Summer 2018

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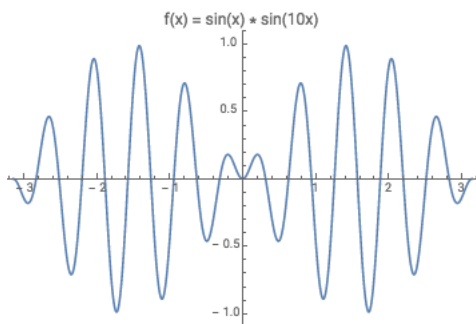### *University of Illinois Laboratory High School*

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1 Introduction

In this document I kind of just want to write down everything that I prove/am interested in over the course of this summer (2018). Hopefully, I will improve my fluency in *LaTeX* and will be motivated to keep to new math.
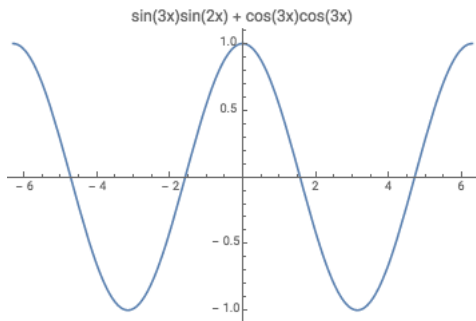
2 June

2.1 Something random I found

So I was trying to find the equation of a graph that looked like this:

I found the equation by noticing that the basic graph was a sine wave, and it was bounded by another sine wave. In school, we leaned that the graph of $h(x) \cdot \sin(x)$ is bounded by $|h(x)|$ and $-|h(x)|$. So, if I wanted my graph to be bounded by $\sin(x)$, then $h(x) = \sin(x)$. Adjusting the frequency of the second sine component yielded the correct graph.

As I was messing around, I came upon the following graph and equation:

This interested me because although the equation was $f(x) = \sin(3x)\sin(2x) + \cos(3x)\cos(2x)$, the graph appeared to be a pure cosine wave. After generalizing the equation to $f(x) = \sin(nx)\sin(mx) + \cos(nx)\cos(mx))$ After trying to guess what the pure cosine equivalent was, I found it: $f(x) = \cos((m-n)x)$.

Of course I had to try and prove this. After trying some things that didn't work, Matty suggesting using induction, and that ended up working. The following is my inductive proof.

2.1.1 The Proof

$$\sin(nx)\sin(mx) + \cos(nx)\cos(mx)) = \cos((m-n)x) \tag{1}$$

The base case of $x = 0$ holds, since

$$\sin(0 \cdot n)\sin(0 \cdot m) + \cos(0 \cdot n)\cos(0 \cdot m) = \cos((m-n)0)$$

$$0 \cdot 0 + 1 \cdot 1 = 1.$$

Suppose (1) is true. Then it should hold that

$$\sin((n+1)x)\sin((m+1)x) + \cos((n+1)x)\cos((m+1)x)) = \cos((m-n)x).$$

$$\sin((nx+x)\sin((mx+x) + \cos((nx+x)\cos((mx+x)) = \cos((mx-nx)$$

$$(\sin(nx)\cos(x) + \cos(nx)\sin(x))\,(\sin(mx)\cos(x) + \cos(mx)\sin(x))$$
$$+ (\cos(nx)\cos(x) - \sin(nx)\sin(x))\,(\cos(mx)\cos(x) - \sin(mx)\sin(x))$$

$$\sin(nx)\sin(mx)\cos^2(x) + \sin(mx)\sin(x)\cos(nx)\cos(nx)\cos(m)$$
$$+\sin(nx)\sin(x)\cos(mx)\cos(x)+\sin^2(x)\cos(nx)\cos(mx)+\cos(nx)\cos(mx)\cos^2(x)-\sin(nx)\sin(x)\cos(mx)\cos(x)$$
$$- \sin(mx)\sin(x)\cos(nx)\cos(x) + \sin(nx)\sin(mx)\sin^2(x) = \cos(mx - nx)$$

$$\sin(nx)\sin(mx)\cos^2(x) + \sin(mx)\sin(x)\cos(nx)\cos(nx)\cos(m)$$
$$+\sin(nx)\sin(x)\cos(mx)\cos(x)+\sin^2(x)\cos(nx)\cos(mx)+\cos(nx)\cos(mx)\cos^2(x)-\sin(nx)\sin(x)\cos(mx)\cos(x)$$
$$- \sin(mx)\sin(x)\cos(nx)\cos(x) + \sin(nx)\sin(mx)\sin^2(x) = \cos(mx - nx)$$

$$\sin(nx)\sin(mx)\cos^2(x)+\sin^2(x)\cos(nx)\cos(mx)+\cos(nx)\cos(mx)\cos^2(x)+\sin(nx)\sin(mx)\sin^2(x) = \cos(mx-nx)$$

$$(\sin(nx)\sin(mx))\,(\sin^2(x) + \cos^2(x)) + (\cos(nx)\cos(mx))\,(\sin^2(x) + \cos^2(x)) = \cos(mx - nx)$$

$$\sin(nx)\sin(mx) + \cos(nx)\cos(mx) = \cos(mx - nx).$$

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2.2 Limits

So basically I was reading the chapter out of the calculus book on limits, and I understood the $\epsilon - \delta$ definition of the limit, but when it came to more complicated things like the Squeeze Theorem, I got confused. So, I decided to slow down and really made sure I understood everything. So, I came across this problem in a Berkeley homework thing: prove

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = 4.$$

Here is my solution to that problem.

2.2.1 The Solution

Let $f(x) = \frac{x^4 - 1}{x - 1}$. Then $\lim_{x \to 1} f(x) = 4$ if for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $|x - 1| < \delta$, then $|f(x) - 4| < \epsilon$.

Because the value of $f(1)$ as x approaches 1 is irrelevant, $f(x)$ can be simplified to $x^3 + x^2 + x + 1$. Thus, we have

$$|x^3 + x^2 + x - 3| < \epsilon$$

$$|(x - 1)(x^2 + 2x + 3)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x^2 + 2x + 3|}.$$

But we also know

$$|x - 1| < \delta$$

So, letting $\delta = \frac{\epsilon}{|x^2 + 2x + 3|}$ will satisfy the definition. However, we do not want to have x in our expression for δ.

To satisfy the definition, we only need to find a value of δ that works. So, if ϵ is sufficiently small (we will find out what the means later), then δ will be less than or equal to 1. Assuming this,

$$|x - 1| \leq 1$$
$$-1 \leq x - 1 \leq 1$$
$$0 \leq x \leq 2$$
$$3 \leq x^2 + 2x + 3 \leq 11.$$

So, $\frac{\epsilon}{|x^2 + 2x + 3|} \leq \frac{\epsilon}{11}$.

Getting back to our assumption that $\delta \leq 1$, we see that this is true if $\epsilon \leq 11$. But, if ϵ is greater than 1, then we can simply let $\delta = 1$. So, solution for δ is $\delta = \min\{1, \frac{\epsilon}{11}\}$.

Proof:

Let $f(x) = \frac{x^4 - 1}{x - 1}$. Then $\lim_{x \to 1} f(x) = 4$ if for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $|x - 1| < \delta$, then $|f(x) - 4| < \epsilon$.

Let $\delta = \min\{1, \frac{\epsilon}{11}\}$. Note that this means $\delta \leq 1$.

$$|x - 1| < \delta.$$

$$|x - 1| \leq 1$$
$$0 \leq x \leq 2$$
$$0 \leq x^2 + 2x + 3 \leq 11.$$

This tells us that $\frac{\epsilon}{|x^2+2x+3|} \leq \frac{\epsilon}{11}$.

In the case that $\min\{1, \frac{\epsilon}{11}\} = \frac{\epsilon}{11}$, we get

$$|x - 1| < \frac{\epsilon}{|x^2 + 2x + 3|} \leq \frac{\epsilon}{11} \leq 1.$$

In the case that $\min\{1, \frac{\epsilon}{11}\} = 1$ we get

$$|x - 1| < \frac{\epsilon}{|x^2 + 2x + 3|} \leq 1 \leq \frac{\epsilon}{11}.$$

In both cases, we arrive at

$$|x - 1||x^2 + 2x + 3| < \epsilon$$

$$|(x^3 + x^2 + x + 1) - 4| < \epsilon$$

$$|f(x) - 4| < \epsilon.$$

2.3 Limit Product Proof

So in the calculus book, one of the questions asked me to prove the product rule for limits: $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$. The book's solution was very complicated, so the following is my simpler proof.

2.3.1 The Proof

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Therefore, we are trying to show that $\lim_{x \to a} f(x)g(x) = LM$.

Let us first prove that $\lim_{x \to a} (f(x) - L)(g(x) - M) = 0$.

Since $\lim_{x \to a} = L$, for every $\epsilon > 0$, there exists some $\delta_f > 0$ such that if $0 < |x-a| < \delta_f$, then $0 < |f(x)-L| < \sqrt{\epsilon}$. Similarly, there must exist some $\delta_g > 0$ such that if $0 < |x - a| < \delta_g$, then $0 < |g(x) - M| < \sqrt{\epsilon}$. Letting $\delta = \min\{\delta_f, \delta_g\}$ satisfies both inequalities, so we also have if $0 < |x-a| < \delta$, then $0 < |f(x)-L||g(x)-M| < \epsilon$, or $0 < |(f(x) - L)(g(x) - M) - 0| < \epsilon$. Therefore, $\lim_{x \to a} (f(x) - L)(g(x) - M) = 0$.

From here the proof is fairly simple. Note that

$$\lim_{x \to a} (f(x) - L)(g(x) - M) = 0$$

$$\lim_{x \to a} f(x)g(x) - Lg(x) - Mf(x) + LM = 0$$

$$\lim_{x \to a} f(x)g(x) - \lim_{x \to a} Lg(x) - \lim_{x \to a} Mf(x) + LM = 0$$

$$\lim_{x \to a} f(x)g(x) - L \lim_{x \to a} g(x) - M \lim_{x \to a} f(x) + LM = 0$$

$$\lim_{x \to a} f(x)g(x) - LM - ML + LM = 0$$

$$\lim_{x \to a} f(x)g(x) = LM.$$

3 July

3.1 Calculus

So, I just finished the chapter on derivatives and they make sense. Mostly. However, I would like to clarify some things, incorporate some concepts from 3Blue1Brown, and prove some results.

3.1.1 What is a derivative?

One interpretation of the derivative is about the rate of change of a function. What this means is for some input x, and some output $f(x)$, the rate of change for this function is how this function changes, some df, given a small change to the input, some dx. To be exact, the rate of change is $\frac{f(x+dx)-f(x)}{dx}$. Then the derivative is whatever this ratio approaches as dx approaches 0 : $\lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx}$.

Graphically, we can see this as the slope of the tangent line to f, since our expression for the derivative can be expressed as df, the change in y, over dx, the change in x. However, always keep in the back of your mind that the derivative is the limit as dx approaches 0.

3.1.2 The Chain Rule

The following is not intended to be a proof of the Chain Rule, rather, just some notes to help it make more intuitive sense.

What the Chain Rule allows us to do is take the derivative of a composition of functions $\frac{d}{dx}g(f(x))$. What this ratio is telling is is how a small change (dx) to x will affect the final output, $g(f(x))$.

First, let us focus on how changing x affects $f(x)$. A small change dx in x will change f by exactly $\frac{df}{dx}\cdot dx$. Let us make the substitution $f(x) = u$, and $\frac{df}{dx}\cdot dx = du$.

Next, we consider how g changes. First is the original input $f(x)$ or u resulting in the output $g(u)$ Then, is input $u + du$, resulting in the output $g(u + du)$. Therefore, the derivative of g at u is $g'(u)du$. Therefore, we have

$$\boxed{\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = g'(f(x))\cdot f'(x).}$$

And this is the chain rule.

3.1.3 Implicit Differentiation

Implicit differentiation is the process of taking the derivative of an expression in which and x and y are interdependent. For example, $x^2 + y^2 = 1$ is an expression where x is not an input, and y is not an output; they are simply two variables linked by that equation.

Say we wanted to find the derivative of that expression: how a small change dx to x changes y by dy. Therefore, the slope we are looking for is $\frac{dy}{dx}$. Keep in mind we are differentiating with respect to x, that is, we are changing x by some dx.

In our circle example, we start by taking the derivative of both sides. What this means is that for some small steps dx and dy, the only way that that value is equal to the original value is it equals the change to the other side when x changes. That is if somethingsomething else, then

$$\frac{d}{dx}\text{something} = \frac{d}{dx}\text{something else.}$$

In our circle example, this means that

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx} = 1.$$

We can split the derivative as such:

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1.$$

We can compute $\frac{d}{dx}x^2$ and $\frac{d}{dx}1$ easily, but how do we compute $\frac{d}{dx}y^2$? That is, how does a small change in x affect y. This sounds like something the chain rule can help us with.

If we let $f(x) = y^2$, then we are trying to compute $\frac{df}{dx}$. However, f takes in x as its input, but outputs y^2. It is not immediately obvious how to compute $\frac{df}{dx}$. But, we can compute $\frac{df}{dy}$, that is, how f changes given a small change dy to y. However, we must link this ratio $\frac{df}{dx}$, since, we want df with respect to dx. So, we can

multiply by $\frac{dy}{dx}$, since this relates dy to dx. This $\frac{dy}{dx}$ also happens to be the derivative we ultimately want to compute. So, we have

$$\frac{dy}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}.$$

Notice that this is simply the Chain Rule.

In our example, this looks like

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x = -y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}.}$$

3.1.4 Leibniz's Rule

Leibniz's Rule gives us a way of quickly computing $\frac{d^n}{dx^n}(fg)(x)$, or the nth derivative of $f(x)g(x)$. We first try to propose a conjecture and then prove it.

Computing the first few, we see that

$$(fg)^0 = fg$$

$$(fg)^1 = f'g + fg'$$

$$(fg)^2 = f^2g + 2f'g' + fg^2$$

from the Product Rule.

The coefficients seem to match that of Pascal's Triangle. Thus, from the example, we conjecture that

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{n-k}g^k. \tag{2}$$

We prove this hold for all positive n by induction.

Base Case: $n = 1$

$$\begin{aligned}
(fg)^1 &= \sum_{k=0}^{1} \binom{1}{k} f^{1-k}g^k \\
&= \binom{1}{0}f^1g^0 + \binom{1}{1}f^0g^1 \\
&= f'g + fg'.
\end{aligned}$$

which is true by the Product Rule.

Induction Step: we assume that (1) holds for $n = n - 1$. Then

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$$(fg)^n = \frac{d}{dx}(fg)^{n-1} = \frac{d}{dx}\sum_{k=0}^{n-1}\binom{n-1}{k}f^{n-1-k}g^k$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d}{dx}f^{n-1-k}g^k$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}f^{n-k}g^k + f^{n-1-k}g^{k+1}$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}f^{n-k}g^k + \sum_{k=0}^{n-1}\binom{n-1}{k}f^{n-1-k}g^{k+1}.$$

We will make use of Pascal's Identity, which states that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

$$= \binom{n-1}{0}f^n g^0 + \binom{n-1}{1}f^{n-1}g^1 + ... + \binom{n-1}{n-2}f^2 g^{n-2} + \binom{n-1}{n-1}f^1 g^{n-1}$$

$$+ \binom{n-1}{0}f^{n-1}g^1 + \binom{n-1}{1}f^{n-2}g^2 + ... + \binom{n-1}{n-2}f^1 g^{n-1} + \binom{n-1}{n-1}f^0 g^n$$

$$= \binom{n-1}{0}f^n g^0 + \left(\binom{n-1}{0} + \binom{n-1}{1}\right)(f^{n-1}g1) + \left(\binom{n-1}{1} + \binom{n-1}{2}\right)(f^{n-2}g^2) + ...$$

$$+ \left(\binom{n-1}{n-2} + \binom{n-1}{n-1}\right)(f^1 g^{n-1}) + \binom{n-1}{n-1}f^0 g^n$$

$$= \binom{n}{0}f^n g^0 + \binom{n}{1}(f^{n-1}g^1) + \binom{n}{2}(f^{n-2}g^2) + ... + \binom{n}{n-1}(f^1 g^{n-1}) + \binom{n}{n}f^0 g^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}f^{n-k}g^k.$$

Therefore, (1) holds for $n = n - 1$. By the principle of induction, (1) is true for all positive integers n.

3.1.5 Fundamental Theorem of Calculus: Part 1

Part one of the FTC states that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$, such that $F'(x) = f(x)$.

You might wonder why antiderivatives have anything to do with the integral of f. The connecting piece lies in the Mean Value Theorem.

Because we are interested in a definite integral, let us create some partition $\mathcal{P}$, where $a = x_0 < x_1 < ... < x_n = b$. Let us look at some z_i belonging to the interval $[x_i, x_{i+1}]$, for $0 \leq i < n$

If we apply the Mean Value Theorem to $F(x)$ on the interval (x_i, x_{i+1}), then we get

$$F'(z_i) = f(z_i) = \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i}$$

$$f(z_i)(x_{i+1} - x_i) = F(x_{i+1}) - F(x_i).$$

Because we want the definite integral, we want the sum over the entire partition, so we take the sum of both sides

$$\sum_{i=0}^{n-1} f(z_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i). \tag{3}$$

Notice what we have on the LHS of (3) : we have the Riemann sum $\mathcal{R}(f, \mathcal{P}, z_i)$. So, this sum approaches the definite integral we want:

$$\sum_{i=0}^{n-1} f(z_i)(x_{i+1} - x_i) = \mathcal{R}(f, \mathcal{P}, z_i) = \int_a^b f(x)dx = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i)$$

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i)$$
$$= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + ... + [F(x_{n-1}) + F(x_{n-2})] + [F(x_n) - F(x_{n-1})]$$
$$= -F(x_0) + F(x_n).$$

But remember $a = x_0$ and $b = x_n$. So,

$$\int_a^b f(x)dx = F(x_n) - F(x_0)$$

$$\boxed{\int_a^b f(x)dx = F(b) - F(a).}$$

3.1.6   Interesting Derivative

In my studying for the Math 220 Proficiency test, I was asked to solve for $\frac{dy}{dx}$ in the following problem:

$$y = (log(x))^{\frac{1}{x^2}}.$$

Here is my solution.
Differentiate both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}(log(x))^{\frac{1}{x^2}}.$$

Now all we have to do is differentiate the RHS and we are done. To do so, we write the RHS equivalently with e^{log}, since these functions undo each other

$$\frac{dy}{dx} = \frac{d}{dx}e^{log(log(x)^{\frac{1}{x^2}})}$$
$$= \frac{d}{dx}e^{\frac{1}{x^2}\log(log(x))},$$
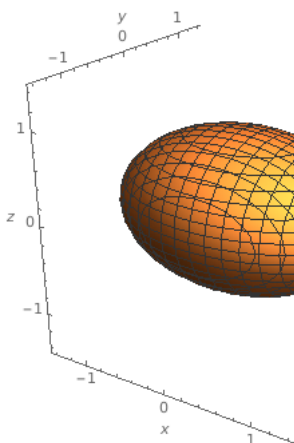
using properties of logarithms.
Now we differentiate the RHS using the Chain Rule and Product Rule:

$$\frac{dy}{dx} = e^{\frac{1}{x^2}\log(log(x))} \cdot \frac{d}{dx}\frac{1}{x^2}\log(log(x))$$
$$= e^{\frac{1}{x^2}\log(log(x))} \cdot (-2x^{-3}\log(log(x)) + \frac{1}{x^2}\frac{1}{x\log(x)})$$
$$= e^{\frac{1}{x^2}\log(log(x))} \cdot (\frac{-2}{x^3}\log(log(x)) + \frac{1}{x^3\log(x)})$$
$$= log(x)^{\frac{1}{x^2}} \cdot (\frac{-2}{x^3}\log(log(x)) + \frac{1}{x^3\log(x)}).$$

3.1.7   Volume of an Ellipsoid

I have wondered how to find the volume of an ellipsoid for a while now, and finally I am able to do so. An ellipsoid is the 3-dimensional analogue of the ellipse, and looks like this:

The equation for an ellipsoid with semi-axis a, b, c along the x, y, z axis, respectively, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The way we will tackle this is by taking cross-sectional areas along the x-axis, parallel to the $y - z$ plane, and integrating them as x varies from $-a$ to a, along the x-axis.
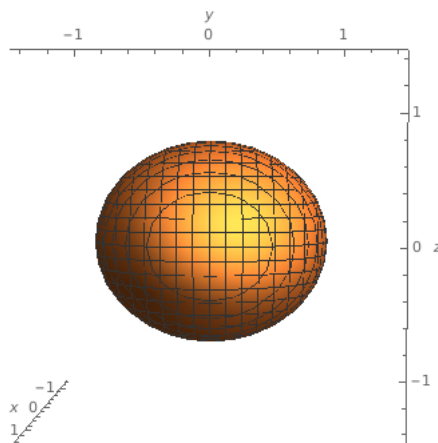

Figure 1: Viewing the ellipsoid in the $y - z$ plane, the same plane as our cross-sectional areas.

Notice that all of the cross-sections are going to be ellipses. To find some particular cross-section at some point along the x-axis, we will need to find the equation of each of the cross-sections. Note that the ellipses are from the $y - z$ plane.

Recall the equation of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

So,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

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$$\frac{y^2}{b^2(1 - \frac{x^2}{a^2})} + \frac{z^2}{c^2(1 - \frac{x^2}{a^2})} = 1.$$

This is the equation of any cross-sectional ellipse parallel to the $y - z$ plane, as the x position varies from $-a$ to a. As a check, we see that when $x = 0$, the equation simply becomes

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which is the equation of an ellipse with semi-axis b and c, which is correct.

Next, we must find the area of each cross-sectional ellipse. Recall that the area of an ellipse with semi-axis a and b is πab. The semi-axis lengths are simply the positive square roots of the denominators in the equation. In our case, the semi-axis have length $b\sqrt{1 - \frac{x^2}{a^2}}$ and $c\sqrt{1 - \frac{x^2}{a^2}}$. So, the area as a function of its x position is

$$A(x) = \pi b\sqrt{1 - \frac{x^2}{a^2}} \cdot c\sqrt{1 - \frac{x^2}{a^2}}$$
$$= \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

Finally, we can integrate all cross-sectional areas from $-a$ to a with the following definite integral:

$$V = \int_{-a}^{a} A(x)dx$$
$$= \pi bc \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx$$
$$= \pi bc \left(x - \frac{1}{a^2} \cdot \frac{1}{3}x^3 \Big|_{-a}^{a}\right)$$
$$= \pi bc \left(\left(a - \frac{1}{a^2} \cdot \frac{1}{3}a^3\right) - \left(-a - \frac{1}{a^2} \cdot \frac{1}{3}(-a)^3\right)\right)$$
$$= \pi bc \left(\left(a - \frac{1}{a^2} \cdot \frac{1}{3}a^3\right) - \left(-a + \frac{1}{a^2} \cdot \frac{1}{3}a^3\right)\right)$$
$$= \pi bc \left(2a - 2\frac{1}{a^2}\frac{1}{3}a^3\right)$$
$$= 2\pi bc \left(a - \frac{1}{3}a\right)$$
$$= \boxed{\frac{4}{3}\pi abc.}$$

4 August

4.1 Putnam Problem

In the calculus book, I came across the following Putnam problem:

Let $n \geq 2$ be an integer, and for any real number $0 \leq \alpha \leq 1$, let $C(\alpha)$ be the coefficient of x^n in the power series expansion of $(1 + x)^\alpha$. Prove that

$$\int_0^1 \left(C(-t - 1) \sum_{k=1}^{n} \frac{1}{t + k}\right) dt = (-1)^n n.$$

Here is my solution.

The power series expansion for $(1+x)^{-t-1}$ is

$$1 + \frac{1}{1!}(-t-1)x + \frac{1}{2!}(-t-1)(-t-2)x^2 + ... + \frac{1}{n!}(-t-1)...(-t-n)x^n + ...$$

The coefficient $\frac{1}{n!}(-t-1)...(-t-n)$ is our $C(-t-1) = \frac{(-1)^n}{n!}(t+1)(t+2)...(t+n)$.
The summation is

$$\sum_{k=0}^{n} = \frac{1}{t+1} + \frac{1}{t+2} + ... + \frac{1}{t+n}.$$

Our integral is now

$$\int_0^1 \left(C(-t-1) \sum_{k=1}^{n} \frac{1}{t+k} \right) dt = \frac{(-1)^n}{n!} \int_0^1 (t+1)(t+2)...(t+n) \left(\frac{1}{t+1} + \frac{1}{t+2} + ... + \frac{1}{t+n} \right) dt$$

$$= \frac{(-1)^n}{n!} \int_0^1 \left((t+2)...(t+n) + (t+1)(t+3)...(t+n) + ... + (t+1)...(t+n-1) \right) dt$$

$$= \frac{(-1)^n}{n!} \int_0^1 \frac{d}{dx}(t+1)...(t+n)dt$$

This last step, rewriting the integrand as the derivative of a function seems sketchy, but it works, considering the smaller cases and inducting upwards. For example,

$$\frac{d}{dx}fgh = \frac{d}{dx}(fg)(h) = (fg)'h + (fg)h' = (f'g + fg')h + fgh' = f'gh + fg'h + fgh',$$

which is exactly what we want.
Continuing,

$$\frac{(-1)^n}{n!} \int_0^1 \frac{d}{dx}(t+1)...(t+n)dt = \frac{(-1)^n}{n!} \left((t+1)...(t+n) \Big|_0^1 \right)$$

$$= \frac{(-1)^n}{n!} \left((2 \cdot 3 \cdot ... \cdot (n+1)) - (1 \cdot 2 \cdot ... \cdot n) \right)$$

$$= \frac{(-1)^n}{n!} (n!(n+1-1))$$

$$= \boxed{(-1)^n n.}$$