Least Possible Volume of a Cone Circumscribed About a Sphere

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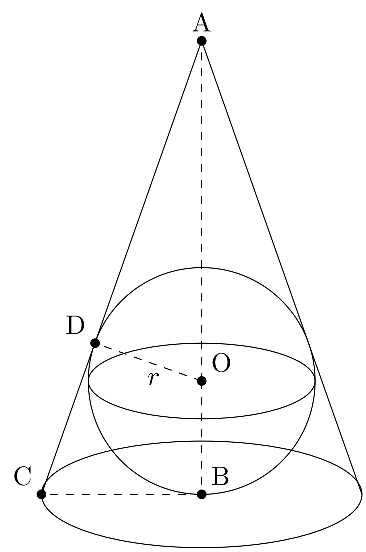
Problem

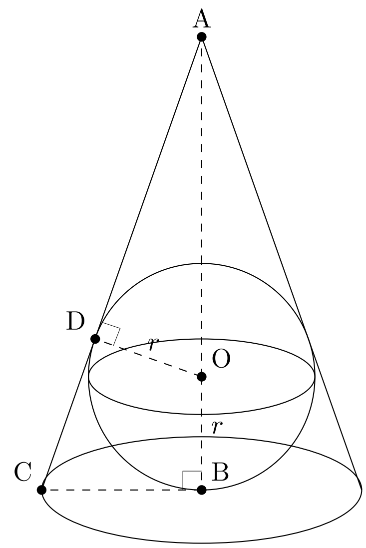
In this paper, I will present a solution to the following problem:

Circumscribe a right circular cone about a sphere of radius r such that the cone has minimum volume. Prove that the cone has exactly double the volume of the sphere.



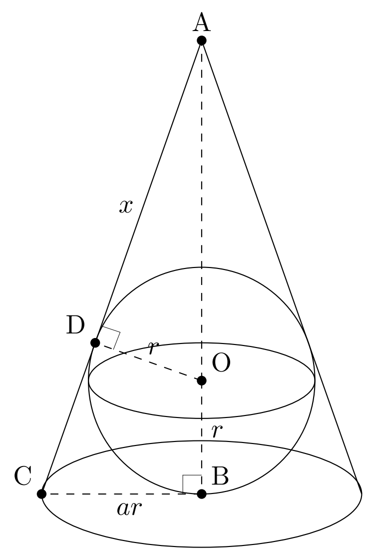
Observations

Firstly, notice that D and B are points of tangency between the sphere and the cone. Therefore, radii $\overline{OD}$ and $\overline{OB}$ are perpendicular to the side and base of the cone, respectively. Secondly, note that $\triangle ADO \sim \triangle ABC$ since they are both right triangles and share angle DAO. Also notice that the radius of the cone must be greater than the radius of the sphere. Finally, note that once the cone's radius has been chosen there is only one possibility for the sides of the cone such that the cone is still circumscribed about the sphere. Therefore, once we have chosen the cone's radius the cone can only have one possible volume.



Solution

Since the radius of the cone must be greater than the radius of the sphere, let the cone's radius be ar, such that $a > 1$ and ar is the volume-minimizing radius of the cone. In addition, let $\overline{AD} = x$.



Using the fact that $\triangle ADO \sim \triangle ABC$, we can write the equation

$$\frac{x}{r} = \frac{\sqrt{x^2 + r^2} + r}{ar}.$$

We solve for x in terms of a and r so that we can find the height of the cone:

$$ax = \sqrt{x^2 + r^2} + r$$

$$(ax - r)^2 = x^2 + r^2$$

$$x = \frac{2ar}{a^2 - 1}.$$

Now we can solve for the height of the cone:

$$\begin{aligned}
&\sqrt{\left(\frac{2ar}{a^2-1}\right)^2+r^2}+r \\
&= \sqrt{\frac{4a^2r^2+r^2(a^2-1)^2}{(a^2-1)^2}}+r \\
&= \frac{r(a^2+1)}{a^2-1}+r. \qquad (1)
\end{aligned}$$

Now we will write a function to represent the volume of the cone in terms of a:

$$f(a)=\frac{1}{3}\pi(ar)^2\left(\frac{r(a^2+1)}{a^2-1}+r\right).$$

Because the cone must have minimum volume, we must find the value of a that minimizes our function. To find the minimum value of a function, we use the fact that at a minimum, the derivative of the function is 0 and the derivatives to the left and right of the minimum are negative and positive, respectively. Taking the derivative of our function:

$$\begin{aligned}
f(a) &= \frac{1}{3}\pi a^2r^2\left(\frac{2a^2r}{a^2-1}\right) \\
f(a) &= \frac{\pi r^3}{3}\left(\frac{2a^4}{a^2-1}\right) \\
\frac{df}{da} &= \frac{\pi r^3}{3}\frac{8a^3(a^2-1)-2a^4(2a)}{(a^2-1)^2} \\
\frac{df}{da} &= \frac{\pi r^3}{3}\frac{4a^3(a^2-2)}{(a^2-1)^2}.
\end{aligned}$$

Setting this derivative equal to 0 and solving for a:

$$4a^3(a^2-2)=0,$$

$$a=0,-\sqrt{2},\sqrt{2}.$$

Since $a>1$, the value of a we must use is $\sqrt{2}$.

Thus, the radius of the cone is $r\sqrt{2}$ and by (1) the height of the cone is $4r$. Therefore, the volume of the cone is

$$\begin{aligned}
&\frac{1}{3}\pi(r\sqrt{2})^2 4r \\
&= \frac{8}{3}\pi r^3.
\end{aligned}$$

Because the volume of the sphere is $\frac{4}{3}\pi r^3$, the volume of the cone is exactly double the volume of the sphere, as desired.