it must be somewhere between the blocks (possibly part-way inside one of them).

This is an easy problem, IF you approach it correctly. If you don't, it's very easy to get it wrong. The approach that is way easier than the others is to consider conservation of momentum. In the beginning of the problem, there is a net external force on the system – the normal force from the wall pushing towards the right. Net force means acceleration, so to begin with, there is an *acceleration towards the right*, while $x_2 < d$ (the spring is compressed), so option (d) is correct.

When block 2 passes $x_2 > d$, the spring starts to pull together, which moves block 1 towards the right. When it moves away from the wall, there is *no longer a net external force* in the horizontal direction, and we can (and should) apply the conservation of momentum to consider what may happen next. No matter what the masses of the two blocks are, momentum must be conserved!

The net momentum of the system is $p_{tot} = m_{tot}v_{cm}$. The mass is not changing, and p_{tot} must be held constant and so v_{cm} *is a constant* after this; option (b) is also correct. All options except (b) and (d) are thus incorrect.

This was demonstrated in lecture, with an extremely similar system, of two air track-carts and a spring. After the system had been set in motion, the center of mass held a constant velocity, despite the oscillating behavior of the two masses. That is exactly what will happen here.

Since the center of mass will hold a constant velocity towards the right, the system will keep moving towards the right until it hits an obstacle (given that we ignore friction).

7.2 Problem 2: Pushing a baseball bat

"The greatest acceleration of the center of mass of a baseball bat will be produced by pushing with a force F at

(a) Position 1 (at the handle)
(b) Position 2 (at the center of mass, around the middle of the bat)
(c) Position 3 (at to the very edge)
(d) Any point. The acceleration is the same.
(e) Not enough information is given to decide."

Honestly, I find this a bit nonintuitive, based an experience – but it's important to note the force F is the same in all cases.

We have found previously that the momentum of a system can be found as $m\vec{v_{cm}}$, where m is the total mass:

$$\vec{p_{tot}} = m_{tot}\vec{v_{cm}} \tag{7.1}$$

If we take the time derivative of this equation, we find

$$\frac{dp_{tot}}{dt}F_{ext} = m_{tot}\vec{a_{cm}} \tag{7.2}$$

The change in momentum of the entire system is the same as the net external force, which is the same as the mass-acceleration product of the center of mass. That gives us, for the acceleration, $a_{cm} = \frac{F_{ext}}{m_{tot}}$. If $F = F_{ext}$ is constant, as it is, and m_{tot} is also constant, then clearly the only possible answer is that the acceleration is the same for all points, the fourth option.

http://www.youtube.com/watch?v=vWVZ6APXM4w has a great demonstration of this effect. Make sure you watch the follow-up video http://www.youtube.com/watch?v=N8HrMZB6_dU and the explanation video http://www.youtube.com/watch?v=BLYoyLcdGPc too. They are a bit less than 15 minutes combined, but the effect is quite nonintuitive and so the videos are rather interesting.

7.3 Problem 3: Jumping off the ground

"A person of mass m jumps off the ground. Suppose the person pushes off the ground with a constant force of magnitude F for T seconds.

What was the magnitude of the displacement of the center of mass of the person while they were in contact with the ground? Express your answer in terms of m, F, T, and g as needed."

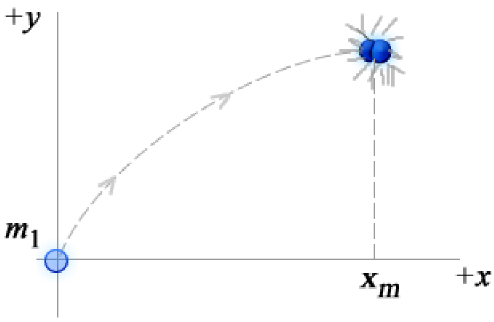
Well, let's see. Since the force is constant, the impulse is simply given by FT. However, I think we should solve this is a different manner than impulse.

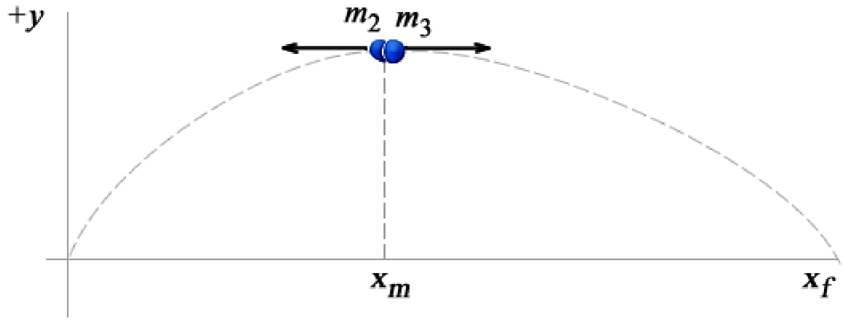
The movement of the center of mass is given by $F_{ext} = ma_{cm}$. With a constant force, and thus a constant acceleration, we can use $\Delta y = \frac{1}{2}at^2$, with $a = a_{cm}$ and $t = T$.
However, let's not forget about gravity. $F_{net} = F - mg$, so $a_{cm} = F/m - g$. That gives us, for the displacement

$$\Delta y = \frac{1}{2}(\frac{F}{m} - g)T^2 \tag{7.3}$$

7.4 Problem 4: Exploding projectile

"An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_m. The projectile breaks into two pieces which fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature.





How far away, x_f, from the original launching point does the larger piece land? Express your answer in terms of some or all of the given variables m_1, x_m, and g."

First, just in case we need them, let's write m_2 and m_3 in terms of m_1:

$$m_2 = \frac{m_1}{4} \tag{7.4}$$

$$m_3 = \frac{3m_1}{4} \tag{7.5}$$

Okay, so what do we know? Ignoring air drag, momentum is conserved in the x direction. After the explosion, $m_2 v_2^{'} + m_3 v_3^{'} = m_1 v_1$.
$v_1 = x_m/t$, but we don't know t. However, we do also know (see below) that $v_2^{'} = -v_1$.

The smaller piece has a certain momentum after the launch, and the exact opposite momentum the other way back. Why? Because $p = mv$, and since it returns to exactly its launch point along the same path, the v must be the same both ways, only in opposite directions. With no air drag, it takes the same amount of time to fall from the top down to the ground, and it must traverse the same horizontal distance back as it did in getting to the top during that same time, which implies having the same horizontal velocity, which for a given mass implies the same momentum (as far as magnitude goes).

The time t taken for m_3 to hit the ground is exactly the same as that of m_2, since there is no air drag that could cause any difference in timing. Using conservation of momentum (equation one), substituting in $v_1 = x_m/t$ (equation two), substituting in the masses (equation three) and finally substituting in $v_3^{'} = (x_f - x_m)/t$:

$$-m_2 v_1 + m_3 v_3^{'} = m_1 v_1 \tag{7.6}$$

$$-m_2\left(\frac{x_m}{t}\right) + m_3 v_3^{'} = m_1 \frac{x_m}{t} \tag{7.7}$$

$$-\frac{m_1}{4}\left(\frac{x_m}{t}\right) + \frac{3m_1}{4} v_3^{'} = m_1 \frac{x_m}{t} \tag{7.8}$$

$$-\frac{m_1}{4}\left(\frac{x_m}{t}\right) + \frac{3m_1}{4}\frac{(x_f - x_m)}{t} = m_1 \frac{x_m}{t} \tag{7.9}$$

All that remains is simplification. First we can eliminate t, followed by m_1 and multiplying it all by 4:

$$-\frac{m_1}{4}(x_m) + \frac{3m_1}{4}(x_f - x_m) = m_1 x_m \tag{7.10}$$

$$-(x_m) + 3(x_f - x_m) = 4x_m \tag{7.11}$$

$$\tag{7.12}$$

And the remainder doesn't need much explanation:

$$3x_f = 8x_m \tag{7.13}$$

$$x_f = \frac{8}{3}x_m \tag{7.14}$$

Just as I hoped, all terms could be written in terms of t, so that it could be eliminated, leaving only known values m_1 (which also cancelled) and x_m, plus the unknown x_f.
Quite a nice result!

7.5 Problem 5: Center of mass of the Earth-Moon system

"The mean distance from the center of the earth to the center of the moon is $r_{em} = 3.84 \times 10^8$ m. The mass of the earth is $m_e = 5.98 \times 10^{24}$ kg and the mass of the moon is $m_m = 7.34 \times 10^{22}$ kg. The mean radius of the earth is $r_e = 6.37 \times 10^6$ m. The mean radius of the moon is $r_m = 1.74 \times 10^6$ m.

How far from the center of the earth is the center of mass of the earth-moon system located?"

We choose a coordinate system centered at the center of the Earth, which is clearly the simplest choice. The definition of the center of mass is then

$$r_{cm} = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{m_e(0) + m_m r_{em}}{m_m + m_e} \approx 4656.2\,\text{km} = 4.6562 \times 10^6\,\text{m} \tag{7.15}$$

The term that is zero is the distance from the center of the coordinate system to the center of the Earth, which is obviously zero given the choice of coordinate system.

7.6 Problem 6: Bouncing ball

"A superball of mass m, starting at rest, is dropped from a height h_i above the ground and bounces back up to a height of h_f. The collision with the ground occurs over a total time t_c. You may ignore air resistance.

(a) What is the magnitude of the momentum of the ball immediately before the collision? Express your answer in terms of m, h_i, and g as needed.
(b) What is the magnitude of the momentum of the ball immediately after the collision? Express your answer in terms of m, h_f, and g as needed.
(c) What is the magnitude of the impulse imparted to the ball? Express your answer in terms of m, h_i, h_f, t_c, and g as needed.
(d) What is the magnitude of the average force of the ground on the ball? Express your answer in terms of m, h_i, h_f, t_c, and g as needed."

The velocity just prior to the collision can be find in several ways, e.g. kinematics or conservation of energy. I will use the latter.
If we choose $U = 0$ at the ground, the initial potential energy is mgh_i, all of which becomes kinetic energy. We set the two equal and solve for v:

$$\frac{1}{2}mv^2 = mgh_i \tag{7.16}$$

$$v = \sqrt{2gh_i} \tag{7.17}$$

The magnitude of the momentum prior to the collision just $p = m\sqrt{2gh_i}$, then.

What about after the collision? Since it returns to a lower height than it was let go from, the collision must have been partially inelastic, so that kinetic energy was lost. The initial kinetic energy must be mgh_f, however. We can then find the new velocity by relating the new kinetic energy and that:

$$\frac{1}{2}m(v')^2 = mgh_f \tag{7.18}$$

$$v' = \sqrt{2gh_f} \tag{7.19}$$

The magnitude of the momentum is then $p' = mv' = m\sqrt{2gh_f}$.

The impulse is just the difference between these, $I = p_f - p_i$; however, since we have magnitudes, we need to consider that the final momentum is really in the opposite direction of the initial momentum. This turns this subtraction into an addition.

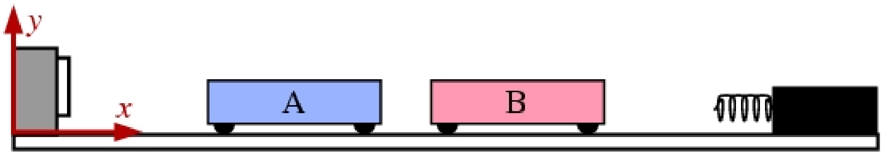
$$I = m(\sqrt{2gh_f} + \sqrt{2gh_i}) = m\sqrt{2g}(\sqrt{h_f} + \sqrt{h_i}) \tag{7.20}$$

Finally, the magnitude of the average force of the ground on the ball. First, we note that $\langle F \rangle = \frac{\Delta p}{\Delta t}$, so the average force due to the collision is just the above answer divided by t_c. However, there is a second force involved! Gravity is pulling the ball down with a force mg, and because it is in contact with the floor, there is a normal force mg, also upwards. The answer is the sum of the two:

$$\Big|\langle F \rangle\Big| = \frac{m\sqrt{2g}(\sqrt{h_f} + \sqrt{h_i})}{t_c} + mg \tag{7.21}$$

7.7 Problem 7: Colliding carts

"The figure below shows the experimental setup to study the collision between two carts.



"In the experiment cart A rolls to the right on the level track, away from the motion sensor at the left end of the track. Cart B is initially at rest. The mass of cart A is equal to the mass of cart B. Suppose the two carts stick together after the collision. Assume the carts move frictionlessly.

The kinetic energy of the two carts after the collision:

(a) is equal to one half the kinetic energy of cart A before the collision.
(b) is equal to one quarter the kinetic energy of cart A before the collision.
(c) is equal to the kinetic energy of cart A before the collision.
(d) is equal to twice the kinetic energy of cart A before the collision.
(e) is equal to four times the kinetic energy of cart A before the collision.
(f) None of the above."

Well, with no other source of energy, we can rule out options (d) and (e) at once. We should also be able to rule out (c) since this is an inelastic collision. However, let's do the math.

Momentum is conserved: $m_A v_A + m_B v_B = (m_A + m_B)v'$. However, $v_B = 0$, so

$$v' = \frac{m_A v_A}{m_A + m_B} \tag{7.22}$$

The initial kinetic energy is

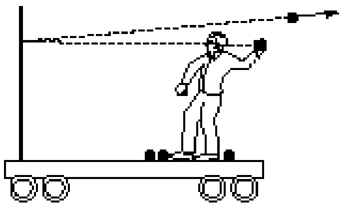
$$K = \frac{1}{2} m_A v_A^2 \tag{7.23}$$

while final kinetic energy is

$$K' = \frac{1}{2}(m_A + m_B)(v')^2 = \frac{1}{2}(m_A + m_B)\frac{m_A^2 v_A^2}{(m_A + m_B)^2} = \frac{m_A^2 v_A^2}{2(m_A + m_B)} \tag{7.24}$$

The ratio between the two is $K'/K = \frac{m_A}{m_A + m_B}$. However, because $m_B = m_A$, we find that the kinetic energy is *half* of the initial, the first choice.

7.8 Problem 8: Man on cart throwing balls



"Suppose you are on a cart, initially at rest on a track with very little friction. You throw balls at a partition that is rigidly mounted on the cart. If the balls bounce straight back as shown in the figure, is the cart put in motion?

(a) Yes, it moves to the right.
(b) Yes, it moves to the left.
(c) No, it remains in place.
(d) Not enough information is given to decide."

The actual title of this problem is "Man on throwing balls", but I assume they simply missed the word "cart".

So this is an interesting problem. It's easy to say that the answer is obviously (c), given that it may appear that all forces are internal, which is in fact not the case. It would be the case if he caught the ball, but he doesn't!

As he throws a ball, it gains momentum towards the left, while he (and the cart, via friction in his shoes) gains momentum towards the right, so that momentum is conserved. Shortly thereafter, the ball bounces, and gives momentum to the cart towards the left, and the ball momentum to the right – except that this change is *twice as large* as when he throw the ball.
In throwing it, he changed the ball's momentum from 0 to mv, while the bounce changed it from mv to $-mv$, a change of $2mv$.

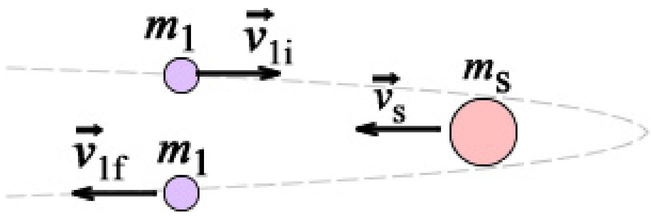
Defining the positive direction to be towards the left:
Before the throw, the cart and ball both have 0 momentum.
After the throw, the ball has momentum mv and therefore the cart $-mv$, so that the sum is zero.
After the bounce, the ball has momentum $-mv$ and therefore the cart $+mv$, so that, again, the sum is zero.

That's when the problem ends – the ball exits the system, and the momentum is never cancelled out, so the cart gains a velocity towards the left.

If he caught the ball, we could add:
After the catch, the ball transfers its momentum $-mv$ to the cart, which then gets a momentum $mv - mv = 0$, and we are back where we began.

A simpler analysis:
Initial momentum of the system is zero, and final momentum of the ball is towards the right. That *must* mean that there is an equal amount of momentum towards the *left* of the cart, or momentum would not be conserved!

7.9 Problem 9: Gravitational slingshot



"A spacecraft of mass $m_1 = 4757$ kg with a speed $v_{1i} = 3 \times 10^3$ m/s approaches Saturn which is moving in the opposite direction with a speed $v_s = 9.6 \times 10^3$ m/s. After interacting gravitationally with Saturn, the spacecraft swings around Saturn and heads off in the opposite direction it approached. The mass of Saturn is $m_s = 5.69 \times 10^{26}$ kg. Find the final speed v_{1f} (in m/s) of the spacecraft after it is far enough away from Saturn to be nearly free of Saturn's gravitational pull."

Spoiler alert: most of the text in this problem is justifying why the solution works, and is only necessary if you don't realize it at once. (I didn't, until it was "too late" to use the simple solution; I'd already solved it in more complex way.)

Considering the two as a system, there are no external forces, so momentum must be conserved. Momentum is a vector though, so we need to be careful with signs. If we take v_{1i} to be positive, the initial velocity of Saturn is negative, and both velocities on the right-hand side are negative.

$$m_1 v_{1i} - m_s v_s = -m_1 v_{1f} - m_s v_{sf} \tag{7.25}$$

We don't know v_{1f} and we don't know v_{sf} (the final velocity of Saturn). The latter must change, even if by an absolutely imperceptible amount.
With two unknowns, we need a second equation.
What more can we say and express as an equation? The total mechanical energy of the system should certainly be constant, since gravity is a conservative force. The mechanical energy here is

$$K_{m1} + U_{m1} + K_s + U_s = K'_{m1} + U'_{m1} + K'_s + U'_s \tag{7.26}$$

Before we try to calculate this, which will clearly not be pretty, let's try to simplify it. Gravitational potential energy depends on two things: the two masses, and the distance between them. (Plus G, which is a constant, of course.) Therefore, if the problem starts and ends at the same distance r, or it starts and ends where r is large enough that $U \approx 0$ (keep in mind that gravitational potential energy is always negative), we can assume that either that $U_{m1} = U'_{m1}$ and $U_s = U'_s$, or that all of those terms are practically zero. This simplifies things a great deal:

$$K_{m1} + K_s + = K'_{m1} + K'_s \tag{7.27}$$

So now, the condition is that the sum of the kinetic energies are the same before and after, i.e. the increase in kinetic energy in the spacecraft comes from a decrease in Saturn's kinetic energy.

With momentum and kinetic energy both conserved, we can solve this in a very simple way: this is an elastic collision. It doesn't matter that the force involved is gravity, instead of contact forces (that are mostly electromagnetic, in the end).

The mass of Saturn is about 10^{23} times greater than that of the satellite, so to an extremely good approximation, a reference frame centered on Saturn is the center of mass frame. For the same reason, the velocity of the COM frame is the velocity of Saturn – the error here is so small that a pocket calculation would round it away entirely; in fact, I couldn't get Mathematica to give me an exact answer! All I can say is that it is much, much less than 1 nanometer per second, which it gives me for m_1 as large as 10^{11} kg. I think we'll be OK with this "approximation"!

All we need to do, then, is transfer ourselves into the center of mass frame, by subtracting the center of mass velocity, find the velocity u_{1f} (using u instead of v in the COM frame), and transfer back. We transfer into it by subtracting the COM velocity:

$$u_{1i} = v_{1i} - v_{cm} = v_{1i} - v_s = 3 \times 10^3\,\text{m/s} + 9.6 \times 10^3\,\text{m/s} = 12.6 \times 10^3\,\text{m/s} \tag{7.28}$$

Since Saturn's velocity is negative in our coordinate system, the subtraction becomes an addition. This makes sense, too: the center of mass, inside Saturn, sees the planets heading towards each other, so the net speed is larger than either of the individual speeds.

Next, we find the velocity after the collision. In the center of mass frame, this is just too easy: the signs flip. $u_{1f} = -u_{1i} = -12.6 \times 10^3\,\text{m/s}$.

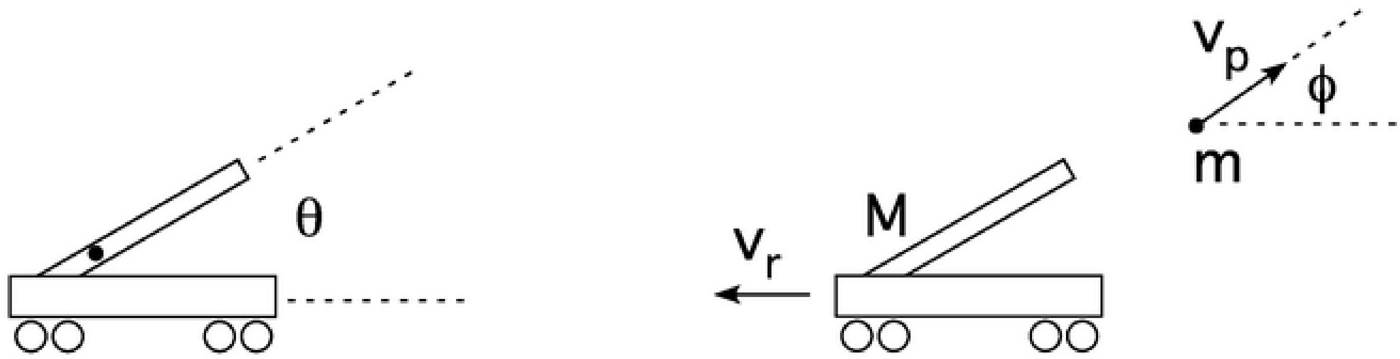
Finally, we convert back to the reference frame of the outside observer by *adding* the COM velocity of $-9.6 \times 10^3\,\text{m/s}$, and end up with

$$v_{1f} = u_{1f} + -9.6 \times 10^3\,\text{m/s} = -22.2 \times 10^3\,\text{m/s} \tag{7.29}$$

They ask for the speed, though, so we need to drop the minus sign, and we are done.

7.10 Problem 10: Railroad gun

"A railroad gun of mass $M = 2.0$ kg fires a shell of mass $m = 1.0$ kg at an angle of $\theta = 45°$ with respect to the horizontal as measured relative to the gun. After the firing is complete, the final speed of the projectile relative to the gun (muzzle velocity) is $v_0 = 130.0\,\text{m/s}$. The gun recoils with speed v_r and the instant the projectile leaves the gun, it makes an angle ϕ with respect to the ground.



What is v_p, the speed of the projectile with respect to the ground (in m/s)?
What is ϕ, the angle that the projectile makes with the horizontal with respect to the ground (in degrees)?"

I have to say that 2 kg for the entire gun and the car seems ridiculously low! If the projectile flies away at 130 m/s, via conservation of momentum, the rail car will move backwards with a speed of at least about a quarter of that (that's just guesswork), which is crazy fast, about the speed of a car on a freeway. I can't see it being less than a tenth, at least. I suppose we'll see soon enough.

Intuitively, I have to admit I thought $\phi = \theta$ and $v_p = v_0$, and thought of the recoil as separate thing, which is clearly not correct. Let's look at a proper analysis.

Clearly, conservation of momentum will be the main way we approach this problem.
Since this is a two-dimensional problem, there will be a bit more work than in the problems we've seen earlier on.

Momentum will be conserved in the x direction, which will be quite a useful fact. What about the y direction? Well, the shell clearly gains upwards momentum, but what about the car/gun? It is pushed down, but can't move downwards. Instead, the momentum is transferred to the Earth. After the launch, gravity acts on the shell, and so the y component of its momentum will change.

Let's first think about this from the reference frame of the car. Not many strange things happen here: the shell launches at an angle θ, and moves away from you at v_0 ($v_0 \cos\theta$ in the horizontal direction, and $v_0 \sin\theta$ upwards). So far, so good.

What happens according to an observer on the ground? The vertical component of the shell's motion is unchanged, since the car is stationary along the y axis. In other words, this observer sees the shell move upwards at $v_0 \sin\theta$ m/s, same as someone on the car.
What about the horizontal component? I find it helpful to take things to extremes (even if they are unrealistic). What if the recoil speed of the car was greater than the shell's speed?
The horizontal component as seen from the ground would shrink, and since the vertical component is unchanged, the angle grows, and v_p moves closer to $v_0 \sin\theta$.
This implies that $\phi > \theta$, and of course that $v_p < v_0$.

What about a more quantitative analysis? Let's first look at the reference frame of the rail gun. The equations for the shell is

$$v_{0x} = v_0 \cos\theta \tag{7.30}$$
$$v_{0y} = v_0 \sin\theta \tag{7.31}$$

Nothing strange going on there.
In the reference frame of an outside observer, standing still on the ground, things change. Such an observer would see the gun speeding towards the left at the same time the shell starts flying to the right. To him, it is clear that the gunner would see the shell move *faster* (in the x direction) than what he sees. In fact, in the limit where the speed of the gun and the speed of the shell are equal, the shell would move straight up to the outside observer.

The relevant equations here are also not very strange, but we can relate the two sets soon. First, the easy part:

$$v_{px} = v_p \cos\phi \tag{7.32}$$
$$v_{py} = v_p \sin\phi \tag{7.33}$$

To the outside observer (and to the gunner), the rail gun is stationary along the y axis. Therefore, $v_0 \sin\theta = v_p \sin\phi$: the two agree on the vertical component. That gives us one useful relationship.

Next, we can relate the x components. The difference there is a simple reference frame shift. As mentioned above, the outside observer sees the shell having a lower speed along the x axis. The difference between the two frames is v_r.

$$v_p \cos\phi = v_0 \cos\theta - v_r \tag{7.34}$$

Next, we can relate the momenta of the two objects. The initial momentum is zero, in both reference frames. Let's write a conservation equation in the outside frame.

$$mv_p \cos\phi - Mv_r = 0 \tag{7.35}$$

Since v_r is a speed in the opposite direction, we need a minus sign there. (Both terms will be positive, and their difference is zero.)

The final answer for v_p has the form

$$v_p = (v_p \sin\phi)\hat{x} + (v_0 \sin\theta)\hat{y} \tag{7.36}$$

... since the y component is the same in both reference frames. However, we only need to find the x component, and then calculate ϕ from that; so we don't really need to think of ϕ as an unknown, as far as solving the system goes. All we need is the x component of the shell, as seen from the outside reference frame.

We have

$$mv_p \cos\phi - Mv_r = 0 \tag{7.37}$$

But $v_p \cos\phi = (v_0 \cos\theta - v_r)$, so

$$m(v_0 \cos\theta - v_r) - Mv_r = 0 \tag{7.38}$$

$$mv_0 \cos\theta = v_r(M + m) \tag{7.39}$$

$$\frac{mv_0 \cos\theta}{M + m} = v_r \tag{7.40}$$

We know all of those variables, so we can find that $v_r = 30.641\,29\,\mathrm{m/s}$. That means we can find the x component:

$$v_{px} = v_0 \cos\theta - v_r = 61.283\,\mathrm{m/s} \tag{7.41}$$

We already had v_{py} in terms of knowns, $v_0 \sin\theta$:

$$v_{py} = v_0 \sin\theta = 91.9239\,\mathrm{m/s} \tag{7.42}$$

We can then finally find v_p and the angle ϕ:

$$v_p = \sqrt{v_{px}^2 + v_{py}^2} = 110.479\,\mathrm{m/s} \tag{7.43}$$

$$\phi = \arctan\frac{v_{py}}{v_{px}} = 56.31^\circ \tag{7.44}$$

Chapter 8: Week 8: No homework

Chapter 9: Week 9: Homework 7

9.1 Problem 1: Rotational kinematics: turntable solutions

"A turntable is a uniform disc of mass m and radius R. The turntable is initially spinning clockwise when looked down on from above at a constant frequency f_0. The motor is turned off at $t = 0$ and the turntable slows to a stop in time t with constant angular deceleration.

(a) What is the magnitude of the initial angular velocity ω_0 of the turntable? Express your answer in terms of f_0.
(b) What is the magnitude of the angular acceleration α of the turntable? Express your answer in terms of f_0 and t.
(c) What is the magnitude of the total angle $\Delta\theta$ in radians that the turntable spins while slowing down? Express your answer in terms of f_0 and t."

Writing down the answers will likely take longer than it takes to solve these questions! We can use the simple equations for rotational kinematics that we saw in the beginning of lecture 19. One was not mentioned there, which is

$$f_0 = \frac{1}{T} = \frac{\omega}{2\pi} \tag{9.1}$$

This implies that $\omega = 2\pi f_0$, which answers part (a).

For part (b), we use $\alpha = \dfrac{\omega_1 - \omega_0}{\Delta t}$, where ω_0 is the initial angular velocity, and ω_1 the final angular velocity. In this case, $\omega_0 > \omega_1$, so α is negative. However, they asked for the magnitude, so we drop the sign, and it comes to a complete halt at time t, so $\omega_1 = 0$:

$$\alpha = \frac{\omega_0}{t} = \frac{2\pi f_0}{t} \tag{9.2}$$

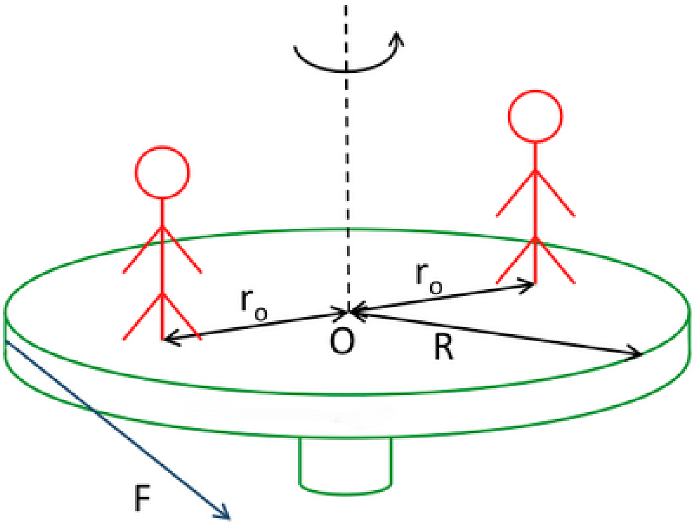
Finally, for part (c), we can use $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, derived from $\theta = \theta_0 + \omega_0 t + \dfrac{1}{2}\alpha t^2$. α is negative, so the addition becomes a subtraction:

$$\Delta\theta = 2\pi f_0 t - \frac{1}{2}\left(\frac{2\pi f_0}{t}\right) t^2 = 2\pi f_0 t - (\pi f_0)\, t = \pi f_0 t \tag{9.3}$$

That's it!

9.2 Problem 2: Angular dynamics

"A playground merry-go-round has a radius of $R = 2$ m and has a moment of inertia $I_{cm} = 5 \times 10^3$ kg m^2 about a vertical axis passing through the center of mass. There is negligible friction about this axis. Two children each of mass $m = 25$ kg are standing on opposite sides at a distance $r_o = 1.4$ m from the central axis. The merry-go-round is initially at rest. A person on the ground applies a constant tangential force of $F = 2 \times 10^2$ N at the rim of the merry-go-round for a time $\Delta t = 10$ s . For your calculations, assume the children to be point masses.



(a) What is the angular acceleration α of the merry-go-round (in rad/s^2)?
(b) What is the angular velocity ω_{final} of the merry-go-round when the person stopped applying the force (in rad/s)?
(c) What average power P_{avg} does the person put out while pushing the merry-go-round (in Watts)?
(d) What is the rotational kinetic energy R.K.Efinal of the merry-go-round when the person stopped applying the force (in kg m^2/s^2)?"

Hmm, I wonder if there is a particular reason why part (d) is not in joules. The dimension is equivalent, but they didn't state "in joules" for whatever reason.

Anyhow, let's see. Unless otherwise specified, I will consider torques and angular momentum relative to the center of mass – though angular momentum should be the same for all points, since this is a rotation about the center of mass, so that disclaimer is probably not necessary.

To start with, we need to calculate the moment of inertia, since we only know it without the children being included.
Considering them as point masses, the total moment of inertia is just the sum of I_{cm} plus mr_o^2 for each of the children:

$$I = I_{cm} + 2mr_o^2 = 5098\,\text{kg m}^2 \tag{9.4}$$

Now, then. The rotational analogue of $F = ma$ is $\tau = I\alpha$. We can find α very easily if we only find the torque relative to the center of mass.
The torque is given by $\tau = \vec{R} \times \vec{F}$, where $\vec{R}$ is the position vector from the origin to point where the force is applied. The force is specified as "tangential", so there is always a right angle between the two, and $\vec{R} \times \vec{F} = RF$, since $\sin(\pi/2) = 1$. The angular acceleration is then

$$\alpha = \frac{\tau}{I} = \frac{RF}{I} = 0.0785\,\text{rad/s}^2 \tag{9.5}$$

Using that, we can find the final angular velocity very easily, using $\omega = \omega_0 + \alpha t$. t is given as 10 seconds in this case, so

$$\omega_{final} = 0 + \alpha t = 0.785\,\text{rad/s} \tag{9.6}$$

In more familiar units, this is 8.00 seconds per rotation (0.125 Hz or 7.5 rpm).

What is the average power of the person pushing the merry-go-round? We should be able to use $W = \vec{F} \cdot \vec{v}$ here, where v is the tangential velocity, $v = \omega R$. The two are always parallel, and so

$$P = Fv = F\omega R \tag{9.7}$$

We could find the average using an integral:

$$P_{avg} = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} P(t)\, dt \tag{9.8}$$

... but surely there is a better way. I looked up the relationship for power and torque, and found that $P = \vec{\tau} \cdot \vec{\omega}$, which also would require an integration (in fact, it would be the same integral), since ω is constantly changing.

I'm not sure if there is an easier way, but this integral shouldn't be very bad, so let's do it.

$$P_{avg} = \frac{1}{\Delta t} \int_0^{\Delta t} FR\omega(t)\, dt \tag{9.9}$$

$$= \frac{FR}{\Delta t} \int_0^{\Delta t} \alpha t\, dt \tag{9.10}$$

$$= \frac{FR\alpha}{\Delta t} \left(\frac{(\Delta t)^2}{2} \right) \tag{9.11}$$

$$= \frac{FR\alpha \Delta t}{2} \tag{9.12}$$

For these values, $P_{avg} = 157$ watts. Quite reasonable.

And at last, the final rotational kinetic energy. The book proves that the work-energy theorem is applicable to rotational energy, so all the work done ($W = P_{avg}\Delta t$) is turned into rotational kinetic energy, so the answer is

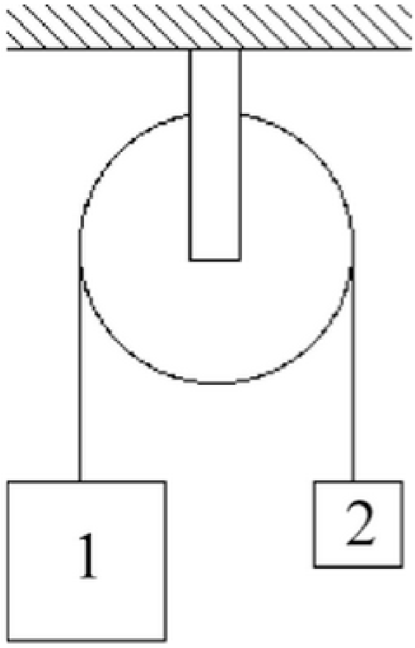
$$W = P_{avg}\Delta t = 1570\,\text{J} \tag{9.13}$$

As as update after the staff solutions are out, this was technically incorrect – but was accepted anyway. They wanted the rotational kinetic energy of the *merry-go-round alone, without the children*, but I don't think that was too clear.
We can find the kinetic energy of that alone as $\frac{1}{2} I_{cm}\omega^2 \approx 1540$ J, instead. Not a lot harder, but I do think the question is a bit vague. Since the previous three questions were all found by considering the children's moments of inertia, I just assumed we should do here, too.

9.3 Problem 3: Atwood machine

"A pulley of mass m_p, radius R, and moment of inertia about the center of mass $I_c = \frac{1}{2} m_p R^2$, is suspended from a ceiling. The pulley rotates about a frictionless axle. An inextensible string of negligible mass is wrapped around the pulley and it does not slip on the pulley. The string is attached on one end to an object of mass m_1 and on the other end to an object of mass with $m_2 < m_1$.

At time $t = 0$, the objects are released from rest.



(a) Find the magnitude of the acceleration of the two objects. Express your answer in terms of m_1, m_2, m_p, R and acceleration due to gravity g.
(b) How long does it take the objects to move a distance d? Express your answer in terms of m_1, m_2, m_p, d and acceleration due to gravity g."

Ah, interesting stuff: a non-massless pulley! Granted, we don't allow for any slipping, and the string is still of negligible mass... but this is still a considerable step towards some realism.
The string is *not massless*, however. Remember that when a string is massless, we can prove that the tension at two different points along the string must have the same magnitude... but in this case, a difference in tension is the cause of the torque that rotates the pulley! More on that in a second.

First off, since the string is inextensible, the acceleration and velocity of both masses and the rope (and the pulley, i.e. the tangential velocity at its edge) must all be the same.
Since $m_1 > m_2$, the system will accelerate such that m_1 goes downwards, m_2 upwards, and the pulley rotates counterclockwise, as seen from the direction we see it. This means $\vec{\omega}$ for the pulley will be out of the screen.

The forces on each block are easy to find. Each block has gravity and tension acting on it. I will take downwards to be the positive direction for block 1, and upwards for block 2, which then yields a common acceleration a without any trouble with signs and directions.
Let's then write Newton's second law equations for the two blocks:

For block 1, $m_1 a = m_1 g - T_1$.
For block 2, $m_2 a = T_2 - m_2 g$.

The differences in tension will cause a tangential force on the pulley, which causes a torque relative to its center, which I will call point C. This torque causes a rotation via $\tau_C = I\alpha_C$.
$a = \alpha_C R$ (this is just the time derivative of $v = \omega R$), so we can also say that $\tau = \dfrac{Ia}{R}$, so that $a = \dfrac{\tau R}{I}$.
The dimension works out to be that of acceleration, which is always a good sign!

What is the torque, then? Well, the tension is tangential, and so the moment arm into the center becomes the radius R, and the angle is always 90 degrees. That gives us $\tau_C = (T_1 - T_2)R$.
We know that the rotation will be counterclockwise, so the torque must be directed out of the screen. Using $\vec{R} \times \vec{F}$ where $F = T_1 - T_2$ with a leftwards direction, the direction of positive torque, according to the cross product, is out of the screen – as it should be! (That is assuming that $T_1 > T_2$, which it should be in this case.)

This means we have three equations and three unknowns:

$$m_1 a = m_1 g - T_1 \tag{9.14}$$
$$m_2 a = T_2 - m_2 g \tag{9.15}$$
$$a = \frac{2(T_1 - T_2)}{m_p} \tag{9.16}$$

I substituted in $I = \frac{1}{2} m_p R^2$ in the last equation, which removed the dependence on R.

I'm never a fan of solving systems of three equations. Can we simplify the task? Solving the first two for T_1 and T_2 respectively, we can find $T_1 - T_2$ by subtracting the other sides of those equations:

$$T_1 - T_2 = m_1 g - m_1 a - m_2 a - m_2 g \tag{9.17}$$

We can then stick this into the third equation, and solve for a:

$$a = 2\frac{g(m_1 - m_2) - a(m_1 + m_2)}{m_p} \tag{9.18}$$
$$a\left(1 + \frac{2(m_1 + m_2)}{m_p}\right) = 2\frac{g(m_1 - m_2)}{m_p} \tag{9.19}$$
$$a = 2\frac{g(m_1 - m_2)}{m_p} \frac{1}{\left(1 + \frac{2(m_1+m_2)}{m_p}\right)} \tag{9.20}$$
$$a = 2\frac{g(m_1 - m_2)}{m_p + 2(m_1 + m_2)} \tag{9.21}$$

Not bad!

Next up, how long does it take to move a distance d? a is clearly constant, since there are only constants in the above equation. Therefore, we can use $d = v_0 t + \frac{1}{2} a t^2$ here. $v_0 = 0$, so the first term disappears. We solve the rest for t:

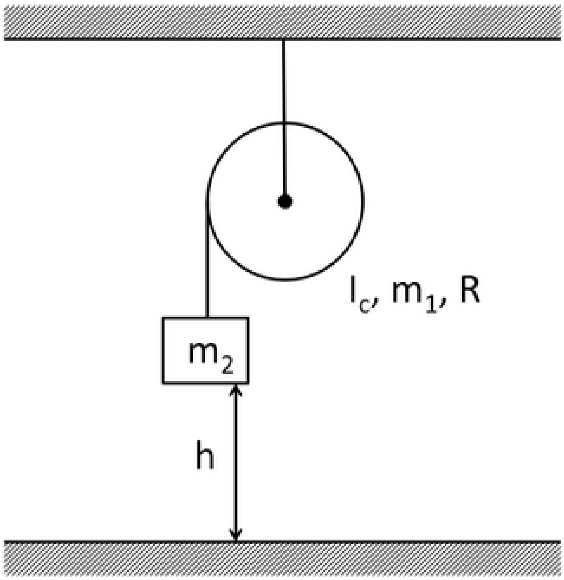
$$\frac{1}{2} a t^2 = d \tag{9.22}$$
$$t^2 = \frac{2d}{a} \tag{9.23}$$
$$t = \sqrt{\frac{2d}{a}} \tag{9.24}$$

All that remains is then to stick the above, semi-complex expression into the square root:

$$t = \sqrt{\frac{2d}{2\frac{g(m_1 - m_2)}{m_p + 2(m_1 + m_2)}}} \tag{9.25}$$
$$t = \sqrt{d \cdot \frac{m_p + 2m_1 + 2m_2}{g(m_1 - m_2)}} \tag{9.26}$$

9.4 Problem 4: Pulley-object rotational dynamics

"A light inflexible cable is wrapped around a cylinder of mass m_1, radius R, and moment of inertia about the center of mass I_c. The cylinder rotates about its axis without friction. The cable does not slip on the cylinder when set in motion. The free end of the cable is attached to an object of mass m_2. The object is released from rest at a height h above the floor. You may assume that the cable has negligible mass. Let g be the acceleration due to gravity.



(a) Find the acceleration a of the falling object. Express your answer in terms of m_2, R, I_c and g.
(b) Find the tension T in the cable. Express your answer in terms of m_2, R, I_{cm} and g.
(c) Find the speed v of the falling object just before it hits the floor. Express your answer in terms of m_2, R, I_{cm}, h and g."

Hmm, this looks as if it should be easier than the previous problem.
The cable has negligible mass, so the tension ought to be zero without the mass there. Therefore, the tension is all due to gravity acting on the block.
Newton's second low on the block, taking downwards to be positive, is

$$m_2 a = m_2 g - T \tag{9.27}$$

The tension then acts on the pulley, in a tangential fashion (as in the last problem, though on the side this time, instead of the top), so that the torque relative to its center is $\tau_C = \vec{R} \times \vec{T} = RT$, with the direction being out of the screen (since the rotation will be counterclockwise). $\vec{R}$ is then the position vector from the center to the point where the force acts, so the $\sin\theta$ term is again always 1, due to the 90 degree angle between the two vectors.

This torque causes an acceleration of the pulley via

$$\tau_C = I_c \alpha \Rightarrow \alpha = \frac{\tau_C}{I_c} = \frac{RT}{I_c} \tag{9.28}$$

$a = \alpha R \Rightarrow \alpha = \dfrac{a}{R}$, so

$$\frac{a}{R} = \frac{RT}{I_c} \tag{9.29}$$

$$a = \frac{R^2 T}{I_c} \tag{9.30}$$

Two equations, with a and T as two unknowns. We can solve both for a, sen them equal, and find T:

$$g - \frac{T}{m_2} = \frac{R^2 T}{I_c} \tag{9.31}$$
$$\frac{T}{m_2} = g - \frac{R^2 T}{I_c} \tag{9.32}$$
$$T + \frac{m_2 R^2 T}{I_c} = m_2 g \tag{9.33}$$
$$T\left(1 + \frac{m_2 R^2}{I_c}\right) = m_2 g \tag{9.34}$$
$$T = m_2 g \frac{1}{\left(1 + \frac{m_2 R^2}{I_c}\right)} \tag{9.35}$$
$$T = \frac{m_2 g}{1 + \frac{m_2 R^2}{I_c}} \tag{9.36}$$
$$T = \frac{m_2 g I_c}{I_c + m_2 R^2} \tag{9.37}$$

That then answers part (b). Let's stick in into the other equation and find a:

$$a = \frac{R^2}{I_c} \frac{m_2 g I_c}{I_c + m_2 R^2} \tag{9.38}$$
$$a = g \frac{m_2 R^2}{I_c + m_2 R^2} \tag{9.39}$$

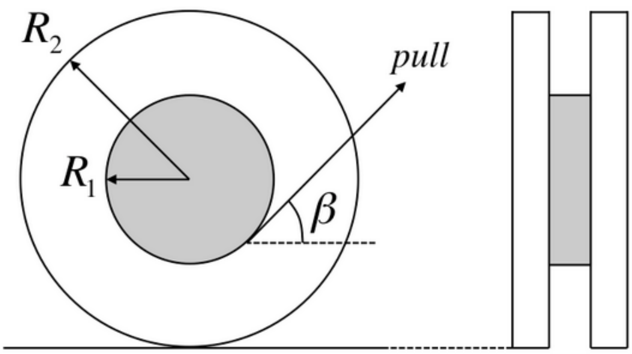
Finally, the speed of the object as it hits the floor.
As previously, $v_0 = 0$ and a is a constant, so we can use basic kinematics equations... only that those involve both t and h.
We can solve $v = at$ for t, and find $t = \frac{v}{a}$. Substitute that into the one that relates acceleration to distance:

$$\frac{1}{2} a \left(\frac{v}{a}\right)^2 = h \tag{9.40}$$
$$\frac{1}{2} \frac{v^2}{a} = h \tag{9.41}$$

$$v = \sqrt{2ha} = \sqrt{2hg \frac{m_2 R^2}{I_c + m_2 R^2}} \tag{9.42}$$

And that's it for this one!

9.5 Problem 5: Yo-yo



A yo-yo of mass m rests on the floor (the static friction coefficient with the floor is μ). The inner (shaded) portion of the yo-yo has a radius R_1, the two outer disks have radii R_2. A string is wrapped around the inner part. Someone pulls on the string at an angle β (see sketch). The "pull" is very gentle, and is carefully increased until the yo-yo starts to roll without slipping. Try it at Home; it's Fun!

For what angles of β will the yo-yo roll to the left and for what angles to the right?

1. Yo-Yo rolls to the left if $\sin\beta < \dfrac{R_1}{R_2}$, and to the right if $\sin\beta > \dfrac{R_1}{R_2}$.
2. Yo-Yo rolls to the left if $\sin\beta > \dfrac{R_1}{R_2}$, and to the right if $\sin\beta < \dfrac{R_1}{R_2}$.
3. Yo-Yo rolls to the left if $\cos\beta < \dfrac{R_1}{R_2}$, and to the right if $\cos\beta > \dfrac{R_1}{R_2}$.
4. Yo-Yo rolls to the left if $\cos\beta > \dfrac{R_1}{R_2}$, and to the right if $\cos\beta < \dfrac{R_1}{R_2}$.

Hmm. Well, unfortunately I don't have a yo-yo (nor anything similar, like a spool of sewing thread), so I can't really try it out! I also have no real intuition of how it behaves here, though I do know that it rolls away when β is large, and towards you when β is small.

Since $\sin\beta \approx \beta$ for small angles, option 1 cannot be true; it is more likely to roll to the left if $\sin\beta$ is large. Option 2 could be true.
$\cos\beta$ becomes smaller as the angle grows. Larger angle means more likely to roll to the left, so smaller cosine also means that. This means we can rule out option 4.

Left are options 2 and 3, though I don't see any obvious way to choose between the two without actually making the calculations! Let's have a look at that.

What can we say about the yo-yo? There are external forces, which also causes external torques. R_1 acts as a moment arm for our pull, for the torque relative to the center of the yo-yo.
There is also the force due to friction. Friction acts along R_2, and also causes a torque on the yo-yo, in the opposite direction to the torque due to the pull.

I will use a coordinate system where leftwards motion is positive.

If we draw a free-body diagram (considering only the center of mass; we should not do this for torques, since distances matter there) and use P to notate the force due to our pull, we find $P\cos\beta$ in the rightwards direction (negative, in this coordinate system), and F_{fr} towards the left.

Using Newton's second law, we can write for the center of mass,

$$ma_{cm} = F_{fr} - P\cos\beta \tag{9.43}$$

We can then calculate the torque due to this pulling force, as $\vec{R} \times \vec{P}$; the angle to the position vector is always 90 degrees, and $\vec{R}$, the moment arm, is R_1, since the string is wrapped around R_1:

$$\tau_P = R_1 P \text{ (direction: out of the screen / causes CCW rotation)} \tag{9.44}$$

There is also a torque due to friction. Again, the angle is always 90 degrees, so $\vec{R} \times \vec{F_{fr}}$ is just the magnitude of the two multiplied together, where the moment arm is now R_2 (friction acts on the outside of the yo-yo):

$$\tau_{fr} = R_2 F_{fr} \text{ (direction: in to the screen / causes CW rotation)} \tag{9.45}$$

When $\tau_{fr} > \tau_P$, there is clockwise rotation, and the yo-yo rolls towards the right. When τ_P wins, it moves towards the right.
Since the torque must reverse direction between these two cases, there is also the possibility that the net torque is zero.

Net torque (CCW/left): $\tau = \tau_P - \tau_{fr} = R_1 P - R_2 F_{fr}$

Using the condition that the torque is zero, we can relate F_{fr} to P:

$$R_1 P - R_2 F_{fr} = 0 \tag{9.46}$$
$$F_{fr} = \frac{R_1}{R_2} P \tag{9.47}$$

Making this substitution into the Newton's second law equation:

$$a = \frac{P}{m}\left(\frac{R_1}{R_2} - \cos\beta\right) \tag{9.48}$$

This acceleration is positive when the yo-yo accelerates to the left, due to the choice of coordinate system, so the condition for moving towards the left is that the above expression is greater than zero. We set up the inequality and solve:

$$\frac{P}{m}\left(\frac{R_1}{R_2} - \cos\beta\right) > 0 \tag{9.49}$$

That happens when

$$\frac{R_1}{R_2} > \cos\beta \tag{9.50}$$

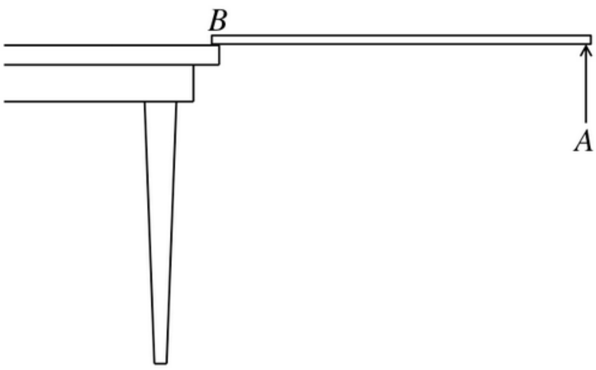
which of course is the same as one of the answer options,

$$\cos\beta < \frac{R_1}{R_2} \tag{9.51}$$

So the answer is option 3,
Yo-Yo rolls to the left if $\cos\beta < \frac{R_1}{R_2}$, and to the right if $\cos\beta > \frac{R_1}{R_2}$.

9.6 Problem 6: Stick on table

"A uniform stick of mass m and length ℓ is suspended horizontally with end B at the edge of a table as shown in the diagram, and the other end A is originally held by hand.
The hand at A is suddenly released.



At the instant immediately after the release:

(a) What is the magnitude of the torque (τ_B) about the end B at the edge of the table? Express your answer in terms of m, ℓ and acceleration due to gravity g as needed.
(b) What is the magnitude of the angular acceleration α about the end B at the edge of the table? Express your answer in terms of m, ℓ and acceleration due to gravity g as needed.
(c) What is the magnitude of the vertical acceleration a of the center of mass? Express your answer in terms of m, ℓ and acceleration due to gravity g as needed.
(d) What is the magnitude of the vertical component of the hinge force (N) at B? Express your answer in terms of m, ℓ and acceleration due to gravity g as needed."

Torque is the force times the moment arm length, which is easy in this case. The relevant force is mg, which acts on the center of mass. Since the stick is uniform, the center of mass is at $\frac{\ell}{2}$. The torque relative to point B is simply

$$\tau_B = mg\frac{\ell}{2} \tag{9.52}$$

since the angle between the two vectors is 90 degrees (just after the stick is released).

We can now find the angular acceleration by knowing the torque, via $\tau = I\alpha$. The moment of inertia in question is the one for the rod, about its end, since that is the pivot point.
Since the pivot point is clearly not at the center of mass in this case, we need to use the parallel axis theorem.

I remember that $I_c = \frac{1}{12}m\ell^2$ for a rod, but we then need to add a term due to the parallel axis theorem. The distance between the center of mass and this new axis is half the rod's length, so via the parallel axis theorem,

$$I_B = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 \tag{9.53}$$

Now, using $\tau_B = I_B\alpha$, we can solve for α:

$$mg\frac{\ell}{2} = \alpha\left(\frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2\right) \tag{9.54}$$

$$g\frac{\ell}{2} = \alpha\frac{1}{12}\ell^2 + \alpha\frac{\ell^2}{4} \tag{9.55}$$

$$g = \alpha\frac{1}{6}\ell + \alpha\frac{\ell}{2} \tag{9.56}$$

$$g = \alpha\ell\left(\frac{1}{6} + \frac{1}{2}\right) \tag{9.57}$$

$$\frac{g}{\ell} = \alpha\left(\frac{4}{6}\right) \tag{9.58}$$

$$\frac{3g}{2\ell} = \alpha \tag{9.59}$$

Next, they want to know the vertical acceleration of the center of mass. α describes the angular acceleration about point B, that the center of mass undergoes. We can use the relationship $a = \alpha R$, and in this case, $R = \frac{\ell}{2}$.

$$a = \frac{3g}{2\ell}\frac{\ell}{2} \tag{9.60}$$

$$a = \frac{3g}{4} \tag{9.61}$$

Finally, what is the magnitude of the vertical component of the hinge force at B?
Well, first up, what is hinge force? I haven't seen that term before, but I assume it is the normal force from the table on the end of the stick, especially as they give it as N.

It's clearly not zero, or the stick would just fall right through.
What we do here is to remember the videos and demonstration of an impulse on a ruler. No matter *where* on the ruler the force is exerted, the acceleration of the center of mass is affected in the same way. Therefore, we can use Newton's second law to relate the net downwards force, $mg - N$, with the mass times acceleration of the stick:

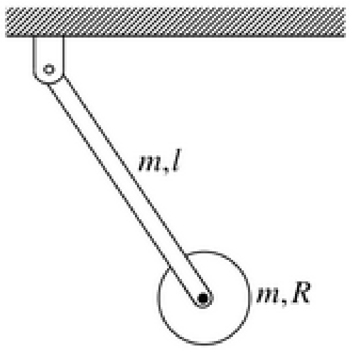
$$mg - N = ma \tag{9.62}$$
$$N = m(g - a) \tag{9.63}$$

We know a from above, so we can substitute than in there:

$$N = m(g - \frac{3g}{4}) \tag{9.64}$$
$$N = \frac{mg}{4} \tag{9.65}$$

9.7 Problem 7: Physical pendulum

"A physical pendulum consists of a disc of radius R and mass m fixed at the end of a rod of mass m and length ℓ.



(a) Find the period of the pendulum for small angles of oscillation. Express your answer in terms of m, R, ℓ and acceleration due to gravity g as needed.
(b) For small angles of oscillation, what is the new period of oscillation if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin? Express your answer in terms of m, R, ℓ and acceleration due to gravity g as needed."

This one took me a while, in large part because of a silly math error that I didn't spot for an hour or so... An ℓ disappeared when I added two expressions. Very frustrating!

Anyway. In part (a), the disk is fixed. We begin by calculating the total moment of inertia for rotating about what I will call point P, the point where the rod is mounted to the roof.

For the rod, we use the parallel axis theorem:

$$I_{rod,end} = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{m\ell^2}{3} \tag{9.66}$$

We also use the parallel axis theorem for the disc. About the disc's own center of mass, $I_{cm} = \frac{1}{2}mR^2$. We need to add to that the distance to the new axis, which is ℓ away.

$$I_{disc} = \frac{1}{2}mR^2 + m\ell^2 = m\left(\frac{R^2}{2} + \ell^2\right) \tag{9.67}$$

The total moment of inertia for rotation about the pivot point, for the combination is then

$$I_P = \frac{1}{6}m(8\ell^2 + 3R^2) \tag{9.68}$$

Let's now consider the torque (relative to the pivot point, P). There is a torque due to the rod (because of gravity acting on its center of mass), and a torque due to the wheel (again, due to gravity acting on its center of mass). These torques depend on the moment arm length, the force of gravity, and the sine of the angle between the two, via the cross product definition:

$$\tau_{P,rod} = \vec{r_P} \times \vec{F_g} = \frac{\ell}{2}mg\sin\theta \tag{9.69}$$

$$\tau_{P,disc} = \vec{r_P} \times \vec{F_g} = \ell mg\sin\theta \tag{9.70}$$

$$\tau_P = \tau_{P,rod} + \tau_{P,disc} = \frac{3}{2}mg\ell\sin\theta \tag{9.71}$$

This is a restoring torque, that is always trying to get things back to equilibrium. Using Newton's second law, or perhaps rather its rotational equivalent $\tau = I\alpha$, only with a negative sign in front since it is a restoring torque:

$$\alpha = -\frac{3}{2I_P}mg\ell\sin\theta \tag{9.72}$$

Using $\alpha = \ddot{\theta}$, and a small angle approximation $\sin\theta \approx \theta$, we get

$$\ddot{\theta} + \frac{3}{2I_P}mg\ell\theta = 0 \tag{9.73}$$

... which is simple harmonic oscillator. This is of course what we wanted all along. The period is then given by $\frac{2\pi}{\omega}$, where ω^2 is the stuff multiplying the square root. We flip that upside down, take the square root, and multiply by the 2π:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2I_P}{3mg\ell}} \tag{9.74}$$

$$T = 2\pi\sqrt{\frac{m(8\ell^2 + 3R^2))}{9mg\ell}} \tag{9.75}$$

$$= \frac{2\pi}{3}\sqrt{\frac{8\ell^2 + 3R^2}{g\ell}} \tag{9.76}$$

What happens for part (b)? When the disc is free to spin, it is also free to stay stationary, so to speak. That is, when it is *fixed*, it is *forced* to rotate along with the motion. If we made a vertical mark at the top of the disk, that mark would turn at an angle θ together with the rod and the rest of the disc.

Because of this, it has a spin component of moment of inertia of $I_{cm,disk} = \frac{1}{2}mR^2$, in addition to the orbital component of mR^2.

With a frictionless bearing, on the other hand, that vertical mark on the disk would be vertical at all times, which means it is not spinning any more.
There is no torque acting on the disc: gravity acts equally on all points, and since it is attached in the center with *no friction*, there can be no torque due to the pin there, either.

This means that the term for the disc's moment of inertia that is due to the spin disappears, and $I_{disc} = mR^2$ – only the orbital part remains.
So we can think of the motion of the disc as having two components: one "orbital", and one "spin". In the previous case, both were present. In this case, when the disc can stay stationary (have no spin motion at all), only the orbital motion remains, and so only the orbital part of the moment of inertia remains.

The torque is unchanged, since we calculated that based on the center of mass. What changes is I_P; the part due to the rod is unchanged, but that due to the disc changes, so that

$$I_{disc} = m\ell^2 \tag{9.77}$$

The total moment of inertia about the pivot point is again the sum of the two moments of inertia:

$$I_P = \frac{m\ell^2}{3} + m\ell^2 = \frac{4m\ell^2}{3} \tag{9.78}$$

That is the only thing that changes, so we stick that into the equation for the period:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2I_P}{3mg\ell}} \tag{9.79}$$

$$T = 2\pi\sqrt{\frac{2(\frac{4m\ell^2}{3})}{3mg\ell}} \tag{9.80}$$

$$T = 2\pi\sqrt{\frac{8m\ell^2}{9mg\ell}} \tag{9.81}$$

$$T = \frac{2\pi}{3}\sqrt{\frac{8\ell}{g}} \tag{9.82}$$

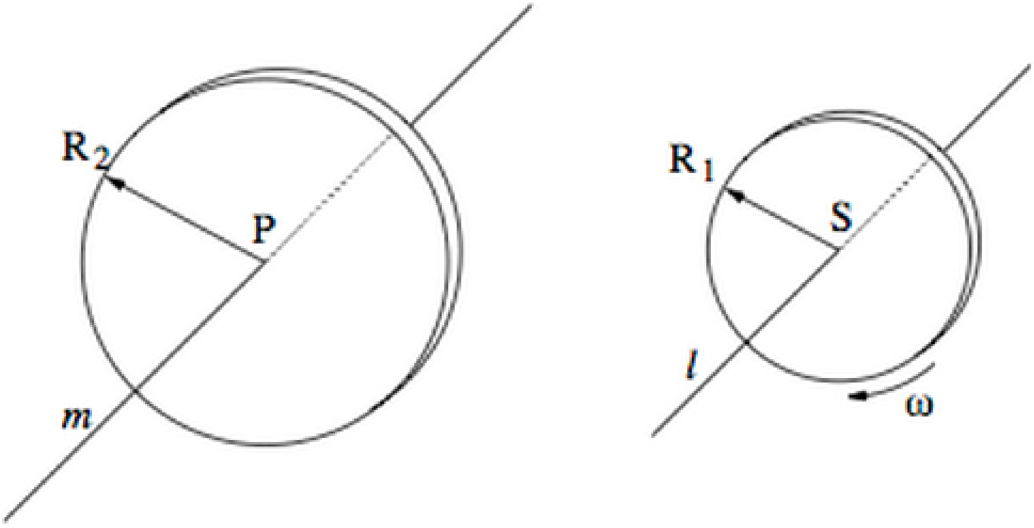
That solves this problem!

9.8 Problem 8: Two rotating disks

"A solid disk 1 with radius R_1 is spinning freely about a frictionless horizontal axle ℓ at an angular speed ω initially. The axle ℓ is perpendicular to disk 1, and goes through the center S of disk 1.

The circumference of disk 1 is pushed against the circumference of another disk (disk 2). Disk 2 has the same thickness and density as disk 1, but has a radius R_2, and it is initially at rest. Disk 2 can rotate freely about a horizontal axle m through its center P. Axles m and ℓ are parallel. The friction coefficient between the two touching surfaces (disk circumferences) is μ.

We wait until an equilibrium situation is reached (i.e. the circumferences of the two disks are no longer slipping against each other). At that time, disk 1 is spinning with angular velocity ω_1, and disk 2 is spinning with angular velocity ω_2.



Calculate the magnitude of the angular velocities $|\omega_1|$ and $|\omega_2|$ in terms of R_1, R_2 and ω.

It is quite remarkable that ω_1 and ω_2 are independent of μ, and it is also independent of the time it takes for the equilibrium to be reached (i.e independent of how hard one pushes the disks against each other)."

Unlike most cases, I'm writing almost all the text for this problem after having solved it. (I usually write while solving, then clean up the text when I have everything correct, and feel I understand the solution fully.)
This problem was certainly the most confusing of the week for many, including myself until I thought about it for quite a long while, while following the forum discussions.

First: angular momentum will *not be conserved*! This is an extremely important point, of course – solving this by assuming it *is* conserved does not work. (Except a side note, below.)

It is clear that there is friction between the disks, or they could not affect one another. Friction is proportional to the normal force, but since the disks are at the side of one another, there is no natural force to push them together.
This force must be provided by something *external* to the system, such as a person holding the two axles. In addition, the force due to friction acts "upwards" and "downwards" on the two disks, respectively (in the order shown in the figure). With a net force upwards or downwards on an object, the center of mass must accelerate upwards! $a_{cm} = \dfrac{F_{ext}}{m}$ must hold for the center of mass. Therefore, in order for the disks to stay where they are, another *external force* comes in: the leftwards disk must be forced down, and the rightwards disk must be forced up, or they will not stay put.

Now, in a bit of a freak coincidence, the correct solution *can* be found by assuming angular momentum is conserved, and by assuming that $\dfrac{m_1}{m_2} = \dfrac{R_1}{R_2}$, which is incorrect! Since mass is proportional to volume, and volume is $\pi R_i^2 h$, the correct equation is $\dfrac{m_1}{m_2} = \dfrac{R_1^2}{R_2^2}$.

Combining this *correct* equation with conservation of momentum, and you can find an answer which looks like the correct ones, only that all exponents (on R_1 and R_2) are one too large! If you then also use the incorrect formula for the masses above, *the error cancels out*, and you find the correct answer!
To be clear, this does not imply that the *method* is correct – it is trivial to show that the total angular momentum must change! See the end notes below, after my solution.

9.8.1 My solution

Okay, so let's consider this in more detail. To begin with, note that below, any time I say the leftmost disk, I mean the leftmost disk in the figure above, which is disk 2 (since it has radius R_2 and ends up spinning at ω_2, I call it disk 2). The rightmost disk is disk 1.

Okay. First, we can write two equations regarding the change in angular momentum of each disk on its own. By the way, because we also deal with objects spinning about an axis through their center of mass, we don't need to specify the point relative to which we find the angular momentum, as the answer is the same for all such points.
The two equations relating these changes are

$$\Delta L_1 = I_1(\omega_1 - \omega) \tag{9.83}$$
$$\Delta L_2 = I_2(\omega_2 - 0) \tag{9.84}$$

Disk 2 starts with 0 initial angular momentum, so its final angular momentum $I_2\omega_2$ equals the change.

The most important forces involved will be the frictional forces due to the contact of the two disks. The magnitude of these forces is unknown (they depend on how hard the disks are pushed together, which we are not told), but that doesn't matter for the solution, as the problem sort-of states.

Disk 1 spins clockwise to begin with. When it comes in contact with disk 2, there is a frictional force on disk 1, due to disk 2. This frictional force must oppose the relative motion, and so it acts downwards (counterclockwise) on disk 1, slowing its rotation. (Anything else would be crazy!)
Via Newton's third law, there is an equal but opposite force on disk 2 (which is still stationary), due to disk 1. This means that force is upwards, i.e. causes counterclockwise rotation.

These forces must cause torque on the two disks, or their rotation would be unaffected (since torque causes change in rotational motion, just as force causes change in linear motion).
For disk 1, there is friction on the left side, acting downwards tangentially along the disk. The torque caused by this, relative to the disk's center, is the cross product of the position vector from the center and the friction vector:

$$\tau_1 = \vec{R_1} \times \vec{F_{fr}} = -R_1 F_{fr} \tag{9.85}$$

As for direction, via the right-hand rule, it is out of the screen, i.e. acts counterclockwise. Again, anything else would be crazy, since the opposite torque would speed the disk's rotation up.
I notate this with a minus sign, as I use clockwise rotation (into the screen) as positive. That is the initial rotation, so I figured it would make sense to call that positive.

For disk 2, we do the same process. Friction is on the right side, acting upwards, tangentially. The torque relative to this disk's center is

$$\tau_2 = \vec{R_2} \times \vec{F_{fr}} = -R_2 F_{fr} \tag{9.86}$$

The direction of this torque is also out of the screen, i.e. it acts counterclockwise. This is also clear if you consider the direction of the motion; the disk starts to spin such that the tangential velocity is reduced, so that slipping is reduced. This is only possible if it spins up counterclockwise.

Note that both torques act counterclockwise, which means angular momentum is increasing in the CCW direction for both disks, and therefore for the system of the two disks combined. This can clearly not be the case if angular momentum is conserved/held constant; if it were held constant, the increase in one disk must be matched by a decrease in the other.

I used F_{fr} for both frictional forces, since they have the same magnitude via Newton's third law. Their directions do differ, however.

Say that this frictional force acts for an unknown time Δt. We can then also write the changes in angular momenta as

$$\Delta L_1 = -F_{fr}\Delta t R_1 \tag{9.87}$$
$$\Delta L_2 = -F_{fr}\Delta t R_2 \tag{9.88}$$

using the relationship $\frac{dL}{dt} = \tau$, which becomes $\Delta L = \tau \Delta t$ if we bring it out of the differential form, and rearrange.

So, we have four equations; two per disk, both of which define the change in angular momentum. If we set them equal in pairs, we get two equations, with many unknowns (F_{fr}, Δt, I_1, I_2, ω_1 and ω_2 – wow). Not to worry, as we can eliminate many of those. First, we can eliminate I_2 by writing it in terms of I_1. It is specified that the disks have the same density and thickness, so we can relate their masses and/or moments of inertia by comparing the radii.

The mass of a disk with some density ρ is $\pi R_i^2 h\rho$. The moment of inertia is then $\frac{1}{2}mR_i^2 = \frac{1}{2}(\pi R_i^2 h\rho)R_i^2$, and the ratio of the two moments of inertia becomes

$$\frac{I_2}{I_1} = \frac{\frac{1}{2}(\pi R_2^2 h\rho)R_2^2}{\frac{1}{2}(\pi R_1^2 h\rho)R_1^2} = \frac{R_2^4}{R_1^4} \tag{9.89}$$

which gives us $I_2 = I_1 \frac{R_2^4}{R_1^4}$. It is proportional to R^4 because both the mass and the moment of inertia are, on their own, proportional to R^2.

Combining the two pairs of ΔL equations, and making the substitution for I_2 using the relationship above, we have

$$I_1\omega_1 - I_1\omega = -F_{fr}\Delta t R_1 \tag{9.90}$$
$$I_1 \frac{R_2^4}{R_1^4}\omega_2 = -F_{fr}\Delta t R_2 \tag{9.91}$$

We can divide the two equations – note how this gets rid of F_{fr}, Δt and I_1 all at once!

$$\frac{I_1\omega_1 - I_1\omega}{I_1 \frac{R_2^4}{R_1^4}\omega_2} = \frac{-F_{fr}\Delta t R_1}{-F_{fr}\Delta t R_2} \tag{9.92}$$
$$R_1^4 \frac{\omega_1 - \omega}{R_2^4 \omega_2} = \frac{R_1}{R_2} \tag{9.93}$$
$$R_1^3 \frac{\omega_1 - \omega}{R_2^3 \omega_2} = 1 \tag{9.94}$$
$$R_1^3\omega_1 - R_1^3\omega = R_2^3\omega_2 \tag{9.95}$$
$$R_1^3\omega_1 - R_2^3\omega_2 = R_1^3\omega \tag{9.96}$$

A bit of a prettier way to write this would be to consider the relative magnitudes of the two torques instead (the torques are *not* the same in magnitude, but the frictional force that causes them *are*). The end result is the same; it is simply a different way to write the equations.

Another relationship we can use is that of the linear velocities of the two disks, which need to match for there to be no slipping.

$$\omega_1 R_1 = -\omega_2 R_2 \tag{9.97}$$

We then have two equations and two unknowns:

$$R_1^3\omega_1 - R_2^3\omega_2 = R_1^3\omega \tag{9.98}$$
$$\omega_1 R_1 = -\omega_2 R_2 \tag{9.99}$$

The solutions are

$$\omega_1 = \frac{R_1^2\omega}{R_1^2 + R_2^2} \tag{9.100}$$
$$\omega_2 = -\frac{R_1^3\omega}{R_2(R_1^2 + R_2^2)} \tag{9.101}$$

They asked for the magnitudes, though, so we need to drop the minus sign in front of ω_2.

9.8.2 Aftermath

So with the solutions in mind, what happens in terms of angular momentum?

$$L_{initial} = I_1\omega \tag{9.102}$$
$$L_{final} = I_1\omega_1 + I_2\omega_2 \tag{9.103}$$

... keeping in mind that ω_2 is negative. We know that $\omega > \omega_1$, and that the moments of inertia don't change. The change in angular momentum is

$$\Delta L_{sys} = L_{final} - L_{initial} = (I_1\omega_1 + I_2\omega_2) - I_1\omega \tag{9.104}$$

Which is, using the expressions for the solutions ω_1 and ω_2, and using $I_2 = I_1\frac{R_2^4}{R_1^4}$:

$$\Delta L_{sys} = -\frac{I_1 R_2^2(R_1 + R_2)}{R_1(R_1^2 + R_2^2)}\omega \tag{9.105}$$

Not a very pretty expression (I think simplification might have made it uglier), but we can consider the simpler case when $R_2 = R_1$:

$$\Delta L_{sys,R_1=R_2} = -I_1\omega \tag{9.106}$$

In this special case, the change in angular momentum is exactly the negative of the *initial* angular momentum: the net angular momentum is ZERO afterwards.
This does actually make a whole lot of sense. If the disks are identical (same thickness, radii and density implies same mass and same moment of inertia), they will rotate at the same angular speed... but opposite directions! Since $L_C = I_c\omega$, and both disks have the same magnitude (but opposite direction) of ω, and the same I_c, the angular momentum of disk 1 is exactly the opposite of disk 1, and the sum is zero.

The solution for angular velocities in this special case is $\omega_1 = \omega/2$ and $\omega_2 = -\omega/2$, so.

$$L_{final,R_1=R_2} = I\frac{\omega}{2} + I\left(-\frac{\omega}{2}\right) = 0 \tag{9.107}$$

9.9 Problem 9: Translation and rotation

"A rod is lying at rest on a perfectly smooth horizontal surface (no friction). We give rod a short impulse (a hit) perpendicular to the length direction of the rod at X. The mass of the rod is 3 kg, and its length is 50 cm. The impulse is 4 kg m/s. The distance from the center C of the rod to X is 15 cm.

(a) What is the translational speed $|\vec{v_{cm}}|$ of C after the rod is hit (in m/s)?
(b) What is the magnitude of the angular velocity ω of the rod about C (in rad/s)?
(c) How far (distance D in meters) has the center C of the rod moved from its initial position 8 seconds after it was hit?
And what is the angle θ (in radians) between the direction of the rod at 8 seconds after it was hit, and its initial direction (before it was hit)? Give the smaller angle.
(d) What is the total kinetic energy K of the rod after it was hit? (in joules)"

Hmm, this appears to be exactly as in the problem solving session. I will try to re-derive everything, though, since looking up the equations and entering the numbers doesn't teach you much.

The motion of the center of mass is very easy to derive. Say the rod is hit by an impulse I. It has zero momentum to begin with, so its new total momentum is I.
$\vec{p_{tot}} = m_{tot}\vec{v_{cm}}$ must hold, and so

$$v_{cm} = \frac{I}{m} = 4/3\,\mathrm{m/s} \tag{9.108}$$

In the absence of external forces, this is held constant.
Part (c) is also extremely simple, then:

$$D = v_{cm}t = (4/3\,\mathrm{m/s})(8\,\mathrm{s}) = 32/3\,\mathrm{m} \tag{9.109}$$

The rotational motion is bit more tricky.
We can choose to consider torques relative to the center of mass, or relative to a point along the line of the impulse. (We *can* choose differently, but why would we?)
I'm not sure which is easier in the end, but I find it easier to visualize it relative to the center of mass, point C.

The torque is then $\tau_C = Fd$, where F is the magnitude of the force, and d the distance between C and X. It we multiply both sides by the (unknown) impact time, we get $\tau_C \Delta t = (F\Delta t)d$, which is the same as saying $L_C = Id$. The initial angular momentum relative to point C is zero, so this is the total angular momentum after the hit.

The angular momentum relative to point C is about the center of mass; so $L_C = I_c\omega$ also holds (where I_c is the moment of inertia of the rod around the center). Setting the two equal,

$$I_c\omega = Id \tag{9.110}$$

$$\omega = \frac{Id}{I_c} = \frac{Id}{\frac{1}{12}m\ell^2} \tag{9.111}$$

For the numbers given, $\omega = 9.6$ rad/s.

After 8 seconds, it has rotated 76.8 radians, which about 12.22 rotations; the angle should be a bit less than 90 degrees (0.22 radians), in other words.
To find the angle,

$$\theta = 76.8 \bmod (2\pi) = 1.402\,\mathrm{rad} = 80.32^\circ \tag{9.112}$$

where mod gives the remainder after a division. The result is the same as $76.8 - (2\pi \times \lfloor \frac{76.8}{2\pi} \rfloor)$.

Finally, the total kinetic energy. This is simply the sum of the translational (linear) kinetic energy, and the rotational:

$$K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_c \omega^2 = 2.6667\,\mathrm{J} + 2.88\,\mathrm{J} = 5.5467\,\mathrm{J} \tag{9.113}$$

Chapter 10: Weeks 10+11: Homework 8

10.1 Problem 1: Going to the Sun

"A spacecraft of mass m is first brought into an orbit around the earth. The earth (together with the spacecraft) orbits the sun in a near circular orbit with radius R (R is the mean distance between the earth and the sun; it is about 150 million km).

(a) What is the speed v_0 (in m/s) of the earth in its orbit of radius $R = 1.5 \times 10^{11}$ m around the sun with a mass $M = 1.99 \times 10^{30}$ kg? Take the gravitational constant $G = 6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$."

All right, this is a long problem (at least in regards to word count), so I will split it up instead of doing the usual all questions first, all answers later.

First, if we treat the orbit as circular (as they clearly want us to: it is "near circular", and they ask for *the* orbital speed; elliptical orbits don't have a single speed, but one that varies over time.

I tend to not always remember the equation here, but I do always remember that the total mechanical energy is $K_e + U = \frac{1}{2}U$. We can rearrange that prior to substitution of the actual values, and then solve for v_0:

$$K_e + U = \frac{1}{2}U \tag{10.1}$$

$$K_e = -\frac{1}{2}U \tag{10.2}$$

$$\frac{1}{2}mv_0^2 = \frac{mMG}{2R} \tag{10.3}$$

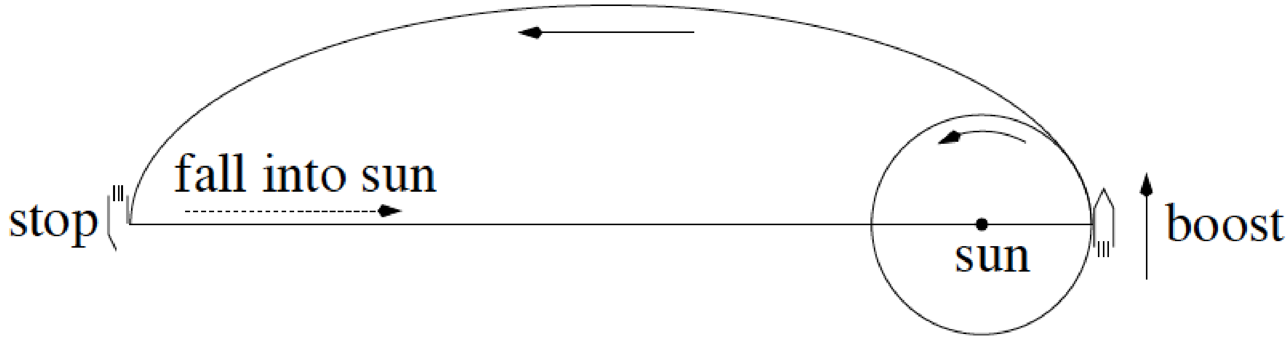
$$v_0 = \sqrt{\frac{MG}{R}} \approx 29\,756\,\mathrm{m/s} \tag{10.4}$$

"We want the spacecraft to fall into the sun. One way to do this is to fire the rocket in a direction opposite to the earth's orbital motion to reduce the spacecraft's speed to zero (relative to the sun).

(b) What is the total impulse I_0 that would have to be given by the rocket to the spacecraft to accomplish this? You may ignore the effect of the earth's gravitation as well as the orbital speed of the spacecraft around the earth as the latter is much smaller than the speed of the earth around the sun. Thus, you may assume that the spacecraft, before the rocket is fired, has the same speed in its orbit around the sun as the earth. Express your answer in terms of m and v_0."

Given that we can neglect almost everything, this is very easy. We have an initial momentum mv_0 (if we indeed neglect the orbital speed around the Earth), and we need to get that speed down to zero, which implies getting our momentum to zero. The change is simply $-mv_0$.
The answer that is accepted is mv_0, however. A bit strange, to me – they don't ask for any magnitudes, and since mv_0 is clearly the current momentum, I would argue that an impulse of $I = mv_0$ would double the current momentum (and thus speed) in the direction the spacecraft is currently moving.
Ah well.

"We will now show that there is a more economical way of doing this (i.e., a much smaller rocket can do the job). By means of a brief rocket burn the spacecraft is first put into an elliptical orbit around the sun; the boost is provided tangentially to the earth's circular orbit around the sun (see figure). The aphelion of the new orbit is at a distance r from the sun. At aphelion the spacecraft is given a backward impulse to reduce its speed to zero (relative to the sun) so that it will subsequently fall into the sun.



(c) Calculate the impulse I_1 required at the first rocket burn (the boost). Express your answer in terms of I_0, R and r."

Okay, so aphelion is the furthest it ever comes from the Sun (perihelion is the closest). If we call aphelion point A, perihelion point P and the Sun point Q, then we have AQ + PQ = $2a$, where a is the orbit's semi-major axis.
If the distance AQ is r, and the current distance PQ from us to the Sun is R, then via the diagram provided, clearly $2a = R + r$, where a is the semi-major axis of the new, elliptical orbit.

Combined with the next question, we need to find the impulse required to move into an elliptical orbit with new speed v_1, such that $a = \dfrac{R+r}{2}$.

We make a burn so that the new speed is v_1, and the new (linear) momentum mv_1. The impulse is then $m(v_1 - v_0)$, but we don't know v_1 yet.
We can figure out v_1 by conservation of energy. *After* the burn, energy is conserved (but not during, of course). The new kinetic energy, plus the new (same as before) potential energy must equal half of the potential energy of the new, elliptical orbit:

$$\frac{1}{2}mv_1^2 - \frac{mMG}{R} = -\frac{mMG}{2a}v_1^2 = \frac{2MG}{R} - \frac{MG}{a} \tag{10.5}$$

$$v_1^2 = 2MG\left(\frac{1}{R} - \frac{1}{2a}\right) \tag{10.6}$$

$$v_1 = \sqrt{2MG\left(\frac{1}{R} - \frac{1}{R+r}\right)} = \sqrt{\frac{2GMr}{R(R+r)}} \tag{10.7}$$

Now, here's the slightly tricky part... We know that $v_0 = \sqrt{\frac{MG}{R}}$, and we need to write the above in terms of v_0. Thankfully, with the simplification done, that is in fact now the opposite of tricky. It could have been! We simply remove those variables from inside the square root, and tack on v_0 outside:

$$v_1 = v_0\sqrt{\frac{2r}{(R+r)}} \tag{10.8}$$

Next, we need to write this in terms of impulse. $I_0 = v_0/m$, and $I_1 = m(v_1 - v_0)$.

$$I_1 = mv_0\sqrt{\frac{2r}{(R+r)}} - mv_0 = I_0\left(\sqrt{\frac{2r}{(R+r)}} - 1\right) \tag{10.9}$$

"(d) What is the speed v_2 of the spacecraft at aphelion? Express your answer in terms of v_0, R and r."

Finally, we need to convert v_1 into v_2. v_1 at perihelion, and v_2 is at aphelion. The speed at perihelion is much greater than that at aphelion.

Angular momentum is the same at both locations. Therefore, $Rmv_1 = rmv_2$, or $Rv_1 = rv_2 \Rightarrow v_2 = \frac{R}{r}v_1$.

$$v_2 = \frac{R}{r}v_0\sqrt{\frac{2r}{R+r}} \tag{10.10}$$

"(e) Calculate the impulse I_2 required at the second rocket burn (at aphelion). Express your answer in terms of I_0, R and r."

This shouldn't be too bad now. We need to bring v_2 down to zero, so

$$I_2 = mv_2 = mv_0\frac{R}{r}\sqrt{\frac{2r}{R+r}} = I_0\frac{R}{r}\sqrt{\frac{2r}{R+r}} \tag{10.11}$$

Again, they want a positive value.

"(f) Compare the impulse under b) with the sum of the impulses under c) and e) (i.e find $I_0 - (I_1 + I_2)$), and convince yourself that the latter procedure is more economical. Express your answer in terms of I_0, R and r."

I will call this ΔI for a lack of a better name.

$$\Delta I = I_0 - \left(I_0\left(\sqrt{\frac{2r}{(R+r)}} - 1\right) + I_0\frac{R}{r}\sqrt{\frac{2r}{R+r}}\right) \tag{10.12}$$

$$\Delta I = I_0 - I_0\left(\sqrt{\frac{2r}{(R+r)}} - 1 + \frac{R}{r}\sqrt{\frac{2r}{R+r}}\right) \tag{10.13}$$

To convince ourselves, we need to find that the expression in parenthesis is always such that $\Delta I > 0$ (otherwise, it's equally or even less efficient).

$$\Delta I = I_0 - I_0\left(\sqrt{\frac{2r}{(R+r)}}\left(1 + \frac{R}{r}\right) - 1\right) \tag{10.14}$$

$$\Delta I = 2I_0 - I_0\left(\sqrt{\frac{2r}{(R+r)}}\left(1 + \frac{R}{r}\right)\right) \tag{10.15}$$

$$\Delta I = I_0\left(2 - \sqrt{2}\sqrt{\frac{R+r}{r}}\right) \tag{10.16}$$

Finally, we can truly convince ourselves by solving this for r manually:

$$2 - \sqrt{2}\sqrt{\frac{R+r}{r}} > 0 \tag{10.17}$$

$$\sqrt{2}\sqrt{\frac{R+r}{r}} < 2 \tag{10.18}$$

$$2\frac{R+r}{r} < 4 \tag{10.19}$$

$$2R < 2r \tag{10.20}$$

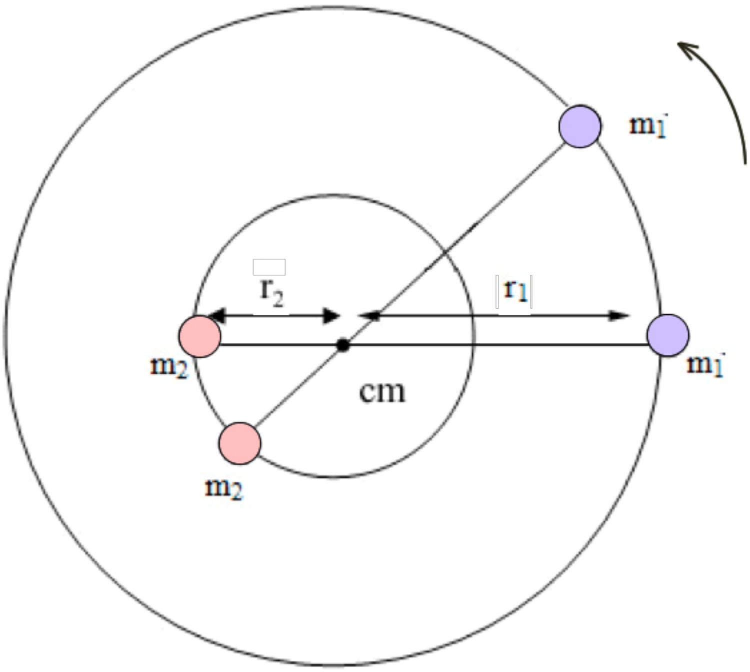
$$R < r \tag{10.21}$$

So indeed, for *any* chosen $r > R$, this is more efficient. Of course, we need to remain in orbit for the result to be useful; we could of course make a ridiculous burn to reach an extremely high speed and escape, which would be less efficient, but in that case, we would have any r as we would not be in an elliptical orbit.

Phew! This took a very long time for me – a while to figure out how to solve part (c), and a *very* long time to figure out where I was going wrong. I got v_0 correct at once, but then accidentally wrote down an incorrect expression in my notes: $v_0 = \sqrt{\frac{2MG}{R}}$. You may notice that is the escape velocity for Earth's orbit, not v_0 – I did too, only the day after I started working on this problem. Once I noticed, everything else went rather smoothly.

10.2 Problem 2: Black hole in X-ray binary

"An X-ray binary consists of 2 stars with masses m_1 (the accreting compact object) and m_2 (the donor). The orbits are circular with radii r_1 and r_2 centered on the center of mass.



(a) Find the orbital period T of the binary following the guidelines given in lectures. Express your answer in terms of $(m_1 + m_2)$, $(r_1 + r_2)$ and G.
(b) In the case of Cyg X-1 (as discussed in lectures), the orbital period is 5.6 days. The donor star is a "supergiant" with a mass 30 times that of the sun. Doppler shift measurements indicate that the donor star has an orbital speed v_2 of about 148 km/sec. Calculate r_2 (in meters).
(c) Calculate r_1 (in meters).
(d) Now calculate the mass m1 of the accreting compact object (express that as ratio to the mass of the sun m_1/M_{Sun})."

Well, part (a) is easy, at least. We even saw that expression, exactly as-is, during the lecture (indeed, in the part about Cygnus X-1, i.e. this system).
We use the equation for periods of elliptical orbits, sometimes known as Kepler's third law (though Kepler only said $T^2 \propto a^3$; the rest was calculated later), only we substitute in $m_1 + m_2$ for the mass, and $r_1 + r_2$ for the orbital radius:

$$T = \sqrt{\frac{4\pi^2 (r_1 + r_2)^3}{G(m_1 + m_2)}} \tag{10.22}$$

For part (b), they tell us the period T, and the velocity v_2. Finding r_2 is a piece of cake, then, if we don't get wrapped up in complex thinking!

$$\frac{2\pi r_2}{T} = v_2 \tag{10.23}$$

$$r_2 = \frac{v_2 T}{2\pi} = \frac{(148 \times 10^3\,\mathrm{m/s})(5.6\,\mathrm{days})}{2\pi} \approx 1.1397 \times 10^{10}\,\mathrm{m} \tag{10.24}$$

We now know T and r_2, but not m_1, m_2 or r_1. For finding r_1, they gave us a hint, though:

"Hint: Your calculations will be greatly simplified if instead of r_1 you set up your equations in terms of r_1/r_2, and using some relation between the distances and the masses. Once you express your equation in terms of r_1/r_2, you will find a third order equation in r_1/r_2. Only one solution is real; the other two are imaginary. There are various ways to find an approximation for r_1/r_2. You can find the solution by trial and error using your calculator, or you can plot the function."

Hmm. Well, via the center of mass definition,

$$m_1 r_1 = m_2 r_2 \tag{10.25}$$

We can certainly find r_1/r_2 from that:

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \tag{10.26}$$

They also tell us that $m_2 = 30 M_{Sun}$.

$$r_1 = \frac{30 M_{Sun}}{m_1} r_2 \tag{10.27}$$

Here is where we need to start applying the hint given. I will copy the staff solution a bit here (i.e. I'm writing this part after the deadline has passed to clean up). We can assign a variable $x = r_1/r_2$.
This then implies that $m_1 = m_2/x$ using the above relationships.

We can now start rewriting the period equation. First, we square it to get rid of the square root on the right-hand side. Then, we factor out r_2^3 and m_2, respectively, out of the parenthesis, to get the insides in fraction form:

$$T^2 = \frac{4\pi^2 r_2^3 (\frac{r_1}{r_2} + 1)^3}{G m_2 (\frac{m_1}{m_2} + 1)} \tag{10.28}$$

Next, we write this in terms of x:

$$T^2 = \frac{4\pi^2 r_2^3 (x + 1)^3}{G m_2 (\frac{1}{x} + 1)} \tag{10.29}$$

Finally, we isolate x on the right hand side:

$$\frac{G m_2 T^2}{4\pi^2 r_2^3} = \frac{(x+1)^3}{\frac{1}{x} + 1} \tag{10.30}$$

We can now approximate this function. We know everything on the left-hand side: $m_2 = 30 M_{Sun} = 30 \times 2 \times 10^{30}\,\mathrm{kg}$, $T = 5.6$ days times 86400 seconds and $r_2 = 1.1397 \times 10^{10}\,\mathrm{m}$.
The left-hand side is approximately equal to 16.04.

We can then plot the two functions

$$y = 16.04 \tag{10.31}$$

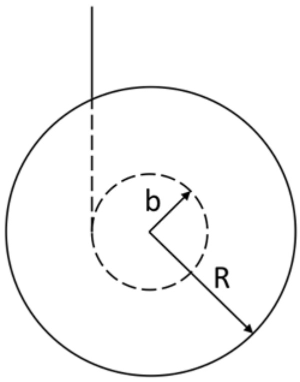
$$y = \frac{(x+1)^3}{\frac{1}{x}+1} \tag{10.32}$$

and see where they intersect. That happens at approximately $x = 1.9031$.

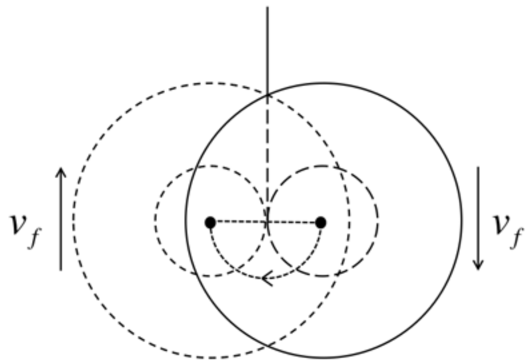
With that value in hand, we can now find $r_1 = xr_2 = 2.169 \times 10^{10}\,\mathrm{m}$ and $m_1 = m_2/x = 15.764$ solar masses.

10.3 Problem 3: Torque, rotation and translation

"A Yo-Yo of mass m has an axle of radius b and a spool of radius R. Its moment of inertia about the center can be taken to be $I = \dfrac{1}{2}mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest. You will need to assume that the center of mass of the Yo-Yo descends vertically, and that the string is vertical as it unwinds.



(a) What is magnitude of the tension in the cord as the Yo-Yo descends? Express your answer in terms of m, b, R and acceleration due to gravity g.
(b) Find the angular speed of the Yo-Yo when it reaches the bottom of the string, when a length ℓ of the string has unwound. Express your answer in terms of m, b, R, ℓ and acceleration due to gravity g.
(c) Find the magnitude of the average tension in the string over the course of the Yo-Yo reversing its direction at the bottom of its descent (see figure below). Express your answer in terms of m, b, R, ℓ and acceleration due to gravity g."



Let's see. First, we can write an equation for the acceleration of the center of mass, in terms of string tension acting upwards, and gravity acting downwards. We choose downwards to be the positive direction, and find

$$ma = mg - T \tag{10.33}$$

Next, we can consider the torque. I will do so considering the center of the yo-yo, call it point C:

$$\tau_C = I_C \alpha \tag{10.34}$$

The torque is due to the tension acting on the inner spool of radius b, and is $\tau_C = Tb$.
We can also use the relationship $a = \alpha R$, which holds if there is no slipping. With these two things in mind, we can rewrite the above equation as

$$Tb = I_C \frac{a}{b} \tag{10.35}$$

We can solve for the tension by solving these for A and setting them equal.

$$a = g - \frac{T}{m} \tag{10.36}$$

$$a = \frac{Tb^2}{I_C} \tag{10.37}$$

$$g - \frac{T}{m} = \frac{Tb^2}{I_C} \tag{10.38}$$

$$g = T\left(\frac{b^2}{I_C} + \frac{1}{m}\right) \tag{10.39}$$

$$\frac{g}{\frac{b^2}{I_C} + \frac{1}{m}} = T \tag{10.40}$$

We are given that $I_C = \frac{1}{2}mR^2$, so we can stick that in there and simplify to find the tension in terms of the variables they want:

$$\frac{g}{\frac{b^2}{(1/2)mR^2} + \frac{1}{m}} = T \tag{10.41}$$

$$\frac{mg}{2\frac{b^2}{R^2} + 1} = T \tag{10.42}$$

$$\frac{mgR^2}{2b^2 + R^2} = T \tag{10.43}$$

Next, they want the angular speed when the Yo-Yo reaches the bottom.
Now, we have a situation equivalent to pure roll, which means that the tangential velocity is always equal to the velocity of the center of mass.
We can therefore solve this more easily (I believe it's easier, anyway) by using a, using that to find the velocity of the center of mass, which then is equal to the tangential velocity, and converting that to angular speed.

We have an expression for the acceleration as a function of T, and we know T, so

$$a = g\left(1 - \frac{R^2}{2b^2 + R^2}\right) \tag{10.44}$$

Since the acceleration is clearly constant in time, the velocity as a function of acceleration is just $v = at$, but we don't know t.
We can use a second constant acceleration kinematics equation, though: $\ell = \frac{1}{2}at^2$. We solve that one for t:

$$\ell = \frac{1}{2}at^2 \tag{10.45}$$

$$\sqrt{\frac{2\ell}{a}} = t \tag{10.46}$$

Combining the two,

$$v_f = a\sqrt{\frac{2\ell}{a}} = \sqrt{2\ell a} = \sqrt{2\ell}\sqrt{g\left(1 - \frac{R^2}{2b^2 + R^2}\right)} \tag{10.47}$$

Finally, to convert to angular speed, we simply use $v_f = \omega b$, so $\omega = \frac{v_f}{b}$:

$$\omega = \frac{\sqrt{2\ell}}{b}\sqrt{g\left(1 - \frac{R^2}{2b^2 + R^2}\right)} \tag{10.48}$$

This can be simplified quite a bit further (I used Mathematica for this one):

$$\omega = 2\sqrt{\frac{g\ell}{2b^2 + R^2}} \tag{10.49}$$

We should be able to solve the last part in terms of impulse. If the speed v_f going back up is the same as the speed down, as the diagram shows, the impulse is $2mv_f$.
The average force acting on the yo-yo is found via

$$I = \langle F \rangle \Delta t \tag{10.50}$$

However, the average force and the average tension are not the same thing. Regardless of the tension, there is clearly a constant downwards force mg acting on the yo-yo, due to gravity. Let's take care of that last.

$$\frac{2mv_f}{\Delta t} = \langle F \rangle \tag{10.51}$$

Of course, this causes a new problem: what is Δt? We know the speed v_f just prior to and just after, but what about during this turnaround?
Because the angular velocity is about the same for the entire turnaround (it doesn't switch directions), v_f is also approximately constant, since the two are linearly proportional.

In that case, $\Delta t = d/v_f$, where d is the distance traveled during this time. So what is *that*, then? I would think it is half the circumference of the inner spool, which is πb. We can then find the time as the distance divided by the tangential velocity, $\Delta t = (\pi b)/v_f$, so using that, plus our expression of the velocity v_f as the string is unwrapped:

$$\frac{2mv_f^2}{\pi b} = \langle F \rangle \tag{10.52}$$

$$\frac{2m}{\pi b}2\ell g\left(1 - \frac{R^2}{2b^2 + R^2}\right) = \langle F \rangle \tag{10.53}$$

(Side note added afterwards: we can just as easily, probably more easily, consider that it moves π radians about the inner spool, and use ω at the turnaround point to calculate the time taken.)

Let's now not forget that $\langle F \rangle$ is the average *net* force on the object. Gravity is pulling it down, which the tension is trying to counteract. Therefore, we *add* mg, $\langle F_{gravity} \rangle$ (which is thankfully a constant) to the above to find the average tension:

$$mg + \frac{2m}{\pi b} 2\ell g \left(1 - \frac{R^2}{2b^2 + R^2}\right) = \langle T_r \rangle \tag{10.54}$$

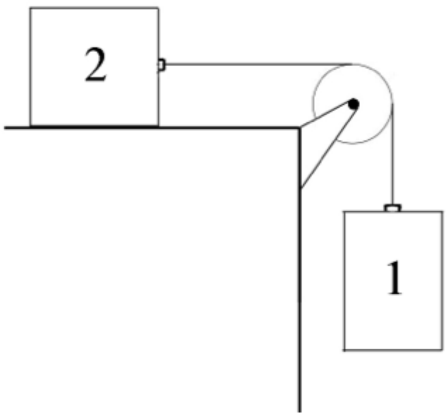
$$mg + \frac{4m\ell g}{\pi b} \left(1 - \frac{R^2}{2b^2 + R^2}\right) = \langle T_r \rangle \tag{10.55}$$

Again, this can be simplified quite a bit, and again, I used Mathematica for that part:

$$\langle T_r \rangle = mg \left(\frac{8b\ell}{2\pi b^2 + \pi R^2} + 1\right) + m \tag{10.56}$$

10.4 Problem 4: Double block pulley

"A pulley of mass m_p, radius R, and moment of inertia about its center of mass I_c, is attached to the edge of a table. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to block 1 that hangs over the edge of the table. The other end of the string is attached to block 2 which slides along a table. The coefficient of sliding friction between the table and the block 2 is μ_k . Block 1 has mass m_1 and block 2 has mass m_2, with $m_1 > \mu_k m_2$. At time $t = 0$, the blocks are released from rest. At time $t = t_1$, block 1 hits the ground. Let g denote the gravitational acceleration near the surface of the earth.



(a) Find the magnitude of the linear acceleration of the blocks. Express your answer in terms of m_1, m_2, I_c, R, μ_k and g as needed.
(b) How far did the block 1 fall before hitting the ground? Express your answer in terms of m_1, m_2, I_c, R, μ_k, t_1 and g as needed."

All right, time to look at some forces, to begin with!

Block 2 has four forces acting on it: mg downwards, $N = mg$ upwards (since there is no acceleration along the y axis, they must cancel), a tension T_2 towards the right, and friction $F_f = \mu_k m_2 g$ towards the left. Block 1 has only two: mg downwards, and T_1 upwards.

Newton's second law for the two gives us, taking downwards (block 1) = rightwards (block 2) as positive:

$$m_2 a = T_2 - \mu_k m_2 g \tag{10.57}$$

$$m_1 a = m_1 g - T_1 \tag{10.58}$$

Next, we can consider the torque and angular acceleration of the pulley. Relative to the center C of the pulley, the torque is $I_c \alpha$. As usual, we use $a = \alpha R$ to rewrite this in terms of the linear acceleration a, and assume there is no slipping or such going on.

$$\tau_C = I_c \frac{a}{R} \tag{10.59}$$

So what is the torque? Well, we can write it as the torque due to T_1 (which causes clockwise rotation) minus the torque due to T_2. Both act at 90 degree angles with the center, so

$$R(T_1 - T_2) = I_c \frac{a}{R} \tag{10.60}$$

We now have three equations and three unknowns: a, T_1 and T_2. If we solve the tension equations for T_2 and T_1 respectively, we can find $T_1 - T_2$ easily, and therefore a.
First, I will solve the above equation for a:

$$a = \frac{R^2}{I_c}(T_1 - T_2) \tag{10.61}$$

Solving the two is also easy:

$$m_2 a + \mu_k m_2 g = T_2 \tag{10.62}$$
$$m_1 g - m_1 a = T_1 \tag{10.63}$$

All that remains is to combine the three as mentioned, and solve for a:

$$a = \frac{R^2}{I_c}(m_1 g - m_1 a - m_2 a - \mu_k m_2 g) \tag{10.64}$$
$$a = \frac{R^2}{I_c}(m_1 g - \mu_k m_2 g) - \frac{R^2}{I_c} m_1 a - \frac{R^2}{I_c} m_2 a \tag{10.65}$$
$$a\left(1 + \frac{R^2 m_1}{I_c} + \frac{R^2 m_2}{I_c}\right) = \frac{R^2}{I_c}(m_1 g - \mu_k m_2 g) \tag{10.66}$$
$$a = \frac{\frac{R^2}{I_c}(m_1 g - \mu_k m_2 g)}{1 + \frac{R^2 m_1}{I_c} + \frac{R^2 m_2}{I_c}} \tag{10.67}$$
$$a = \frac{g R^2 (m_1 - \mu_k m_2)}{I_c + R^2(m_1 + m_2)} \tag{10.68}$$

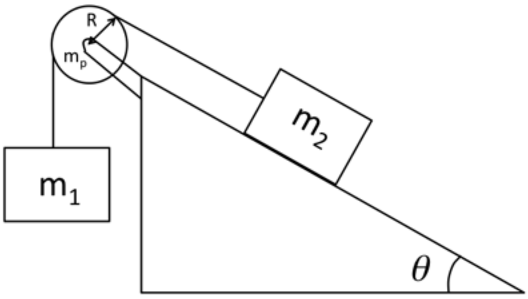
Well then! Let's see about part (b).
Is the acceleration constant? Yes, it is; nothing in there should change over time. Therefore, we can answer this one using some very basic kinematics:

$$d = \frac{1}{2} a t_1^2 = \frac{1}{2} \frac{g R^2 (m_1 - \mu_k m_2)}{I_c + R^2(m_1 + m_2)} t_1^2 \tag{10.69}$$

10.5 Problem 5: Wheel, inclined plane, two masses and a rope

"A wheel in the shape of a uniform disk of radius R and mass m_p is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass $I_{cm} = \frac{1}{2} m_p R^2$. A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass m_2 that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass m_1 that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle θ. Once the objects are released from rest, the cord moves without slipping around the disk. Find the magnitude of accelerations of each object, and the magnitude of tensions in the string on either side of the pulley.

Assume that the cord doesn't stretch ($a_1 = a_2 = a$). Express your answers in terms of the masses m_1, m_2, m_p, angle θ and the gravitational acceleration due to gravity near earth's surface g."



They then ask for a, T_1 (tension at m_1) and T_2 (tension at m_2).
This certainly looks like the slightly more complex brother of the previous problem!

To begin with, we can't know which of the masses will "win", if any. If static friction wins, then $a = 0$, which is the trivial solution and one that I will not even attempt to submit. What happens otherwise? Well, getting the sign correct is guesswork, as far as I can tell; according to forum discussions, this seems to be the consensus. I will call downwards (for m_1) and uphill (for m_2) positive in this solution.

Well then! We yet again have a bunch of forces. The forces on the hanging mass are unchanged, so we get the same equation there:

$$m_1 a = m_1 g - T_1 \tag{10.70}$$

Block 2 changes the game a little. We have the same four forces, but we now need to decompose the gravitational force into the normal force component and the "downhill" component. Performing the decomposition, we find the normal force as $m_2 g \cos\theta$, which the downhill force is $m_2 g \sin\theta$.
The incline is frictionless, so gravity is the only downhill force. There is a tension T_2 uphill, however, All in all, the normal force cancels out the component of gravity perpendicular to the incline, while T_2 uphill and $m_2 g \sin\theta$ battles where the block should move.
Using the directions I chose,

$$m_2 a = T_2 - m_2 g \sin\theta \tag{10.71}$$

Two equations, three unknowns. We need to consider the pulley next, as usual in these problems.
As in the previous problem, the tensions cause a torque, and both are perpendicular to the center of the wheel. The torque relative to the pulley's center is $\tau_C = R(T_1 - T_2)$, which again is equal to $I_{cm}\alpha = I_{cm}\frac{a}{R}$.
We were given I_{cm} in terms of mass and radius:

$$R(T_1 - T_2) = \frac{1}{2} m_p R a \tag{10.72}$$

$$\frac{2(T_1 - T_2)}{m_p} = a \tag{10.73}$$

I will again find $T_1 - T_2$ by solving those two equations individually and subtracting them:

$$m_1 g - m_1 a = T_1 \tag{10.74}$$

$$m_2 a + m_2 g \sin\theta = T_2 \tag{10.75}$$

$$T_1 - T_2 = m_1 g - m_1 a - m_2 a - m_2 g \sin\theta \tag{10.76}$$

Substitute that in to the torque equation:

$$a = \frac{2}{m_p}\left(m_1 g - m_1 a - m_2 a - m_2 g \sin\theta\right) \tag{10.77}$$

$$a = \frac{2}{m_p}\left(m_1 g - m_2 g \sin\theta\right) - \frac{2m_1}{m_p}a - \frac{2m_2}{m_p}a \tag{10.78}$$

$$a\left(1 + \frac{2m_1}{m_p} + \frac{2m_2}{m_p}\right) = \frac{2}{m_p}\left(m_1 g - m_2 g \sin\theta\right) \tag{10.79}$$

$$a = \frac{\frac{2}{m_p}\left(m_1 g - m_2 g \sin\theta\right)}{1 + \frac{2m_1}{m_p} + \frac{2m_2}{m_p}} \tag{10.80}$$

$$a = \frac{2g\left(m_1 - m_2 \sin\theta\right)}{m_p + 2m_1 + 2m_2} \tag{10.81}$$

$$\tag{10.82}$$

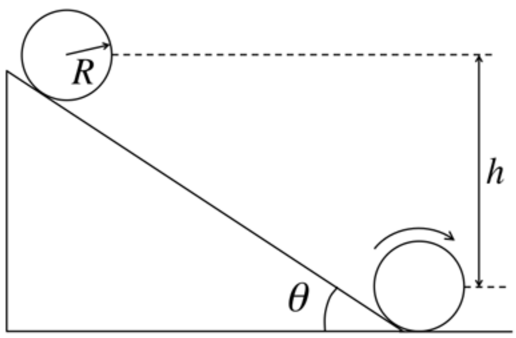
We can then find the tensions easily, since we solved for them earlier. Whether it will be pretty is a different matter!

$$T_1 = m_1\left(g - \frac{2g\left(m_1 - m_2 \sin\theta\right)}{m_p + 2m_1 + 2m_2}\right) \tag{10.83}$$

$$T_2 = m_2\left(g\sin\theta + \frac{2g\left(m_1 - m_2 \sin\theta\right)}{m_p + 2m_1 + 2m_2}\right) \tag{10.84}$$

10.6 Problem 6: Rolling object on an incline

"A hollow cylinder of outer radius R and mass M with moment of inertia about the center of mass $I_{cm} = MR^2$ starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the acceleration due to gravity. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. The goal of this problem is to find an expression for the smallest possible value of μ_s such that the cylinder rolls without slipping down the incline plane and the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



(a) What is the magnitude of the acceleration a of the center of mass of the cylinder on the incline? Express your answer in terms of θ and g as needed.
(b) What is the minimum value for the coefficient of static friction μ_s such that the cylinder rolls without slipping down the incline plane? Express your answer in terms of θ.
(c) What is the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline? Express your answer in terms of g and h as needed."

Okay, so let's see. First, what happens with zero friction? Clearly, there is no rolling at all, since there will be no torque on the cylinder.
For there to be pure roll, a condition that must be fulfilled is that the tangential speed ωR is the same as

the velocity at the center of mass. Since the angular acceleration is $\alpha = a/R$, we must have $\alpha R = a$ for pure roll to hold.

Let's start out with part (a) and see where that leads.
If there is is no slipping, then there is no kinetic friction. There is, however, *static* friction. Without that, the cylinder would slide down without turning at all.

If we choose a coordinate system where y is perpendicular to the incline, while x is downhill, we can write the the normal component of gravity as $Mg\cos\theta$, while the downhill component is $Mg\sin\theta$. Static friction acts upwards: the friction must be such that the torque causes clockwise rotation (or it would roll the wrong way!). This implies an uphill frictional force, $\mu_s N = \mu_s Mg\cos\theta$. (Another way to think of it is that the cylinder wants to slide downhill. Therefore, static friction acts uphill, since friction always *opposes* relative motion between surfaces.)

$$Ma = Mg\sin\theta - \mu_s Mg\cos\theta \tag{10.85}$$
$$a = g(\sin\theta - \mu_s\cos\theta) \tag{10.86}$$

This would seem to answer part (a), but we're not allowed to use μ_s in the answer, so we need to keep working.

As mentioned earlier, in this analysis, the cylinder rolls due to the torque caused by friction. Friction acts uphill, and the magnitude of the torque, relative to the center of the cylinder, is $RF_f = RMg\mu_s\cos\theta$.

A useful relation is then that $\tau = I_{cm}\alpha = I_{cm}\dfrac{a}{R}$ (the latter part holds for pure roll only), and we are given that $I_{cm} = MR^2$, so

$$RMg\mu_s\cos\theta = aMR \tag{10.87}$$

M and R both cancel.
We can solve this for μ_s:

$$g\mu_s\cos\theta = a \tag{10.88}$$
$$\mu_s = \frac{a}{g\cos\theta} \tag{10.89}$$

This gives us the acceleration, now that we can write μ_s in terms of g and $\cos\theta$:

$$a = g(\sin\theta - \frac{a}{g\cos\theta}\cos\theta) \tag{10.90}$$
$$a = g\sin\theta - a \tag{10.91}$$
$$a = \frac{g\sin\theta}{2} \tag{10.92}$$

Next, we substitute this back into μ_s to get it in terms of θ:

$$\mu_s = \frac{\frac{g\sin\theta}{2}}{g\cos\theta} \tag{10.93}$$
$$\mu_s = \frac{\tan\theta}{2} \tag{10.94}$$

Very nice and simple! This is the *minimum* amount of friction required for pure roll. More friction wouldn't hurt; as the acceleration equation shows, more friction doesn't cause less acceleration, but it does prevent sliding.

Finally, what is the velocity of the center of mass as it reaches the bottom? Well, we know the acceleration. We could use the work-energy theorem, but there will be both linear kinetic energy and rotational kinetic energy, so that seems like it would be harder. Then again, we don't know the *time* which we need for kinematics, so I will go the energy route anyway.
The final velocity v causes an angular velocity $v = \omega/R$ with no sliding. The total kinetic energy can be written down as being equal to Mgh, which is the total energy available to be converted:

$$K_{lin} + K_{rot} = Mgh \tag{10.95}$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\omega^2 = Mgh \tag{10.96}$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}(MR^2)\left(\frac{v}{R}\right)^2 = Mgh \tag{10.97}$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mgh \tag{10.98}$$

$$v^2 = gh \tag{10.99}$$

$$v = \sqrt{gh} \tag{10.100}$$

Nice! The intermediate results were semi-complex at times, but the answers are all dead simple.

There are several interesting things in this result, at least two of which I didn't realize until a few days after solving this. One is that the rotational kinetic energy in this case is exactly equal to the linear kinetic energy – the expression on the right in equation 4 above simplifies to $\frac{1}{2}Mv^2$!
Without this term, the velocity would be $\sqrt{2gh}$ instead, i.e. exactly a factor $\sqrt{2}$ greater, regardless of much of anything else.
(I rewrote that equation after realizing this; I previously had it in a form which made this hard to see.)

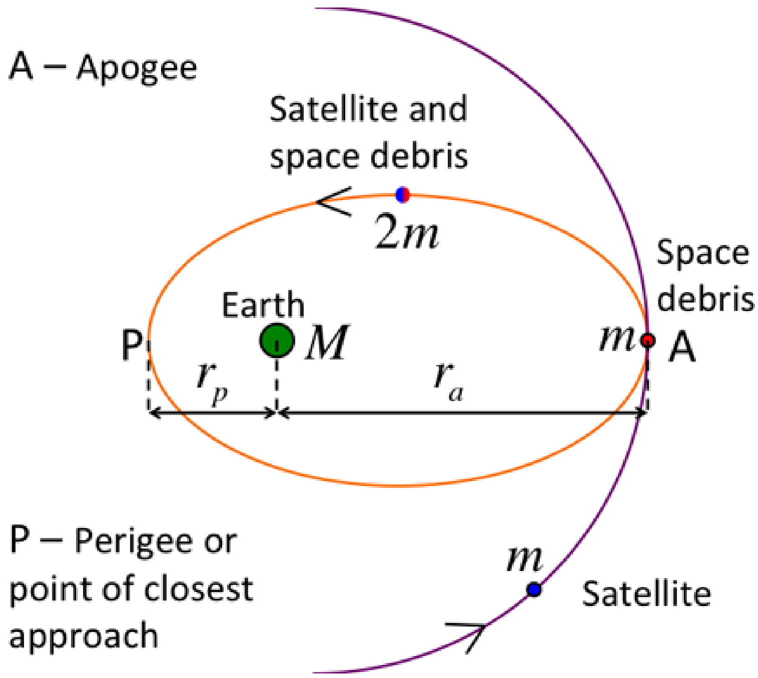
Second, I chose to analyze this relative to the center, which means that static friction provides a torque. How can there be an increase in rotational kinetic energy without a torque that does positive work? As far as I know, there certainly cannot. Therefore, according to this analysis, static friction provides this positive work!
However, it still does no net work, which is the amazing thing: it is a linear force uphill, which therefore fights with $Mg\sin\theta$ about the linear acceleration. It therefore acts to reduce the final linear velocity by a factor $1/\sqrt{2}$, and the final kinetic energy by a factor $1/2$; this is instead turned into rotational kinetic energy here.

So while static friction appears to does positive work increasing the rotational kinetic energy, it appears to do an equal amount of negative work in the linear motion, for a net of zero work – as it must be.

If we instead analyze this problem relative to the point of contact, friction can provide no torque (as it acts through that point), and we will instead find gravity providing the torque and therefore doing the work that gets the cylinder rolling. For other points, where both forces can cause a torque relative to the center, we should find some combination of the two effects, but with the same end result.

10.7 Problem 7: Space debris collision

"A satellite of mass m is orbiting the earth, mass M, in a circular orbit of radius r_a. Unfortunately a piece of space debris left by a passing rocket lies directly in the satellite's path. The piece of debris has the same mass m as the satellite. The debris collides with the satellite and sticks to the satellite. Assume that the debris has negligible speed just before the collision. After the collision, the satellite and debris enter an elliptical orbit around the earth. The distance of closest approach to the earth of the satellite and the debris is r_p. Let G be the universal constant of gravity. You may assume that $M \gg m$.



(a) Find an expression for the speed v_0 of the satellite before the collision. You may express your answer in terms of M, r_a and G as needed.
(b) Calculate the ratio r_a/r_p."

I wonder how realistic the answers will be – a piece of debris with negligible speed (relative to the Earth) wouldn't stay in place for very long!

The satellite begins with linear momentum mv_0. After the hit, the mass doubles, and so velocity is cut in half. Call this post-hit velocity v_a (a for apogee); using conservation of (linear) momentum, we then have $mv_0 + 0 = 2mv_a$, so indeed $v_a = \frac{v_0}{2}$.
We could also find this relationship using conservation of angular momentum relative to the center of the Earth, by the way.

Given this "initial" velocity v_a and the initial distance to the Earth, we could find the orbital parameters for the new elliptical orbit, but I don't believe we will need all of them. Clearly, the apogee distance r_a is simply the initial radius of the circular orbit, which is even given in the problem, only they don't mention it explicitly, but use the same variable for the two (and draw the graphic showing the two are equal).

Now, then. How can we calculate v_0? Well, we know the orbital radius, and for a circular orbit, each orbital radius has unique velocity. This velocity can be derived by remembering that the total mechanical energy is always $\frac{1}{2}U$, but I'm confident that I remember the quite simple velocity equation, so:

$$v_0 = \sqrt{\frac{MG}{r_a}} \tag{10.101}$$

$r_a + r_p = 2a$, where a is the elliptical orbit's semi-major axis. Via an energy calculation, we can relate the new velocity v_a plus the current potential energy with the total mechanical energy for an elliptical orbit, which depends on $2a$, so we can find a.

$$\frac{1}{2}(2m)v_a^2 - \frac{2mMG}{r_a} = -\frac{2mMG}{2a} \tag{10.102}$$

$$v_a^2 - \frac{2MG}{r_a} = -\frac{2MG}{2a} \tag{10.103}$$

$$\frac{1}{v_a^2 - \frac{2MG}{r_a}} = -\frac{a}{MG} \tag{10.104}$$

$$\frac{MG}{\frac{2MG}{r_a} - v_a^2} = a \tag{10.105}$$

$$\frac{r_a MG}{2MG - r_a v_a^2} = a \tag{10.106}$$

Since we know that $r_a + r_p = 2a$, the above must be equal to $\frac{r_a+r_p}{2}$. We can also substitute in the value for $v_a = v_0/2 = \frac{1}{2}\sqrt{\frac{MG}{r_a}}$:

$$\frac{r_a MG}{2MG - r_a \frac{MG}{4r_a}} = \frac{r_a + r_p}{2} \tag{10.107}$$

$$\frac{4r_a MG}{8MG - MG} = \frac{r_a + r_p}{2} \tag{10.108}$$

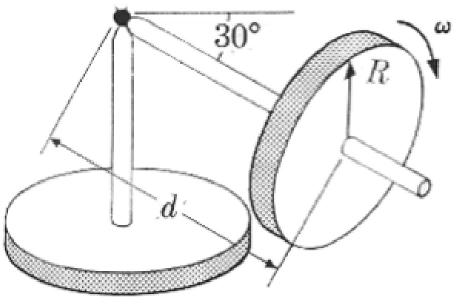
$$\frac{8r_a}{7} - r_a = r_p \tag{10.109}$$

$$\frac{r_a}{7} = r_p \tag{10.110}$$

Well, that sure became simple. The ratio is then $r_a/r_p = 7$ – showing that the apogee is at a (much) greater than distance than the perigee, as one would expect.

And that's it for this problem!

10.8 Problem 8: Turntable solutions



"A gyroscope consists of a uniform disc of mass radius $M = 2$ kg and radius $R = 0.3$ m. The disc spins with an angular speed $\omega = 400$ rad/s as shown in the figure [above]. The gyroscope precesses, with its axle at an angle 30° below the horizontal (see figure). The gyroscope is pivoted about a point $d = 0.6$ m from the center of the disc. What is the magnitude of the precessional angular velocity Ω (in radians/sec)?"

Time for a short break to read the textbook! I'm unsure whether we can use $\Omega = \dfrac{\tau}{I_c\omega}$ here (after vector decomposition), or not.
They indeed seem to consider that we can ignore any angular momentum due to the orbital motion, and therefore, this approximation should be valid. Very well, then.
My solution will be less rigorous than the quite technical discussion in the textbook; if you want more detail, I recommend having a look there. Actually, I would recommend that either way!

The torque relative to what I will call point P, the pivot point where the axle meets the stand, is $\tau_P = \vec{d} \times \vec{F_{gr}} = (\vec{d} \times \vec{g})M$. Unlike what we have seen previously, the angle is not 90 degrees. Gravity is always straight downwards, of course, but as the angle the axle makes with the horizontal grows (downwards), the torque goes down. It is at a maximum with $\theta = 0$, and zero when the axle is pointing straight down (which makes sense: the two vectors are then anti-parallel, so the cross product must be zero). The equation then becomes

$$\tau_p = \vec{d} \times \vec{F_g} = dMg\cos\theta \tag{10.111}$$

(where θ is the angle that is marked as 30 degrees).

Why a cosine, in a cross product? Because the angle *between the two vectors* is not equal to the 30° degrees shown, but instead is $90° - 30°$ degrees. It makes intuitive sense that when the angle shown is zero, the torque is at a maximum, and when the axle is vertical, there is zero torque.
We could write the cross product as $dMG\sin\alpha$, where α is the angle between the vectors, followed by $\alpha + \theta = 90°$. This then makes it clear that we need $\sin\alpha = \sin(90° - \theta) = \cos\theta$. I will write it in terms of the cosine of θ, since that gives us a simple expression in terms of the given variables ($\theta = 30°$).

The spin angular momentum due to the disk spinning about its center of mass can be written as $I_c\omega$, where $I_c = \frac{1}{2}MR^2$ for a solid disk.
The direction of this is "inwards" along the axle, no matter the axle's angle; so radially inwards and partially upwards, in this case.

We now know torque and the spin angular momentum. The spin angular momentum needs to be decomposed, though, as only the radial portion matters for the precession.
Consider the time when the system has rotated such that the view from the angle the figure is shown is now such that the axle is in the plane of the page, and we see the disk head-on, on the right side of the pivot point.
The torque is then pointed into the page, while spin angular momentum points left/upwards, at an angle with the horizontal due to the non-horizontal axle.

Left/upwards in more mathematical terms would mean $-\hat{r}$ (left) and $+\hat{k}$ (upwards), using cylindrical coordinates, where $+\hat{\theta}$ is into the page.

As the disk/gyroscope precesses, only the direction of the radial component changes, with the center of mass of the disk tracing out a circle in a horizontal plane. The angle, and therefore the upwards/z component does not change as long as ω (the disk's spin angular velocity) is held constant. Neither does the magnitude of the spin angular momentum change; the only change in its direction, as mentioned.
The time derivative of $\hat{r}$ is given as $\frac{d\theta}{dt}\hat{\theta} = \Omega\hat{\theta}$, i.e. into the page. However, if we treat this more rigorously, we will find that Ω is negative, and so the system will move "towards us" as seen here (clockwise as seen from above).

For a more rigorous treatment, see chapter 22 in the textbook (the end of page 22-14 and onwards).

All in all, we have

$$|\Omega| = \frac{|\tau|}{|L_{spin}|} = \frac{dMg\cos\theta}{0.5MR^2\omega\cos\theta} \tag{10.112}$$

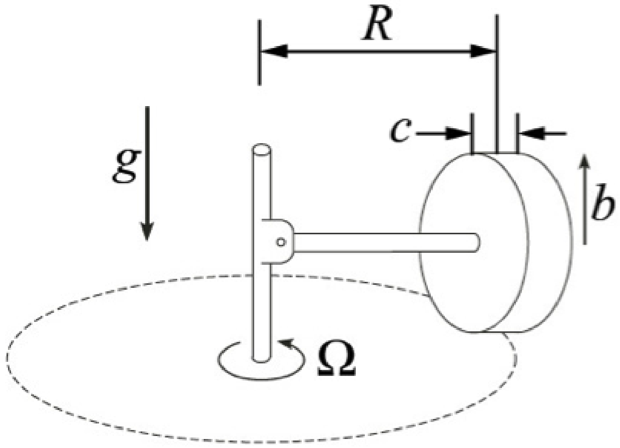
$$= \frac{2dg}{R^2\omega} \tag{10.113}$$

For these values, $\Omega = \frac{1}{3}$ rad/s (using $g = 10\,\mathrm{m/s^2}$, which it appears we are supposed to), which is accepted as correct!
I was a bit worried when the cosines cancelled out, as I expected the angle to matter. Apparently, the

effect is indeed cancelled out, as both the torque component and the spin angular momentum component that matter are smaller (by the same factor).

10.9 Problem 9: Grain mill

"In a grain mill, grain is ground by a massive wheel which rolls without slipping in a circle on a flat horizontal surface driven by a vertical shaft. The rolling wheel has radius b and is constrained to roll in a horizontal circle of radius R at angular speed Ω. Because of the stone's angular momentum, the contact force with the surface can be considerably greater than the weight of the wheel. In this problem, the angular speed Ω about the shaft is such that the contact force between the ground and the wheel is equal to twice the weight. The goal of the problem is to find Ω. Assume that the wheel is closely fitted to the axle so that it cannot tip, and that the width of the wheel $c \ll R$. Neglect friction and the mass of the axle of the wheel. Let g denote the acceleration due to gravity.



(a) How is the angular speed ω of the wheel about its axis related to the angular speed Ω about the shaft? Express your answer in terms of Ω, b and R.
(b) What is the horizontal component of the angular momentum vector about the point P in the figure above? Although we have not shown this, for this situation it is correct to compute the horizontal component of the angular momentum by completely ignoring the rotation of the mill wheel about the vertical axis, taking into account only the rotation of the mill wheel about its own axle. Express your answer in terms of Ω, M, b and R.
(c) What is the magnitude of the torque about the joint (about the point P in the figure above) due to the forces acting on the axle-wheel combination? Express your answer in terms of N, R, M, g.
(d) What is the value of Ω if the contact force between the stone and the ground $N = 2Mg$? Express your answer in terms of b and g."

Having read the section in the book (chapter 22) on exactly this problem, I feel like I'm cheating here! I will do what I can to derive everything I use, in order to ensure I understand it all, at least.

All right. The first part is rather easy, at least: the center of mass of the wheel must move with speed $v_{cm} = \omega b$ if there is no slip (this is a condition of pure roll). Meanwhile, the entire wheel is also rotating about the center axis with angular speed Ω, which can be used to find $v_{cm} = \Omega R$ separately from the previous relationship.
We can then simply set the two equal and solve for ω, since Ω is allowed in the answer:

$$\omega b = \Omega R \tag{10.114}$$

$$\omega = \frac{\Omega R}{b} \tag{10.115}$$

Next, the horizontal component of the angular momentum relative to point P. Given the hint in the problem, this is very easy. The angular momentum about the axle's axis due to the rotation (about the wheel's center of mass) is just $I_c\omega$, where we use $I_c = \frac{1}{2}Mb^2$ for a solid disk of radius b (not R in this problem!):

$$L_P = I_c\omega = (\frac{1}{2}Mb^2)\left(\frac{\Omega R}{b}\right) = \frac{\Omega RMb}{2} \tag{10.116}$$

Part (c) is regarding the magnitude of the torque about the center axle (point P is not in the figure, but it is in the book; it is where the axle connects to the vertical bar, at the hinge).

Well, what forces could cause a torque? Gravity acting on the wheel certainly counts; the torque at P (see above) due to gravity acting on the wheel is $\tau_{P,gravity} = (\vec{R} \times \vec{g})M = RMg$ (there is a 90 degree angle, so $\sin\theta = 1$), with the direction being into the page (causing rotation as shown for Ω).
Next, there is the normal force $N = 2Mg$ causing a torque $\vec{R} \times \vec{N} = RN = 2RMg$, with the direction being out of the page, opposing the previous torque.
If the axle is taken to be massless, there are no other forces that act such that they cause a torque relative to point P.

The net torque, or at least the magnitude of it, is just the torque due to the normal force minus the torque due to gravity:

$$|\tau_{P,net}| = RN - RMg \tag{10.117}$$

We could write this in terms of $2Mg$ instead of N, but the grader really wants it in terms of N, according to the forum discussions. I submitted the above as the first attempt, and it was indeed accepted.

Finally, what is Ω, in terms of only b and g, at a time where $N = 2Mg$?
This is the precession frequency – note how the system looks a lot like a gyroscope. (It's even as an example in the gyroscope section in the book.)

We learned in lecture that $\Omega = \dfrac{\tau}{L_{spin,cm}}$, but this only holds if $L_{spin,cm} \gg L_{orbital}$, which doesn't appear to be the case here. In the case of a typical gyroscope, the spin could be several thousand rpm (200π rad/s or more), while the orbits was closer to 5 per minute or even less.
Here, the two are much closer together.

We can solve this in (at least?) two ways. One is, in fact, to use the above equation:

$$\Omega = \frac{2RMg - RMg}{\frac{1}{2}\Omega RMb} \tag{10.118}$$

$$\Omega^2 = \frac{g}{\frac{1}{2}b} \tag{10.119}$$

$$\Omega = \sqrt{\frac{2g}{b}} \tag{10.120}$$

The second is to find the torque as $\dfrac{dL}{dt}$ (i.e. take the time derivative of L above) and set that equal to the torque we found earlier. However, to do this properly, we need to consider the directions properly too. Check the book (chapter 22) for a proper derivation. The result is:

$$\frac{dL}{dt} = \frac{\Omega^2 RMb}{2} = RMg \tag{10.121}$$

$$\Omega^2 b = 2g \tag{10.122}$$

$$\Omega = \sqrt{\frac{2g}{b}} \tag{10.123}$$

The source of the extra Ω is tricky, since I have not written all this in terms of components and unit vectors. The source of it is due to the differentiation of the $\hat{r}$ unit vector:

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta} \tag{10.124}$$

where θ is the position along the circle, and $\hat{\theta}$ is the unit vector in the azimuthal direction (in cylindrical coordinates). Ω is just the time rate of change of this angle, by definition, so that $\Omega = \frac{d\theta}{dt}$. Therefore, in terms of vectors,

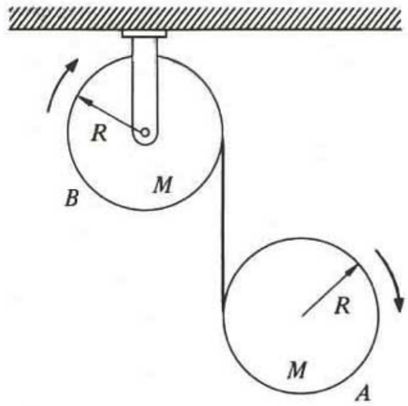
$$\vec{L}_{spin,cm} = \frac{\Omega RMb}{2}(-\hat{r}) \tag{10.125}$$

$$\frac{d\vec{L}_{spin,cm}}{dt} = \frac{\Omega RMb}{2}\left(-\frac{d\hat{r}}{dt}\right) = \frac{\Omega RMb}{2}\left(-\Omega\hat{\theta}\right) \tag{10.126}$$

The magnitude is therefore multiplied by Ω in this differentiation. I apologize for the sloppiness here; again, check the book if you're looking for a rigorous treatment of this problem.

10.10 Problem 10: Double drums rotating

"A drum A of mass M and radius R is suspended from a drum B also of mass M and radius R, which is free to rotate about its axis. The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind. Gravity is directed downwards. Both drums are initially at rest. Consider the drums to be uniform disks.



Find the initial acceleration of drum A, assuming that it moves straight down. Express your answer in terms of M, R and acceleration due to gravity g as needed."

Because drum B (the one at the top) is free to rotate, this problem is not quite as easy as it might look to begin with. We must assume that it too rotates, and that the tape is unrolled from *both* drums at the same time.

Okay then, let's see. First, let's consider the linear acceleration of drum A, which will certainly give us more than one unknown. Using downwards as the positive direction,

$$Ma = Mg - T \tag{10.127}$$

The string (tape?) will unroll, which means we can also consider the angular acceleration, due to the torque provided by this tension. The torque relative to the center of drum A $\tau_A = I_c\alpha$, which is also simply $\vec{R} \times \vec{T}$, where $\vec{R}$ is the position vector from the center (since we take that as our origin for the torque) to the edge of drum A.

$$RT = \left(\frac{1}{2}MR^2\right)\alpha_A \tag{10.128}$$

Here is where we must be very careful. We can *not* use $a = \alpha R$ here! That holds when the drum unrolls such that 100% of the added length of tape comes from the drum – but both drums are unrolling at the same time! In other words, we don't have pure roll in this situation. Instead, we must consider the torque and angular acceleration of drum B. Since both radius, mass and tension are all the same, we find

$$RT = \left(\frac{1}{2}MR^2\right)\alpha_B \tag{10.129}$$

By comparing these two last equations, we don't even need to solve either so find $\alpha_A = \alpha_B$; everything except those variable names are the same in both equations.

Finally, we can consider the position (and change in position) considering how much tape is unrolled. Following the book's approach, an amount $R\Delta\theta_A$ is unrolled from the first drum in some time Δt, and the same thing except with a B index holds for drum B. The distance fallen for drum A is the sum of the two, i.e. the total amount of tape unwound. If we take the time derivative of these expressions, we get

$$\frac{dy}{dt} = R\frac{d\theta_A}{dt} + R\frac{d\theta_B}{dt} \tag{10.130}$$

... and again:

$$\frac{d^2y}{dt^2} = a = R(\alpha_A + \alpha_B) \tag{10.131}$$

The values of $\alpha_A = \alpha_B$ in terms of the tension is

$$\frac{2T}{MR} = \alpha_A = \alpha_B \tag{10.132}$$

And using the first equation we found, $T = M(g - a)$, so

$$\alpha_A = \alpha_B = \frac{2}{MR}M(g - a) \tag{10.133}$$

$$\alpha_A = \alpha_B = \frac{2}{R}(g - a) \tag{10.134}$$

So at this stage, we have two equations:

$$\alpha_A = \alpha_B = \frac{2}{R}(g - a) \tag{10.135}$$

$$a = R(\alpha_A + \alpha_B) \tag{10.136}$$

Substitute the top one into the lower one:

$$a = 2R(\frac{2}{R}(g - a)) \tag{10.137}$$

$$a = 4g - 4a \tag{10.138}$$

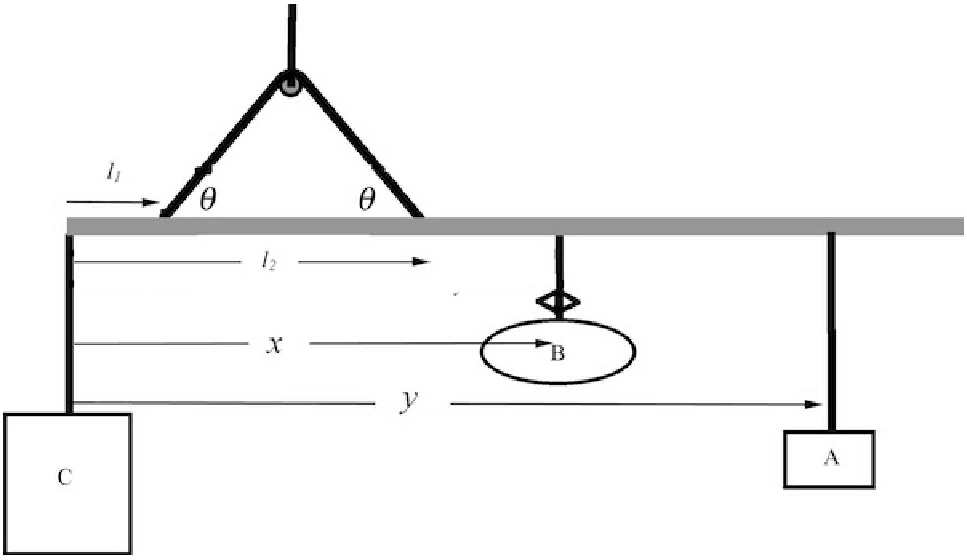
$$a = \frac{4}{5}g \tag{10.139}$$

The acceleration is higher than the $\frac{2}{3}g$ we find if the top drum cannot spin, and we therefore assume pure roll.

Chapter 11: Week 12: Homework 9

11.1 Problem 1: Crane

"A crane is configured as below, with the beam suspended at two points ℓ_1 and ℓ_2 by each end of a cable passing over a frictionless pulley. The two ends of the cable each make an angle θ with the beam. A counterbalance object C with mass m_C is fixed at one end of the beam. A balance object B of mass m_B is attached to the beam and can move horizontally in order to maintain static equilibrium. The crane lifts an object A with mass m_A at a distance y from the counterbalance. For simplicity, assume the pulley, beam and cable to be massless.



(a) What is the tension in the cable that runs over the pulley? Express your answer in terms of m_A, m_B, m_C, θ and acceleration due to gravity g.
(b) At what horizontal position, x, should one put the balance object B such that the crane doesn't tilt? Express your answer in terms of m_A, m_B, m_C, ℓ_1, ℓ_2 and y."

Let's first consider the vertical forces on the beam. We have three weights, balanced by the same tension in two places; the tensions need to be decomposed, though. If the angle was 90 degrees, the vertical component of the tension would clearly be at a maximum, so we need a sine in there (which drawing it out and doing the trigonometry confirms):

$$g(m_A + m_B + m_C) = 2T \sin\theta \tag{11.1}$$

We only need to divide both sides by $2\sin\theta$, and we have the answer to part (a):

$$\frac{g(m_A + m_B + m_C)}{2\sin\theta} = T \tag{11.2}$$

For part (b), we need to consider the torque on the system. We can calculate torques relative to any point of our choosing, but what point would make things the easiest? If we choose $x = 0$, the torque due to mass C disappears. The same argument holds for other points and other masses. Just below the cable, between the two tensions, the torques due to both tensions cancel out.

Because the answer doesn't allow θ and doesn't allow g, we should choose the point where the tensions cause no torque. That way, all disallowed variables should either not enter the equation (θ) or cancel (g). I will call that point $b = \ell_1 + (\ell_2 - \ell_1)/2 = \dfrac{\ell_1 + \ell_2}{2}$, to reduce clutter in the torque equation. I use out of the screen as the positive direction.

$$\tau_b = bgm_C - (x-b)gm_B - (y-b)gm_A \tag{11.3}$$

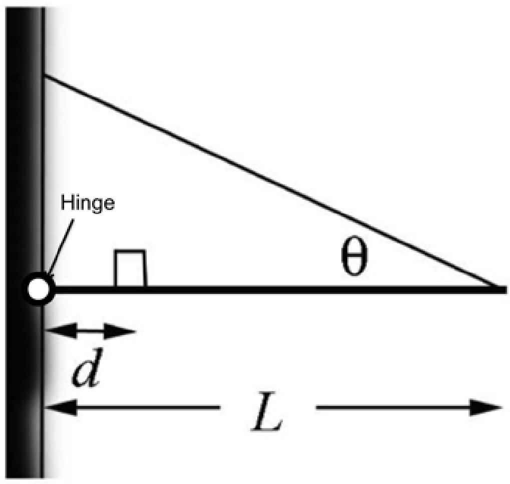
This must be equal to zero. g cancels, as hoped for/expected.

$$0 = bm_C - (x-b)m_B - (y-b)m_A \tag{11.4}$$
$$xm_B = bm_C + bm_B - ym_A + bm_A \tag{11.5}$$
$$x = \frac{b(m_C + m_B + m_A) - ym_A}{m_B} \tag{11.6}$$
$$x = \frac{\frac{\ell_1+\ell_2}{2}(m_C + m_B + m_A) - ym_A}{m_B} \tag{11.7}$$

We would, in the end, find the same answer if we calculated the torque relative to any other point (the torque relative to any point is equal in static equilibrium; the book shows how).

11.2 Problem 2: Steel beam and cable

"A uniform steel beam of mass $m_1 = 150.0\,\text{kg}$ is held up by a steel cable that is connected to the beam a distance $L = 5.0$ m from the wall, at an angle $\theta = 35.0°$ as shown in the sketch. The beam is bolted to the wall with an unknown force $\vec{F}$ exerted by the wall on the beam. An object of mass $m_2 = 60.0\,\text{kg}$ resting on top of the beam, is placed a distance $d = 2.0$ m from the wall. For simplicity, assume the steel cable to be massless. Use $g = 9.8\,\text{m/s}^2$ for the gravitational acceleration.



(a) Find the tension (in Newton) in the cable. Start by drawing a free-body diagram for the beam, then find equations for static equilibrium for the beam (this will involve force equations and torque relations).
(b) Find the horizontal and vertical components of the force (in Newton) that the wall exerts on the beam."

Okay. There are four forces on the beam (with 1 or 2 components each): normal force (2 components) at the hinge, gravity acting purely downwards at the center of mass ($L/2$), gravity acting purely downwards at d and the tension (2 components) at the end of the beam.

The tension clearly acts upwards and inwards, so the normal force must act outwards (towards the right), as they are the only two horizontal forces. Whether the normal force acts upwards or downwards I don't know however, since there is also gravity in the mix. I will guess that it acts upwards, and so if it turns out negative, I guessed wrong.

For the tensions, we have

$$T_x = -T\cos\theta \tag{11.8}$$
$$T_y = T\sin\theta \tag{11.9}$$

using a coordinate system where $+x$ is towards the right. We can now calculate the sum the forces in the vertical direction to zero:

$$N_y + T\sin\theta - g(m_1 + m_2) = 0 \tag{11.10}$$

One equation, two unknowns. Next, we can consider torque. The net torque relative to any point must be zero. If we choose the point right at the hinge, the unknown normal force doesn't cause a torque, so we get

$$g\left(\frac{L}{2}m_1 + dm_2\right) - LT\sin\theta = 0 \tag{11.11}$$

The horizontal forces also cannot cause at torque relative to this point. We now have two equations and two unknowns, though we also need to find N_x later on. That turns out to be trivial, however, so let's begin with T and N_y.

Note that T is the only unknown in this second equation, so we start by finding that:

$$g\left(\frac{L}{2}m_1 + dm_2\right) = LT\sin\theta \tag{11.12}$$
$$\frac{g\left(\frac{L}{2}m_1 + dm_2\right)}{L\sin\theta} = T \tag{11.13}$$

For the given values, $T = 1691.5$ newton. We can then find N_y by solving the previous equation for that, and sticking in this value of T.

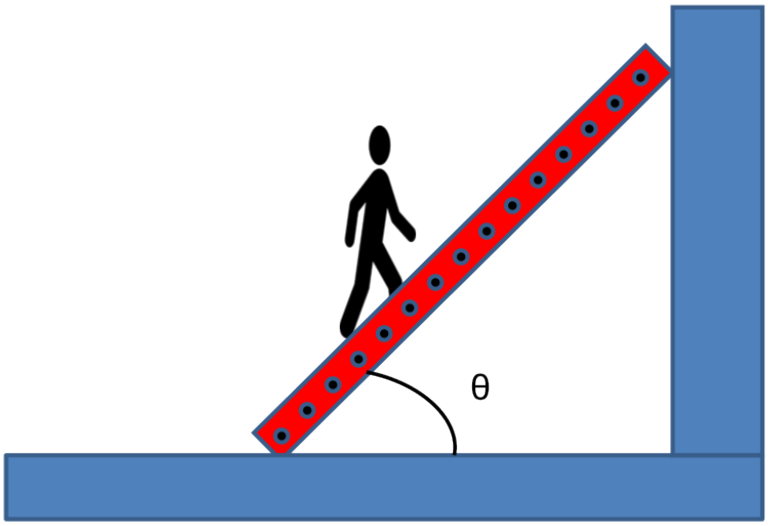
$$N_y = g(m_1 + m_2) - T\sin\theta \tag{11.14}$$
$$N_y = g(m_1 + m_2) - \frac{g\left(\frac{L}{2}m_1 + dm_2\right)}{L} \tag{11.15}$$

For the given values, $N_y = 1087.8$ newton.

As for N_x, it and T_x are the only two horizontal forces. Therefore, they must be equal in magnitude, and so $N_x = T\cos\theta = 1385.6$ N.

11.3 Problem 3: Person on ladder

"A person of mass $m_2 = 85.0$ kg is standing on a rung, one third of the way up a ladder of length $d = 4.0$ m. The mass of the ladder is $m_1 = 15.0$ kg, uniformly distributed. The ladder is initially inclined at an angle $\theta = 40.0°$ with respect to the horizontal. Assume that there is no friction between the ladder and the wall but that there is friction between the base of the ladder and the floor with a coefficient of static friction μ_s.



Start this problem by drawing a free-body force diagrams showing all the forces acting on the person and the ladder. Indicating a choice of unit vectors on your free-body diagrams may be helpful.

(a) Using the equations of static equilibrium for both forces and torque, find expressions for the normal and horizontal components of the contact force between the ladder and the floor, and the normal force between the ladder and the wall. Consider carefully which point to use for computing the torques. Determine the magnitude of the frictional force (in N) between the base of the ladder and the floor below.
(b) Find the magnitude for the minimum coefficient of friction between the ladder and the floor so that the person and ladder does not slip.
(c) Find the magnitude $C_{ladder,ground}$ (in N) of the contact force that the floor exerts on the ladder. Remember, the contact force is the vector sum of the normal force and friction.
Find the direction of the contact force that the floor exerts on the ladder. i.e. determine the angle α (in radians) that the contact force makes with the horizontal to indicate the direction."

We could probably use the equations from lecture most of the way, but I will re-derive everything here anyway.

The vertical forces consist of the normal force where the ladder touches the ground (I call this point Q), gravity due to the person at $d/3$ along the length, and gravity at the ladder's center of mass $d/2$ along the length. Therefore,

$$N_Q = g(m_1 + m_2) \tag{11.16}$$

In the horizontal direction, we have the normal force from the wall (point P) N_P towards the left, and a frictional force $f_s \leq \mu N_Q$ at point Q towards the right (since the ladder wants to slip towards the left). This gives us, just at the edge of slipping ($f_s = \mu_s N_Q$, i.e. the maximum friction possible):

$$N_P = f_s = \mu_s N_Q \tag{11.17}$$

Next, we can consider the torque. I will calculate them relative to point Q, so that two out of the five forces/force components "disappear" (they can't cause torque through that point). I will use into the screen (clockwise rotation) as positive, since that is how the ladder wants to rotate.

Now, these cross products depend on the angle, but the angle between the position vector from Q to where gravity acts, and the gravitational force vector, is not θ. Indeed, it's easy to see that if $\theta = 0$, the angle between them would be 90 degrees. The relevant angle is $90° - \theta$, so that is what we need for the cross products; also, $\sin(90° - \theta) = \cos(\theta)$.
θ is the relevant angle for the normal force at P, however, so that one remains a sine.

Alternatively, we can try to find the perpendicular distance of either vector, and multiply that by the full magnitude of the other, which is the same thing.

$$\tau_Q = \frac{d}{3}m_2 g\cos\theta + \frac{d}{2}m_1 g\cos\theta - dN_P\sin\theta \tag{11.18}$$

This needs to be equal to zero. We can set it equal to zero, solve for N_P (which we earlier said was equal to f_s in magnitude) and find the answer for part (a):

$$0 = \frac{d}{3}m_2 g\cos\theta + \frac{d}{2}m_1 g\cos\theta - dN_P\sin\theta \tag{11.19}$$

$$N_P = \frac{\frac{d}{3}m_2 g\cos\theta + \frac{d}{2}m_1 g\cos\theta}{d\sin\theta} \tag{11.20}$$

$$f_s = N_P = g\cot\theta\left(\frac{m_2}{3} + \frac{m_1}{2}\right) \tag{11.21}$$

(Since $f_s = N_P$.)
All variables above are known, so we can calculate $f_s = 418.5$ N.

Next, we need to find μ_s. $f_s = \mu_s N_Q$, and we know N_Q to be the sum of the two weights, $g(m_1 + m_2)$.

$$\mu_s = \frac{1}{g(m_1+m_2)}g\cot\theta\left(\frac{m_2}{3} + \frac{m_1}{2}\right) \tag{11.22}$$

$$\mu_s = \frac{1}{m_1+m_2}\cot\theta\left(\frac{m_2}{3} + \frac{m_1}{2}\right) \tag{11.23}$$

$$\mu_s = \frac{cot\theta(2m_2 + 3m_1)}{6(m_1+m_2)} \tag{11.24}$$

In terms of numbers, $\mu_s \geq 0.427$ will meet this condition, so that there is no sliding.

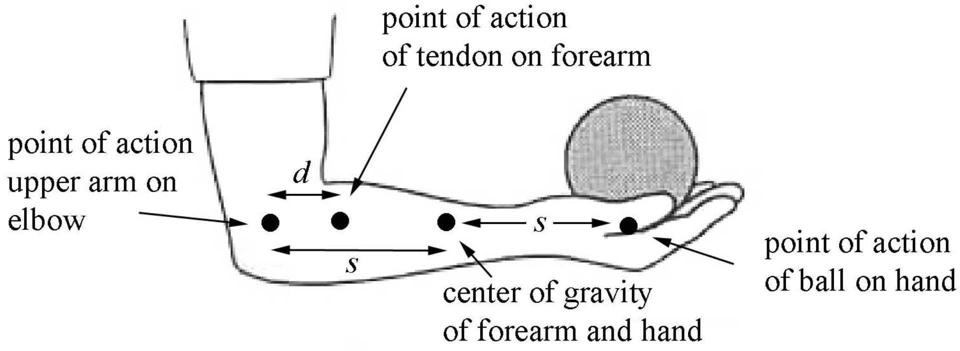
Next, they want the magnitude and angle of the contact force. $N_Q = g(m_1+m_2) = 980$ N, and $f_s = 418.5N$. In terms of unit vectors,

$$C_{ladder,ground} = f_s\hat{x} + N_Q\hat{y} \tag{11.25}$$

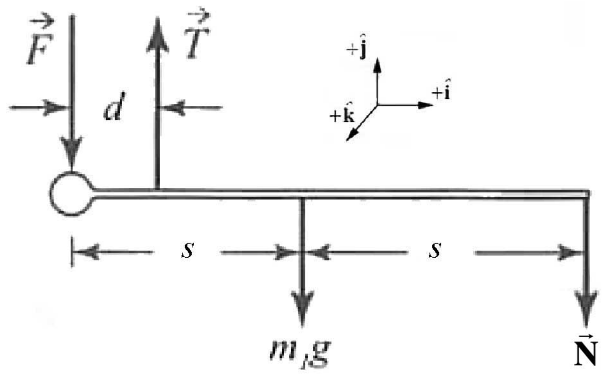
The magnitude of this vector is $C_{ladder,ground} = \sqrt{418.5^2 + 980^2} = 1065.6$ N. The angle α must be less than 90 degrees, or the friction would point towards the left. It is found as $\alpha = \arctan\frac{N_Q}{f_s}$, which is about 1.167 radians, or 66.88 degrees.

11.4 Problem 4: Static equilibrium arm

"You are holding a ball of mass m_2 in your hand. In this problem you will solve for the upward force $\vec{T}$ that the tendon of your biceps muscle exerts to keep the forearm horizontal and the downward force $\vec{F}$ that the upper arm exerts on the forearm at the elbow joint. Assume the outstretched arm has a mass of m_1, the center of mass of the outstretched arm is a distance s from the elbow, the tendon attaches to the bone a distance d from the elbow, and the ball is a distance $2s$ from the elbow. (Taking $\vec{T}$ to be upward and $\vec{F}$ to be downward, with no horizontal components, indicates that this is a simplified model.)
A schematic representation of this situation is shown below:



Hint: The forces can be modeled as shown in the following Free Body Diagram:



(a) What is the magnitude of the tension $T \equiv |\vec{T}|$ in the tendon? Express your answer in terms of s, m_1, m_2, d and g as needed.
(b) What is the magnitude of the force that the upper arm exerts on the forearm at the elbow joint? Express your answer in terms of s, m_1, m_2, d and g as needed."

The problem description certainly sounds complex, but given the diagram and even a free body diagram, this should be one of the easier problems of the week. I choose a coordinate system with $x = 0$ and $y = 0$ at the elbow joint, with $+x$ to the right and $+y$ upwards (which I just noticed is marked in the free body diagram).

We need a net force of zero in the vertical direction, which gives us our first equation (equating upwards and downwards forces):

$$T = F + m_1 g + m_2 g \tag{11.26}$$

where $m_2 g$ is equal in magnitude to the normal force from the hand to the ball.

Next, the torques must be zero, relative to any point of our choosing. I choose the center of the coordinate system, so that F causes no torque. Downwards forces then cause a counterclockwise (into the screen) torque, which I denote as positive.

$$\tau = -dT + s m_1 g + 2 s m_2 g \tag{11.27}$$

This must be equal to zero; we can set it as such and solve for T:

$$0 = -dT + sg(m_1 + 2m_2) \tag{11.28}$$

$$T = \frac{sg(m_1 + 2m_2)}{d} \tag{11.29}$$

This answers part (a); for part (b), we solve the force equation for F and substitute in T.

$$F = T - g(m_1 + m_2) \tag{11.30}$$

$$F = \frac{sg(m_1 + 2m_2)}{d} - g(m_1 + m_2) \tag{11.31}$$

Indeed quite easy compared to the previous ones. (Not to mention compared to last week's problems.)

11.5 Problem 5: Specific strength

"A metal meter stick made of steel rotates about its midpoint. The angular speed is slowly increased. At what value of the angular speed will the stick break apart at the center? Give your answer in rad/s.

Hint: find a relationship between the maximum angular frequency and the breaking (ultimate tensile strength) of steel. Use the values that are given in this table in the handout of lecture 26 [link not copied]."

The possibly relevant values in the handout are (all values for steel, of course):
$Y = 20 \times 10^{10}\,\mathrm{N/m^2}$
Ultimate tensile strength $= 5.2 \times 10^8\,\mathrm{N/m^2}$
Density: $\rho = 8 \times 10^3\,\mathrm{kg/m^3}$

This problem is fairly similar to problem 9, which I solved prior to this one.

First, we need to calculate the tension at the center. The book has a derivation in chapter 9. The result is

$$T(r) = \frac{m\omega^2}{2L}(L^2 - r^2) \tag{11.32}$$

$$T(0) = \frac{1}{2}Lm\omega^2 \tag{11.33}$$

as r is the distance from the center. (m is the total mass of the rod, while L is the length *assuming we rotate it about its end.*)

We can write for the total mass $m = AL\rho$, where A is the unknown cross-sectional area of the stick. That gives us, for the tension at the center,

$$T(0) = \frac{1}{2}AL^2\rho\omega^2 \tag{11.34}$$

The ultimate tensile stress is a pressure, $P_{ult} = F/A$. We need to multiply it by the cross-sectional area to get a force, that we can compare with the tension. We can then set the two equal and solve for ω.

$$\frac{1}{2}AL^2\rho\omega^2 = P_{ult}A \tag{11.35}$$

A cancels, and we can solve to find

$$\omega = \sqrt{\frac{2P_{ult}}{L^2\rho}} \tag{11.36}$$

However, in this equation, L is not the one meter length of the meter stick! It is half that: it is the length that sticks out from the center, and since we rotate the stick about its midpoint, we get half a meter for L. This then gives $\omega \approx 721$ rad/s, which is about 6900 rpm.

11.6 Problem 6: Static friction of stick leaning against a wall

"A stick of length $\ell = 60.0$ cm rests against a wall. The coefficient of static friction between stick and the wall and between the stick and the floor are equal. The stick will slip off the wall if placed at an angle greater than $\theta = 40.0$ degrees. What is the coefficient of static friction, μ_s, between the stick and the wall and floor?"

There is a diagram, but it's too simple to include, really. The stick is leaning towards a wall on the left, and θ is measured between the vertical and the stick, so that it would be 0 if the stick was upright.

This problem is very similar to the one with the leaning ladder, only that there is now a frictional force along the wall also.
I will use the same naming scheme of point Q touching the ground (normal force N_Q) and point P touching the wall (normal force N_P). As for friction, I will use F_Q and F_P.
Aside from those four, there is only one force remaining: gravity, acting on the center of mass. Apparently, this must cancel out (the mass is not given), but I will call it m while solving.

The frictional force on the wall must be upwards, since the stick wants to slide down. The frictional force on the floor is towards the left, since the stick wants to slide to the right. I will use a standard coordinate system with $+x$ being towards the right and $+y$ being upwards.
The problem notes that the stick is just about to slide at the wall, so $F_P = \mu_s N_P$ holds there.

However, how could it slide at the wall without also sliding on the floor? It's a rigid stick; unless it goes off into the third dimension, it cannot slide at the wall while staying in place on the floor. Not only that, but this might just be a statically indeterminate problem if we don't consider it to be about to slip in both places at once. That is, if we don't assume that, we will have more unknowns than equations, and need extra information. We haven't learned about those in the course, so in short, I assume that it is about to slip in *both* places, so that also $F_Q = \mu_s N_Q$ holds, rather than the general case $F_Q \le \mu_s N_Q$ which doesn't help us a whole lot.

First off, we need the sum of forces in both directions to be zero. Starting with the vertical forces,

$$F_P + N_Q - mg = 0 \tag{11.37}$$
$$\mu_s N_P + N_Q = mg \tag{11.38}$$

Next, the horizontal forces:

$$N_P - F_Q = 0 \tag{11.39}$$
$$N_P = \mu_s N_Q \tag{11.40}$$

And finally, the torque, relative to point Q (or any other point, but I choose point Q), must be zero.
F_Q and N_Q act through this point, and cannot cause any torque relative to it. The torque due to gravity is $(\ell/2)mg\sin\theta$; the others are in the opposite direction, with $F_P = \mu_s N_P$ causing a torque $\ell\mu_s N_P \sin\theta$, and N_P causing a torque $\ell N_P \cos\theta$.

$$(\ell/2)mg\sin\theta - \ell N_P(\mu_s \sin\theta + \cos\theta) = 0 \tag{11.41}$$

So, three equations, with μ_s, N_P and N_Q as unknowns. We only really care about μ_s, though. We can eliminate N_P using $N_P = \mu_s N_Q$, which leaves two equations and two unknowns:

$$(\ell/2)mg\sin\theta - \ell\mu_s N_Q(\mu_s \sin\theta + \cos\theta) = 0 \tag{11.42}$$
$$\mu_s^2 N_Q + N_Q = mg \tag{11.43}$$

We can solve the second one for N_Q:

$$\mu_s^2 N_Q + N_Q = mg \tag{11.44}$$
$$N_Q(1+\mu_s^2) = mg \tag{11.45}$$
$$N_Q = \frac{mg}{1+\mu_s^2} \tag{11.46}$$

We can then combine the two equations; in the second equation below, mg cancels, ℓ cancels, and we can divide through by $\sin\theta$. The rest is just simplification to get it into a standard form for a quadratic:

$$\frac{\ell mg}{2}\sin\theta - \frac{\ell\mu_s mg}{1+\mu_s^2}(\mu_s\sin\theta + \cos\theta) = 0 \tag{11.47}$$
$$\frac{1}{2} - \frac{\mu_s}{1+\mu_s^2}(\mu_s + \cot\theta) = 0 \tag{11.48}$$
$$\frac{1}{2} - \frac{\mu_s^2 + \mu_s\cot\theta}{1+\mu_s^2} = 0 \tag{11.49}$$
$$\frac{1-\mu_s^2 - 2\mu_s\cot\theta}{2(1+\mu_s^2)} = 0 \tag{11.50}$$
$$1-\mu_s^2 - 2\mu_s\cot\theta = 0 \tag{11.51}$$
$$\mu_s^2 + 2\cot(\theta)\mu_s - 1 = 0 \tag{11.52}$$

Finally, after all that massaging, we can solve this for μ.

$$\mu_s = \frac{-2\cot\theta \pm \sqrt{4\cot^2\theta + 4}}{2} \tag{11.53}$$
$$\mu_s = -\cot\theta \pm \sqrt{\cot^2\theta + 1} \tag{11.54}$$
$$\mu_s = -\cot\theta + \frac{1}{\sin\theta} \tag{11.55}$$

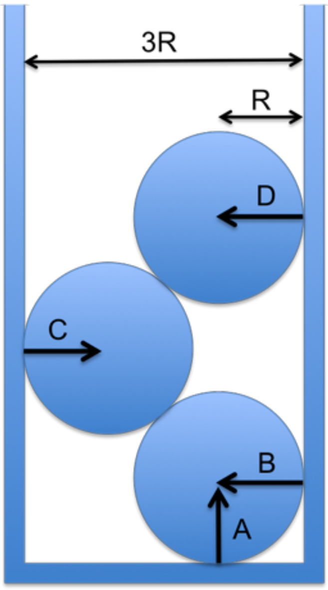
Only the positive root gives a meaningful answer (the other one gives $\mu_s < 0$ which is unphysical).
We can simplify this even one step further:

$$\mu_s = \tan\frac{\theta}{2} \tag{11.56}$$

Lots of work if you do the math manually (unless I missed some obvious simplifications), but the result is certainly very elegant!

Sidenote: this problem was graded incorrectly until November 27-28 (depending on timezones etc); the grader was set such that $\tan(\theta/2)$ was correct if you specified θ as the number given *in degrees*, despite the calculator using radians. As such, the accepted μ_s was about 2.24(!) in my case, rather than the actually correct 0.36 or so that is now accepted.

11.7 Problem 7: Three balls in a tube



"Three smooth balls of iron of mass m and radius R are placed inside a tube of diameter $3R$ (see Figure). Find the magnitude of the forces (A, B, C and D) exerted by the sides of the container on each ball. Write your answers in terms of m, g and R."

I will begin by assuming that there is no friction. That means that forces D, C and B are purely horizontal, and that force A is purely vertical. It also means that the middle ball must provide both an upwards and a rightwards force on the top ball.

Drawing this out (anything else might just be insanity; see partial drawing below), it's clear that $A = 3mg$, or there cannot be equilibrium, if it A is the only upwards force.

The distance between the center of the bottom ball and the center of the middle ball is exactly $2R$ (same for the middle and top balls).
The distance from the right side to the center of the bottom ball is R; the distance from the left side to the center of the middle ball is also R. Therefore, since the entire tube is $3R$, the horizontal distance between the two centers must also be R.

Using the Pythagorean theorem, the vertical distance between the centers must then be $\sqrt{3}$ times R (for both the top-middle and the middle-bottom balls).

So, forces... forces...
Consider the forces on the top ball. There is a force to the left, which cannot cause a torque relative to its center, since the angle between the position vector and the force vector would be 180 degrees.
Likewise, mg due to gravity cannot cause a torque, as it acts on the center.
This means that only the contact force due to the middle ball remains, which must therefore *create no torque*, or the top ball would have a net torque! There is no other force that could possibly create an opposing torque and cancel it out.
The only way this can happen is if the net normal force is pointing straight towards the center of the top ball!

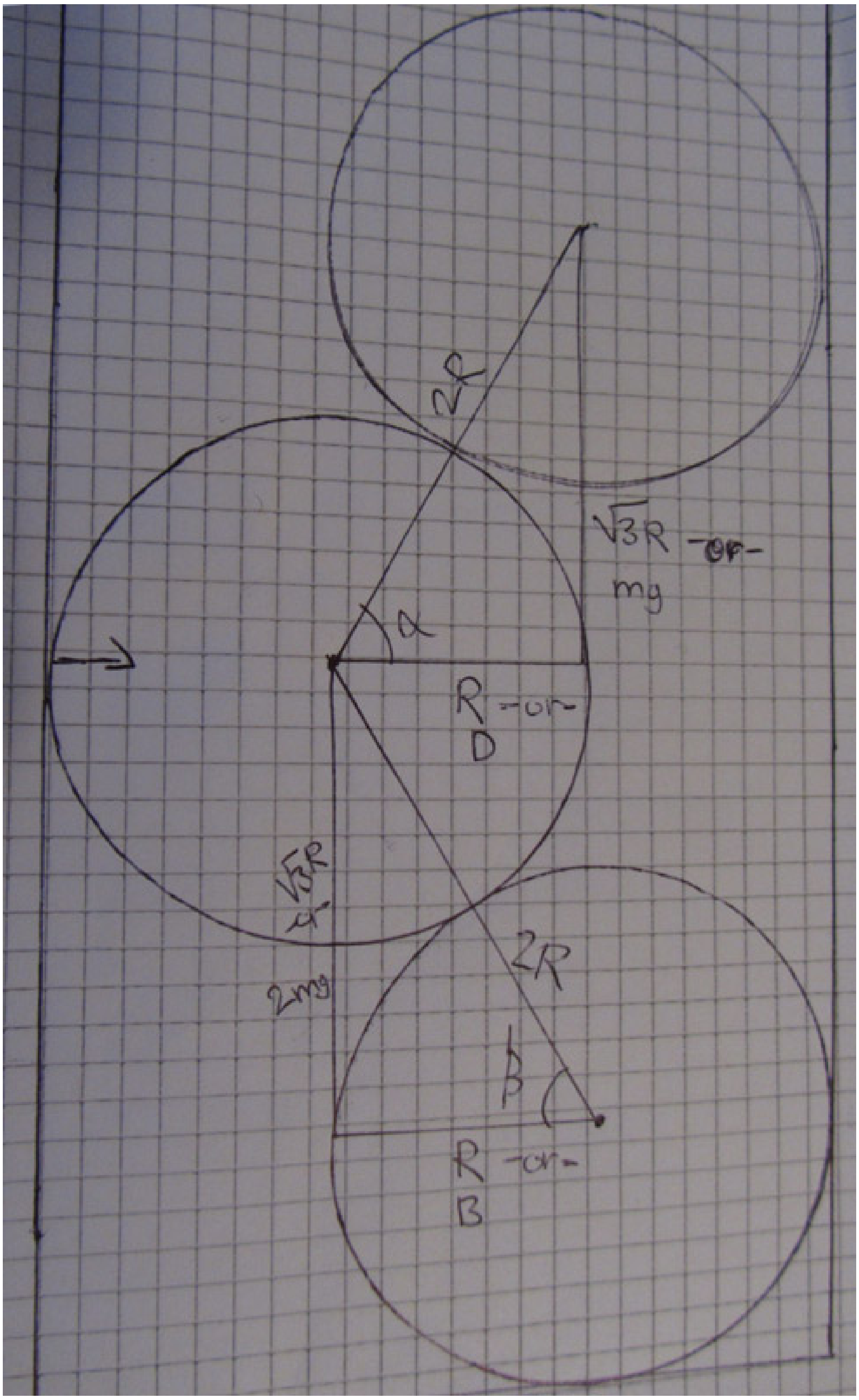
This then puts another constraint on the normal force, so we now know: it must be D in magnitude to the right (or there is a net horizontal force on the top ball), mg up (or there is a net downwards force on the top ball), *and* be at the correct angle, or there is a net torque.

We can draw a triangle showing the angle; as mentioned, it is R wide and $\sqrt{3}R$ high, with a $2R$ hypotenuse (between the two balls' centers). Drawing the angle, we find

$$\tan \alpha = \frac{\sqrt{3}R}{R} = \sqrt{3} \tag{11.57}$$

We then draw a vector triangle for the forces; the angle must be the same, or the net force won't point towards the center of the top ball! For the same α, clearly $\tan \alpha$ must also be the same. Relating the forces instead, we have D horizontally and mg on the vertical side, so

$$\tan \alpha = \frac{mg}{D} \tag{11.58}$$



I didn't label the forces here, since it make it very difficult to get it at all readable. Doing so is practically mandatory to solve this though, in my opinion; this was my second, simplified drawing.

(This is perhaps the cleanest thing I've drawn in years, which is why I don't post hand-drawn stuff often. It's usually much harder to read, which says something!)

α must then be the same for the net force vector, or that force will create a torque on the top ball. We can set the two tangents equal and find D:

$$\frac{mg}{D} = \sqrt{3} \tag{11.59}$$

$$D = \frac{mg}{\sqrt{3}} \tag{11.60}$$

Nice! What about the bottom ball? We have a very similar situation there! There is an upwards force $2mg$ to the middle ball instead of mg, since the bottom ball supports both of those above it.

For the sides, we again find:

$$\tan\beta = \frac{\sqrt{3}R}{R} = \sqrt{3} \tag{11.61}$$

The forces again need the same angle, so we can find the tangent for the forces, and set the two equal again:

$$\tan\beta = \frac{2mg}{B} \tag{11.62}$$

$$\sqrt{3} = \frac{2mg}{B} \tag{11.63}$$

$$B\sqrt{3} = 2mg \tag{11.64}$$

$$B = \frac{2mg}{\sqrt{3}} \tag{11.65}$$

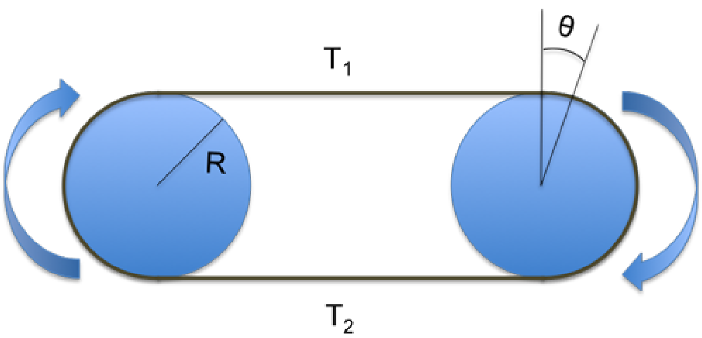
Finally, for the middle ball, we can simply sum the horizontal forces; the one to the right needs to be equal to the sum of those to the right, or there is a net force. C to the right must cancel with $B + D$ to the left, and we know those two.

$$C = B + D \tag{11.66}$$

$$C = \frac{2mg}{\sqrt{3}} + \frac{mg}{\sqrt{3}} = \frac{3mg}{\sqrt{3}} = \frac{3\sqrt{3}mg}{3} = \sqrt{3}mg \tag{11.67}$$

And that's it! Easy once I found the trick, but I have to admit it took a while. If I hadn't drawn it out, it would have been way harder.

11.8 Problem 8: Two flywheels and a drive belt



"The flywheel of a motor is connected to the flywheel of an electric generator by a drive belt. The flywheels are of equal size each of radius R. While the flywheels are rotating the tension in the upper and lower portions of the drive belt are T_1 and T_2 respectively. The drive belt exerts a torque $\tau = (T_2 - T_1)R$ on the generator (around its center). The coefficient of static friction between the drive belt and each flywheel is μ_s. Assume the tension is as high as possible with no slipping between the belt and the flywheel, and that the drive belt is massless.

(a) Derive a differential expression representing the change of tension along the portion of the belt in contact with one of the flywheels. That is find the value of dT/T for one of the two flywheels. $dT/T =$"

1. $\dfrac{1}{\mu_s} d\theta$
2. $\dfrac{1}{\mu_s R} d\theta$
3. $\mu_s d\theta$
4. $R\mu_s d\theta$

What is T_1?

1. $\dfrac{\tau}{R}\dfrac{1}{e^{\mu_s \pi} - 1}$
2. $\dfrac{\tau}{R}\dfrac{1}{1 - e^{-\mu_s \pi}}$
3. $\dfrac{\tau}{R}e^{\mu_s \pi}$
4. $\dfrac{\tau}{R}e^{-\mu_s \pi}$
5. $\dfrac{\tau}{R}(1 - e^{\mu_s \pi})$

What is T_2?

1. $\dfrac{\tau}{R}\dfrac{1}{e^{\mu_s \pi} - 1}$
2. $\dfrac{\tau}{R}\dfrac{1}{1 - e^{-\mu_s \pi}}$
3. $\dfrac{\tau}{R}e^{\mu_s \pi}$
4. $\dfrac{\tau}{R}e^{-\mu_s \pi}$
5. $\dfrac{\tau}{R}(1 - e^{\mu_s \pi})$

The equations look like capstan equations, which is not entirely unexpected: we have differing tensions in something wound around a cylinder (or two).
Indeed, the recommended reading is the book's derivation of the capstan equation.

Let's start by looking at part one. I will look at the rightmost wheel, and basically assume the other one doesn't exist.
$T_2 > T_1$, or the torque would be in the opposite direction of the rotation, and so it wouldn't be in any kind of equilibrium. Therefore, the frictional force is counterclockwise along the wheel, "helping" T_1, so that there can be equilibrium.
We therefore have the same situation as the book, and don't need to think of the opposite case (reversing directions or such).

Since the derivation is fairly complex, and the book derivation applies to this situation, I will use some results from there, to get started. There is a sign difference that we can ignore if we only keep track of directions/which tension is the larger one.

$$\frac{dT}{T} = \mu_s d\theta \tag{11.68}$$

for one wheel, which answers part (a) as-is.
Part (b) is not as straightforward, with or without the book's help. First, we have one useful relationship given to us in the question:

$$\tau = R(T_2 - T_1) \Rightarrow \frac{\tau}{R} = T_2 - T_1 \tag{11.69}$$

We'll need that later.
If we integrate the previous equation, from T_1 to T_2 on the left-hand side, and from 0 to π on the right, we find

$$\ln\frac{T_2}{T_1} = \mu_s\pi \Rightarrow \frac{T_2}{T_1} = e^{\mu_s\pi} \tag{11.70}$$

And so, indeed, T_2 will be larger than T_1. Solved for T_2, we have, of course

$$T_2 = T_1 e^{\mu_s\pi} \tag{11.71}$$

We now have two equations and two unknowns, so we can solve the rest from here.

$$\frac{\tau}{R} = T_2 - T_1 \tag{11.72}$$
$$T_2 = T_1 e^{\mu_s\pi} \tag{11.73}$$

We can find T_1 by substitution; we stick $T_1 e^{\cdots}$ in for T_2 in the first equation and solve:

$$T_1 e^{\mu_s\pi} - T_1 = \frac{\tau}{R} \tag{11.74}$$
$$T_1\left(e^{\mu_s\pi} - 1\right) = \frac{\tau}{R} \tag{11.75}$$
$$T_1 = \frac{\tau}{R}\frac{1}{e^{\mu_s\pi} - 1} \tag{11.76}$$

We have a simple relationship between T_2 and T_2 above, so finding T_2 is trivial now – at least getting it mathematically equivalent. To get it to look like one of the answer options (as this was the week's only multiple choice question), we need to divide through by the exponential, and use $1/e^x = e^{-x}$:

$$T_2 = T_1 e^{\mu_s\pi} = \frac{\tau}{R}\frac{e^{\mu_s\pi}}{e^{\mu_s\pi} - 1} \tag{11.77}$$
$$= \frac{\tau}{R}\frac{1}{1 - \frac{1}{e^{\mu_s\pi}}} \tag{11.78}$$
$$= \frac{\tau}{R}\frac{1}{1 - e^{-\mu_s\pi}} \tag{11.79}$$

11.9 Problem 9: Hanging rod length

"A long rod hangs straight down from one end. How long (in meters) can the rod be before its weight causes it to break off at the end if it is made of iron? Titanium? Give your answer in meters.

Use the following values for densities and tensile strengths:

The densities of iron and titanium are $7.8 \times 10^3\,\mathrm{kg/m^3}$ and $4.5 \times 10^3\,\mathrm{kg/m^3}$ respectively.

The breaking - ultimate tensile strength: 350 MPa for iron and 450 MPa for titanium (MPa $= 10^6$ N/m^2)."

Hmm, I wonder if this can be solved in the very naive way. If we consider it attached at the very top, then essentially 100% of the weight is below that point. Therefore, we only need to find the stress due to the weight of the entire bar, $mg = (AL\rho)g$.

The ultimate tensile stress is given as a pressure, force per unit area; $P_{ult} = F/A$. We need to multiply it by the cross-sectional area A to find a force (comparable to a weight, since both are in newtons):

$$AL\rho g = P_{ult}A \tag{11.80}$$

A cancels:

$$L\rho g = P_{ult} \tag{11.81}$$

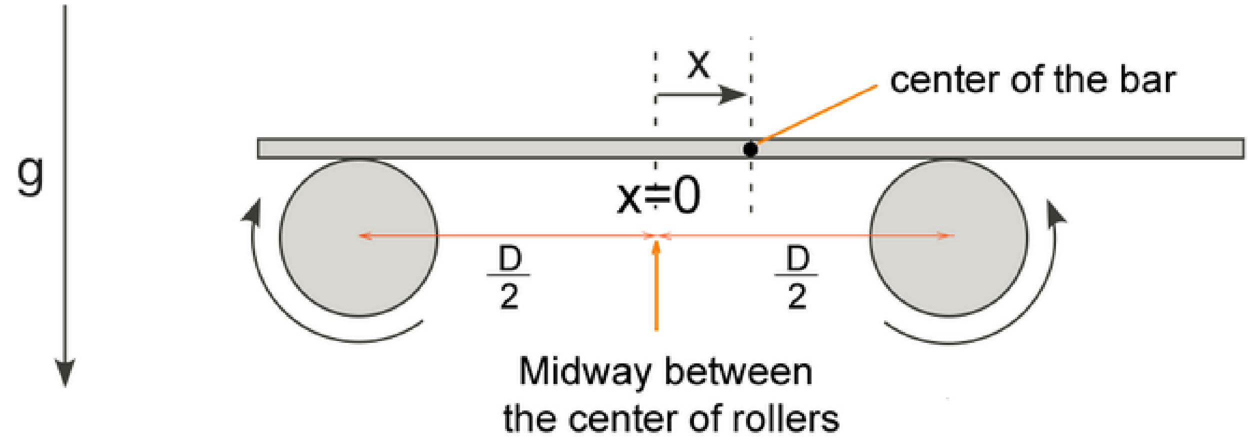
$$L = \frac{P_{ult}}{\rho g} \tag{11.82}$$

And indeed, plugging in the values, this is correct! The answers are 4574 m (4.6 km) for iron, and 10194 m (10.2 km) for titanium.

Chapter 12: Week 13: No homework

Chapter 13: Week 14: Homework 10

13.1 Problem 1: Bar on rollers



"A bar of mass m and negligible height is lying horizontally across and perpendicular to a pair of counter rotating rollers as shown in the figure. The rollers are separated by a distance D. There is a coefficient of kinetic friction μ_k between each roller and the bar. Assume that the bar remains horizontal and never comes off the rollers, and that its speed is always less than the surface speed of the rollers. Take the acceleration due to gravity to be g.

(a) Find the normal forces N_L and N_R exerted by the left and right rollers on the bar when the center of the bar is displaced a distance x from the position midway between the rollers. Express your answers in terms of m, x, d and g."

b) Find the differential equation governing the horizontal displacement of the bar $x(t)$. Express your answer in terms of x, d, μ_k and g.

c) The bar is released from rest at $x = x_0$ at $t = 0$. Find the subsequent location of the center of the bar, $x(t)$. Express your answer in terms of x_0, d, μ_k, t and g."

The recommended reading gives us a not-so-small hint that this is a simple harmonic oscillation.
With the condition given, there will always be slipping, and therefore always kinetic friction. We know nothing about the speed of the rotation, but since the frictional force is given by $\mu_k N$, that shouldn't matter, as long as there is always slipping.

Newton's second law in the horizontal direction (with rightwards as positive) gives us

$$ma_x = \mu_k N_L - \mu_k N_R = \mu_k(N_L - N_R) \tag{13.1}$$

Rewritten,

$$\ddot{x} = \frac{\mu_k}{m}(N_L - N_R) \tag{13.2}$$

Vertically (with upwards as positive):

$$0 = N_L + N_R - mg \tag{13.3}$$

Two equations, three unknowns. Now, if the center of the bar is at $x > 0$, it's clear that $N_R > N_L$, and vice versa if $x < 0$. The above equations doesn't account for that. The net torque on the bar (about the center, say) must also be zero, or it won't remain horizontal. We can capture that as

$$0 = (x + D/2)N_L - (D/2 - x)N_R \tag{13.4}$$

since gravity acting at the center of mass can cause no torque relative to the center of mass. It's unfortunate that we need to find N_L and N_R too, or there would certainly be less algebra involved. We begin by finding N_L and N_R; for that, we only need the last two equations. After that, we have one (differential) equation and one unknown left.

The vertical force equation easily gives us

$$N_L = mg - N_R \tag{13.5}$$

Solving the torque equation for N_R gives us

$$\frac{(x + D/2)}{(D/2 - x)} N_L = N_R \tag{13.6}$$

Substitute that back:

$$N_L = mg - \frac{(x + D/2)}{(D/2 - x)} N_L \tag{13.7}$$

$$N_L \left(1 + \frac{(x + D/2)}{(D/2 - x)}\right) = mg \tag{13.8}$$

$$N_L = \frac{mg}{1 + \frac{(x+D/2)}{(D/2-x)}} \tag{13.9}$$

$$N_L = \frac{mg(D - 2x)}{2D} \tag{13.10}$$

And, substitute that into the equation for N_R, below:

$$N_R = mg - N_L \tag{13.11}$$

$$N_R = mg - \frac{mg(D - 2x)}{2D} \tag{13.12}$$

For part (b), we substitute this back into the $\ddot{x}$ equation:

$$\ddot{x} = \frac{mu_k}{m} \left(\frac{mg(D - 2x)}{2D} - mg + \frac{mg(D - 2x)}{2D}\right) \tag{13.13}$$

$$\ddot{x} = \mu_k g \left(\frac{D}{D} - \frac{2x}{D} - 1\right) \tag{13.14}$$

$$\ddot{x} = -\frac{2\mu_k g x}{D} \tag{13.15}$$

The sign changes in step 1, since we get a double negative on the fraction when calculating $N_L - N_R$. Finally, for part (c), we notice that this is a simple harmonic motion, and solve it accordingly.

$$\ddot{x} + \mu_k g \frac{2}{D} x = 0 \tag{13.16}$$

$$x = x_0 \cos(\omega t) \tag{13.17}$$

$$\omega = \sqrt{\frac{2\mu_k g}{D}} \tag{13.18}$$

So, all in all,

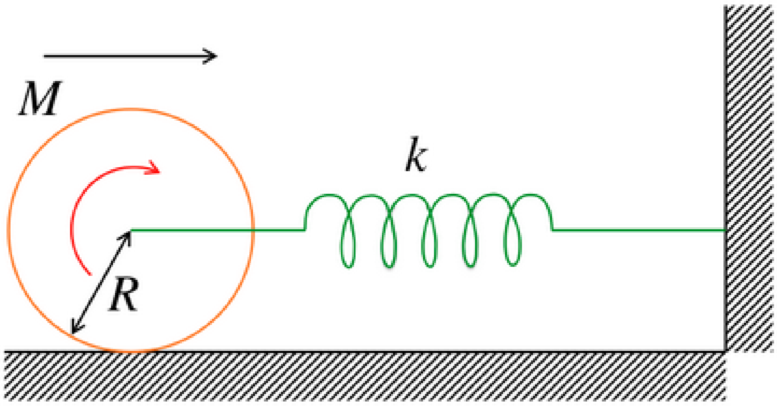
$$x = x_0 \cos\left(\sqrt{\frac{2\mu_k g}{D}}t\right) \tag{13.19}$$

If we write x as $x = \cos(\omega t + \varphi)$ and set $t = 0$, we find

$$x_0 = x_0 \cos(\varphi) \tag{13.20}$$

and so $\cos(\varphi) = 1 \Rightarrow \varphi = 0$, which is why I didn't include it above. (I figured as much since it was released from rest, not to mention they didn't ask for it.)

13.2 Problem 2: Table problem: Rolling solution



"Attach a solid cylinder of mass M and radius R to a horizontal massless spring with spring constant k so that it can roll without slipping along a horizontal surface. If the system is released from rest at a position in which the spring is stretched by an amount x_0 what is the period T of simple harmonic motion for the center of mass of the cylinder? Express your answer in terms of M and k."

First, let's identify the forces present. There's the spring force of magnitude kx, and the frictional force F_f.
When the spring is stretched, the spring force is towards the right, in the direction of the acceleration. The frictional force is opposite that, and will provide a torque that causes the cylinder to roll.
If we use rightwards as positive (since the acceleration will begin in that direction), kx will begin negative, since the initial position is $x = -x_0$. As usual, then, we must write $-kx$ for the spring force. The frictional force also has a negative, since it's towards the left when the acceleration is positive:

$$m\ddot{x} = -kx - F_f \tag{13.21}$$

Next, since there is pure roll, we can use $a = \ddot{x} = \alpha R$. We also have that $\tau = I\alpha$, which leads us to (via $\tau = RF_f$ and $I = \frac{1}{2}MR^2$):

$$RF_f = (\frac{1}{2}MR^2)(\ddot{x}/R) \tag{13.22}$$

$$F_f = \frac{1}{2}M\ddot{x} \tag{13.23}$$

We could also write an equation relating vertical forces, but it turns out we don't need to.
If we substitute the value of F_f into the previous equation,

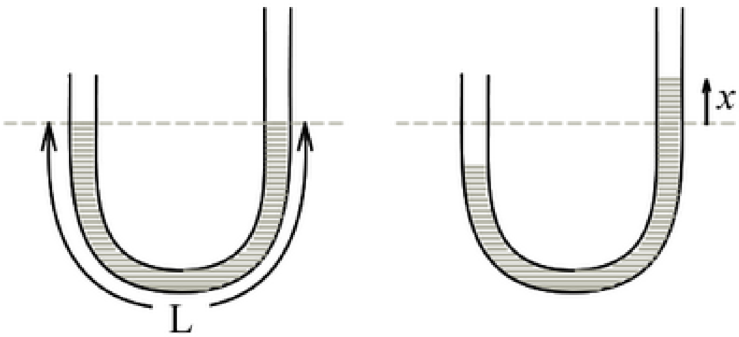
$$M\ddot{x} = -kx - \frac{1}{2}M\ddot{x} \tag{13.24}$$
$$\frac{3}{2}M\ddot{x} = -kx \tag{13.25}$$
$$\ddot{x} + \frac{2k}{3M}x = 0 \tag{13.26}$$

A simple harmonic oscillation, as we would expect. The solution is then

$$x = x_0\cos(\omega t + \pi) \tag{13.27}$$
$$\omega = \sqrt{\frac{2k}{3M}} \tag{13.28}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{2k}} \tag{13.29}$$

where I wrote the phase as π since at $t = 0$, we need $x = -x_0$. I could also have written the entire right-hand side as negative.

13.3 Problem 3: U-tube



Before introducing piston
Right after introducing piston

"A U-tube open at both ends to atmospheric pressure P_0 is filled with an incompressible fluid of density ρ. The cross-sectional area A of the tube is uniform and the total length of the column of fluid is L. A piston is used to depress the height of the liquid column on one side by a distance x_0, and then is quickly removed. What is the frequency of the ensuing simple harmonic motion? Assume streamline flow and no drag at the walls of the U-tube. (Hint: use conservation of energy). Express your answer in terms of L and acceleration due to gravity g."

Hmm, we've done this in lecture already, but let's re-derive it, then. The liquid has a velocity that is the same everywhere (under these conditions), $\dot{x}$. Therefore, the liquid as a whole has a kinetic energy of

$$\frac{1}{2}M\dot{x}^2 = \frac{1}{2}AL\rho\dot{x}^2 \tag{13.30}$$

There is also gravitational potential energy. We define $U = 0$ at the equilibrium point. The change is then that a height of fluid x of mass $m = Ax\rho$ is moved upwards a distance x. (It's essentially taken from the left side and moved upwards on the right side, gaining potential energy.)
The sum of these two energies must be a constant:

$$\frac{1}{2}AL\rho\dot{x}^2 + Ax\rho gx = \text{constant} \tag{13.31}$$

using $mgh = (Ax\rho)gx$.

We take the time derivative of this; the rate of change in the energy must be zero if it's constant, which the differentiation takes of for us.

$$\frac{1}{2}AL\rho\dot{x}^2 + A\rho g x^2 = \text{constant} \tag{13.32}$$

$$\frac{1}{2}AL\rho 2\dot{x}\ddot{x} + A\rho g 2x\dot{x} = 0 \tag{13.33}$$

$$L\ddot{x} + 2gx = 0 \tag{13.34}$$

$$\ddot{x} + \frac{2g}{L}x = 0 \tag{13.35}$$

$\dot{x}$, A and ρ cancel, and we end up with a simple harmonic oscillation, as expected (and as usual, at this point!). The solution is

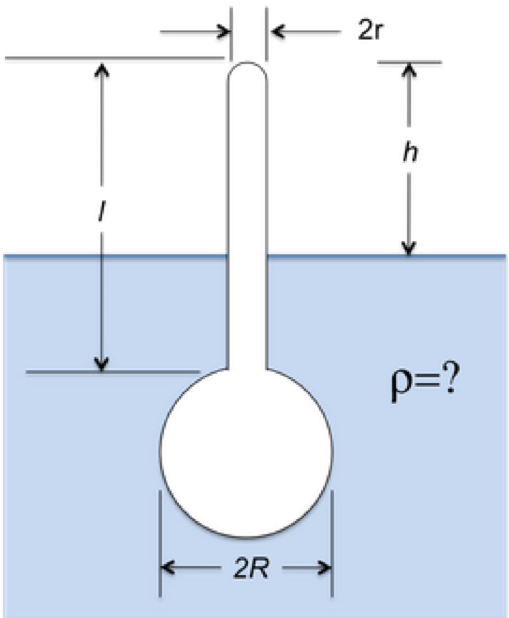
$$x = x_0 \cos(\omega t) \tag{13.36}$$

$$\omega = \sqrt{\frac{2g}{L}} \tag{13.37}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{L}} \tag{13.38}$$

... though in reality there will be losses which cause damping, so T will be longer, and the amplitude will decrease rather rapidly, rather than stay constant forever as this solution predicts.

13.4 Problem 4: Liquid density



"A hydrometer is a device that measures the density of a liquid. The one shown in the figure has a spherical bulb of radius R attached to a cylindrical stem of radius r and length ℓ. When placed in a liquid, the device floats as shown in the figure with a length h of stem protruding. Given that the mass of the hydrometer is M, find the density ρ of the liquid. Express your answer in terms of M ,R, r, ℓ and h."

The total volume of the hydrometer is

$$V_{sphere} + V_{cylinder} = \frac{4}{3}\pi R^3 + \pi r^2 \ell \tag{13.39}$$

while the submerged part is

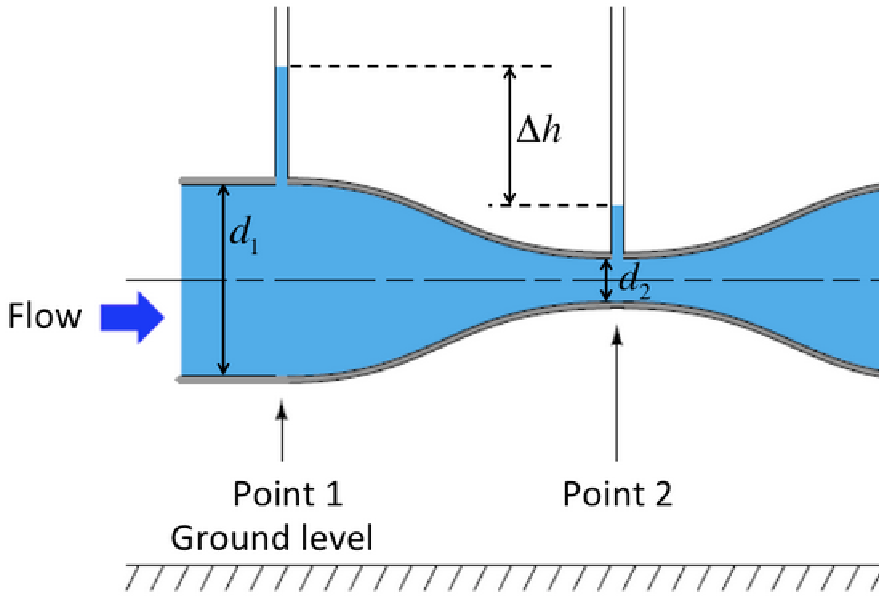
$$\frac{4}{3}\pi R^3 + \pi r^2(\ell - h) \tag{13.40}$$

Since it floats, the upwards buoyant force must be equal to the downwards gravitational force Mg.
The buoyant force is equal to the weight of the displaced water, which is the submerged volume times ρ (which is its mass) times g. That is,

$$Mg = \rho g \left(\frac{4}{3}\pi R^3 + \pi r^2(\ell - h)\right) \tag{13.41}$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3 + \pi r^2(\ell - h)} \tag{13.42}$$

13.5 Problem 5: Venturi flow meter



"A Venturi flow meter is used to measure the the flow velocity of a water main. The water main has a diameter of $d_1 = 40.0$ cm, and the constriction has a diameter of $d_2 = 20.0$ cm. The two vertical pipes are open at the top, and the difference in water level between them is $\Delta h = 2.0$ m. Find the velocity v_m (in m/s), and the volumetric flow rate Q (in m^3/s), of the water in the main."

The volumetric flow rate must be the same both the thick part at d_1 and the thinner at d_2, since water is practically incompressible.
Therefore, the velocity must be greater at point 2 than at point 1.
I will, for consistency, use v_1 for the velocity at point 1; $v_1 = v_m$.

$$Q = v_1 A_1 = v_2 A_2 \tag{13.43}$$

$$Q = v_1 \pi \left(\frac{d_1}{2}\right)^2 = v_2 \pi \left(\frac{d_2}{2}\right)^2 \tag{13.44}$$

This gives us

$$v_1 d_1^2 - v_2 d_2^2 = 0 \tag{13.45}$$

We can also relate the energies at the two points via Bernoulli's equation. We have kinetic energy (per unit volume), gravitational potential energy (per unit volume), and pressure. The GPE is equal at the two points, as they are at equal height with equal ρ, so if we wrote it down it would simply cancel.

$$\frac{1}{2}\rho v_1^2 + P_1 = \frac{1}{2}\rho v_2^2 + P_2 \tag{13.46}$$

We don't know v_1, v_2, P_1 or P_2, so we have four unknowns. We can rewrite this a bit, though.

$$P_1 - P_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \tag{13.47}$$

We can use the height of the water columns to figure out the pressure difference.

The air at the top of the water columns are at atmospheric pressure, call it $P_0 = 1\,\mathrm{atm}$.
The height of the left column, measured from the horizontal center line, depends on $P_1 - P_0$, via Pascal's law:

$$P_1 - P_0 = \rho g h_1 \tag{13.48}$$

The right column is similar.

$$P_2 - P_0 = \rho g h_2 \tag{13.49}$$

We don't know h_1 or h_2, but we know $h_1 - h_2 = \Delta h$. If we subtract the two equations,

$$(P_1 - P_0) - (P_2 - P_0) = \rho g h_1 - \rho g h_2 \tag{13.50}$$

$$P_1 - P_2 = \rho g \Delta h \tag{13.51}$$

We use this in equation (13.47). That gives us these two equations (after ρ cancels):

$$g\Delta h = \frac{1}{2}\left(v_2^2 - v_1^2\right) \tag{13.52}$$

$$v_1 d_1^2 - v_2 d_2^2 = 0 \tag{13.53}$$

Since we don't care about v_2, we can solve the second equation for it, substitute that into the first, and then just forget about v_2 altogether.

$$v_2 = v_1 \frac{d_1^2}{d_2^2} \tag{13.54}$$

$$2g\Delta h = \left(v_1 \frac{d_1^2}{d_2^2}\right)^2 - v_1^2 \tag{13.55}$$

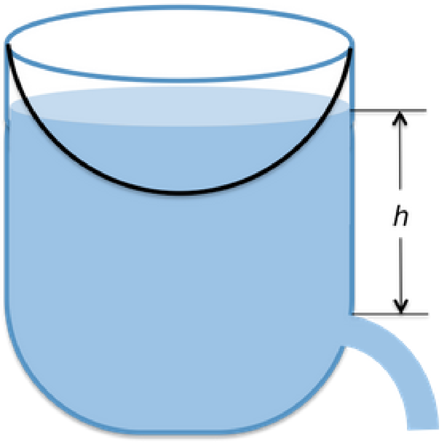
$$2g\Delta h = v_1^2\left(\frac{d_1^4}{d_2^4} - 1\right) \tag{13.56}$$

$$\sqrt{\frac{2g\Delta h}{\frac{d_1^4}{d_2^4} - 1}} = v_1 \tag{13.57}$$

$$\sqrt{\frac{2g\Delta h\ d_2^4}{d_1^4 - d_2^4}} = v_1 \tag{13.58}$$

For the number we were given, this gives us $v_1 = v_m = 1.6174\,\text{m/s}$.
Using the simple relationship $Q = v_1 A_1 = v_1 \left(\frac{d_1}{2}\right)^2$ we find a flow rate of $Q = 0.203\,\text{m}^3/\text{s}$.

13.6 Problem 6: Bucket with a hole



"A cylindrical bucket has a small hole at the bottom. The water exiting the hole has velocity v. What is the depth, h, of the water in the bucket?"

I suppose the question is rather "at what depth h is the hole located", according to the picture.
Let's see, what facts do we have? Not a whole lot in terms of given facts, but if we add to that the things discussed in lecture (and in the book), we have a lot more.

We can solve this in multiple ways, I noticed.

13.6.1 Solution 1

The pressure at that depth is $P_1 = 1\,\text{atm} + \rho g h$. The pressure difference between inside and outside the bucket is then simply $\rho g h$.
We can apply Bernoulli's equation here, again while ignoring the term related to gravitational potential energy, as there is no height difference involved (if we consider a point at that depth, but at the container's left side, as being inside). Using P_1 for the pressure inside the bucket at depth h, and P_2 for the pressure outside:

$$\frac{1}{2}\rho v_{inside}^2 + P_1 = \frac{1}{2}\rho v^2 + P_2 \tag{13.59}$$

$$\frac{1}{2}\rho v_{inside}^2 + 1\,\text{atm} + \rho g h = \frac{1}{2}\rho v^2 + 1\,\text{atm} \tag{13.60}$$

$$\frac{1}{2} v_{inside}^2 + g h = \frac{1}{2} v^2 \tag{13.61}$$

$$h = \frac{v^2}{2g} \tag{13.62}$$

Here, I consider v_{inside} to be negligible compared to v, so I ignore it. It we consider v_{inside} to be the velocity *just* inside the hole, that is clearly not correct. However, the rest of the equation is equally valid at the leftmost edge of the container.

13.6.2 Solution 2

I feel a bit funny about the assumption $v_{inside} = 0$ while considering a point at depth h in the liquid, as the equation doesn't specify where that point is: near the hole, or far from it.

We can solve this in a slightly different way. We again begin with Bernoulli's equation, but this time, we consider a point at the surface of the liquid (above the hole), and a point just outside the hole. Both are exposed to the atmosphere, so $P_1 = P_2 = 1\,\text{atm}$ and we don't need to specify that in the equation, as it will simply cancel.

Instead, we have the gravitational potential energy per unit volume, $\rho g y$, in the equation. On the left side, we have at the top of the container, where it is $\rho g h$; I define the zero level to be at the hole, so the term only exists on the left-hand side.

$$\frac{1}{2}\rho v_{surface}^2 + \rho g h = \frac{1}{2}\rho v^2 \tag{13.63}$$

$$2gh = v^2 \tag{13.64}$$

$$h = \frac{v^2}{2g} \tag{13.65}$$

As before, we approximate the other velocity, this time at the surface, to be zero. We find exactly the same result using this method.

13.7 Problem 7: Buoyant force of a balloon

"Helium balloons are used regularly in scientific research. A typical balloon would reach an altitude of 40.0 km with an air density of $4.3 \times 10^{-3}\,\text{kg/m}^3$. At this altitude the helium in the balloon would expand to $540\,000.0\,\text{m}^3$. Take $g = 10\,\text{m/s}^2$. Find the buoyant force on the balloon."

The buoyant force is given by the weight of the displaced fluid – air in this case – so this should be very simple. Weight is given by mass times g, while mass is ρV, so $F_B = V\rho_{air}g$:

$$F_B = (540\,000.0\,\text{m}^3)(4.3 \times 10^{-3}\,\text{kg/m}^3)(10\,\text{m/s}^2) \approx 23220N \tag{13.66}$$

Very simple indeed.

Part IV

Exam questions

Chapter 1: Midterm 1

1.1 Problem 1: Derivatives and vectors

"A point particle has a position vector $\vec{r}(t)$ as a function of time t, given by

$$\vec{r}(t) = (2 - t^2)\hat{x} - 2t(t+4)\hat{y} + 10(t+2)\hat{z} \tag{1.1}$$

where distances are in meters, and time t is in seconds. Now, let $t = t_1 = 14$ s.

(a) What is the distance of the particle to the origin at time t1 ? (in meters)"

In other words, what is the magnitude of $\vec{r}(t)$ when we substitute $t = 14\,\mathrm{s}$ into the equation:

$$\vec{r}(t_1) = (2 - 14^2)\hat{x} - 2 \cdot 14(14+4)\hat{y} + 10(14+2)\hat{z} \tag{1.2}$$
$$= -194\hat{x} - 504\hat{y} + 160\hat{z} \tag{1.3}$$

The magnitude is found as $\sqrt{r_x^2 + r_y^2 + r_z^2}$, so

$$|\vec{r}(t_1)| = \sqrt{317252} \approx 563.25\,\mathrm{m} \tag{1.4}$$

"(b) What is the speed of the particle at at time t_1? (in m/s)"

We can differentiate the position equation with respect to t:

$$\vec{r}(t) = (2 - t^2)\hat{x} - (2t^2 + 8t)\hat{y} + (10t + 20)\hat{z} \tag{1.5}$$
$$\frac{d}{dt}\vec{r}(t) = \vec{v}(t) = (-2t)\hat{x} - (4t+8)\hat{y} + (10)\hat{z} \tag{1.6}$$

We then again make the substitution for $t = 14\,\mathrm{s}$, and then take the magnitude, and find

$$\vec{v}(t_1) = (-2 \cdot 14)\hat{x} - (4 \cdot 14 + 8)\hat{y} + (10)\hat{z} \tag{1.7}$$
$$|\vec{v}(t_1)| = \sqrt{(-28)^2 + (-64)^2 + 10^2} = \sqrt{4980} \approx 70.57\,\mathrm{m/s} \tag{1.8}$$

"(c) What is the (smaller) angle between the velocity vector at time t_1 and the $\hat{z}$ axis? (in degrees)"

Hmm. I got this answer right on the exam, but when having a closer look, I noticed that my answer was actually off by 0.9% (less than 1 degree, but read on).
Comparing my solution and the staff's, it's clear that my solution can give much, much greater errors for other components, so I got "lucky", getting it marked correct on my first try, despite an invalid method. Had v_z been much larger, I would've gotten it wrong (though would have had a second try remaining). Anyway, long story short, I rewrote this answer to use a proper solution.

The staff's solution used the dot product – I didn't think of that, clever. Let's try to calculate it that way. We know that the dot product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, but also $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ so if we try this with $\vec{a} = \vec{v}$ and $\vec{b} = \hat{z}$, the unit vector for the z axis, we should be able to solve for that angle:

$$\sqrt{4980}|\hat{z}|\cos\theta = 10 \tag{1.9}$$
$$\theta = \arccos\frac{10}{\sqrt{4980}} \tag{1.10}$$

The magnitude of $\hat{z}$ is 1 by definition, so that gives us $\theta = 81.85°$.

"(d) What are the components of the particle's acceleration vector $\vec{a} = (a_x, a_y, a_z)$ at time t_1?"

Yet again we take the time derivative, this time of the velocity vector:

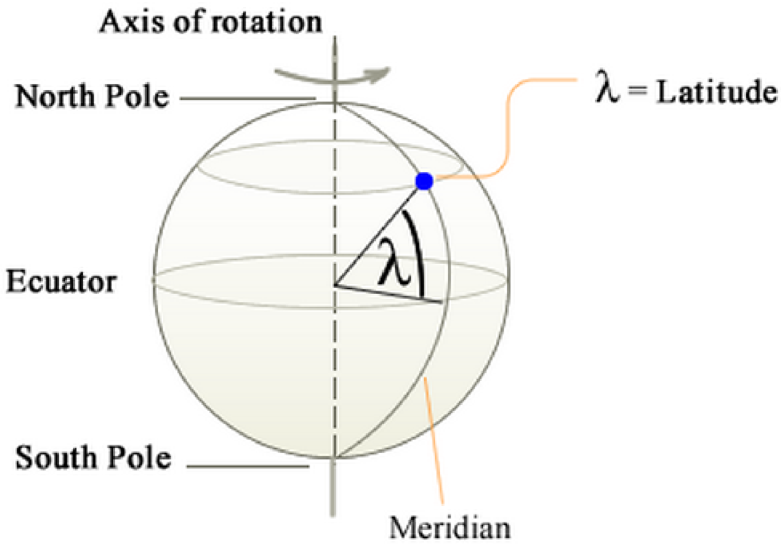
$$\frac{d}{dt}\vec{v}(t) = \vec{a}(t) = (-2)\hat{x} - (4)\hat{y} \tag{1.11}$$

So the answers are

$$a_x = -2\,\mathrm{m/s^2} \tag{1.12}$$
$$a_y = -4\,\mathrm{m/s^2} \tag{1.13}$$
$$a_x = 0\,\mathrm{m/s^2} \tag{1.14}$$

1.2 Problem 2: Rotating Earth

"Every point on Earth uniformly rotates once a day in a circular path about Earth's axis. Suppose that the Earth is a perfect sphere with radius $R_E = 6380\,\mathrm{km}$ and that the rotational period of the Earth is 23 hours 56 min and 4 sec. Calculate the speed (in m/s) and the acceleration (in m/s^2) due to the Earth's rotation for



(a) a point on the Equator."

Okay, first off, all of these problems will require the radius in meters, and the period in seconds, so let's start out by finding those two. The radius is then 6.38×10^6 m, and the period $23 \cdot 3600 + 56 \cdot 60 + 4 = 86164$ seconds.

Next, let's find symbolic formulas for the two quantities they want. Both depend on the "effective" radius, which is the largest at the equator at $r = r_E$ and smallest at the poles, at $r = 0$ (at one single point). This radius depends on the angle from the equator, i.e. the latitude. The relationship can be derived with trigonometry, or essentially by guessing – it is a trig function of the angle, such that the function is maximized at an angle of zero (from the equator), and minimized at 90 degrees... In other words, the cosine:

$$r = R_E \cos \lambda \tag{1.15}$$

Not the most rigorous derivation, but I'm already completely sure that it's correct, so I won't really bother deriving it under an exam.

The velocity (or speed, rather) is found as

$$v = \frac{2\pi r}{T} = \frac{2\pi R_E \cos\lambda}{T} \tag{1.16}$$

while the centripetal acceleration is found as

$$|a_c| = \frac{v^2}{r} = \frac{4\pi^2 r^2}{rT^2} = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 R_E \cos\lambda}{T^2} \tag{1.17}$$

As functions of λ alone, we find

$$v(\lambda) = \frac{2\pi(6.38 \times 10^6\,\mathrm{m})\cos\lambda}{86\,164\,\mathrm{s}} \tag{1.18}$$

$$|a_c(\lambda)| = \frac{4\pi^2(6.38 \times 10^6\,\mathrm{m})\cos\lambda}{(86\,164\,\mathrm{s})^2} \tag{1.19}$$

Finally, we can simply plug in the numbers, and find, at the equator ($\lambda = 0$):

$$v = 465.24\,\mathrm{m/s} \tag{1.20}$$

$$|a_c| = 0.0339\,\mathrm{m/s^2} \tag{1.21}$$

"(b) Zurich (latitude $\lambda = 47.40°$ N)."

North or south doesn't matter, the radius "tapers off' equally in both directions. We stick the numbers in, and find

$$v = 314.908\,\mathrm{m/s} \tag{1.22}$$

$$a = 0.022\,963\,\mathrm{m/s^2} \tag{1.23}$$

"(c) Melbourne (latitude $\lambda = 37.80°$ S).

$$v = 367.61\,\mathrm{m/s} \tag{1.24}$$

$$a = 0.026\,807\,\mathrm{m/s^2} \tag{1.25}$$

"(d) the South Pole."

Both are zero for $\lambda = 90°$.

1.3 Problem 3: Bucket in rotation

"A bucket of water is swung in a vertical plane at the end of a rope of length $\ell = 3$ m. The mass of the bucket plus water is 5 kg and the gravitational acceleration is $g = 10\,\mathrm{m/s^2}$. We assume that the mass of the rope can be neglected.

(a) What is the minimal speed of the bucket at its highest point in the circular motion, such that the water does not fall out? (in m/s)"

The condition we need to meet is essentially $|a_c| > g$.

Since

$$|a_c| = \frac{v^2}{r} \tag{1.26}$$

we can solve for v, and find

$$v = \sqrt{r|a_c|} \tag{1.27}$$

Substitute in $r = \ell = 3\,\text{m}$ and $|a_c| = g = 10\,\text{m/s}^2$ and we find that

$$v = \sqrt{30} = 5.477\,\text{m/s} \tag{1.28}$$

"(b) For this speed, what is the magnitude of the centripetal acceleration that the water in the bucket experiences at the highest point?"

The centripetal acceleration must cancel out gravity, so the answer is simply $g = 10\,\text{m/s}^2$.

"(c) At the lowest point,

(1) the speed is higher and the centripetal acceleration is lower than at the highest point of the circular motion.
(2) the speed is lower and the centripetal acceleration is higher than at the highest point of the circular motion.
(3) the speed is higher and the centripetal acceleration is higher than at the highest point of the circular motion.
(4) the speed is lower and the centripetal acceleration is lower than at the highest point of the circular motion.
(5) speed and centripetal acceleration are the same as at the highest point of the circular motion."

This was the only (sub)question I missed on this exam, solely because I didn't interpret the question correctly and pretty much guessed at the answer. If we assume uniform circular motion, the speed and centripetal acceleration will be the same everywhere, but that interpretation doesn't make a whole lot of sense. It's incorrect, though.

A second interpretation is even worse: if we think of this as an extension of the previous two parts, we find

"What is the minimal speed of the bucket at its highest point in the circular motion, such that the water does not fall out?
At the lowest point: ..."

So it can be interpreted to be asking what the minimum speed and centripetal acceleration necessary is at the bottom, which doesn't make a lot of sense, either. This, too, will give the wrong answer.

Embarrassingly, I didn't quite get the question until after the exam. The *intended* interpretation is that *gravity is the only other force acting on the bucket*, so after it has fallen down *due to gravity*, will the speed have increased or decreased? What about the centripetal acceleration?

Well, duh! All of a sudden I find this as easy as I expect most students did at once...
Gravity accelerates it downward, so clearly the speed must have increased after the "fall". The centripetal acceleration is proportional to v^2, so clearly that too must have increased.

1.4 Problem 4: Elevator problem

"An elevator is stopped at the ground floor. It starts moving upwards at constant acceleration $a > 0$ for 5 seconds. It then keeps a constant speed for 35 seconds. Finally, it slows down with an acceleration of the same magnitude (but opposite direction) $-a$, until it comes to a halt at the top floor. The top floor is 410 meters above the ground floor.

(a) What is the maximal speed v of the elevator ? (in m/s)
(b) What is the acceleration a? (in m/s^2)"

First, I will use a coordinate system where y is positive upwards. I use y instead of x despite there being only one dimension, since I'm used to having y upwards.

Okay, so there are three phases: constant acceleration at a for 5 seconds, constant velocity for 35 seconds, and constant acceleration (or deceleration) at $-a$ for 5 seconds: since a is the same in either case, it must come to a halt in the same time it took to accelerate up to that velocity in the first place.

For the first phase, we set $t = 0$ and $y_0 = 0$. We find the distance covered to be

$$y_{acc} = \frac{1}{2}at^2 = (12.5\,\mathrm{s}^2)a \tag{1.29}$$

For the second, we know that the time taken is 35 seconds, and that the velocity must be given by $v = at$, where $t = 5\,\mathrm{s}$ (the time it spends to accelerate). We reset the clock, and find

$$y_{const} = v_0 t = (5\,\mathrm{s})a \cdot (35\,\mathrm{s}) = (175\,\mathrm{s}^2)a \tag{1.30}$$

Finally, it comes to a halt; we again reset the clock, and use both v_0 (above) and a here:

$$\begin{aligned} y_{dec} &= (5\,\mathrm{s})a \cdot (5\,\mathrm{s}) - \frac{1}{2}a(5\,\mathrm{s})^2 && (1.31)\\ &= (25\,\mathrm{s}^2) - (12.5\,\mathrm{s}^2)a = (12.5\,\mathrm{s}^2)a && (1.32)\end{aligned}$$

We add all of these displacements up:

$$\begin{aligned} y_{total} &= (12.5\,\mathrm{s}^2)a + (175\,\mathrm{s}^2)a + (12.5\,\mathrm{s}^2)a && (1.33)\\ &= (200\,\mathrm{s}^2)a && (1.34)\end{aligned}$$

We know from the problem that this distance covered must equal 410 m, so we set it equal to that and solve for a.

$$\begin{aligned} (200\,\mathrm{s}^2)a &= 410\,\mathrm{m} && (1.35)\\ a &= \frac{410\,\mathrm{m}}{200\,\mathrm{s}^2} = 2.05\,\mathrm{m/s}^2 && (1.36)\end{aligned}$$

Does this answer make sense? Let's try it out. The distance covered during both the acceleration phases would be 51.25 meters, which leaves 358.75 meters for the constant velocity phase. That would require a constant velocity of 10.25 meters per second, which is indeed the velocity you would reach at $2.05\,\mathrm{m/s}^2$ for 5 seconds. The answers are indeed marked as correct.

1.5 Problem 5: Vertically thrown stones

"A stone is thrown up vertically from the ground (the gravitational acceleration is $g = 10\,\mathrm{m/s}^2$). After a time $\Delta t = 2$ s, a second stone is thrown up vertically. The first stone has an initial speed $v_1 = 18.0\,\mathrm{m/s}$, and the second stone $v_2 = 18.0\,\mathrm{m/s}$.

(a) At what time t after the first stone is thrown will the two stones be at the same altitude h above ground? (in seconds)
(b) At what altitude h above ground will the two stones meet? (in meters)"

Let's set y_0 at the point they are thrown from, and $t = 0$ when the first is thrown. Therefore, the second is thrown at $t = \Delta t$, and we need to use $(t - \Delta t)$ in the kinematics equations for the second object for it to work out.

With all that in mind, the two position equations are

$$y_1(t) = v_1 t - \frac{1}{2} g t^2 \qquad = v_1 t - 5t^2 \tag{1.37}$$
$$y_2(t) = v_2(t - \Delta t) - \frac{1}{2} g (t - \Delta t)^2 \qquad = v_2 t - v_2 \Delta t - 5(t - \Delta t)^2 \tag{1.38}$$

Sanity check: if Δt is 2, and $t = 2$ as well, $y_2(t) = 0$ as it should be.

We can now set the two equal, substitute in some values, and find t:

$$v_1 t - 5t^2 = v_2 t - v_2 \Delta t - 5(t - \Delta t)^2 \tag{1.39}$$
$$18t - 5t^2 = 18t - (18)(2) - 5(t - 2)^2 \tag{1.40}$$
$$-5t^2 = -36 - 5(t^2 - 4t + 4) \tag{1.41}$$
$$0 = -36 + 20t - 20 \tag{1.42}$$
$$t = \frac{56}{20} = 2.8\,\mathrm{s} \tag{1.43}$$

I originally wrote these equations with units, and got an incredible mess, and an incorrect answer (I didn't have to submit it to realize it was wrong, either!), so I re-did it without units, and got a number that looked much more reasonable.

Part (b) should now be easy, at least. We know the time, and so we can use either kinematic equation (since they are at the same location). I choose the first one, of course, since it's less complex.

$$h = v_1 t - (5\,\mathrm{m/s^2}) t^2 \tag{1.44}$$
$$h = (18\,\mathrm{m/s})(2.8\,\mathrm{s}) - (5\,\mathrm{m/s^2})(2.8\,\mathrm{s})^2 \tag{1.45}$$
$$h = 50.4\,\mathrm{m} - 39.2\,\mathrm{m} = 11.2\,\mathrm{m} \tag{1.46}$$

And we are done.

1.6 Problem 6: Stone off a cliff

"A person is standing on the edge of a cliff of height $h = 22$ m. She throws a stone of mass $m = 0.2$ kg vertically down with speed $v_0 = 11$ m/s (stone 1) and another stone of the same mass vertically up at the same speed (stone 2). The gravitational acceleration is $g = 10\,\mathrm{m/s^2}$.

(a) What is the speed of stone 1 at the bottom of the cliff? (in m/s)
(b) What is the speed of stone 2 at the bottom of the cliff? (in m/s)
(c) What is the time of flight of stone 1 when it hits the bottom of the cliff? (in s)
(d) What is the time of flight of stone 2 when it hits the bottom of the cliff? (in s)
(e) What is the average speed of stone 1 during its flight? (in m/s)
(f) What is the average speed of stone 2 during its flight? (in m/s)
(g) What is the magnitude of the average velocity of stone 1 during its flight?
(h) What is the magnitude of the average velocity of stone 2 during its flight?"

Goodness! I might need the multiple tries just for typo correction with so many things to work out!

Since the stones are independent of each other, I will do a/c/e/g first, and b/d/f/h later.

We should know at this point that the mass is completely irrelevant for this problem, at least if we ignore air resistance.
Relevant kinematic equations for the first stone are, with $+y$ chosen upwards and $y = 0$ at the bottom of the cliff:

$$y(t) = 22\,\mathrm{m} - (11\,\mathrm{m/s})t - \frac{1}{2}gt^2 \tag{1.47}$$

$$v(t) = -11\,\mathrm{m/s} - gt \tag{1.48}$$

We first need to know the time t when it hits the ground. We can find it from the first equation, set equal to 0.

$$0 = 22\,\mathrm{m} - (11\,\mathrm{m/s})t - \frac{1}{2}(10\,\mathrm{m/s^2})t^2 \tag{1.49}$$

$$0 = (5\,\mathrm{m/s^2})t^2 + (11\,\mathrm{m/s})t - 22\,\mathrm{m} \tag{1.50}$$

Besides the units, this is just a simple quadratic equation, with the solution

$$t = \frac{-11 \pm \sqrt{11^2 - 4 \cdot 5 \cdot (-22)}}{10} = -1.1 \pm 2.3685 = 1.2685\,\mathrm{s} \tag{1.51}$$

... if we neglect the negative solution, as we should! That answers (c), then. Now, back to (a), the speed.

$$v(t) = \Big| - 11\,\mathrm{m/s} - (10\,\mathrm{m/s^2})(1.2685\,\mathrm{s})\Big| = 23.685\,\mathrm{m/s} \tag{1.52}$$

"(e) What is the average speed of stone 1 during its flight? (in m/s)"

Average speed is simply distance divided by time. It travels h meters in t seconds, so we find

$$\bar{v}_1 = \frac{h}{t} = \frac{22\,\mathrm{m}}{1.2685\,\mathrm{s}} \approx 17.343\,\mathrm{m/s} \tag{1.53}$$

"(g) What is the magnitude of the average velocity of stone 1 during its flight?"

Because it has traveled in one direction only, the distance is equal to the displacement, and so the magnitude of the average velocity is equal to the average speed.

Next up: the second stone. I will simply solve this as a separate problem, instead of relating the two. Since the workings will be about the same as above, I'll keep in briefer, though.

$$y(t) = 22\,\mathrm{m} + (11\,\mathrm{m/s})t - \frac{1}{2}gt^2 \tag{1.54}$$

$$v(t) = 11\,\mathrm{m/s} - gt \tag{1.55}$$

Note that the initial velocity is now upwards. We again solve for the time, and find $t = 3.468\,54\,\mathrm{s}$ to hit the bottom, which answers (d). Plugging that into $v(t)$, we find the speed as it hits, which answers (b): $|v(3.46854)| = 23.685\,\mathrm{m/s}$. Same as the other one – I suspected as much, but in the interest of avoiding silly mistakes, I chose to solve them separately anyhow.

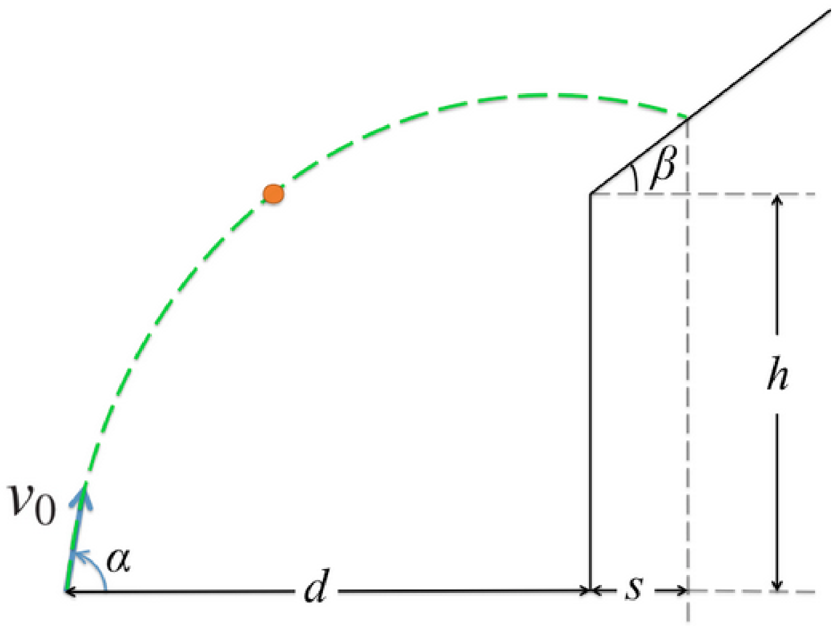
Next, average speed. This one differs: we now have the upwards movement PLUS the downwards movement, divided by the time.
How far above the throw point did it reach? With the launch velocity, it must have reached the top at $t = 1.1\,\mathrm{s}$, which translates into reaching 6.05 meters up. It then falls back down 6.05 meters, plus the 22 meters below the starting point, for a total of 34.1 meters, divided by the time:

$$\bar{v}_2 = \frac{34.1\,\mathrm{m}}{3.468\,54\,\mathrm{s}} = 9.83\,\mathrm{m/s} \tag{1.56}$$

And, finally, magnitude of the average velocity. Here, only displacement matters, and it ends up 22 meters below where it started:

$$\widetilde{v}_2 = \frac{22\,\mathrm{m}}{3.468\,54\,\mathrm{s}} = 6.34\,\mathrm{m/s} \tag{1.57}$$

1.7 Problem 7: Stone on roof, find distance



"We are standing at a distance $d = 15$ m away from a house. The house wall is $h = 6$ m high and the roof has an inclination angle $\beta = 30°$. We throw a stone with initial speed $v_0 = 20\,\mathrm{m/s}$ at an angle $\alpha = 51°$. The gravitational acceleration is $g = 10\,\mathrm{m/s^2}$. (See figure)

(a) At what horizontal distance from the house wall is the stone going to hit the roof (s in the figure)? (in meters)
(b) What time does it take the stone to reach the roof? (in seconds)"

This problem is scaring me a bit: there have been *many* reports on the wiki from students who claim it's failing their correct answers. The staff insist that all such answers are incorrect, though. So the question is: when (or if) I think I've solved it, will I have made the same mistake they all did, or will I have *actually* solved it? I suppose there's only one way to find out...

Among the staff hints are

- Make sure you use $g = 10\,\mathrm{m/s^2}$
- Make sure you don't round answers to less than 2-3 decimals
- Solve (a) independently from (b).
- Make sure you are not making any uncalled for approximations.
- Do not waste attempts by plugging the value of $d + s$, or simply rounding your answer.

Okay, let's see. I will begin by writing down the kinematics equations, and we'll see where that gets us. This is of course the easy part. I choose a coordinate system centered at the throw, with $+\hat{x}$ to the right and $+\hat{y}$ upwards.

$$x(t) = (v_0 \cos\alpha)t \tag{1.58}$$
$$v_x(t) = v_0 \cos\alpha \tag{1.59}$$
$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \tag{1.60}$$
$$v_y(t) = v_0 \sin\alpha - gt \tag{1.61}$$

Both the x and y position of where it hits the roof depend on β. If $\beta = 0$, clearly it will hit at $y = h$, and with a large value of s. If $\beta = 90°$, the "roof" is more like a wall, and it hits at $x = d$, with $s = 0$. In between these extremes, $x > d$ (so $s > 0$) and $y > h$.

I'm not really sure how to solve this, but one way that may work is to try to find the roof height (above h) as a function of x beyond the house edge, where it starts. Clearly, it should be 0 just at the house edge, and go toward infinity for ridiculously high values of x (since it has no defined end, it just keeps going in the direction shown in the figure).

If we draw just a simple triangle as the "roof", and mark out β, and a point along the hypotenuse we call (x, y), we find that

$$\tan\beta = \frac{y}{x} \tag{1.62}$$
$$y = x\tan\beta \tag{1.63}$$

This makes sense – if $\beta = 45°$, $y = x$, i.e. they increase at the same rate as you go towards the right. If β is really large, y grows very fast as x grows a little, and if β is very low, y barely grows at all as you go further to the right.

What I just called x is really the same as s in the problem, so I will use that from now on. The y coordinate where it hits, as measured with $x = 0$ and $y = 0$ centered on the throw, is then $y = h + s\tan\beta$ and $x = d + s$. I'm not sure what the staff meant by not "wast[ing] attempts by plugging the value of $d + s$", but I don't see how that could be incorrect, so I will try this out.

Using the kinematics equations, we then have

$$(v_0 \cos\alpha)t = d + s \tag{1.64}$$
$$(v_0 \sin\alpha)t - \frac{1}{2}gt^2 = h + s\tan\beta \tag{1.65}$$

The unknowns are t and s, and those are exactly the values the problem asks for us to find. Awesome!

Since it is allowed, I solved these equations in Mathematica, and found $s = 9.301\,730\,802\,\mathrm{m}$ and $t = 1.930\,791\,624\,\mathrm{s}$. Ridiculous precision, but since so many students had trouble with it, I decided to submit with all those decimals instead of possibly rounding too much.

I figured I would also solve the equations manually, but honestly, it turned out to be a bit too bad, at least with the first method I tried (solve first equation for t, substitute into second, solve for s). Tons of terms, $\tan\alpha$ and $\cos\alpha$ everywhere, s in 3-4 different terms, some inside squared expressions that you'd have to multiply out, etc, etc.

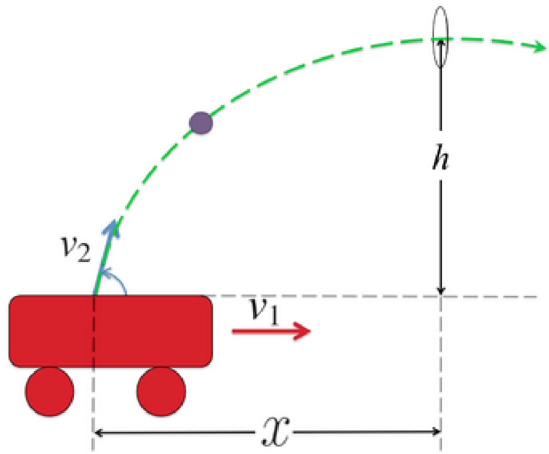
The staff solution does this better by using $x - d$ instead of s, which simplifies things a bit. The full answer is still not pretty, though. This is what Mathematica gives me, fully simplified taking into account things such as $g > 0$, $0 < \alpha < \pi/2$ etc.:

$$s = \frac{v_0 \cos(\alpha)\left(\sqrt{2dg\tan(\beta) - 2gh + v_0^2 \sec^2(\beta)\sin^2(\alpha - \beta)} + v_0 \sec(\beta)\sin(\alpha - \beta)\right) - dg}{g} \tag{1.66}$$

$$t = \frac{\sqrt{2dg\tan(\beta) - 2gh + v_0^2\sec^2(\beta)\sin^2(\alpha-\beta)} + v_0\sec(\beta)\sin(\alpha-\beta)}{g} \tag{1.67}$$

Wow.

1.8 Problem 8: Man on a flatcar with ball



"A person is riding on a flatcar traveling at a constant speed $v_1 = 15\,\text{m/s}$ with respect to the ground. He wishes to throw a ball through a stationary hoop in such a manner that the ball will move horizontally as it passes through the hoop. The hoop is at a height $h = 4\,\text{m}$ above his hand. He throws the ball with a speed $v_2 = 14\,\text{m/s}$ with respect to the flatcar. Let $g = 10\,\text{m/s}^2$ and neglect air drag completely. (see figure)

(a) At what horizontal distance x in front of the hoop must the person release the ball? (in meters)
(b) When the ball leaves his hand, what is the direction of the velocity vector of the ball as seen from the flatcar? (angle α_{car} with respect to the horizontal in degrees)
(c) When the ball leaves his hand, what is the direction of the velocity vector of the ball as seen from the ground? (angle α_{ground} with respect to the horizontal in degrees)"

I'll try to do most of the work in a coordinate system fixed to the ground, such that the flatcar is moving forward at v_1.

So, from this frame, at what velocity is the ball thrown? The car moves horizontally forward, so clearly the y component (which I choose to be positive upwards) will not change. Instead, using prime notation for the launch velocity as seen from the ground,

$$v_2' = (v_2\cos\alpha_{car} + v_1)\hat{x} + (v_2\sin\alpha_{car})\hat{y} \tag{1.68}$$

A bit of a cumbersome way to write it, but the point is that only the x component will change.

Writing these as components, then, we we of course have

$$v_{2x}' = v_2\cos\alpha_{car} + v_1 \tag{1.69}$$
$$v_{2y}' = v_2\sin\alpha_{car} \tag{1.70}$$

We can now calculate the distance x as if we were standing still on the ground and throwing the ball, instead, with less of a headache... at least hopefully.
The condition is that it must move horizontally through the hoop. Since it moves in a parabola, this means (as is evident from the drawing, too) that it must be at its apex when it moves through. That happens when $v_{2y} = 0$, for one.

In fact, we can ignore x motion almost completely, unless I'm missing something! We find when $v_{2y} = 0$, and extract the time from that equation. Unfortunately, the angle is also unknown. We can find it by also relating the y *position* to the height h, though.
Anyway, let's set up the kinematics equations. I will write them from the reference frame fixed to the ground. $x_0 = 0$ is where the ball is launched, which is x meters in front of the hoop.

$$x(t) = (v_1 + v_2 \cos \alpha_{car})t \tag{1.71}$$
$$v_x(t) = v_1 + v_2 \cos \alpha_{car} \tag{1.72}$$
$$y(t) = (v_2 \sin \alpha_{car})t - \frac{1}{2}gt^2 \tag{1.73}$$
$$v_y(t) = v_2 \sin \alpha_{car} - gt \tag{1.74}$$

So at some value for t, we should have $y(t) = h$ and $v_y(t) = 0$. The equations have α_{car} and t as unknowns, then, so this should be solvable. Let's do it. Let's solve the simple one for α_{car}:

$$v_2 \sin \alpha_{car} - gt = 0 \tag{1.75}$$
$$\sin \alpha_{car} = \frac{gt}{v_2} \tag{1.76}$$
$$\alpha_{car} = \arcsin \frac{gt}{v_2} \tag{1.77}$$

Next, we need to find an expression for t:

$$h = (v_2 \sin \alpha_{car})t - \frac{1}{2}gt^2 \tag{1.78}$$
$$(2v_2 \sin \alpha_{car})t - gt^2 = 2h \tag{1.79}$$
$$gt^2 - (2v_2 \sin \alpha_{car})t = -2h \tag{1.80}$$

Before we move on, we can substitute in the angle. We get the sine of the arcsine, so we end up without trig functions:

$$gt^2 - (2v_2 \frac{gt}{v_2})t = -2h \tag{1.81}$$
$$gt^2 - (2gt)t = -2h \tag{1.82}$$
$$-gt^2 = -2h \tag{1.83}$$
$$t = \sqrt{\frac{2h}{g}} \approx 0.8944\,\mathrm{s} \tag{1.84}$$

We can now find α_{car} using the expression we had above:

$$\alpha_{car} = \arcsin \frac{gt}{v_2} \approx 0.693012 \text{ rad} \approx 39.707° \tag{1.85}$$

Nice. We now have what we need to find x, since we have a kinematic equation for it:

$$x = (v_1 + v_2 \cos \alpha_{car})t = (15\,\mathrm{m/s} + (14\,\mathrm{m/s}) \cos(39.707°))(0.8944\,\mathrm{s}) = 23.049\,\mathrm{m} \tag{1.86}$$

Finally, we need to find the angle as seen from the ground. Intuitively, how would the angle change? I think it should become less steep, i.e. a smaller angle. If we imagine the car moving really, really fast (to be silly, say 10 km/s), so that $v_1 \gg v_2$, the throw is almost exclusively in the x direction as seen from the ground; he might throw it upwards at 10-20 m/s and forward at 10-20 m/s, but with the 10 *km* per second relative movement, his throw really doesn't matter. The angle must be very, very small in that case.

Now, how do we calculate it? The easiest way I can think of is to calculate the velocity vector (as seen from the ground), and then do some basic trigonometry. We have the four kinematics equations above; at $t = 0$, their values are

$$v_x(0) = v_1 + v_2 \cos \alpha_{car} = 15\,\mathrm{m/s} + (14\,\mathrm{m/s})\cos(39.707°) = 25.77\,\mathrm{m/s} \tag{1.87}$$
$$v_y(0) = v_2 \sin \alpha_{car} = (14\,\mathrm{m/s})\sin(39.707°) = 8.944\,\mathrm{m/s} \tag{1.88}$$

The angle can then be found by drawing this up and realizing that we need the arctangent of v_y over v_x:

$$\alpha_{ground} = \arctan \frac{v_y}{v_x} \approx 19.140° \tag{1.89}$$

Indeed, if v_x grows, the angle gets smaller, as predicted.
That's it for this exam!

Chapter 2: Midterm 2

2.1 Problem 1: Gravitational potential, kinetic energy, conservation of mechanical energy

"A meteorite of mass $m = 2 \times 10^4$ kg is approaching head-on a planet of mass $M = 7 \times 10^{29}$ kg and radius $R = 3 \times 10^4$ km. Assume that the meteorite is initially at a very large distance from the planet where it has a speed $v_0 = 4 \times 10^2$ km/s. Take $G = 6.67 \times 10^{-11}$.

Determine the speed of the meteorite v (in m/s) just before it hits the surface of the planet. (The planet has no atmosphere, so we can neglect all friction before impact)"

Because the two start out separated by a "very large distance", I assume that $U_{initial} = 0$ (that is, we treat the separation as infinitely large). If we find gravitational potential energy at the planet's surface, and then calculate the change in potential energy, we can apply the conservation of mechanical energy to find the meteorite's kinetic energy, and from that, the impact speed.
In doing so, we assume that the planet's movement and change in kinetic energy is negligible.

Initial kinetic energy is $\frac{1}{2}mv_0^2$, while initial potential energy is zero.
The final kinetic energy is $\frac{1}{2}mv^2$ where v is the impact velocity; final potential energy is $-\frac{GMm}{R}$. Note that the final potential energy is *negative*, and therefore *smaller* than the initial, despite the initial being zero.

Conservation of mechanical energy gives us $K + U = K' + U'$, using the prime notation to mean "after" (or at the collision, rather), so we get

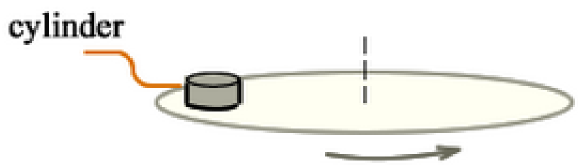
$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) \tag{2.1}$$

$$mv_0^2 + \frac{2GMm}{R} = mv^2 \tag{2.2}$$

$$v = \sqrt{v_0^2 + \frac{2GM}{R}} \approx 1.809 \times 10^6 \text{ m/s} \tag{2.3}$$

Since this is an exam, and since 1800 km/s is a very high speed, let's just ensure that the total energy didn't change.
Initial kinetic energy of the meteorite was 1.6×10^{15} J (holy smokes), which equals the initial energy.
Final kinetic energy is 3.27×10^{16} J, while final potential energy is -3.11×10^{16} J, so the final energy total is 1.6×10^{15} J.
Excellent.

2.2 Problem 2: Cylinder on turntable



"Suppose the metal cylinder shown above has a mass of $m = 0.10$ kg and that the coefficient of static friction between the surface and the cylinder is $\mu = 0.12$. If the cylinder is $x = 0.20$ m from the center of the turntable, what is the maximum speed v_{max} that the cylinder can move along its circular path without slipping off the turntable? Choose the range that includes your answer."

There needs to be a centripetal force on the cylinder for it to stay where it is. This force is provided by the contact force between the cylinder and the surface; friction, in other words. The force required is $\frac{mv^2}{x}$. The maximum possible frictional force is given by μmg. No other forces are relevant, so the condition is

$$\frac{mv_{max}^2}{x} = \mu mg \tag{2.4}$$

$$v_{max} = \sqrt{x\mu g} \tag{2.5}$$

The equation is dimensionally consistent, and it says that $v_{max} = 0.4898\,\mathrm{m/s}$ using $g = 10\,\mathrm{m/s^2}$, or very slightly less using $g = 9.81\,\mathrm{m/s^2}$.

In either case, the answer is in the range $0.0 < v_{max} \leq 0.5$ m/s, so that's the answer.

2.3 Problem 3: Woman in elevator

"A woman weighs $F_g = 550\,\mathrm{N}$ when standing on a stationary scale. Now, the woman is riding an elevator from the 1st floor to the 10th floor. As the elevator approaches the 10th floor, it decreases its upward speed from 6 m/s to 1 m/s in a time interval of 2 seconds. What is the average force exerted by the elevator floor on this woman during this 2 s interval? Use $g = 10\,\mathrm{m/s^2}$."

This shouldn't be too hard. With only one attempt however, since it is multiple choice, I still saved this (and problem 2) for last.

As the speed decreases, her momentum changes from $(55\,\mathrm{kg})(6\,\mathrm{m/s}) = 330$ kg m/s to $(55\,\mathrm{kg})(1\,\mathrm{m/s}) = 55$ kg m/s. That's an impulse of $I = p_f - p_i = -275$ kg m/s.
(Her mass is $\frac{F_g}{g} = \frac{550\,\mathrm{N}}{10\,\mathrm{m/s^2}} = 55\,\mathrm{kg}$.)

The impulse can be used to find the average force (not the final answer, mind you, we still need to consider mg):

$$\langle F \rangle\,(2\,\mathrm{s}) = -275\,\mathrm{kg\ m/s} \tag{2.6}$$

$$\langle F \rangle = \frac{-275\,\mathrm{kg\ m/s}}{2\,\mathrm{s}} = -137.5\,\mathrm{N} \tag{2.7}$$

She is then 137.5 N lighter than usual as the elevator slows down. Her net weight is $550\,\mathrm{N} + -137.5\,\mathrm{N} = 412.5\,\mathrm{N}$. This is the same as the force the floor exerts on her (the force the floor exerts on you *is* your weight, according to our definitions).

A second way of solving this is to consider acceleration. When the elevator is slowing down, the perceived gravity decreases. The elevator's average acceleration is

$$a_{avg} = \frac{1\,\mathrm{m/s} - 6\,\mathrm{m/s}}{2\,\mathrm{s}} = -2.5\,\mathrm{m/s^2} \tag{2.8}$$

The perceived gravity is then $g + (-2.5\,\mathrm{m/s^2}) = 7.5\,\mathrm{m/s^2}$, and her weight is $mg = 412.5$ N.

2.4 Problem 4: Two skaters

"Two skaters of mass $m_1 = 50\,\mathrm{kg}$ and $m_2 = 70\,\mathrm{kg}$ are standing motionless on a horizontal ice surface. They are initially a distance $L = 8.0$ meters apart. They hold a massless rope between them. After pulling the rope, the skater of mass m_1 has moved a distance $\ell = 1.0$ meters away from his initial position. We can completely neglect friction in this problem.

What is the distance L' between the two skaters when the skater of mass m_1 has moved a distance ℓ? (in meters)"

Oh. I read the problem at least three times before realizing that *both* skaters will move... Silly me.

Okay, so what do we know? With no external forces (such as friction) in the horizontal direction, conservation of momentum holds. Not knowing any final velocity, this might seem to be of limited usefulness, but let's see!
Since the initial velocities are both zero, I will use v_1 and v_2 for the velocities they move at after the fact. Initial momentum is zero and is conserved, so

$$m_1 v_1 + m_2 v_2 = 0 \tag{2.9}$$

From that, we can find

$$\frac{v_1}{v_2} = -\frac{m_2}{m_1} \tag{2.10}$$

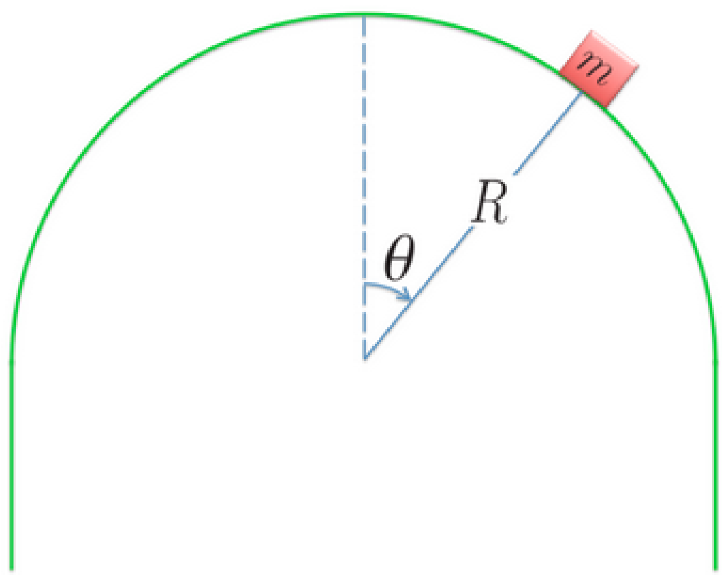
$$v_2 = -\frac{m_1 v_1}{m_2} \tag{2.11}$$

v_1 moved a distance ℓ under some unknown time t; what distance did v_2 move under that same time? He moved a distance m_1/m_2 times as great (the ratio of their speeds), and clearly they both went towards each other.

They came $\ell = 1.0$ meters closer due to the movement of m_1, and $\frac{m_1}{m_2}\ell = 5/7$ meters closer due to the moment of m_2. Therefore,

$$L' = L - \ell\left(1 + \frac{m_1}{m_2}\right) \approx 6.286\,\mathrm{m} \tag{2.12}$$

2.5 Problem 5: Sliding down a dome



“A small object of mass $m = 20\,\text{kg}$ slides down a spherical dome of radius $R = 12\,\text{m}$ without any friction. It starts off at the top (polar angle $\theta = 0$) at zero speed. Use $g = 10\,\text{m/s}^2$. (See figure)

(a) What is the magnitude of the force (in Newtons) exerted by the dome on the mass when it is at the top, at $\theta = 0°$?
(b) What is the magnitude of the force (in Newtons) exerted by the dome on the mass when it is at $\theta = 30°$?
(c) At what angle θ_0 does the sliding mass take off from the dome? Answer in degrees ($0 \leq \theta_0 \leq 90°$).”

Hmm, I assume this problem is meant to be similar to one previously, which stated that it started with a negligible but nonzero speed. It should be in a stable equilibrium at the top, so with exactly zero speed, it should stay there forever!

At $\theta = 0$, the normal force should be simply $mg = 200$ N. It is at rest, there are no forces other than gravity and the normal force, and the two must balance out exactly or it wouldn't be at rest.

What is the normal force for other values of θ, though? At 90 degrees, it's clearly zero, as there's nothing to make it stick to the dome, and no reason for the two to still be in contact at that point. However, it becomes zero earlier: when the object loses contact with the surface (question c).

When does it “take off”, though? How can we find a simple criterion to calculate at which angle that happens?
Well, for it to move in along this circular (in cross section, at least) dome, it needs to have a certain centripetal force inwards.

The criterion for falling off is then that the centripetal force is no longer strong enough to keep the tangential velocity changing along a circular path. The centripetal force required is $\frac{mv^2}{R}$.
The forces that can provide this force is gravity and the normal force. In decomposing the force of gravity, we find $mg\cos\theta$ in the radially inwards direction (perpendicular to the surface) and $mg\sin\theta$ in the tangential direction; we only need the radial part here, though.

The centripetal force is always radially inwards; the radial component of gravity $mg\cos\theta$ is also inwards, but the normal force is radially *outwards*, and so contributes a minus sign:

$$\frac{mv^2}{R} = mg\cos\theta - N \tag{2.13}$$

$$N = mg\cos\theta - \frac{mv^2}{R} \tag{2.14}$$

When the object falls off, N must be zero (there can't be any contact forces without contact!). That gives us the condition

$$g\cos\theta_0 = \frac{v_{off}^2}{R} \tag{2.15}$$

$$\theta_0 = \arccos\frac{v_{off}^2}{gR} \tag{2.16}$$

(I call the speed v_{off} specifically because I accidentally used it as a general value for part (b), which gave me the wrong answer. I thankfully figured that out before submitting, but it took a while to realize my mistake!)

Unfortunately, we don't know v_{off}! We can find the tangential acceleration, but since there are only conservative forces involved, mechanical energy is conserved, so we can use an energy approach and forget about the kinematics.

Say we define $U = 0$ at the point where $\theta = 90°$. That means the initial gravitational potential energy is mgR; what about the final energy? It is not zero, since the object will fly off prior to reaching the zero point.
We can find the height above the zero point in terms of θ_0. Drawing it out, some trigonometry shows that $h = R\cos\theta_0$. This is consistent with h being a maximum at $\theta = 0$ and zero at $\theta = 90°$, which is always a good sign!

The gain in kinetic energy must equal the loss in potential energy. Since the kinetic energy started out at zero, we find:

$$\frac{1}{2}mv_{off}^2 = mgR - mgR\cos\theta \tag{2.17}$$

However, we have an expression for θ_0 in equation (2.16) which is the only angle of θ we care about for part (c); taking the cosine of both sides gives us $\cos\theta_0 = \frac{v_{off}^2}{gR}$, so we substitute that in the velocity equation above:

$$\frac{1}{2}mv_{off}^2 = mgR - mv_{off}^2 \tag{2.18}$$

$$v_{off}^2 = 2gR - 2v_{off}^2 \tag{2.19}$$

$$v_{off} = \sqrt{\frac{2gR}{3}} \tag{2.20}$$

This is then the speed it has as it falls off.

We can then use this value for θ_0:

$$\theta_0 = \arccos\frac{\frac{2gR}{3}}{gR} \tag{2.21}$$

$$\theta_0 = \arccos\frac{2}{3} \approx 48.1897° \tag{2.22}$$

Incredibly, g, R and m all cancel at some point, and the angle is a constant! Honestly, this is a bit mind-blowing to me. I would at least expect g to matter, but nope.

We can find a general value of v, which is valid for all angles, by going back to the conservation of energy equation:

$$\frac{1}{2}mv^2 = mgR - mgR\cos\theta \tag{2.23}$$

$$v^2 = 2gR(1 - \cos\theta) \tag{2.24}$$

$$v = \sqrt{2gR(1 - \cos\theta)} \tag{2.25}$$

We need that to find the normal force at 30 degrees. We found an expression for that earlier, but now we also know v, so we can find N in terms of only known values:

$$N = mg\cos\theta - \frac{mv^2}{R} \tag{2.26}$$

$$N = mg\cos\theta - \frac{2mgR(1 - \cos\theta)}{R} \tag{2.27}$$

$$N = mg(\cos\theta - 2(1 - \cos\theta)) \tag{2.28}$$

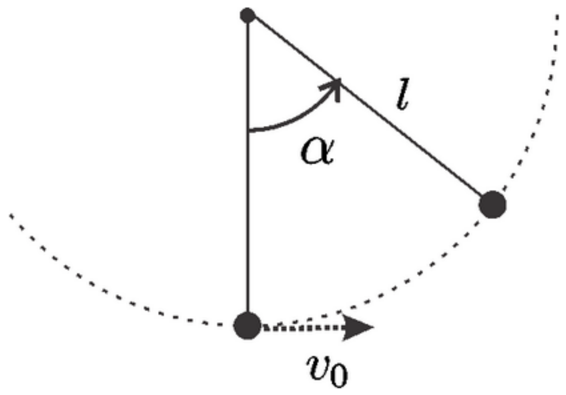
$$N = mg(3\cos\theta - 2) \approx 119.615\,\mathrm{N} \tag{2.29}$$

So the answers are

(a) Normal force at $\theta = 0$: $mg = 200$ N
(b) Normal force at $\theta = 30°$: 119.615 N
(c) Angle where it slides off: $\theta_0 = 48.1897°$

2.6 Problem 6: Pendulum with cut string

A small ball of mass $m = 0.60$ kg hangs from a massless string of length $\ell = 1.4$ m. The ball travels in a vertical circle and its speed at the bottom is $v_0 = 7.0$ m/s (see figure). Neglect all friction and air drag, and use $g = 10\,\mathrm{m/s^2}$ for the gravitational acceleration. The ball is so small that we can approximate it as a point.



(a) Find the speed of the ball (in m/s) when the string is at $\alpha = 50°$.
(b) What is the tension in the string (in Newton) when it is at $\alpha = 50°$?
(c) The string of the pendulum is cut when it is at $\alpha = 50°$. First, we want to neglect all air drag during the trajectory of the ball.
What is the maximal height h (in meters) the ball reaches above its point of release?
What time t_{up} (in s) does it take the ball to reach the highest point from the instant the string is cut?
What time t_{dn} (in s) does it take the ball to go from the highest point back to the altitude it was released from the string?

(d) Next, we want to take air drag into account for point (c). Let $\widetilde{h}$ be the maximal height of the ball above the point it was released, $\widetilde{t_{up}}$ is the time to get there, and $\widetilde{t_{dn}}$ is the time to get back to the altitude it was released (with air drag). Which of the following is true? (neglect the effect of air drag before the string is cut)

(d1)

- $\widetilde{h}$ is always greater than h
- $\widetilde{h}$ is always smaller than h
- $\widetilde{h}$ is the same as h
- We need to know if the ball is is the pressure dominated regime or in the viscous regime to tell whether $\widetilde{h} > h$, $\widetilde{h} < h$ or $\widetilde{h} = h$.

(d2)

- $\widetilde{t_{up}} > \widetilde{t_{dn}}$ and $\widetilde{t_{dn}} < t_{dn}$
- $\widetilde{t_{up}} < \widetilde{t_{dn}}$ and $\widetilde{t_{up}} < t_{up}$
- $\widetilde{t_{up}} = \widetilde{t_{dn}}$ and $\widetilde{t_{up}} < t_{up}$
- $\widetilde{t_{up}} > \widetilde{t_{dn}}$ and $\widetilde{t_{up}} > t_{up}$

- The answer depends on whether the initial speed is larger or smaller than the terminal speed.

WOW! That took a while to typeset properly (copy/paste doesn't work for the math notation, unfortunately)!

Okay, so let's start. The string tension is perpendicular to the ball's movement at all times, and therefore cannot do work. This means that gravity is the only force that *can* do work. Because of that, 100% of the loss in kinetic energy is converted to gravitational potential energy.
We can use conservation of energy, considering only kinetic energy and gravitational potential energy.

We define the zero point of potential energy to be at $\alpha = 0$, for simplicity. We can then calculate the potential energy at $\alpha = 50°$, relate the initial and final total energies: $K + U = K' + U'$, using prime notation for the "after" energies (at $\alpha = 50°$).

First, however, we need to find a way to calculate the height above the zero point (I'll call it h) in terms of ℓ and α.
Drawing it out, it can be seen that $h = \ell - \ell\cos\alpha = \ell(1 - \cos\alpha)$ (in a way identical to what was done for the pendulum in lecture).

Knowing that, we can now relate the initial energy (left-hand side) and final energy (right-hand side) as the string is cut:

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + mg\ell(1 - \cos\alpha) \tag{2.30}$$
$$v_0^2 = v^2 + 2g\ell(1 - \cos\alpha) \tag{2.31}$$
$$v^2 = v_0^2 - 2g\ell(1 - \cos\alpha) \tag{2.32}$$
$$v = \sqrt{v_0^2 - 2g\ell(1 - \cos\alpha)} \tag{2.33}$$

We can then answer part (a). With the numbers given, $v(\alpha = 50°) = 6.2448\,\text{m/s}$.

Next, they want to know the tension at this point. We can find it by relating all the forces in the radial direction. The tension is always perpendicular to the pendulum's movement, so there is no tension in the tangential direction.

In order to move along the circular path, there needs to be a centripetal force $ma_c = \frac{mv^2}{\ell}$ radially inwards. Gravity and tension are the two forces that can help provide this.
We first need to decompose the gravitational force, since we only want the radial component. The radial component is $mg\cos\theta$ in magnitude, and is radially outwards at $\theta = 0$. Newton's second law in the radially inwards direction gives us

$$\frac{mv^2}{\ell} = T - mg\cos\alpha \tag{2.34}$$
$$T = \frac{mv^2}{\ell} + mg\cos\alpha \tag{2.35}$$
$$T = \frac{m}{\ell}\left(v_0^2 - 2g\ell(1 - \cos\alpha)\right) + mg\cos\alpha \tag{2.36}$$
$$T = \frac{m}{\ell}v_0^2 - 2gm(1 - \cos\alpha) + mg\cos\alpha \tag{2.37}$$
$$T = \frac{m}{\ell}v_0^2 - mg(2 - 3\cos\alpha) \tag{2.38}$$

This gives us a tension, at $\alpha = 50°$, of $20.57\,\text{N}$.
The equation also makes intuitive sense: higher v_0 means higher tension, and there are v_0-α combinations where v_0 is not great enough for the tension to be positive – that is, if v_0 is too small, it will never reach that angle.

Now, then, on to the interesting stuff. The string is cut at the above point. What the the height h it reaches, measured *above the point of release*, if we neglect air drag?

Okay, so we can use an energy based approach here, too, since we neglect air drag. I will re-use the variable name h, and re-define $U = 0$ to be at the height it is now, and call that height zero as well. We know v, so we can easily find the kinetic energy. Again, gravity is the only force that will reduce the kinetic energy, and so the entire reduction in kinetic energy will be converted into gravitational potential energy. Initial kinetic energy depends on v, but final kinetic energy on $v\cos\theta$. The y component, $v\sin\theta$, will have gone down to zero, while the x component remains untouched. Using conservation of energy (and keep in mind that this h is unrelated to everything prior to this):

$$\frac{1}{2}mv^2 + 0 = \frac{1}{2}m(v\cos\alpha)^2 + mgh \tag{2.39}$$

$$mv^2 + 0 = mv^2\cos^2\alpha + 2mgh \tag{2.40}$$

$$\frac{v^2(1-\cos^2\alpha)}{2g} = h \tag{2.41}$$

$$\frac{v^2\sin^2\alpha}{2g} = h \tag{2.42}$$

This gives us a height of $h = 1.1442$ m. It doesn't give us the time, though... Should've thought of that. We can find the same result using kinematics, by finding the time where the velocity becomes 0. We can then substitute that time into the displacement equation $h = x_0 + vt_{up} + \frac{1}{2}at_{up}^2$ to find the height that way, too:

However, we must keep in mind that the upwards velocity is not v, but $v\sin\alpha$.

$$v\sin\alpha - gt_{up} = 0 \Rightarrow t_{up} = \frac{v\sin\alpha}{g} \tag{2.43}$$

$$h = v\sin\alpha t_{up} - \frac{1}{2}gt_{up}^2 \Rightarrow h = \frac{v^2\sin^2\alpha}{2g} \tag{2.44}$$

We find the same height, but also the time taken: the upwards velocity, divided by g, a familiar result. In terms of numbers, $t_{up} = 0.47838$ seconds.
And after that, the time for the ball to fall back down. Without air drag, this is a symmetric problem, so the time must be the same. Let's verify via kinematics just to be sure:

$$h - \frac{1}{2}gt_{dn}^2 = 0 \tag{2.45}$$

$$\frac{1}{2}gt_{dn}^2 = h \tag{2.46}$$

$$t_{dn} = \sqrt{\frac{2h}{g}} \approx 0.478\,38\,\mathrm{s} \tag{2.47}$$

Finally, the scary-looking part. Without quantitative answers, it shouldn't be that bad, though. First out:

- $\widetilde{h}$ is always greater than h
- $\widetilde{h}$ is always smaller than h
- $\widetilde{h}$ is the same as h
- We need to know if the ball is is the pressure dominated regime or in the viscous regime to tell whether $\widetilde{h} > h$, $\widetilde{h} < h$ or $\widetilde{h} = h$.

Well, what will happen with air drag? Since they ask us to neglect air drag before the string is cut, the previous results are all valid. All we need to do is compare the trajectory with and without drag.

With drag, there will be a resistive force opposing the motion (relative to the air), which means a downwards force, and a "backwards" force (to the left, as the figure shows the problem). This clearly means it cannot go as high as it would otherwise (the downwards force, and so the downwards acceleration, is greater), so $\widetilde{h}$ must always be smaller than h.
This is equally true in both regimes (though in air, we are clearly pressure dominated). In both regimes, the force opposes the motion, and so in either, the result will be a lower maximum height.

Next, part 2:

1. $\widetilde{t_{up}} > \widetilde{t_{dn}}$ and $\widetilde{t_{dn}} < t_{dn}$
2. $\widetilde{t_{up}} < \widetilde{t_{dn}}$ and $\widetilde{t_{up}} < t_{up}$
3. $\widetilde{t_{up}} = \widetilde{t_{dn}}$ and $\widetilde{t_{up}} < t_{up}$
4. $\widetilde{t_{up}} > \widetilde{t_{dn}}$ and $\widetilde{t_{up}} > t_{up}$
5. The answer depends on whether the initial speed is larger or smaller than the terminal speed.

Okay, let's try to rule out some of these.
Number 3, $\widetilde{t_{up}} = \widetilde{t_{dn}}$ is not true. With air resistance, a projectile launched with an initial upwards takes *longer* to fall back down, than to come up in the first place. This makes number number 1 and 4 false, too.

Left are number 2 and number 5. Let's tackle them one by one.
$\widetilde{t_{up}} < \widetilde{t_{dn}}$ is true, as we have seen. Is $\widetilde{t_{up}} < t_{up}$ also true? If so, this must be the answer.
With air drag, it doesn't reach as high, so it makes sense that it takes less time with air drag.
On the other hand, with air drag, the speed is constantly lowered by the drag (plus gravity, in either case), which means it takes longer time to reach a certain height... But there's a simple way to show that the the time taken to reach the maximum height must be *less* with air drag acting.

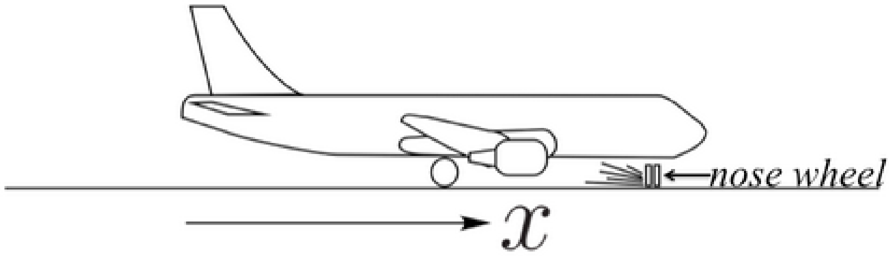
The initial velocity upwards is the same; the time we're looking for is when the net acceleration (or deceleration, if you prefer) has made the upwards velocity 0. With the *same* initial velocity, the case with the greatest downwards force (or acceleration) is clearly the one to stop first, and that is the case *with* air drag. $\widetilde{t_{up}} < t_{up}$ must be the case!

What about option 5? Does the time taken to move upwards depend on whether the initial speed is larger or smaller than the terminal speed? I don't see why it would. That leaves option 2 as the only one possible answer, it looks like! It's always nice to be able to not only find a correct option, but rule out the others, too.

Option 1 is wrong because $\widetilde{t_{up}} < \widetilde{t_{dn}}$, and the option has it the other way around.
Option 2 looks good!
Option 3 is wrong because $\widetilde{t_{up}} < \widetilde{t_{dn}}$
Option 4 is wrong because $\widetilde{t_{up}} < \widetilde{t_{dn}}$
Option 5 looks wrong, because the terminal speed has to do with when the drag force (upwards) and gravity (downwards) balance out in a *downwards* fall. There is never any such balance in an upwards motion though air; there will always be deceleration.

These answers are correct, by the way. (I did get h, t_{up} and t_{dn} wrong the first time, as I used v instead of $v \sin\alpha$ by accident. D'oh!)

2.7 Problem 7: Emergency landing of a plane



"An airliner makes an emergency landing with its nose wheel locked in a position perpendicular to its normal rolling position. The forces acting to stop the airliner arise from friction due to the wheels and from the breaking effort of the engines in reverse thrust mode. The force of the engine on the plane is constant, $F_{engine} = -F_0$. The sum of the horizontal forces on the airliner (in its forward direction) can be written as

$$F(t) = -F_0 + \left(\frac{t}{t_s} - 1\right) F_1 \tag{2.48}$$

from touchdown at time $t = 0$ to the final stop at time $t_s = 28$ s ($0 \le t \le t_s$).
The mass of the plane is $M = 80$ tonnes (one tonne is 1000 kg). We have $F_0 = 260$ kN and $F_1 = 41$ kN. Neglect all air drag and friction forces, except the one stated in the problem.

(a) Find the speed v_0 of the plane at touchdown (in m/s).

(b) What is the horizontal acceleration of the plane at the time t_s?
What is the acceleration at the time of touchdown? (absolute values; in m/s^2)

(c) What distance s does the plane go between touchdown and its final stop at time t_s? (in meters)

(d) What work do the engines in reverse thrust mode do during the emergency landing? (magnitude in Joules; the force due to engines is $(-F_0)$)

(e) How much heat energy is absorbed by the wheels during the emergency landing? (magnitude in Joules; the force due to wheels is $(F(t) + F_0)$)"

For part (a), we can use the impulse:

$$\langle F \rangle \Delta t = p_f - p_i = -p_i = -Mv_0 \tag{2.49}$$

$$-Mv_0 = \langle F \rangle t_s \tag{2.50}$$

$$v_0 = -\frac{\langle F \rangle t_s}{M} \tag{2.51}$$

The force is linear, so finding the average force should be easy.

$$\langle F \rangle = \frac{F(0) + F(t_s)}{2} = \frac{-F_0 - F_1 + (-F_0)}{2} = \frac{-2F_0 - F_1}{2} = -280\,500\,\mathrm{N} \tag{2.52}$$

We can then find v_0:

$$v_0 = \frac{(280\,500\,\mathrm{N})(28\,\mathrm{s})}{80\,000\,\mathrm{kg}} = 98.175\,\mathrm{m/s} \tag{2.53}$$

... or about 353.5 km/h, or 220 mph. We can also find this by realizing that $a = F/M$, and integrating that to find the change in velocity from $t = 0$ to $t = t_s$. That yields exactly the above answer.

Next up: acceleration. The only horizontal forces are the forces we're given, so setting up a Newton's second law equation is simple:

$$Ma = -F_0 + \left(\frac{t}{t_s} - 1\right) F_1 \tag{2.54}$$

$$a = -\frac{F_0}{M} + \left(\frac{t}{t_s} - 1\right) \frac{F_1}{M} \tag{2.55}$$

Plugging in numbers,

$$a(t = 0) = -3.7625\,\mathrm{m/s^2} \tag{2.56}$$

$$a(t = t_s) = -3.25\,\mathrm{m/s^2} \tag{2.57}$$

They ask for *magnitudes* though, so we need to get rid of the minus signs.
Do the values make sense? Yes, they do. $v_0 - at_s = 0$, if we use the average of the two as a; that is, $a = -3.506\,25\,\mathrm{m/s^2}$.

What distance s does the plane move between touchdown and its stop at $t = t_s$?

This is where the problem gets harder. I originally calculated s incorrectly, and then used that to find an incorrect value for part (d) and part (e). s and part (d) *were accepted*, despite being incorrect. The worst part, though, was that part (e) was *not* accepted. The solution was consistent, though – my answer for (e) was the only one possible if (a) and (d) had been correct, since I related the energies (initial kinetic energy = work by engines + loss to friction). The exact same incorrect answer was found by integrating the work over the distance s, since that distance was incorrect!

Let's instead have a look at integrating the acceleration to find the correct answers, which I did for my second try.

If we integrate the acceleration, we find a function for the change in the velocity. Δv, you might call it.

$$\Delta v = \int a\,dt = \int \left(-\frac{F_0}{M} + \frac{F_1}{M}\frac{t}{t_s} - \frac{F_1}{M}\right) dt \tag{2.58}$$

$$= \int \left(-\frac{F_0 + F_1}{M} + \frac{F_1}{Mt_s}t\right) dt \tag{2.59}$$

$$= -\frac{t(F_0 + F_1)}{M} + \frac{F_1 t^2}{2Mt_s} \tag{2.60}$$

This can be used to find the answer for the initial velocity, as well, by plugging in $t = t_s$ and all numeric values in the above equation. That gives you $\Delta v = -v_0$. The velocity as a function of time is then given by $v(t) = v_0 + \Delta v$:

$$v(t) = v_0 - \frac{t(F_0 + F_1)}{M} + \frac{F_1 t^2}{2Mt_s} \tag{2.61}$$

We can then integrate this over from $t = 0$ to $t = t_s$ to find the distance traveled:

$$s = \int_0^{t_s} \left(v_0 - \frac{t(F_0 + F_1)}{M} + \frac{F_1 t^2}{2Mt_s}\right) dt \tag{2.62}$$

$$s = v_0 t_s + \left[-\frac{t^2(F_0 + F_1)}{2M} + \frac{F_1 t^3}{6Mt_s}\right]_0^{t_s} \tag{2.63}$$

$$s = v_0 t_s + \left[-\frac{t_s^2(F_0 + F_1)}{2M} + \frac{F_1 t_s^2}{6M}\right] \tag{2.64}$$

Plugging in the numbers, $s \approx 1340.97\,\mathrm{m}$.

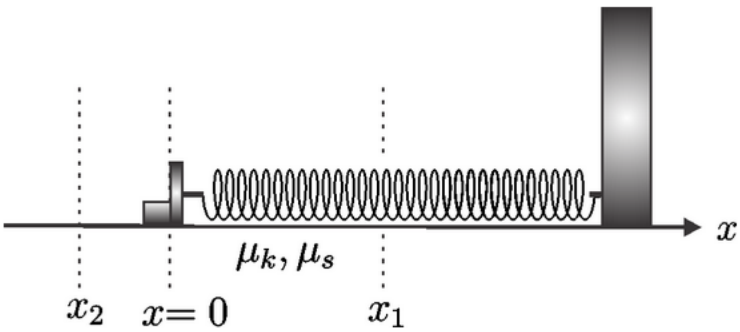
Finding the work by the engines is trivial now: $W_{engines} = -F_0 s = -348\,652\,200\,\mathrm{J}$. They ask for the magnitude, though, so we need to drop the minus sign.

Next, the work done by friction. This is a bit trickier, since we need to integrate in order to solve it by force times distance. Our force is also specified as a function of time, not distance.
We can solve it via an energy approach, however.
The initial kinetic energy is $\frac{1}{2}Mv_0^2$, and the final kinetic energy is exactly zero. Most of the kinetic energy is removed by the engines in reverse thrust, but everything that isn't must be removed as heat by the wheels.

$$|W_{friction}| = \frac{1}{2}Mv_0^2 - F_0 s = 385\,533\,225\,\mathrm{J} - 348\,652\,200\,\mathrm{J} = 36\,881\,025\,\mathrm{J} \tag{2.65}$$

2.8 Problem 8: Mass pushed by a spring

"A block of mass $m = 2$ kg on a horizontal surface is connected to a spring connected to a wall (see figure). The spring has a spring constant $k = 14$ N/m. The static friction coefficient between the block and the surface is $\mu_s = 0.5$, and the kinetic friction coefficient is $\mu_k = 0.2$. Use $g = 10\,\mathrm{m/s^2}$ for the gravitational acceleration.



(a) The spring is initially uncompressed and the block is at position $x = 0$. What is the minimum distance x_1 we have to compress the spring for the block to start moving when released? (in meters)

(b) Find the distance $|x_2 - x_1|$ between the point of release x_1 found in (a), and the point x_2 where the block will come to a stop again. (in meters)

(c) What time t_{12} does it take the block to come to a rest after the release? (i.e., the time of travel between points x_1 and x_2; in seconds)

(d) What will happen after the block has come to a rest at point x_2?

1. The block will move back towards x_1, and it will oscillate with constant frequency and exponentially decreasing amplitude.
2. The block will move back towards x_1, and it will oscillate while decreasing both frequency and amplitude.
3. The block will start moving back towards x_1, and it will come to a final halt before reaching it.
4. The block will stay at its resting position x_2.
5. The answer depends on whether $x_2 > 0$ or $x_2 < 0$.

Hmm, damped oscillations... something which we haven't seen in the course yet, on an exam! I wonder if we can get by withing solving the differential equations, which from what I recall (from similar cases in electronics, from 6.002x) is not easy at all.

At least part (a) should be easy. We need to overcome the maximum possible static friction, $\mu_s N = \mu_s mg$. The spring force is kx in magnitude, so

$$kx_1 > \mu_s mg \tag{2.66}$$
$$x_1 > \frac{\mu_s mg}{k} \tag{2.67}$$

For our values, $x_1 > 10/14$ m for the system to not just stay in place.

Okay, so if we release the block at that point (or a micrometer past it), where does the block come to a stop?
I would guess that the energy approach is (probably by far) the easiest way to solve this.
The spring has potential energy $U = \frac{1}{2}kx_1^2$ stored to begin with.
Some of it is turned to kinetic energy, and some of it wasted due to friction.
After that, it comes to a halt at x_2, at which point there is again energy stored in the spring, $\frac{1}{2}kx_2^2$ this time.

Thankfully, the kinetic friction is constant at $\mu_k N = \mu_k mg$, and so it does work which is simply $-\mu_k mg|x_2 - x_1|$.
Adding it all up, energy after and energy before minus losses:

$$\frac{1}{2}kx_2^2 = \frac{1}{2}kx_1^2 - \mu_k mg|x_2 - x_1| \tag{2.68}$$
$$\tag{2.69}$$

In order to get rid of the absolute value signs, we can think for a bit. Will $x_2 > x_1$, always? No, it will move in the opposite direction. $x_1 > x_2$ always, on the other hand. Therefore, we can negate the expression to $x_1 - x_2$ and remove the absolute value signs.

$$\frac{1}{2}kx_2^2 = \frac{1}{2}kx_1^2 - \mu_k mg(x_1 - x_2) \tag{2.70}$$
$$kx_2^2 = kx_1^2 - 2\mu_k mg(x_1 - x_2) \tag{2.71}$$
$$kx_2^2 - 2\mu_k mgx_2 = kx_1^2 - 2\mu_k mgx_1 \tag{2.72}$$
$$x_2^2 - x_2\frac{2\mu_k mg}{k} - x_1^2 + \frac{2\mu_k mgx_1}{k} = 0 \tag{2.73}$$

This is a bit too bad for me to rearrange and solve symbolically, so let's try this:

$$x_2^2 - x_2(4/7\,\mathrm{m}) - 10/14\,\mathrm{m} + 20/49\,\mathrm{m} = 0 \tag{2.74}$$

$$x_2 = \frac{4}{2\times 7} \pm \frac{1}{2}\sqrt{(4/7)^2 - 4(-5/49)} \tag{2.75}$$

This yields two answers: $x_2 = x_1$ which is clearly not the answer we want, and $x_2 = -1/7$.
The distance is then $|-1/7 - 10/14| = 0.85714$ m.

Next, they want us to find the amount of time that this movement took. The force follows a single equation the entire journey, so we should be able to solve this.

$+x$ is to the right, so $+ma$ in Newton's second law will be on the left side. On the right, we have the spring force $-kx$ and the friction $F_f = +\mu_k mg$ (which is constant). We can write a as $\ddot{x}$.

$$m\ddot{x} = \mu_k mg - kx \tag{2.76}$$
$$\ddot{x} + \frac{k}{m}x = \mu_k g \tag{2.77}$$

This is a bit familiar. For a vertical oscillator like this, we have a very similar expression. The term $\mu_k g$ turns out to not change the period of the oscillation, but only change the center position and/or amplitude. In other words, we can use the formula for the period we already know (see below for why):

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{2.78}$$

However, they don't ask us for the period, but rather the time it takes the block to come to a rest. I would say that's half a period – from one extreme to the other. We find $t = 1.18741$ seconds using $\omega_0/2$. Can it be trusted?

I also solved the differential equation (with Mathematica), and set set $x_2 = x(t)$ and solved for time, using the values above.
The cosine term in the solution to this equation is $\cos\left(\frac{kt}{\sqrt{km}}\right) = \cos(\omega_t)$. Solving that equation, the end result is then exactly the same as $\omega_0/2$.
Nice!

Finally, what happens next? Well, the block is at rest, and it has a spring force of $|kx| = (14\,\text{N/m})(1/7\,\text{m}) =$ 2 N on it. Is this greater than the maximum possible static friction? That value is $\mu_s mg = 10$ N, so clearly the answer is no.

The block will stay in place at x_2 due to friction.

2.9 Problem 9: Double-well potential

"An object of mass $m = 80$ kg moves in one dimension subject to the potential energy

$$U(x) = \frac{\lambda}{4}(x^2 - a^2)^2 + \frac{b}{2}x^2 \tag{2.79}$$

Here we use $\lambda = 3\,\text{kg/(m}^2\text{ s}^2)$, $a = 9\,\text{m}$ and $b = 223\,\text{kg/s}^2$.

(a) How many equilibrium points (stable and unstable ones) does this potential have?

(b) Find a stable equilibrium point x_0 such that x_0 is positive. (in meters)

(c) Do a Taylor expansion of the force $F(x)$ for x close to the equilibrium point, $x \approx x_0$, that is $F(x) = F_0 - k(x - x_0) + \ldots$ What are the values for F_0 (in Newton) and k (in kg/s^2)?

(d) What is the period T of small oscillations (in seconds) of this mass around the equilibrium point x_0? (Note that the parameter k found in the previous question acts like a spring constant that wants to pull small deviations back to the equilibrium point)"

Okay, let's see. Mathematically, an equilibrium point is where the potential is zero. If the second derivative is positive, it is a stable equilibrium point; if it is less than zero, it is an unstable equilibrium point. These points are very easy to see on a plot of U vs x, which I'll use to check the answers before submitting.

Before that, let's do the actual math.

$$\frac{dU}{dx} = \frac{\lambda}{2}(x^2 - a^2)(2x) + bx = \lambda x(x^2 - a^2) + bx \tag{2.80}$$
$$\frac{d^2U}{dx^2} = 3\lambda x^2 - \lambda a^2 + b \tag{2.81}$$

So how many equilibrium points are there? It's very obvious if you look at the graph, but let's try to find out mathematically. The first derivative is third-order, which implies three roots (though not all of them need to be real, mathematically).

$$\lambda x(x^2 - a^2) + bx = 0 \tag{2.82}$$
$$\lambda x(x^2 - a^2 + \frac{b}{\lambda}) = 0 \tag{2.83}$$

It is clearly zero where $x = 0$. The other two cases can be found by solving the quadratic in parenthesis for its zero points:

$$x^2 + \frac{b}{\lambda} - a^2 = 0 \tag{2.84}$$

$$x = \pm\frac{1}{2}\sqrt{-4(b/\lambda - a^2)} = \pm\sqrt{a^2 - b/\lambda} = \pm\sqrt{243/3 - 223/3} = \pm\sqrt{20/3} = \pm 2\sqrt{5/3} \tag{2.85}$$

With the values found, these three zeroes are $x = 0$, $x = -2\sqrt{5/3}$ and $x = 2\sqrt{5/3}$, about $x \pm 2.582$ m, plus the point at $x = 0$.
The second derivatives for these three points, in the order listed above, are $U''(0) = -20$ (unstable equilibrium point), $U''(-2\sqrt{5/3}) = 40$ (stable), $U''(2\sqrt{5/3}) = 40$ (stable). (Keep in mind that $U''(x) > 0$ means stable, while $U''(x) < 0$ means unstable; the magnitudes don't matter here.)

All of this matches the graph exactly. So far, we have

(a) 3 equilibrium points
(b) $x_0 = 2\sqrt{5/3} \approx 2.58199$ m

Next, they want us to do a Taylor expansion for the *force* around $x = x_0$. First, let's write an exact equation for the force, which is *minus* the first derivative of U:

$$F(x) = -\lambda x(x^2 - a^2) - bx \tag{2.86}$$

The Taylor expansion, in general terms, for the constant and first-order terms only, becomes

$$F(x) = F(x_0) + F'(x_0)(x - x_0) \tag{2.87}$$

where $F_0 = F(x_0)$ and $k = -F'(x_0)$, using the notation in the question. $F(x_0)$ should be zero by definition of the stable equilibrium; let's verify using the full polynomial:

$$F(x_0) = -\lambda x_0(x_0^2 - a^2) - bx_0 \tag{2.88}$$

Indeed, it turns out to be zero, if we substitute in the values given (and found, for the case of the value of x_0 that is greater that zero).

What about $k = -F'(x_0)$? First, let's find $F'(x)$, as

$$F'(x) = -\lambda(3x^2 - a^2) - b \tag{2.89}$$

$k = -F'(x_0)$ is then $-(-40) = 40$.

Finally, what is the period of oscillation? Using $T = 2\pi\sqrt{\frac{m}{k}}$, we find $T = 2\pi\sqrt{2} \approx 8.8857$ s. At first glance, that seems unreasonably high, though the mass is 80 kg and k just 40 N, so I suppose it's sensible after all.

That's it for this exam!

Chapter 3: Midterm 3

3.1 Problem 1: Momentum change

"A block of mass $m = 2\,\mathrm{kg}$ is initially at rest on a horizontal surface. At time $t = 0$, we begin pushing on it with a horizontal force that varies with time as $F(t) = \beta t^2$, where $\beta = 1.2\,\mathrm{N/s^2}$. We stop pushing at time $t_1 = 5\,\mathrm{s}$. [$F(t) = 0$ for $t > t_1$].

(a) First, assume the surface is frictionless. What is the magnitude of the final momentum of the block at $t_1 = 5$ s? (in kg m/s)"

Since the mass is initially at rest, the final momentum equals the impulse, $p_{fin} = I = \displaystyle\int F(t)dt$.

In other words, we need to solve a simple integral, with a constant coefficient.

$$p_{fin} = \beta \int_0^{t_1} t^2\, dt = \beta \left[\frac{t^3}{3}\right]_0^{t_1} = \frac{\beta t_1^3}{3} \tag{3.1}$$

For $\beta = 1.2\,\mathrm{N/s^2}$ and $t_1 = 5\,\mathrm{s}$, we find $p_{fin} = 50\,\mathrm{kg\ m/s}$.

"b) Let us now consider a new situation where the object is initially at rest on a rough surface. The coefficient of static friction is $\mu_s = 0.2$. What is the speed of the block at time $t_2 = 5$ s? For simplicity, we take static and kinetic friction coefficients to be the same, $\mu_s = \mu_k$ and consider $g = 10\,\mathrm{m/s^2}$."

First, let's identify the forces on the block. We have gravity, mg, downwards, and a normal force of equal magnitude $N = mg$ upwards, since there is no vertical acceleration.
Horizontally, there is the external force $F(t)$ in one direction, and a frictional force $F_f \leq \mu_s mg$ in the other. Once the object is moving, $F_f = \mu_k mg$ at all times (since the force never goes below the threshold again).

Before it has started to move, the frictional force equals $F(t)$; this happens until $F(t) > \mu_s mg$, i.e. until the static friction reaches the maximum possible value. When does that happen? Let's see. I will call this time t_1, not to be confused with the one used in part (a).

$$\beta t_1^2 = \mu_s mg \tag{3.2}$$

$$t_1 = \sqrt{\frac{\mu_s mg}{\beta}} \tag{3.3}$$

The dimension works out correctly (as a sanity check), and $t_1 = 1.825\,742\,\mathrm{s}$.
Again, a check (this is an exam!): $F(1.825742) = 4$ N, while $\mu mg = 4$ N also.

After that time has passed, the object is sliding, and we can find the acceleration by applying Newton's second law. Alternatively, we can find the final momentum using the impulse-momentum theorem, after which we simply divide by the mass to find the speed.
Actually, both would involve a time-integral of a force, divided by mass... So I suppose they are very much the same.

The net force is now given by $\beta t^2 - \mu_k mg$. We integrate that from t_1 to t_2, and divide by the mass m.

$$v = \frac{1}{m}\int_{t_1}^{t_2} \beta t^2\, dt - \mu_k g \int_{t_1}^{t_2} dt \tag{3.4}$$

$$v = \frac{\beta}{m}\left[\frac{t^3}{3}\right]_{t_1}^{t_2} - \mu_k g(t_2 - t_1) \tag{3.5}$$

$$v = \frac{\beta(t_2^3 - t_1^3)}{3m} - \mu_k g(t_2 - t_1) \tag{3.6}$$

With $t_2 = 5\,\mathrm{s}$ and t_1 as above we find $v = 17.4343$ m/s.

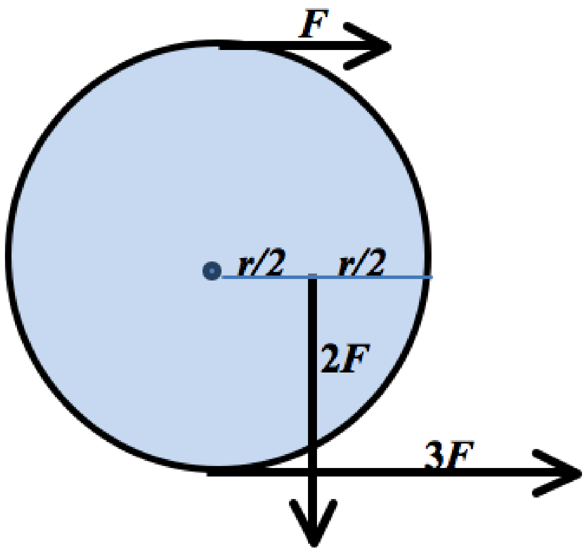
"(c) What is the power P provided by the force $F(t)$ at time $t_3 = 4$ s (in Watts) in the case where there is friction (part (b))?"

Power is given by $P = \vec{F} \cdot \vec{v} = Fv$ (in this case, since the instantaneous force and the instantaneous speed are fully parallel).
First, we calculate the velocity at $t = 4$ s using the above formula, and find $v(t = 4) = 7.23432$ m/s.
Next, we calculate the force at that time: $F(t = 4) = \beta(4\,\mathrm{s})^2 = 19.2$ N.

Finally, the instantaneous power is the product of the two: $P(t = 4) = 138.899$ W.

3.2 Problem 2: Torque

"A uniform solid disc of mass m and radius r is acted upon by three forces of given magnitudes (see the diagram).



(a) If the disc rotates about an axis perpendicular to the screen and passing through the center of the disc (as it is viewed from top as in the figure), the magnitude of the angular acceleration, α, and the sense of rotation of the disc as viewed from top is:"

1. $\alpha = 2F/(mr)$; counterclockwise
2. $\alpha = 2F/(mr)$; clockwise
3. $\alpha = 4F/(mr)$; counterclockwise
4. $\alpha = 4F/(mr)$; clockwise
5. $\alpha = 6F/(mr)$; counterclockwise
6. $\alpha = 6F/(mr)$; clockwise

All right, let's see.
First, the two forces that act on the edges work against each other, for a net torque of $r(3F - F) = 2rF$, counterclockwise (out of the screen).
The third force is also clockwise, working against the above, with moment arm $r/2$ and force $2F$, for a torque rF, clockwise.

The net torque is therefore rF, counterclockwise. The moment of inertia for rotation about this axis is $\frac{1}{2}mr^2$. $\tau = I\alpha$, so $\alpha = \frac{\tau}{I}$:

$$\alpha = \frac{rF}{(1/2)mr^2} = \frac{2F}{mr} \text{ (CCW)} \tag{3.7}$$

"(b) If the disc rotates about an axis perpendicular to the screen and passing through the point of application of force $3F$ (as it is viewed from top as in the figure), the magnitude of the angular acceleration, α, and the sense of rotation of the disc as viewed from top is:"

1. $\alpha = 2F/(mr)$; counterclockwise
2. $\alpha = 2F/(mr)$; clockwise
3. $\alpha = 4F/(mr)$; counterclockwise
4. $\alpha = 4F/(mr)$; clockwise
5. $\alpha = 6F/(mr)$; counterclockwise
6. $\alpha = 6F/(mr)$; clockwise

OK, so first, what is the torque about this point? It's certainly not the same as it was before (and neither is the moment of inertia).
The $3F$ force causes no torque through this point. The force at the top is $2r$ away, causing a torque $2rF$, clockwise.
For the third force, we use the fact that the cross product is given by the the magnitude of one vector, times the perpendicular distance of the other. Here, the perpendicular distance of the position vector is $r/2$, so $\tau = (r/2)(2F) = rF$, also clockwise.

Summed together, there is a net torque $3rF$, clockwise. Next, the moment of inertia. We use the parallel-axis theorem, so the moment of inertia is the same as before, plus md^2 where d is the distance between axes, i.e. r for this problem.

$$\alpha = \frac{3rF}{(1/2)mr^2 + mr^2} = \frac{2F}{mr} \text{ (CW)} \tag{3.8}$$

The rotation is now in the opposite direction, since the only CCW force causes no torque.

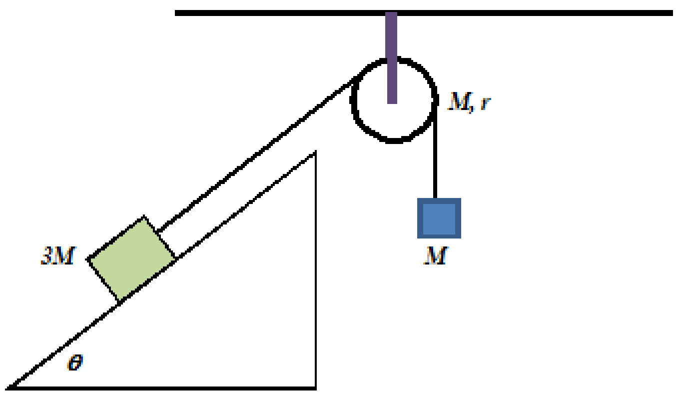
"(c) If the disc rotates about an axis perpendicular to the screen and passing through the point of application of force $2F$ (as it is viewed from top as in the figure), the magnitude of the angular acceleration, α, and the sense of rotation of the disc as viewed from top is:"

1. $\alpha = 4F/(mr)$; counterclockwise
2. $\alpha = 4F/(mr)$; clockwise
3. $\alpha = 2F/(mr)$; counterclockwise
4. $\alpha = 2F/(mr)$; clockwise
5. $\alpha = 8F/(3mr)$; counterclockwise
6. $\alpha = 8F/(3mr)$; clockwise

One more time. For a change, let's calculate the moment of inertia first. Again, using the parallel axis theorem: $I = (1/2)mr^2 + m(r/2)^2 = (3/4)mr^2$
Next, we calculate the torque. The $2F$ force causes no torque relative to its own application point. The perpendicular distance to the $3F$ force is r, so the torque is $3rF$, direction CCW.
The force at the top is has the same perpendicular distance, for a torque of rF, direction CW.
The net torque is then $2rF$, CCW, since they work against each other.

$$\alpha = \frac{2rF}{(3/4)mr^2} = \frac{8F}{3mr} \text{ (CCW)} \tag{3.9}$$

3.3 Problem 3: Massive pulley



"In the diagram, block $3M$ slides downward without friction. The string connecting blocks $3M$ and M is ideal (that is, its mass can be neglected and it is not stretchable). The pulley, a uniform solid disc of mass M and radius r, rotates without slipping. Find the acceleration of block $3M$."

1. $a = (2g/9)(3\cos\theta - 1)$
2. $a = (3g/5)(2\sin\theta + 1)$
3. $a = (g/3)(\sin\theta + 1)$
4. $a = (2g/9)(3\sin\theta - 1)$
5. $a = (3g/5)(2\cos\theta + 1)$
6. $a = (g/3)(\cos\theta + 1)$

Okay. We set up Newton's second law equations for the two blocks, and one of the rotational variants for the pulley, using the no-slip condition, $a = \alpha R \Rightarrow \alpha = \frac{a}{R}$.

Since the $3M$ block slides downwards, I use that as the position direction, as well as counterclockwise rotation for the pulley.
Tension T_1 acts on the $3M$ block, while T_2 acts on the M block. $T_1 > T_2$, or the pulley would have to rotate in the opposite direction

$$3Mg\sin\theta - T_1 = 3Ma \tag{3.10}$$
$$T_2 - Mg = Ma \tag{3.11}$$
$$r(T_1 - T_2) = (1/2)Mr^2\frac{a}{r} \tag{3.12}$$

I will solve this as I usually do: solve the two first equations for T_1 and T_2 respectively, and then substitute those into the torque equation.

The first:

$$3Mg\sin\theta - T_1 = 3Ma \tag{3.13}$$
$$T_1 = 3M(g\sin\theta - a) \tag{3.14}$$

The second:

$$T_2 - Mg = Ma \tag{3.15}$$
$$T_2 = M(a+g) \tag{3.16}$$

And the dirty work:

$$r(3Mg\sin\theta - 3Ma - (Ma + Mg)) = (1/2)Mr^2\frac{a}{r} \tag{3.17}$$
$$3g\sin\theta - 3a - a - g = (1/2)a \tag{3.18}$$
$$g(3\sin\theta - 1) = (9/2)a \tag{3.19}$$
$$(2g/9)(3\sin\theta - 1) = a \tag{3.20}$$
$$\tag{3.21}$$

3.4 Problem 4: Angular collision 2

"A merry-go-round (pictured) is sitting in a playground. It is free to rotate, but is currently stationary. You can model it as a uniform disk of mass 180 kg and radius 130 cm (consider the metal poles to have a negligible mass compared to the merry-go-round). The poles near the edge are 117 cm from the center.

Someone hits one of the poles with a 7 kg sledgehammer moving at 16 m/s in a direction tangent to the edge of the merry-go-round. The hammer is not moving after it hits the merry-go-round.

How much energy $|\Delta E|$ is lost in this collision? (enter a positive number for the absolute value in Joules)"

Alright. The moment of inertia, modeling the merry-go-round as a solid disk, is $(1/2)mr^2 = 152.1\,\text{kg m}^2$.

The sledgehammer has a linear momentum of $(7\,\text{kg})(16\,\text{m/s}) = 112\,\text{kg m/s}$, and a kinetic energy of, using $\frac{1}{2}mv^2$, 896 J. All of the momentum is transferred to the pole, and then to the merry-go-round, causing an angular impulse. All of its kinetic energy is also transferred from it/converted, but not all of it becomes rotational kinetic energy (or the answer to this question would be zero).

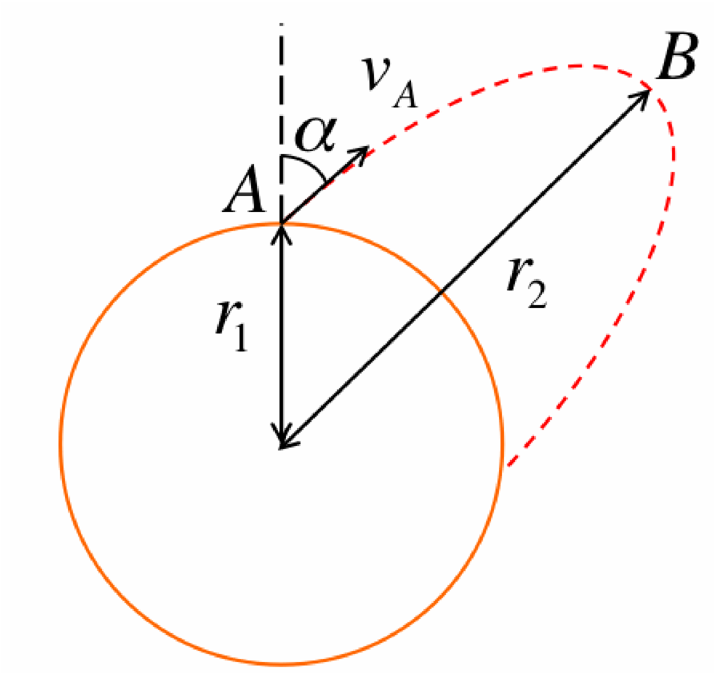
The sledgehammer hits a distance 1.17 m from the center, causing an angular impulse $J = (1.17\,\text{m})(112\,\text{kg m/s}) = 131.04\,\text{kg m}^2/\text{s}$ ($J = \tau\Delta t = rp = rF\Delta t = \Delta L$, so this is also the new angular momentum, relative to the center, since $L_{cm} = 0$ to begin with.)

We can calculate the rotational velocity after the collision using $L_{cm} = I_{cm}\omega \Rightarrow \omega = \frac{L_{cm}}{I_{cm}}$, which is the rotational equivalent to $v = p/m$. Doing that, we find $\omega = 0.86154$ rad/s. The rotational kinetic energy is then

$$E_{post} = \frac{1}{2}I\omega^2 = 56.448\,\mathrm{J} \tag{3.22}$$

So how much energy was lost? Subtracting the initial and final energies, $\Delta E = 896\,\mathrm{J} - 56.448\,\mathrm{J} = 839.552\,\mathrm{J}$ was lost (to heat, vibration, noise, etc).

3.5 Problem 5: Ballistic missile



"A spherical non-rotating planet (with no atmosphere) has mass $m_1 = 4 \times 10^{24}\,\mathrm{kg}$ and radius $r_1 = 5000$ km. A projectile of mass $m_2 \ll m_1$ is fired from the surface of the planet at a point A with a speed v_A at an angle $\alpha = 30°$ with respect to the radial direction. In its subsequent trajectory the projectile reaches a maximum altitude at point B on the sketch. The distance from the center of the planet to the point B is $r_2 = (5/2)r_1$. Use $G = 6.674 \times 10^{-11}\,\mathrm{kg}^{-1}\ \mathrm{m}^3\ \mathrm{s}^{-2}$.

What is the initial speed v_A of the projectile? (in m/s)"

(Note: Having looked at the staff solutions after the exam, before I post these notes, I realize that this solution is overly complex. There's no need whatsoever to find a, and frankly I should've taken a step back to realize that while solving!)

Okay. The trajectory is clearly part of an ellipse; by the looks of it, a focus could certainly be at the center of the planet. In other words, we can treat this as an elliptical orbit. We enter it at some point (the launch site) with a velocity v_A tangential to the orbit. The total energy of the orbit is the kinetic energy at that point, plus the gravitational potential energy at that point. That must always be equal to $-\dfrac{Gm_1m_2}{2a}$, where a is the semi-major axis of the orbit.

$$\frac{1}{2}m_2v_A^2 - \frac{Gm_1m_2}{r_1} = -\frac{Gm_1m_2}{2a} \tag{3.23}$$

We do know the distance to apogee (at point B): it's given as $r_2 = (5/2)r_1$. The question is: does this imply that the distance from the center of the planet to the other edge of the orbit (perigee) is r_1? That is, is the orbit tangent to the planet's surface on the other side, so that $2a = r_1 + r_2$?

Well, we can try to find this out. We can apply both the conservation of mechanical energy and the conservation of angular momentum (about the center of the planet) to the system.
First, we use the conservation of mechanical energy at the launch site and at apogee:

$$-\frac{Gm_1m_2}{2a} = \frac{1}{2}m_2v_B^2 - \frac{Gm_1m_2}{r_2} \tag{3.24}$$

We now have two equations; the unknowns are v_A, a and v_B. (r_2 is known, in terms of r_1.)
Next, we use the conservation of angular momentum; if we can do so without adding any unknowns, things are looking good. We equate initial angular momentum with that at B:

$$r_1m_2v_A\sin\alpha = r_2m_2v_B \tag{3.25}$$

α is known, so this should now be solvable with quite a bit of work... Collecting the equations with trivial simplifications (cancelling m_2, multiplying through by 2 in the top two equations):

$$v_A^2 - \frac{2Gm_1}{r_1} = -\frac{Gm_1}{a} \tag{3.26}$$
$$-\frac{Gm_1}{a} = v_B^2 - \frac{2Gm_1}{r_2} \tag{3.27}$$
$$r_1v_A\sin\alpha = r_2v_B \tag{3.28}$$
$$r_2 = (5/2)r_1 \tag{3.29}$$

Eliminating r_2:

$$v_A^2 - \frac{2Gm_1}{r_1} = -\frac{Gm_1}{a} \tag{3.30}$$
$$-\frac{Gm_1}{a} = v_B^2 - \frac{2Gm_1}{(5/2)r_1} \tag{3.31}$$
$$r_1v_A\sin\alpha = (5/2)r_1v_B \tag{3.32}$$
$$\tag{3.33}$$

We can solve the last equation for v_B in terms of v_A:

$$(2/5)v_A\sin\alpha = v_B \tag{3.34}$$

The remaining equations are

$$v_A^2 - \frac{2Gm_1}{r_1} = -\frac{Gm_1}{a} \tag{3.35}$$
$$-\frac{Gm_1}{a} = ((2/5)v_A\sin\alpha)^2 - \frac{2Gm_1}{(5/2)r_1} \tag{3.36}$$
$$\tag{3.37}$$

Since these equations are equal, we can set the sides containing v_A equal and solve.

$$v_A^2 - \frac{2Gm_1}{r_1} = \frac{4}{25}v_A^2 \sin^2\alpha - \frac{4Gm_1}{5r_1} \tag{3.38}$$

$$v_A^2 - \frac{4}{25}v_A^2 \sin^2\alpha = \frac{2Gm_1}{r_1} - \frac{4Gm_1}{5r_1} \tag{3.39}$$

$$v_A^2(1 - \frac{4}{25}\sin^2\alpha) = \frac{6Gm_1}{5r_1} \tag{3.40}$$

$$v_A = \sqrt{\frac{6Gm_1}{5r_1(1-\frac{4}{25}\sin^2\alpha)}} \tag{3.41}$$

8169.46 m/s is the answer then, and $2a \neq r_1 + r_2$. Good thing I didn't assume that; using that leads to an answer more than 5% too high, so it would most likely have been graded as incorrect (as it should!).

3.6 Problem 6: Rocket acceleration

"Consider a rocket in space that ejects burned fuel at a speed of $v_{ex} = 2.0$ km/s with respect to the rocket. The rocket burns 10% of its mass in 290 s (assume the burn rate is constant).

(a) What is the speed v of the rocket after a burn time of 145.0 s? (suppose that the rocket starts at rest; and enter your answer in m/s)

(b) What is the instantaneous acceleration a of the rocket at time 145.0 s after the start of the engines? (in m/s^2)"

The definition of thrust (derived from conservation of momentum) is that $F = v_{ex}\frac{dm}{dt}$, only instead v_{ex} I'm used to u.
We know that $F = ma$, but here, everything is a function of time, including the mass of the rocket.

Instead, we can use conservation of momentum. The derivation for this is shown in lecture.
We can show that

$$\Delta v = -v_{ex} \ln \frac{m_f}{m_i} \tag{3.42}$$

(sometimes called the rocket equation) where Δv is the change in velocity, m_f the final mass of the rocket, and m_i the initial mass of the rocket; both of the masses include the fuel, of course, or they would be equal.

For part (a), we need simply stick in $m_f/m_i = 0.95$ (since if the burn rate is constant, and it burns 10% in 290 s, it must burn 5% in 145 s).

We need to write this in mathematical form later, though, so let's do that now instead. We find

$$m_f/m_i = 1 - 0.1\frac{t}{290\,\mathrm{s}} \tag{3.43}$$

Anyway, back to the velocity. Using the above,

$$\Delta v = -v_{ex} \ln\left(1 - 0.1\frac{t}{290\,\mathrm{s}}\right) \tag{3.44}$$

We can simply plug the numbers in. The initial velocity is zero, so that implies that $v = \Delta v$:

$$v = 102.58\,\mathrm{m/s} \tag{3.45}$$

for an average acceleration of $0.7\,\mathrm{m/s^2}$; rather pathetic, to be honest!

Next is the instantaneous acceleration. We can find this from the above using a bit of calculus. We can take the derivative "manually", by finding

$$\frac{\Delta v}{\Delta t} = \frac{v_{t+\Delta t} - v_t}{\Delta t} = \frac{1}{\Delta t}\left(-v_{ex}\ln\left(1 - 0.1\frac{t+\Delta t}{290\,\mathrm{s}}\right) + v_{ex}\ln\left(1 - 0.1\frac{t}{290\,\mathrm{s}}\right)\right) \tag{3.46}$$

In the limit where $\Delta t \to 0$, this becomes the derivative of the velocity, which of course is the acceleration. At $t = 145\,\mathrm{s}$, this gives us $a = 0.726\,\mathrm{m/s^2}$ or so. Alternatively, we could simply take the derivative the usual way. We have

$$v = C\ln(u) \tag{3.47}$$

which has the derivative

$$\frac{dv}{dt} = C\frac{1}{u}\frac{du}{dt} = -2000\left(\frac{1}{1 - 0.1\frac{t}{290\,\mathrm{s}}}\right)(-\frac{1}{2900\,\mathrm{s}}) \tag{3.48}$$

Evaluated at $t = 145\,\mathrm{s}$, this also gives us $a = 0.726\,\mathrm{m/s^2}$.

3.7 Problem 7: Doppler shift

"A source of sound emits waves at a frequency $f = 450$ Hz. An observer is located at a distance $d = 160$ m from the source. Use $u = 340$ m/s for the speed of sound.

(a) Assume completely still air. How many wavefronts (full waves) N are there between the source and the observer?

(b) If the observer is moving away from the source at a (radial) velocity $v = 40$ m/s, how does the number of wavefronts N found in part (a) change with time? For the answer, give the rate of change of N, namely $\frac{dN}{dt}$.

(c) By comparing the difference of the rate of wavefronts leaving and wavefronts entering the region between source and observer, calculate the frequency f observed by the moving observer. (in Hz)
hint: how does the difference relate to the rate of change of N you calculated in (b)?

(d) Let us now assume that both source and observer are at rest, but wind blows at a constant speed $v = 20$ m/s in the direction source towards observer. By comparing the difference of the rate of wavefronts leaving and wavefronts entering the region between source and observer, calculate the observed frequency f? (in Hz)"

First, to get us started, wavelength is given by $\lambda = \frac{u}{f}$. Therefore, the wavelength of 450 Hz sound is 0.75555 m (with the five repeating).

Part (a) is easy, then: $N = \frac{160\,\mathrm{m}}{0.755\,555\,\mathrm{m}} = 211.765$ wavefronts.
I would ordinarily round this down to 211, since the question asks for "full waves", but in a clarification on the wiki it was stated no rounding is necessary.

For part (b), it's clear that the number must increase, since the distance is increasing. As the observer is the one moving, there shouldn't be any other effects to consider (if the *source* moves, wavefronts are compressed/spaced out, etc).

The distance is increasing by 40 meters each second; each meter contains a bit more than one wavefront at this frequency, so the answer must be a bit above 40 (40/0.75555). Using the chain rule, with dd/dt as the rate of change of distance (since they labeled it d),

$$\frac{dN}{dt} = \frac{dd}{dt} \cdot \frac{dN}{dd} \tag{3.49}$$

The first number is given as 40 m/s, while the second is just the number of wavefronts per meter (the reciprocal of meters per wavefront, i.e. wavelength), so

$$\frac{dN}{dt} = (40\,\mathrm{m/s})(\frac{1.3235}{1\,\mathrm{m}}) \approx 52.94\,\mathrm{wavefronts/s} = 52.94\,\mathrm{Hz} \tag{3.50}$$

As intuitively expected, this is just the velocity divided by the wavelength (or, equivalently, multiplied by the wavelength's reciprocal). In terms of symbols,

$$\frac{dN}{dt} = \frac{dd}{dt}\frac{f}{u} \tag{3.51}$$

"(c) By comparing the difference of the rate of wavefronts leaving and wavefronts entering the region between source and observer, calculate the frequency f observed by the moving observer. (in Hz)
hint: how does the difference relate to the rate of change of N you calculated in (b)?"

Interesting! I hadn't realized this relationship, I must say. We find

$$f - f' = \frac{dN}{dt} \tag{3.52}$$

$$f' = f - \frac{dN}{dt} \tag{3.53}$$

which I will admit I partly realized this because I knew the formula for Doppler shift to begin with, which gives the same numerical answer.
In the case where the observer is moving towards the source, the time derivative turns negative (the number of wavefronts between the two is going down), so in that case $f' > f$, as it should be.

Using this, we find $f' = 397.06$ Hz; $f' < f$ since the observer is moving away.

This equation is really saying that the number of wavefronts that reach the receiver (f') is the number of wavefronts sent out per second, minus the number of wavefronts per second you outrun, I suppose. Of course, they will catch up eventually, unless you move at a speed greater than u away.

I believe that by combining the two equations above, we should be able to find "the" formula for Doppler shift of a moving observer:

$$f' = f - f\frac{dd}{dt}\frac{1}{u} \tag{3.54}$$

$$f' = f\left(1 - \frac{v_{rad}}{u}\right) \tag{3.55}$$

where $\dfrac{dd}{dt} = v_{rad} = v\cos\theta$ is the radial velocity. Nice!

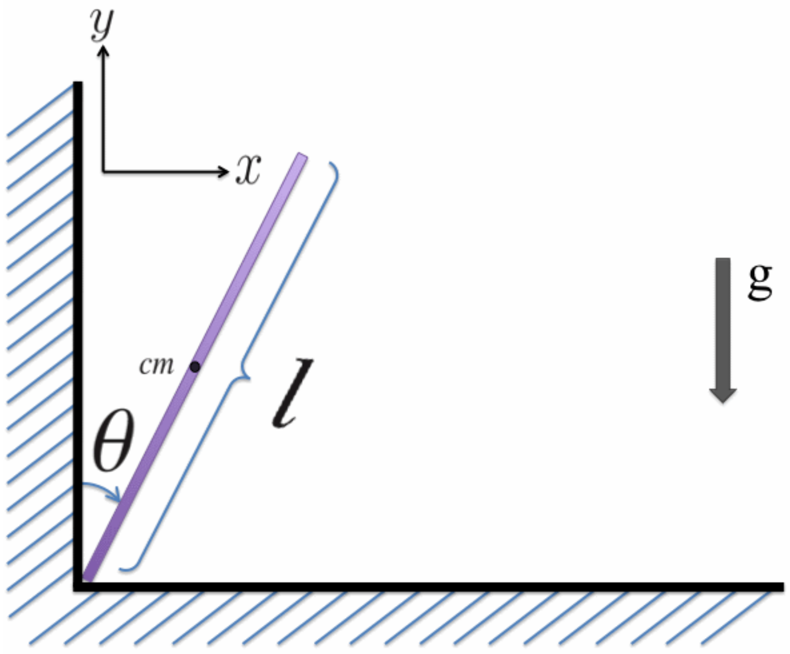
Finally, part (d), repeated for simplicity:

"(d) Let us now assume that both source and observer are at rest, but wind blows at a constant speed $v = 20$ m/s in the direction source towards observer. By comparing the difference of the rate of wavefronts leaving and wavefronts entering the region between source and observer, calculate the observed frequency f? (in Hz)"

Since the waves are pressure fronts in the air, and wind is movement of air, the receiver will clearly receive the sound waves earlier than with no wind. How does the perceived frequency change, though?

Intuitively, I would say it does not change, with the following reasoning: if 450 wavefronts enter per second, and more than 450 exit, where did the rest come from? If more exit than enter, the region would have to run out of wavefronts after a while, which makes no sense at all!
If less exit than enter, the air would be crowded by wavefronts, which makes equally little sense.
The only sensible answer is that wavelength and frequency are unchanged.
I suppose this can be shown in a better way than this, but I leave that for another time (and perhaps for the staff solutions?).

3.8 Problem 8: Falling ruler



"A ruler stands vertically against a wall. It is given a tiny impulse at $\theta = 0°$ such that it starts falling down under the influence of gravity. You can consider that the initial angular velocity is very small so that $\omega(\theta = 0°) = 0$. The ruler has mass $m = 150$ g and length $\ell = 20$ cm. Use $g = 10\,\mathrm{m/s^2}$ for the gravitational acceleration, and the ruler has a uniform mass distribution. Note that there is no friction whatsoever in this problem. (See figure)

(a) What is the angular speed of the ruler ω when it is at an angle $\theta = 30°$? (in radians/sec)

(b) What is the force exerted by the wall on the ruler when it is at an angle $\theta = 30°$? Express your answer as the x component F_x and the y component F_y (in Newton).

(c) At what angle θ_0 will the falling ruler lose contact with the wall? ($0 \le \theta_0 \le 90°$; in degrees) [hint: the ruler loses contact with the wall when the force exerted by the wall on the ruler vanishes.]"

I've left this problem for last, as I'm a bit confused about exactly what it's asking; the wiki clarifications haven't made it easier, either. It seems to me that F_y for the *wall* (not the floor) should be zero at all times if there is no friction, but that doesn't appear to be the case.
One answer also stated that we could treat the entire L-shaped object as the "wall" (i.e. wall+floor).

Anyway, it looks to be like this... For part (b), we ignore the wording "wall" and simply calculate the contact force, ignoring exactly which structure provides it. (I find it a bit confusing that the wall can provide the vertical force if we consider the ruler 1-dimensional, but apparently that is the case.)

For part (c), we find where the contact force from the *vertical wall*, i.e. F_x, is zero. F_y will still be greater than zero when that happens. And for part (a), this discussion is not very relevant.

All right, let's get started. Step one: forces. F_x and F_y, the total contact force, act in the corner, on the very bottom of the rod.
mg acts on the center of mass, a length $\ell/2$ up along the rod.
Only gravity can cause a torque relative to the corner; this torque is given by $\tau = (\ell/2)mg\sin\theta$.

To begin with, this torque is zero, and the stick is in equilibrium. When hit with the tiny (negligible, except that it sets the stick in motion) impulse, θ grows to a tiny angle, and so there is a tiny torque, which is changing with time.

The angular acceleration is given by $\alpha = \tau/I$, so

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{(\ell/2)mg\sin\theta}{I_Q} \tag{3.56}$$

where I_Q is the moment of inertia about that point in the corner, which I choose to label Q.

The moment of inertia is $(1/3)m\ell^2$ for a rod about its end like this; it is found using the moment of inertia about a rod's center of mass $(1/12)m\ell^2$, plus $m(\ell/2)^2$ via the parallel axis theorem. The angular acceleration is therefore given by

$$\alpha = \frac{3g\sin\theta}{2\ell} \tag{3.57}$$

where θ is a function of time. $\alpha = \ddot{\theta}$, so

$$\ddot{\theta} - \frac{3g}{2\ell}\sin\theta = 0 \tag{3.58}$$

Hmm. I don't think this is the intended way to solve this. Let's try an energy approach instead. The stick begins with a total energy of $U = mg(\ell/2)$. As it falls, this is converted into rotational kinetic energy. (All of it, assuming we consider about the contact point, and only before it loses contact.)

As it rotates/falls, the center of mass is lowered down, so that it loses potential energy. The height of the center of mass is given by $h = (\ell/2)\cos\theta$.

$$mg(\ell/2) = mg(\ell/2)\cos\theta + \frac{1}{2}I_Q\omega^2 \tag{3.59}$$

$$mg(\ell/2) = mg(\ell/2)\cos\theta + \frac{1}{2}\left(\frac{1}{3}m\ell^2\right)\omega^2 \tag{3.60}$$

$$g(1/2) = g(1/2)\cos\theta + \frac{1}{6}\ell\omega^2 \tag{3.61}$$

$$g\left(\frac{1}{2} - \frac{1}{2}\cos\theta\right) = \frac{1}{6}\ell\omega^2 \tag{3.62}$$

$$\sqrt{\frac{3g}{\ell}(1-\cos\theta)} = \omega \tag{3.63}$$

A-ha, much more reasonable.
At $\theta = 30°$, we find $\omega = 4.48288$ rad/s.

"(b) What is the force exerted by the wall on the ruler when it is at an angle $\theta = 30°$? Express your answer as the x component F_x and the y component F_y (in Newton)."

I got really, really, really stuck on this one. In fact, I spent most of Sunday and Monday trying to figure out where the heck I was going wrong; I found that F_x vanishes (next question) at 90 degrees. Apparently, it loses contact with the wall earlier, but I couldn't find out how to put that into mathematical terms, even if I understood that it happened.

Below is a solution that is highly inspired by the staff's solution (though all text etc is my own). My alternative solution, which I finally solved the day *after* the exam deadline, follows after that.

3.8.1 Staff solution-inspired answers for parts b/c/d

All forces can be thought of as acting on the center of mass, as far as linear motion is concerned. However, that the center of mass moves towards the right doesn't imply that the end loses contact with the corner; the CoM moves with full contact, to begin with. The ruler couldn't tip over without this initial force (and acceleration), since the center of mass must move towards the right when the ruler tips over.

The center of mass also moves downwards. When these two motions are "balanced", so to speak, and the center of mass traces out an arc (i.e. part of a circle, so that it undergoes circular motion), there is still contact, and $F_x > 0$.

In order to find the forces, we use Newton's second law:

$$ma_x = F_x \tag{3.64}$$
$$ma_y = F_y - mg \tag{3.65}$$

So we need to find the linear acceleration (of the center of mass).

From work above, we already have

$$\omega = \dot{\theta} = \sqrt{\frac{3g}{\ell}(1 - \cos\theta)} \tag{3.66}$$
$$\alpha = \ddot{\theta} = \frac{3g}{2\ell}\sin\theta \tag{3.67}$$

Using basic trigonometry, we can find that $x_{cm} = (\ell/2)\sin\theta$ and $y_{cm} = (\ell/2)\cos\theta$. We then take the time derivative of those, twice each, using the chain rule (since θ is a function of time, and we most certainly can't differentiate with respect to θ to find the acceleration).

For the second differentiation, we need to use both the product rule and the chain rule.

$$v_x = \frac{dx_{cm}}{dt} = (\ell/2)(\cos\theta)\dot{\theta} \tag{3.68}$$
$$a_x = \frac{dv_x}{dt} = (\ell/2)(-\sin(\theta)\dot{\theta}^2 + \cos(\theta)\ddot{\theta}) \tag{3.69}$$

Here, we could either save ourselves some trouble by calculating values for $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$ and sticking them in, or find full expressions in terms of the given variables by substituting them in. If we choose the latter, we get, after simplification,

$$a_x = \frac{3}{4}g(3\cos\theta - 2)\sin\theta \tag{3.70}$$

Multiply this by m, since we found that $ma_x = F_x$, and

$$F_x = \frac{3}{4}mg(3\cos\theta - 2)\sin\theta \tag{3.71}$$

At $\theta = 30°$, this gives us 0.336 N.

We then take a step back and do the same thing for the y component.

$$v_y = \frac{dy_{cm}}{dt} = (\ell/2)(-\sin\theta)\dot{\theta} \tag{3.72}$$
$$a_y = \frac{dv_y}{dt} = (\ell/2)((-\cos\theta)\dot{\theta}^2 - (\sin\theta)\ddot{\theta}) \tag{3.73}$$

Again, we could substitute in values, or do the algebra. In terms of the given variables, and simplified,

$$a_y = -\frac{3}{2}g(1 + 3\cos\theta)\sin^2(\theta/2) \tag{3.74}$$

We can't simply multiply by m to find the force, though: $ma_y = F_y - mg$, so $F_y = m(a_y - g)$.

$$F_y = m\left(g - \frac{3}{2}g(1 + 3\cos\theta)\sin^2(\theta/2)\right) \tag{3.75}$$

which gives us $F_y = 0.957693$ N.

Finally, to find the angle where it loses contact with the wall, we set $F_x = 0$ (as hinted). We can divide away lots of stuff from both sides, which is the nice thing about zero:

$$0 = \frac{3}{4}mg(3\cos\theta - 2)\sin\theta \tag{3.76}$$
$$0 = 3\cos\theta - 2 \tag{3.77}$$
$$\frac{2}{3} = \cos\theta \tag{3.78}$$
$$\arccos\frac{2}{3} = \theta \approx 48.1897° \tag{3.79}$$

This exact angle was the answer to question 5 on exam 5 ("Sliding down a dome"), too, for when a block sliding of a spherical dome loses contact.

3.8.2 My own solution

I'm writing this section the day after the exam, after having realized why my first "solution" was incorrect. In short, I tried to find the linear acceleration of the center of mass using $\vec{a} = \vec{\alpha} \times \vec{R}$; however, that equation gives the *tangential acceleration only.*
My second attempt was to consider the centripetal acceleration (i.e. the radial acceleration), but I never considered the large picture, and so it was only today, the day after the exam closed, that I realized that these equations only give the respective component, and that they are not two different ways to find the net center of mass acceleration... Any time I get stuck like this, if I try to "re-start" the problem, I just end up with the same train of thought again, which is rather frustrating!
Anyway, with no further ado... What I wish I'd found a day earlier:

$a_{tan} = \alpha R$, where in this case, $R = \ell/2$.

$$a_{tan} = \frac{\ell}{2}\frac{3g}{2\ell}\sin\theta = \frac{3g}{4}\sin\theta \tag{3.80}$$

Next, we need to find the radial acceleration, i.e. the inwards (towards the corner) centripetal acceleration, which must equal $(\ell/2)\omega^2$ in magnitude, or the center of mass will not undergo circular motion.

$$a_{rad} = (\ell/2)\omega^2 = \frac{3g}{2}(1 - \cos\theta) \tag{3.81}$$

The net acceleration of the center of mass is the sum of these, but we care about the x and y components rather than their sum, so we take the x and y components of the above accelerations and sum them together, like this:

$$\begin{align} a_x &= a_{tan}\cos\theta - a_{rad}\sin\theta \tag{3.82}\\ &= \frac{3g}{4}\sin\theta\cos\theta - \frac{3g}{2}(1-\cos\theta)\sin\theta \tag{3.83}\\ &= \frac{3g}{4}\left(\sin\theta\cos\theta - 2\sin\theta(1-\cos\theta)\right) \tag{3.84}\end{align}$$

The tangential component is positive (it points towards the right to begin with, and $\hat{x}$ is towards the right), while the radial is negative (a_{rad} points towards the corner, opposite the x axis).

We can find the y component(s) similarly.

$$\begin{align} a_y &= -a_{tan}\sin\theta - a_{rad}\cos\theta \tag{3.85}\\ a_y &= -\frac{3g}{4}\sin\theta\sin\theta - \frac{3g}{2}(1-\cos\theta)\cos\theta \tag{3.86}\\ a_y &= -\frac{3g}{4}\left(\sin^2\theta + 2\cos\theta(1-\cos\theta)\right) \tag{3.87}\\ a_y &= -\frac{3g}{4}\left(\sin^2\theta + 2(\cos\theta - \cos^2\theta)\right) \tag{3.88}\end{align}$$

Here, both components are negative; the tangential acceleration starts to the right, and then points down, while $+y$ is up; the radial points down to begin with (the component is always purely down, of course), and is also negative for that reason.

Now that we know these, we can apply the equations we found earlier relating these to the forces we seek. Solved for the forces, they are

$$\begin{align} ma_x &= F_x \tag{3.89}\\ ma_y + mg &= F_y \tag{3.90}\end{align}$$

So if we make the substitutions,

$$\begin{align} \frac{3mg}{4}\left(\sin\theta\cos\theta - 2\sin\theta(1-\cos\theta)\right) &= F_x \tag{3.91}\\ mg - \frac{3mg}{4}\left(\sin^2\theta + 2(\cos\theta - \cos^2\theta)\right) &= F_y \tag{3.92}\end{align}$$

which gives us, in numbers: $F_x = 0.3364$ N and $F_y = 0.9577$ N.

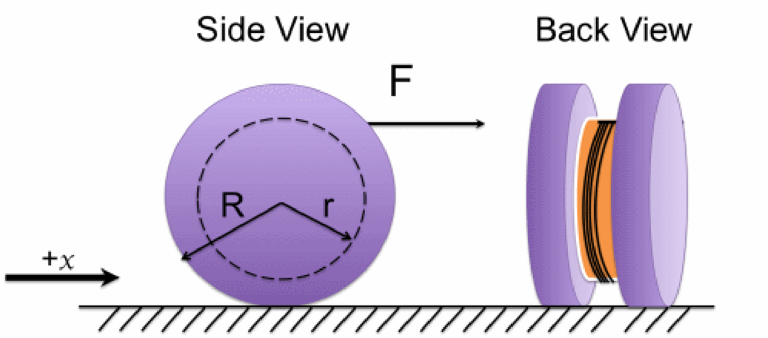
Finally, to find the angle at where it loses contact with the wall, we set $F_x = 0$ and solve. Lots of stuff can be divided away from both sides as a first step.

$$\begin{align} 0 &= \frac{3mg}{4}\left(\sin\theta\cos\theta - 2\sin\theta(1-\cos\theta)\right) \tag{3.93}\\ 0 &= \cos\theta - 2(1-\cos\theta) \tag{3.94}\\ 2 &= 3\cos\theta \tag{3.95}\\ \arccos\frac{2}{3} &= \theta \tag{3.96}\end{align}$$

$\theta \approx 48.19°$ is the angle when it loses contact with the wall, and the center of mass stops undergoing circular motion.

Incredibly, this is independent of both the stick's length *and* mass.

3.9 Problem 9: Yoyo



"A yoyo of mass $m = 2$ kg and moment of inertia $I_{cm} = 0.09\,\text{kg m}^2$ consists of two solid disks of radius $R = 0.3$ m, connected by a central spindle of radius $r = 0.225$ m and negligible mass. A light string is coiled around the central spindle. The yoyo is placed upright on a flat rough surface and the string is pulled with a horizontal force $F = 24$ N, and the yoyo rolls without slipping.

(a) What is the x-component of the acceleration of the center of mass of the yoyo? (in m/s^2)
(b) What is the x-component of the friction force? (in N)"

The horizontal force causes a clockwise torque. I will assume the string is wound such that the force is applied at the top, as drawn, even though it's not actually specified in text. (If it were at the bottom, the torque would be in the other direction.)

Here comes the slightly dangerous part... The frictional force is *not* towards the left. For this reason, $a > F/m$ (where F is the pulling force). However, we don't need to know this in advance (I didn't), so let's set up the equation as if friction is towards the left.

With this in mind, writing an equation for the linear acceleration is very easy:

$$F - F_f = ma \tag{3.97}$$

Next, we look at torques. The pull of the string provides a torque rF, clockwise.
Friction provides a torque RF_f, also clockwise, assuming it is towards the left.

$\tau_{cm} = I_{cm}\alpha$; we can use the no-slip condition $\alpha = a/R$ (the part in contact with the ground). We can not use it for the string, however. Less string will be unrolled than what would happen at pure roll at the central spindle.

Our second equation then becomes, using clockwise torque/rotation as positive:

$$\frac{rF + RF_f}{I_{cm}} = \frac{a}{R} \tag{3.98}$$

From the first equation, we find

$$\frac{F - F_f}{m} = a \tag{3.99}$$

Substituted into the second, and using $I_{cm} = 2(\frac{1}{2}\frac{m}{2}R^2$, which cleans things up a bit:

$$\frac{rF + RF_f}{I_{cm}} = \frac{1}{R}\frac{F - F_f}{m} \tag{3.100}$$

$$\frac{rF + RF_f}{2(1/2)(m/2)R^2} = \frac{1}{Rm}(F - F_f) \tag{3.101}$$

$$\frac{rF + RF_f}{(1/2)R} = F - F_f \tag{3.102}$$

$$\frac{2rF}{R} - F = -3F_f \tag{3.103}$$

$$F\left(1 - \frac{2r}{R}\right) = 3F_f \tag{3.104}$$

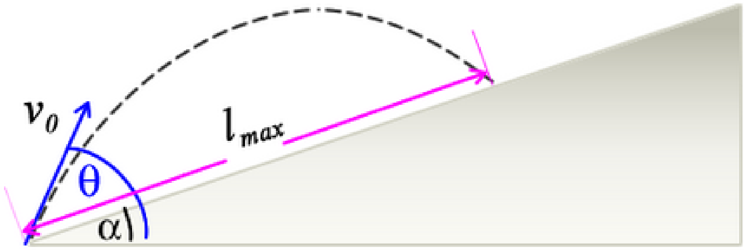
$$\frac{F}{3}\left(1 - \frac{2r}{R}\right) = F_f \tag{3.105}$$

In terms of numbers, $F_f = -4$ N. Since it is negative, it is towards the right – opposite of the assumption.

Finding a is then trivial, using the above equation for a. The total force towards the right is 28 N, m is 2 kg, and so $a = 14\,\mathrm{m/s^2}$.

Chapter 4: Final exam

4.1 Problem 1: Maximal range



"A gunman standing on a sloping ground fires up the slope. The initial speed of the bullet is $v_0 = 370$ m/s. The slope has an angle $\alpha = 23°$ from the horizontal, and the gun points at an angle θ from the horizontal. The gravitational acceleration is $g = 10\,\mathrm{m/s^2}$.

(a) For what value of θ (where $\theta > \alpha$) does the gun have a maximal range along the slope? (In degrees, from the horizontal.)
(b) What is the maximal range of the gun, ℓ_{max}, along the slope? (In meters.)"

Let's see. We want to maximize ℓ_{max}, and also find the value of θ which causes it to be maximized. That is, we find a function $\ell_{max}(\theta)$, take its derivative, set that equal to zero, and look at the point(s) we find to to identify the maximum.

ℓ_{max} is not the total distance the bullet moves, of course; the parabolic trajectory is much longer than the actual distance moved along the slope. In order to find the diagonal distance, we can write down the x and y coordinates where the bullet hits, and use some trigonometry.

We can decompose the bullet's motion into x and y components. Using some simple trigonometry,

$$\sin\alpha = \frac{y}{\ell_{max}} \tag{4.1}$$
$$\cos\alpha = \frac{x}{\ell_{max}} \tag{4.2}$$

So the coordinates where it lands are given by

$$x = \ell_{max}\cos\alpha \tag{4.3}$$
$$y = \ell_{max}\sin\alpha \tag{4.4}$$

assuming the sensible choice of coordinate system, i.e. x positive rightwards, y positive upwards and the origin where the gunman is standing.

v_0 is at an angle θ to the ground; the components are given by

$$v_{0x} = v_0\cos\theta \tag{4.5}$$
$$v_{0y} = v_0\sin\theta \tag{4.6}$$

The bullet coordinates as a function of time are, using $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ where a is constant,

$$x(t) = v_0 \cos(\theta) t \tag{4.7}$$
$$v_x(t) = v_0 \cos\theta \tag{4.8}$$
$$y(t) = v_0 \sin(\theta) t - \frac{1}{2} g t^2 \tag{4.9}$$
$$v_y(t) = v_0 \sin\theta - g t \tag{4.10}$$

The height at which it hits, as a function of x (more simple trigonometry) is

$$y = x \tan\alpha \tag{4.11}$$

We can then write that, at the point/time of collision,

$$v_0 \cos(\theta) t = x \tag{4.12}$$
$$v_0 \sin(\theta) t - \frac{1}{2} g t^2 = x \tan\alpha \tag{4.13}$$

x, θ and t are all unknown. We can substitute using $x = \ell_{max} \cos\alpha$, so that we have

$$v_0 \cos(\theta) t = \ell_{max} \cos\alpha \tag{4.14}$$
$$v_0 \sin(\theta) t - \frac{1}{2} g t^2 = \ell_{max} \cos\alpha \tan\alpha \tag{4.15}$$

That doesn't change the number of unknowns, but it does get rid of the x. We can eliminate the time from the second equation by using the first,

$$t = \frac{\ell_{max} \cos\alpha}{v_0 \cos\theta} \tag{4.16}$$

Substituting that in and simplifying,

$$v_0 \sin(\theta) \left(\frac{\ell_{max} \cos\alpha}{v_0 \cos\theta} \right) - \frac{1}{2} g \left(\frac{\ell_{max} \cos\alpha}{v_0 \cos\theta} \right)^2 = \ell_{max} \cos\alpha \tan\alpha \tag{4.17}$$
$$\ell_{max} \cos\alpha \tan\theta - \frac{1}{2} g \frac{\ell_{max}^2 \cos^2\alpha}{v_0^2 \cos^2\theta} = \ell_{max} \cos\alpha \tan\alpha \tag{4.18}$$
$$\tan\theta - \frac{1}{2} g \frac{\ell_{max} \cos\alpha}{v_0^2 \cos^2\theta} = \tan\alpha \tag{4.19}$$

We can then solve for ℓ_{max}.

$$\ell_{max}(\theta) = \frac{2 v_0^2 \cos^2\theta}{g \cos\alpha} (\tan\theta - \tan\alpha) \tag{4.20}$$
$$\ell_{max}(\theta) = \frac{2 v_0^2}{g \cos\alpha} \left(\cos^2\theta \tan\theta - \cos^2\theta \tan\alpha \right) \tag{4.21}$$

Our goal is now to maximize this. v_0 is a constant, g is a constant and α is a constant. What we really want to maximize is therefore simply (well, it's not that simple, but still) this expression:

$$\cos^2\theta \tan\theta - \cos^2\theta \tan\alpha \tag{4.22}$$

i.e. the expression in parenthesis. $\tan\alpha$ is a constant, which also helps.
By far the easiest way to do this is to graph that, and read off the answer, by the way! I did that to verify,

but will try to carry out the full calculation.
We calculate the derivative of this with respect to θ and set that equal to zero, to find the maxima:

Part one:

$$\frac{d}{d\theta}\left(\cos^2\theta\tan\theta\right) = \cos^2\theta\sec^2\theta + \tan\theta(-2\sin\theta\cos\theta) \tag{4.23}$$

Part two:

$$\frac{d}{d\theta}\left(\cos^2\theta\tan\alpha\right) = -2\sin\theta\cos\theta\tan\alpha \tag{4.24}$$

So all in all,

$$\cos^2\theta\sec^2\theta + \tan\theta(-2\sin\theta\cos\theta) + 2\sin\theta\cos\theta\tan\alpha \tag{4.25}$$
$$1 - 2\sin^2\theta + 2\sin\theta\cos\theta\tan\alpha \tag{4.26}$$

$\sec\theta = \frac{1}{\cos\theta}$, so those cancel. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\tan\theta\sin\theta\cos\theta = \sin^2\theta$.

This is still not pretty, but let's try. We set this equal to zero and try to solve for θ:

$$1 - 2\sin^2\theta + 2\sin\theta\cos\theta\tan\alpha = 0 \tag{4.27}$$
$$1 - 2\sin^2\theta + \sin(2\theta)\tan\alpha = 0 \tag{4.28}$$
$$2\sin^2\theta - \sin(2\theta)\tan\alpha = 1 \tag{4.29}$$

This is where I gave up and used Mathematica; these more hairy trig expressions aren't my favorite. This can apparently be simplified down to

$$\cos(\alpha - 2\theta) = 0 \tag{4.30}$$

which is easier to work with. Take the arccosine of both sides, and

$$\alpha - 2\theta = -\frac{\pi}{2} \tag{4.31}$$
$$\alpha + \frac{\pi}{2} = 2\theta \tag{4.32}$$
$$\frac{\alpha}{2} + \frac{\pi}{4} = \theta \tag{4.33}$$
$$\tag{4.34}$$

OK, so I cheated a bit here; I first tried $\arccos(0) = \pi/2$, which gave me a negative angle for θ, and so I tried a different choice. Choosing $-\pi/2$ instead gives a value for θ in the only possible range, $0 < \theta < \pi/2$.

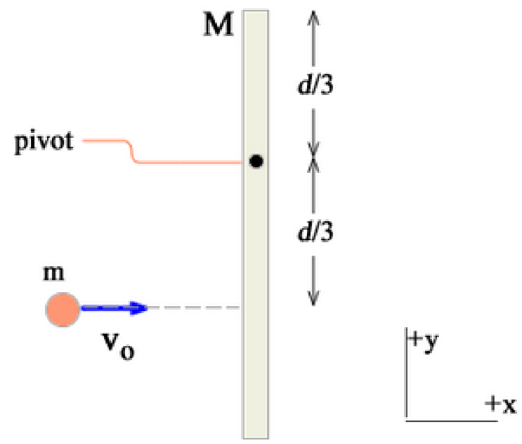
This gives us $\theta \approx 0.98611$ radians, or 56.5 degrees. Plugging this value back into ℓ_{max}, we find $\ell_{max} =$ 9843.74 meters.

This answer looks unreasonably large to me, especially given the travel time with gravity in mind, but it does seem sensible if we compare it to a formula from lecture. The maximum horizontal distance it could travel (if the ground was flat) is $v_0^2\sin(2\theta)/g$ meters, which evaluates to about 12600 m.
If this answer is correct, what are the coordinates where it hits? Using the formulas we found, it is about $x = 9061$ m and $y = 3846$ m(!).

The maximum height the bullet could reach is $(v_0\sin\theta)^2/(2g) \approx 4760$ meters, so this answer could be correct... and it is!

Well, that was a pain. This is one of those questions where I pretty much expect that there is a simpler solution. Granted, if I'd used Mathematica the entire way it wouldn't have been that hard, but surely that shouldn't be necessary.

4.2 Problem 2: Angular collision



"A uniform rod of mass M and length d is initially at rest on a horizontal and frictionless table contained in the xy plane, the plane of the screen. The figure is a top view, gravity points into the screen. The rod is free to rotate about an axis perpendicular to the plane and passing through the pivot point at a distance $d/3$ measured from one of its ends as shown. A small point mass m, moving with speed v_0, hits the rod and stick to it at the point of impact at a distance $d/3$ from the pivot.

(a) If the mass of the rod is $M = 4m$, what is the magnitude of the angular velocity of the rod+small mass system after the collision?"

1. $\omega = 2v_0/(3d)$
2. $\omega = 3v_0/(2d)$
3. $\omega = 4v_0/(3d)$
4. $\omega = 3v_0/(4d)$
5. $\omega = 8v_0/(3d)$
6. $\omega = 3v_0/(8d)$
7. $\omega = 5v_0/(3d)$
8. $\omega = 3v_0/(5d)$

My first try was way incorrect – I misread the question completely. I read it as a collision a distance $d/3$ from the center of mass, and didn't even realize it wasn't completely free to move. Geez – reading carefully is important, and I should know better than to *not* read very carefully.

So there's a pivot point, which is located a distance $d/3$ from the *end* of the rod. The end is located a distance $d/2$ from the center of the rod, and so the pivot point is located $d/2 - d/3 = d/6$ from the rod's center (of mass).

We can use the relationship $L = I\omega$ to find ω. The angular momentum of the rod prior to the collision is zero, so the angular impulse from the collision will equal its final angular momentum L, about the pivot point.
(Alternatively, we can consider the angular momentum of the system to be constant about the pivot point; just prior to the collision, the total angular momentum is $mv_0(d/3)$ from the small mass and 0 from the rod; this is conserved for the system after the collision.)

What is the moment of inertia I about this point though? We start off with the moment of inertia of a rod, about its center of mass, which is $\frac{1}{12}Md^2$. Next, we need to increase that by an amount $M(d/6)^2$ via the parallel axis theorem, as we aren't rotating it about its center of mass.
Finally, we must add the moment of inertia due to the point mass, which is mr^2, where r is the distance between the rotation axis and the point where it is stuck. (r is *not* the distance from the center of mass to that point – if it were stuck at the rod's center of mass, it would still have a nonzero contribution!) That distance is $r = d/3$ as given in the problem, so we add $mr^2 = m(d/3)^2 = \frac{1}{9}md^2$, for a total moment of inertia of

$$I = I_{cm,rod} + I_{parallelaxis} + I_{pointmass} = \frac{1}{12}Md^2 + \frac{1}{36}Md^2 + \frac{1}{9}md^2 \tag{4.35}$$

Using $M = 4m$, this simplifies to

$$I = \frac{1}{3}md^2 + \frac{1}{9}md^2 + \frac{1}{9}md^2 = \frac{5}{9}md^2 \tag{4.36}$$

Next, what is the angular impulse about the pivot point? The linear impulse is simply mv_0; to find the angular impulse, we multiply this by the distance between the impact point and the pivot point, $d/3$. As mentioned previously, the post-collision angular momentum will be equal to this angular impulse, so:

$$\omega = \frac{L}{I} = \frac{\frac{mv_0 d}{3}}{\frac{5}{9}md^2} = \frac{9mv_0 d}{15md^2} = \frac{3v_0}{5d} \tag{4.37}$$

"(b) Using again $M = 4m$. What is the speed of the center of mass of the rod right after collision?"

1. $v_{cm} = v_0$
2. $v_{cm} = v_0/2$
3. $v_{cm} = v_0/3$
4. $v_{cm} = v_0/5$
5. $v_{cm} = v_0/10$
6. $v_{cm} = v_0/20$

We can simply use $v = \omega R$, where R is the distance between the pivot point and the point we care about, i.e. the center of mass. We said earlier that this distance was $d/6$, so

$$v_{cm} = \frac{3v_0}{5d} \times \frac{d}{6} = \frac{v_0}{10} \tag{4.38}$$

4.3 Problem 3: Atmospheric pressure

"In the lecture, we discussed the case of an isothermal atmosphere where the temperature is assumed to be constant. In reality, however, the temperature in the Earth's atmosphere is not uniform and can vary strongly and in a non-linear way, especially at high altitude. To a good approximation, the temperature T drops almost linearly with altitude up to 11 km above sea level, at a constant rate:

$$\frac{dT}{dz} = -\alpha \text{ for } z \leq 11 \text{ km} \tag{4.39}$$

where $\alpha = 6.5$ K/km (Kelvin per km) and z is the height above the sea level. The temperature stays then approximately constant between 11 km and 20 km above sea level.

Assume a temperature of 15°C and a pressure of 1 atm at sea level (1 atm = $1.013\,25 \times 10^5\,\mathrm{N/m^2}$). Furthermore, take the molecular weight of the air to be (approximately) 29 g/mol. The universal gas constant is $R = 8.314\,\mathrm{J\ K^{-1}\ mol^{-1}}$ and the acceleration due to gravity is $g = 10\,\mathrm{m/s^2}$ (independent of altitude). Assume that air can be treated as an ideal gas.

(a) Under the assumptions above, calculate the atmospheric pressure p (in atm) at $z = 10$ km above sea level for the case of a linear temperature drop."

Let's see! In lecture, for calculating this in the case of an isothermal atmosphere, we calculated the density as

$$\rho = \frac{Nm}{V} \tag{4.40}$$

where N is the number of molecules, m the mass (in kg) of each molecule, V the volume and ρ the density of that (small) volume. We also used the ideal gas law in the form $PV = NkT$, as we also will here. We rearrange it a bit

$$\frac{P}{kT} = \frac{N}{V} \tag{4.41}$$

Using $\rho = \frac{Nm}{V}$, we substitute a bit to find

$$\rho = \frac{Pm}{kT(z)} \tag{4.42}$$

Still as in lecture, but with the important change I noted just above, that the temperature is now a function of z.
We we substitute this into the separable differential equation that we previously used to derive Pascal's law:

$$\frac{dP}{dz} = -\rho(z)g = -\frac{Pmg}{kT(z)} \tag{4.43}$$

We rearrange this,

$$\frac{dP}{P} = -\frac{mg}{kT(z)}\,dz \tag{4.44}$$

This is where things change. We now need to consider T(z), where $T'(z) = -\alpha$ up to 11 km, and then zero ($T(z) =$ constant) up to 20 km.

Since the temperatures is 15 degrees C at sea level, and it drops by 6.5 K/km, the temperature at 11 km to 20 km must be -56.5 C. In between, the temperature is $15 - \alpha z = 15 - 6.5z$ degrees C. The problem only asks about stuff up to 10 km, however, and what happens above is of little concern to us; it doesn't enter into the equations.

We need to convert the temperature to kelvin, which we do by adding 273.15 to the number. I also find it more sensible to work in terms of meters, not kilometers. Therefore, $\alpha = 6.5 \times 10^{-3}\,\mathrm{K/m}$, and

$$T(z) = 288.15 - (6.5 \times 10^{-3}\,\mathrm{K/m})z \tag{4.45}$$

which is valid up to 11 000 m.

Moving on, we substitute this into our equation:

$$\frac{dP}{P} = -\frac{mg}{k(288.15 - (6.5 \times 10^{-3}\,\mathrm{K/m})z)}\,dz \tag{4.46}$$

Or, in terms of symbols where T_0 is the temperature at sea level,

$$\frac{dP}{P} = -\frac{mg}{k(T_0 - \alpha z)}\,dz = C\frac{1}{T_0 - \alpha z}\,dz \tag{4.47}$$

where $C = -\frac{mg}{k}$.

I prefer this form for the integration, since the constants and units look messy. For the first time, I chose to use the rather quaint method of a lookup table. I found

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(|ax+b|) \tag{4.48}$$

Since C is a constant, we can move that outside the integral. We then just map $a = -\alpha$ and $b = T_0$, so the result is

$$C\frac{1}{a}\ln(|ax+b|) = -\frac{C}{\alpha}\ln(|T_0 - \alpha z|) \tag{4.49}$$

$$= \frac{mg}{k\alpha}\ln(|T_0 - \alpha z|) \tag{4.50}$$

So we can finally go to calculate the definite integrals. The left-hand side is easy (and unchanged since lecture). On the right-hand side, we do what we did above, and substitute for h and 0:

$$\frac{dP}{P} = -\frac{mg}{k(T_0 - \alpha z)}\,dz \tag{4.51}$$

$$\int_{P_0}^{P_h}\frac{dP}{P} = -\frac{mg}{k}\int_0^h \frac{1}{T_0 - \alpha z}\,dz \tag{4.52}$$

$$\ln P_h - \ln P_0 = \frac{mg}{k\alpha}\ln(|T_0 - \alpha h|) - \frac{mg}{k\alpha}\ln(T_0) \tag{4.53}$$

$$\ln\frac{P_h}{P_0} = \frac{mg}{k\alpha}(\ln(T_0 - \alpha h) - \ln(T_0)) \tag{4.54}$$

$$\ln\frac{P_h}{P_0} = \frac{mg}{k\alpha}\ln(1 - \frac{\alpha h}{T_0}) \tag{4.55}$$

I removed the absolute value bars since $T_0 > \alpha h$ for all values that we use.

We can now exponentiate both sides of this equation.

$$\frac{P_h}{P_0} = e^{\frac{mg}{k\alpha}\ln(1-\frac{\alpha h}{T_0})} \tag{4.56}$$

$$P_h = P_0 e^{\frac{mg}{k\alpha}\ln(1-\frac{\alpha h}{T_0})} \tag{4.57}$$

This should be a useful answer, but we can manipulate it a bit further, using $e^{a\ln b} = b^a$, so that

$$P_h = P_0\left(1 - \frac{\alpha h}{T_0}\right)^{\frac{mg}{k\alpha}} \tag{4.58}$$

Plotting this versus the lecture's $P_h = P_0 e^{-h/H_0}$ with $H_0 = 8000$ m, it's clear that the two are very similar. For heights less than 1 km, they are *very similar* (hard to see a difference at all on a plot). At 10 km, where the error is the greatest, the difference is still only some 10-14%.

Finally, to answer question (a): using this formula, the pressure at $z = 10$ km, using $P_0 = 1$ atm, the pressure is $P_h = 0.253613$ atm. For comparison, the lecture's equation gives $P_h = 0.286505$ atm, about 13% more.

To find the answer, I used $\alpha = 6.5 \times 10^{-3}\,\mathrm{K/m}$, $T_0 = 288.15\,\mathrm{K}$, $m = 29 \times 1.66 \times 10^{-27}\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$ and $k = R/N_A \approx 1.38 \times 10^{-23}\,\mathrm{J/K}$.

"(b) The cruising altitude of a commercial aircraft is about 33'000 ft (or 10 km). Assume that the cabin is pressurized to 0.8 atm at cruising altitude. What is the minimal force Fmin (in Newton) per square meter that the walls have to sustain for the cabin not to burst? Use the atmospheric pressure found in (a)."

Well, this is certainly easy compared to all the above!
The internal pressure is 0.8 atm, but the external pressure only 0.2536 atm. The pressure difference is what gives rise to the force on the walls, and that difference is 0.5464 atm, or 55362.7 pascal (newton per square meter), which answers this question.

"(c) We close a plastic bottle full of air inside the cabin when the aircraft is at cruising altitude of $z = 10$ km. The volume of the bottle is V_1, the pressure and temperature inside the cabin are 0.8 atm and $T_1 = 27\,^\circ\mathrm{C}$, respectively. Assume that at sea level the atmospheric pressure is 1 atm, and the temperature is decreased by 10 Kelvin with respect to the cabin's temperature.

What is the magnitude of the percentage change in volume of the air inside the bottle when it is brought to sea level? Enter the magnitude of the percentage change in volume in

$$\left|\frac{\Delta V}{V_1}\right| \times 100 \tag{4.59}$$

OK. We begin with a certain volume, and then increase the pressure and decrease the temperature. Using the ideal gas law, the volume for the two cases is

$$V_1 = \frac{nRT_1}{P_1} \tag{4.60}$$

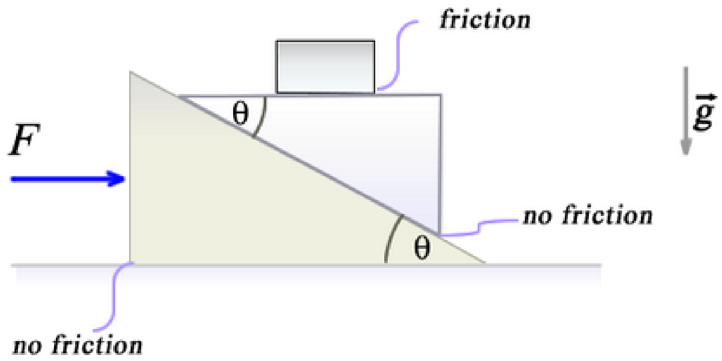
$$V_2 = \frac{nRT_2}{P_2} \tag{4.61}$$

$\Delta V = V_2 - V_1$, so

$$\left|\frac{\Delta V}{V_1}\right| \times 100 = \left|\frac{V_2}{V_1} - 1\right| \times 100 = \left|\frac{P_1 T_2}{P_2 T_1} - 1\right| \times 100 \tag{4.62}$$

We substitute in $P_1 = 0.8$ atm, $P_2 = 1$ atm, $T_1 = 300.15$ K and $T_2 = 290.15$ K and find an answer of 22.6%.

4.4 Problem 4: Prisms



"In the diagram, both the prisms and the block have equal masses m. Angle θ is given. Both surfaces of the larger prism are frictionless; however, there is friction between the horizontal surface of the smaller prism and the block. A horizontal and constant force of unknown magnitude F is exerted on the larger prism. As a result, the three objects remain at rest relative to each other.

(a) Find the magnitude of the acceleration of the larger prism a."

1. $a = 2g\tan\theta$
2. $a = g\tan\theta$
3. $a = g\sin\theta$
4. $a = 2g\cos\theta$
5. $a = g\cos\theta$
6. $a = 2g\sin\theta$

"(b) Find the value of the pushing force F."

1. $F = 6mg\tan\theta$
2. $F = 6mg\sin\theta$
3. $F = 6mg\cos\theta$
4. $F = 3mg\cos\theta$
5. $F = 3mg\sin\theta$
6. $F = 3mg\tan\theta$

"(c) Find the minimum coefficient of static friction μ_s between the block and the smaller prism that makes it possible for the block to stay at rest relative to the prism."

1. $\mu_s = 2\tan\theta$
2. $\mu_s = \tan\theta$
3. $\mu_s = 2\sin\theta$
4. $\mu_s = \sin\theta$
5. $\mu_s = 2\cos\theta$
6. $\mu_s = \cos\theta$

OK. So this problem is a slightly strange one, I think; when I just glanced over it as I copied it down, I figured the system was standing still, and the force was on the "middle prism"; turns out the force is on the leftmost prism instead, so that the entire thing must be constantly accelerating, sliding faster and faster, for this to work!

If $F = 0$, the lower prism would glide towards the left, while the upper prism would glide down and towards the right.

a for one object must be the same for all three, since they are at rest relative to each other.
The total mass of the three objects is $3m$, and since F in the only horizontal external force, $a = F/(3m)$ must hold.

My first take on this one was incorrect – the answer I found wasn't among the options, so I didn't waste any submissions, however.
I noticed that if I wrote the normal force in terms of $\sec\theta$ instead of $\cos\theta$, the answer was listed, though I couldn't really figure out *why* that would be correct. I did come up with the solution after a while, and got it right. A few hours later, I came up with this solution, which is really what I had the entire time, only I confused the expressions for N and that of $N\cos\theta$ earlier.

Anyway, let's look at the forces involved. On the top block, there are three forces: gravity at mg straight down, a normal force of that same magnitude straight up, and a frictional force $\mu_s mg$, towards the right. (This is the force that provides the acceleration of the block.)
This is, by the way, using the condition that the block is about to slip; $F_f \leq \mu_s mg$ in general, but in the about-to-slip case, it is exactly equal.

On the top prism (or middle, if you prefer), the weight of the top block is acting downwards with magnitude mg, and the weight of the prism itself too, for a total downward force $2mg$. Then there's a force $\mu_s mg$ towards the *left* – the Newton's third law pair of the friction acting on the block.
Finally, there's the normal force from the bottom prism, acting diagonally upwards. The horizontal component of this, $N\sin\theta$ is the source of acceleration for this prism. The vertical component must be $N\cos\theta = 2mg$, or there would be vertical acceleration of the prism.

As for the bottom prism, we can restrict ourselves to horizontal forces. That leaves F, the source of the acceleration, and N acting diagonally downwards/towards the left. The horizontal component of magnitude $N\sin\theta$ opposes the motion.

All in all, this gives us a bunch of equations:

$$F - N\sin\theta = ma \quad \text{Newton's second law, bottom prism} \tag{4.63}$$
$$N\sin\theta - \mu_s mg = ma \quad \text{Newton's second law (horizontal), top prism} \tag{4.64}$$
$$N\cos\theta = 2mg \quad \text{Newton's second law (vertical), top prism} \tag{4.65}$$
$$\mu_s mg = ma \quad \text{Newton's second law, block} \tag{4.66}$$

Note that I use N to denote the normal force acting on the top prism; there are several other normal forces, but they aren't as important. Via Newton's third law, this force also acts on the bottom prism.

From the last equation, we have $\mu_s = a/g$ – that should be very helpful once we have a.
If we rewrite the $N\sin\theta - \dots$ equation using this relationship for μ_s, we get $N\sin\theta = 2ma$. We can then divide that by the equation right below, and get

$$\tan\theta = \frac{2ma}{2mg} \tag{4.67}$$
$$g\tan\theta = a \tag{4.68}$$

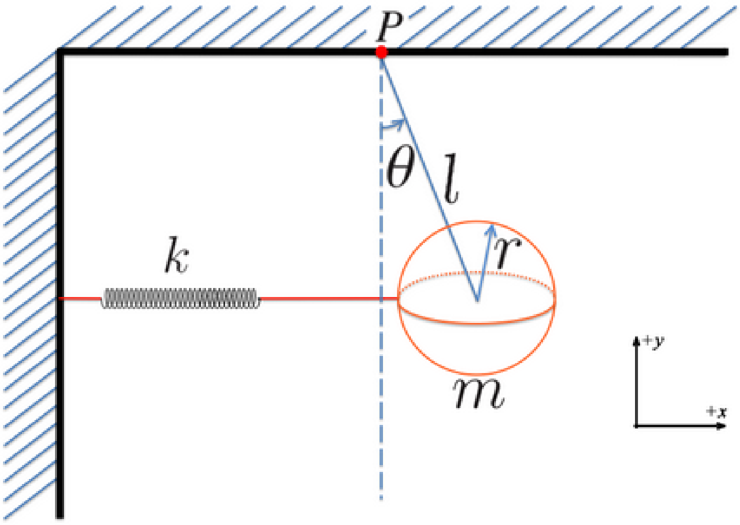
Excellent! We then know a, and also $\mu_s = a/g = \tan\theta$. Only F remains. Now, the easy way is to use $F/(3m) = a$ as I mentioned earlier, but we can use these equations as well. From the first equation in the group above, using the now-known value for a and that of $N\sin\theta = 2ma$,

$$F - N\sin\theta = ma \tag{4.69}$$
$$F - 2ma = ma \tag{4.70}$$
$$F = 3ma \tag{4.71}$$
$$F = 3mg\tan\theta \tag{4.72}$$

And we are done! To summarize, the answers are

$$a = g\tan\theta \tag{4.73}$$
$$F = 3mg\tan\theta \tag{4.74}$$
$$\mu_s = \tan\theta \tag{4.75}$$

4.5 Problem 5: A harmonic oscillator



"A pendulum of mass $m = 0.9$ kg and length $\ell = 1$ m is hanging from the ceiling. The massless string of the pendulum is attached at point P. The bob of the pendulum is a uniform shell (very thin hollow sphere) of radius $r = 0.4$ m, and the length ℓ of the pendulum is measured from the center of the bob. A spring with spring constant $k = 10\,\mathrm{N/m}$ is attached to the bob (center). The spring is relaxed when the bob is at its lowest point ($\theta = 0$). In this problem, we can use the small-angle approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Note that the direction of the spring force on the pendulum is horizontal to a very good approximation for small angles θ. (See figure)
Take $g = 10\,\mathrm{m/s^2}$.

(a) Calculate the magnitude of the net torque on the pendulum with respect to the point P when $\theta = 5°$. (magnitude; in Nm)
(b) What is the magnitude of the angular acceleration $\alpha = \ddot{\theta}$ of the pendulum when $\theta = 5°$? (magnitude; in radians/s^2)
(c) What is the period of oscillation T of the pendulum? (in seconds)"

First of all, let's not forget that the moment of inertia of a spherical shell is *not* $\frac{2}{5}mr^2$ – that holds for solid spheres only! The relevant moment of inertia is $\frac{2}{3}mr^2$.

With that in mind, lets try to analyze the situation. I started by drawing it all out.
The torque on the bob is given by the torque due to gravity, $\vec{\ell} \times \vec{F_g}$ plus the torque due to the spring, $\vec{\ell} \times \vec{F_{spr}}$.
The former is $\ell mg\sin\theta$; $\sin\theta$ because of the cross product, while there other terms should be rather clear.

So what is the torque due to the spring force? Well, first, what *is* the spring force? It's clear that it will be leftwards at this moment, just as the torque due to gravity. The magnitude is simply k times the extension of the spring. Since it is at equilibrium when $\theta = 0$, it is stretched an amount $\ell \sin\theta$ past that, *if we neglect the vertical displacement*, as hinted ("Note that the direction of the spring force on the pendulum is horizontal to a very good approximation for small angles θ").
To find the torque, we take ℓ (the lever arm; there is no torque relative to the bob's center since the spring is fastened there, but there is a torque relative to point P) and multiply that by the spring force $k\ell\sin\theta$. We should then multiply this by the sine of the angle between the two vectors; that angle is $\theta + 90°$, which gives us a $\cos\theta$ term via $\sin(90° + \theta) = \cos\theta$. We can approximate this term as 1 and ignore it, since we are allowed to use the small angle approximation.

We then know the torque relative to point P, which is

$$\tau_P = \ell mg \sin\theta + k\ell^2 \sin\theta \tag{4.76}$$

We apply the small angle approximation to the sine terms as well, and find

$$\tau_P = \ell mg\theta + k\ell^2\theta \tag{4.77}$$

Next, we calculate the moment of inertia of the bob. About its center of mass, it is simply $\frac{2}{3}mr^2$ (moment of inertia for a hollow sphere/thin spherical shell), but we want the moment of inertia about point P. We must therefore add a term $m\ell^2$ via the parallel axis theorem. The total moment of inertia about point P is

$$I_P = \frac{2}{3}mr^2 + m\ell^2 \tag{4.78}$$

Finally, we use the relationship $\tau = -I\alpha$ (not forgetting the minus sign for the restoring torque) to find α as the ratio of these two quantities (I also wrote α as $\ddot{\theta}$):

$$\ddot{\theta} = -\frac{\ell mg + k\ell^2}{\frac{2}{3}mr^2 + m\ell^2}\theta \tag{4.79}$$

(Without the minus sign, this equation would not make sense: it would state that as you move the bob towards the right, the angular acceleration would grow in that same direction.)

This is in the form $\ddot{\theta} = -\omega^2\theta$ (with the angular frequency ω being a constant), which is a simple harmonic oscillation. We can calculate the answer to part (b) using the above equation, and then use the solution to the above differential equation to find the period, part (c).

The period is given by $T = \dfrac{2\pi}{\omega}$, where ω is (as noted above) the square root of the term multiplying θ.

$$\omega = \sqrt{\frac{\ell mg + k\ell^2}{\frac{2}{3}mr^2 + m\ell^2}} \tag{4.80}$$

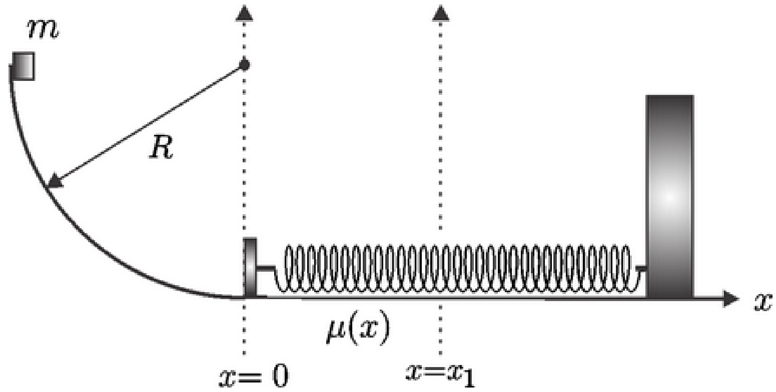
$$T = 2\pi\sqrt{\frac{\frac{2}{3}mr^2 + m\ell^2}{\ell mg + k\ell^2}} \tag{4.81}$$

We then have the answers to all the questions. The magnitude of the torque is easily calculated as 1.656 newton-meters using the equation above; I used the one *without* the small angle approximation, but they are very close together.

Next, $\alpha = \ddot{\theta}$ is also rather easy to calculate using the equation we have for that (with the small angle approximation). There, I find $|\ddot{\theta}|$ as $1.665\,\mathrm{rad/s^2}$. Finally, the period is calculated as 1.439 seconds.

4.6 Problem 6: Gliding mass stopped by spring

"A small block of mass $m = 1$ kg glides down (without friction) a circular track of radius $R = 2$ m, starting from rest at height R. At the bottom of the track it hits a massless relaxed spring with spring constant $k = 7$ N/m, which starts to be compressed as the block continues to move horizontally. Note that we assume no energy loss during this 'collision'. There is friction between the block and the horizontal surface, and it is not uniform. As a function of distance, the friction coefficient varies like $\mu(x) = \alpha x$, with $\alpha = 0.6\mathrm{m}^{-1}$. Assume for simplicity that static and dynamic friction coefficients are the same, and use $g = 10\,\mathrm{m/s^2}$. (See figure)



(a) What is the maximal distance x_1 that the block moves horizontally away from the track at $x = 0$? (in meters)
(b) What time t_1 does it take for the block to travel between $x = 0$ (relaxed spring) and $x = x_1$ (block at first stop)? (in seconds)
(c) What will happen after the block reaches point x_1?"

1. The block will move back and get catapulted up the circular track.
2. The block will move back and reach a second stop somewhere between $x = 0$ and $x = x_1$.
3. The block will move back and reach a second stop exactly at $x = 0$.
4. The block will stay put forever at $x = x_1$

All right, this looks like an interesting problem.
With no frictional losses on the circular track, we can use conservation of energy to calculate its "initial" velocity (at the "collision"). If we take the horizontal surface as $y = 0$ and $U = 0$, the initial gravitational potential energy is mgR, and the initial kinetic energy 0. The sum of the two must equal the kinetic energy as it exits the circular track, plus the gravitational potential energy at that point, which we defined as zero.

$$\frac{1}{2}mv^2 = mgR \tag{4.82}$$

$$v = \sqrt{2gR} \tag{4.83}$$

Once it reaches the horizontal surface, we have two forces opposing our motion: the spring force of magnitude kx, and the frictional force of magnitude $F_f = \mu_s mg = \alpha x mg$. This causes a leftwards acceleration (i.e. our velocity decreases) of

$$a = \frac{\sum |}{m} = -\frac{k + \alpha mg}{m}x \tag{4.84}$$

At first, I didn't write it down like this, and so I didn't realize that this is a simple harmonic oscillation! ($a = \ddot{x}$.)
We have friction, but other than that, it should be rather clear.
I first wanted to solve it in terms of integration acceleration, but the force is a function of x, which makes that harder. Next, I considered an energy analysis (or power, rather) where I had similar problems. Let's look at simple harmonic motion instead. Above, we have

$$\ddot{x} = -\frac{k + \alpha mg}{m} x \tag{4.85}$$

This equation is clearly not always valid; the oscillation will die out. We can use this for the "compression phase", though, i.e. until the mass stops. The solution is

$$x = x_1 \cos(\omega t + \varphi) \tag{4.86}$$
$$\dot{x} = -x_1 \omega \sin(\omega t + \varphi) \tag{4.87}$$
$$\omega = \sqrt{\frac{k + \alpha mg}{m}} \tag{4.88}$$
$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k + \alpha mg}} \tag{4.89}$$

The amplitude (that we usually call x_{max}) is what we call x_1 in this problem.
We need to use $\dot{x}$ to find the "amplitude" x_1 and the phase angle φ. We should be able to calculate the time taken (part b) as $T/4$, but let's take one step at a time.

By setting the x and $\dot{x}$ equations to their respective values at $t = 0$ (our initial conditions), we find

$$0 = x_1 \cos(\varphi) \tag{4.90}$$
$$\sqrt{2gR} = -x_1 \sqrt{\frac{k + \alpha mg}{m}} \sin(\varphi) \tag{4.91}$$

The first equation must imply that $\varphi = \pi/2$ or $\varphi = 3\pi/2$, since $x_1 \neq 0$. If that's the case, the sine term in the second equation becomes 1, and so

$$\sqrt{2gR} = -x_1 \sqrt{\frac{k + \alpha mg}{m}} \tag{4.92}$$
$$x_1 = -\sqrt{2gR} \frac{1}{\sqrt{\frac{k+\alpha mg}{m}}} \tag{4.93}$$

This has a sign error, so the phase must be $3\pi/2$ so that the sine is -1 and x_1 is positive.

$$x_1 = \frac{\sqrt{2mgR}}{\sqrt{k + \alpha mg}} \tag{4.94}$$

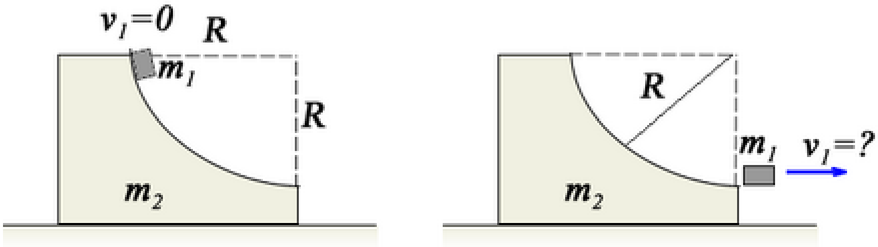
This gives us $x_1 = 1.754$ meters.
Via the period formula, $T = 1.74264$ seconds, though we are interested in a quarter of that value, $T/4 = 0.43566$ seconds.
(In the first quarter, the mass moves from $x = 0$ to $x = x_1$. In the second, it moves back. In the third, it moves to $x = -x_1$, and in the fourth, it moves back to $x = 0$. Assuming no friction, that is; in this case, things will change.)

What happens next? Well, the maximum static friction at this location is $\mu_s mg = \alpha x mg = 10.52$ N. The spring force is $kx = 12.278$ N, so static friction will be overcome, and the system will start to move. That leaves options 1, 2 and 3.

The spring has a stored energy of $\frac{1}{2}kx^2 = 10.77$ J, compared to the initial kinetic energy $\frac{1}{2}mv^2 = mgR = 20$ J. Just less than half the energy was wasted due to friction as the block came to a temporary halt. Does this make it safe to assume it will move past $x = 0$ again, though? In terms of energy, I'm not sure.
However, the spring force is always larger than the frictional force, as $k > \alpha mg$ (and so, of course, $kx > \alpha mgx$), meaning that not only can't it come to a halt prior to $x = 0$, it cannot *slow down*, either; it will have a positive acceleration until it loses contact with the spring. Therefore it must be catapulted back up!

4.7 Problem 7: Sliding blocks

"A small cube of mass $m_1 = 1.0$ kg slides down a circular and frictionless track of radius $R = 0.4$ m cut into a large block of mass $m_2 = 4.0$ kg as shown in the figure below. The large block rests on a horizontal and frictionless table. The cube and the block are initially at rest, and the cube m_1 starts from the top of the path. Find the speed of the cube v_1 as it leaves the block. Take $g = 10.0\,\mathrm{m/s^2}$. Enter your answer in m/s."



All right. First, if the larger block was fastened to the table and did not move at all, the velocity could be found as $\sqrt{2gR} \approx 2.8284$ m/s as in the previous problem. The answer we find here must be smaller, since some of the energy will go towards moving the bigger block in the opposite direction. In other words, the number above is an upper bound.

My solution was more complex than it needed to be, literally: when I reached the end, I had three equations with three unknowns. I solved the first equation for N, the normal force on the cube by the ramp, but realized that equation was the only one of the three to contain that term! That equation turned out to be useless, and the other two alone solved the problem. With that in mind, I cut out everything I wrote regarding forces and centripetal force. Instead, I used the conservation of mechanical energy and conservation of momentum alone.

First, we can write a conservation of energy equation. I will use v_B for the velocity of the large block/ramp.

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_B^2 = m_1 g R \tag{4.95}$$

The total energy available at the start is $m_1 g R$, and since no energy is lost to e.g. friction, that must be a constant throughout.
v_1 and v_B are both unknown, so we need a second equation. If we define it to be zero at the height where the cube flies off, 100% of the initial energy has turned into kinetic energy at that point.

What more can we use? Well, we can also apply the conservation of momentum in the horizontal direction. Since the initial momentum of the block+cube system is zero, it must always be zero, since there are no external horizontal forces (e.g. friction between block and table).

$$m_1 v_1 - m_2 v_B = 0 \tag{4.96}$$

I wrote $m_2 v_B$ as negative, so that $v_B > 0$ while the momentum is towards the left.
We then have two equations and two unknowns – very simple.

I'll start by solving the second one of these for v_B, which gives me $v_B = (m_1/m_2)v_1$.
We can substitute this into the larger equation, and solve for v_1.

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_2}v_1\right)^2 = m_1 g R \tag{4.97}$$

$$v_1^2 + \frac{m_1}{m_2}v_1^2 = 2gR \tag{4.98}$$

$$v_1^2\left(1 + \frac{m_1}{m_2}\right) = 2gR \tag{4.99}$$

$$v_1^2 = \frac{2gRm_2}{m_1 + m_2} \tag{4.100}$$

$$v_1 = \sqrt{\frac{2gRm_2}{m_1 + m_2}} \tag{4.101}$$

I find $v_1 = 2.529$ m/s, about 90% of what we would find if the ramp was immovable. The ramp itself is moving at $v_B = 0.632$ m/s towards the left, with momentum 2.529 kg m/s, as expected – the cube's mass is exactly 1 kg, so its momentum has the same value as its velocity, as can be seen here; the two are identical in magnitude, but opposite in direction, so the total horizontal momentum is indeed zero.

That ends this exam, this course and these notes! Thanks for reading.

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