ASYMPTOTIC ANALYSIS
BIG OH (O)
Big omega (Ω)
Theta notation (θ)
UNIT # 1

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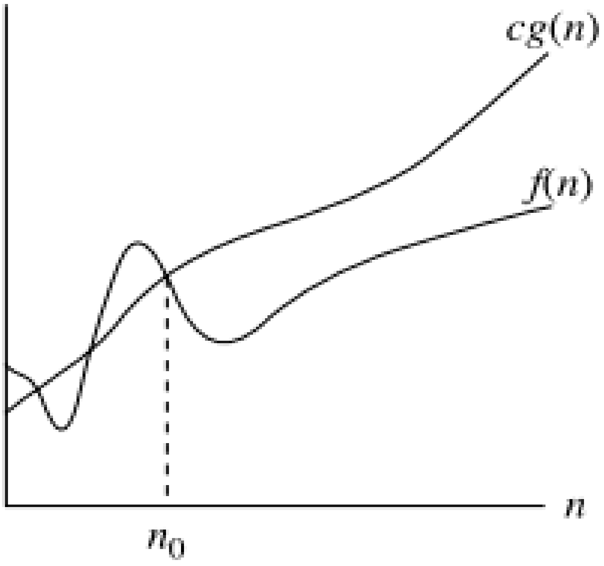
BIG OH (O)

O-NOTATION (UPPER BOUNDS):

O-notation

$O(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$.



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

QUESTION # 1

- $F(n) = 2n + 5$, find $g(n)$, C and n_o

By the definition of big Oh:

$$f(n) = Og(n)$$

$$\text{iff} \quad f(n) \le C.g(n) \qquad \text{for all } n \ge n_o$$

$$2n + 5 \le 3g(n) \qquad \text{say } C = 3$$

$$2n + 5 \le 3n$$ Highest power of f(n)

For n =1 : $2(1) + 5 \le 3 = 7 \le 3$ **False!**

For n =2 : $2*2 + 5 \le 3*2 = 9 \le 6$ **False!**

For n =3 : 2*3 + 5 $\le 3*3 = 11 \le 9$ **False!**

For n =4 : 2*4 + 5 $\le 3*4 = 13 \le 12$ **False!**

For n =5 : 2*5 + 5 $\le 3*5 = 15 \le 15$ **TRUE!**

WE can say f(n) = O(n) for C=3 and n >=5

QUESTION # 2

- $F(n) = 10n^2 + 4n + 2$, find $g(n)$, C and n_o

By the definition of big Oh:

$$f(n) = Og(n)$$

$$\text{iff} \quad f(n) \leq C.g(n) \quad \text{for all } n \geq n_o$$

$$g(n) = n^2$$

$$10n^2 + 4n + 2 \leq 11n^2 \quad \text{say } C = 11$$

For n =1 : $10*1 + 4*1 + 2 \leq 11 \quad = \quad 16 \leq 11$ **False!**

For n =2 : $10*2 + 4*2 + 2 \leq 11*4 \quad = \quad 50 \leq 44$ **False!**

For n =3 : $10*3 + 4*3 + 2 \leq 11*9 \quad = \quad 104 \leq 99$ **False!**

For n =4 : $10*4 + 4*4 + 2 \leq 11*16 \quad = \quad 178 \leq 176$ **False!**

For n =5 : $10*5 + 4*5 + 2 \leq 11*25 \quad = \quad 272 \leq 275$ **TRUE!**

WE can say f(n) = O g(n) for C=11 and n >=5

WHICH SHOULD WE CONSIDER IS UPPER BOUND?

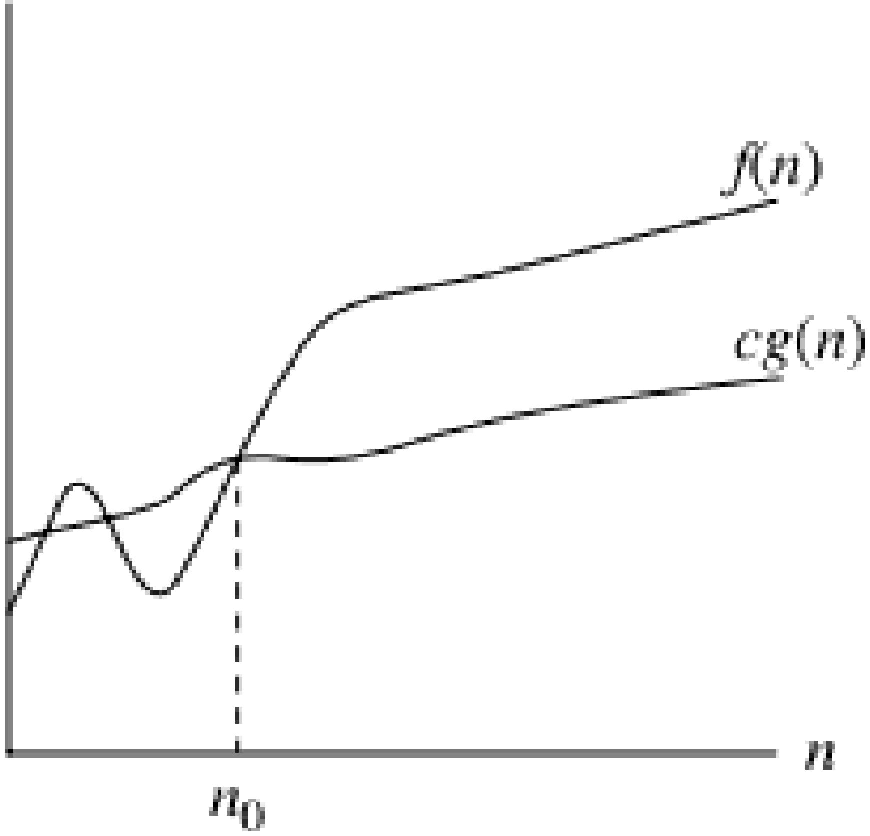
- Here, both n^2 and $n^3$are the upper bound of the function f(n)
- Both the answers are correct
- But always go for $n^{2.}$

We go by the least upper bound always.

Big omega (Ω)

Ω-notation (Lower bounds):

- □□$g(n)) = \{ f(n)$: there exist constants $C > 0, n_0 > 0$ such that 0 □ $cg(n)$ □ $f(n)$ for all n □ n_0 $\}$



Example # 1

- $f(n) = 2n + 5$, find the Ω notation, c and n_o

By the definition of big omega :

$$f(n) = \Omega g(n)$$

$$\text{iff } f(n) \geq Cg(n) \text{ for all } C>0,\ n \geq n_o$$

$$2n + 5 \geq c.\ g(n)$$

$$C=2 \text{ and } n_o \geq 1$$

$$2n + 5 \geq 2.n$$

For $n = 1 : 2*1 + 5 \geq 2*1 = 7 \geq 2$ **TRUE**

We can say that $f(n) = \Omega(n)$ for $c=2$ and $n>=1$

Example # 2

$F(n) = 10n^2 + 4n + 2$, find $g(n)$, C and n_o

By the definition of big OMEGA:

$$f(n) = \Omega\ g(n)$$

$$\text{iff} \quad f(n) \geq C.g(n) \quad \text{for all } n \geq n_o$$

$$g(n) = n^2$$

$$10n^2 + 4n + 2 \geq 10n^2 \quad \text{say } C = 10$$

For $n = 1$: $10*1 + 4*1 + 2 \geq 10 = 16 \geq 10$ **TRUE!**

We can say that $f(n) = \Omega\ g(n)$

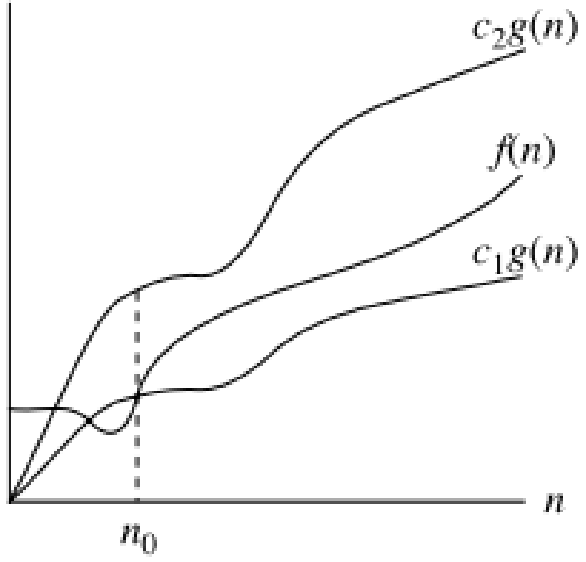
For $C=10$ and $n_o \geq 1$

Theta notation (θ)

θ-notatioN

Θ-notation

$$\Theta(g(n)) = \{f(n) : \text{ there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .$$



$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Example 1

- $f(n) = 2n + 5$, find Θ-notation.

By the definition of Theta Notation:

$$f(n) = \Theta(g(n))$$

$$\text{iff } C_1 g(n) \leq f(n) \leq C_2 g(n)$$

$$2.n \leq 2n + 5 \leq 3.n$$

That is we have to prove both O and Ω

For n = 1 : $2*1 \leq 2*1 + 5 \leq 3*1 = 2 \leq 7 \leq 3$ FALSE

For n = 2 : $2*2 \leq 2*2 + 5 \leq 3*2 = 4 \leq 9 \leq 6$ FALSE

For n = 3 : $2*3 \leq 2*3 + 5 \leq 3*3 = 6 \leq 11 \leq 9$ FALSE

For n = 4 : $2*4 \leq 2*4 + 5 \leq 3*4 = 8 \leq 13 \leq 12$ FALSE

For n = 5 : $2*5 \leq 2*5 + 5 \leq 3*5 = 10 \leq 15 \leq 15$ TRUE

Therefore, $f(n) = \Theta(n)$,**for** $C_1 = 2$, $C_2 = 3$, $n >= 5$

So we can say that $f(n) = \Theta(n)$