Basel Problem

Akash Pardeshi

University of Illinois Laboratory High School

November 2018

1 Introduction

I present a proof for two classic summations:

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{2}$$

I consider two definite integrals, represent them as the desired sums, and evaluate the integrals, completing the proofs.

2 Proof of (1)

Consider the following integral:

$$I = \int_0^1 \frac{dx}{1+x^2}.$$

We recognize the integrand as the sum of an infinite geometric series of first term 1 and common ratio $r = -x^2$. We know the series converges since $r \in (0, 1)$.

Thus, we may write

$$\begin{aligned} I &= \int_0^1 1 - x^2 + x^4 - x^6 + \cdots dx \\ &= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \right) \Bigg|_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \end{aligned}$$

To evaluate I, we note that it yields the inverse tangent function:

$$\begin{aligned} I &= \int_0^1 \frac{dx}{1+x^2} \\ &= \tan^{-1}(x) \Big|_0^1 \\ &= \frac{\pi}{4}. \end{aligned}$$

Therefore, we have

$$I = \int_0^1 \frac{dx}{1+x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

3 Proof of (2)

Consider the following integral:

$$I = \int\limits_{x=0}^{1} \int\limits_{y=0}^{1} \frac{1}{1-xy}\, dy\, dx.$$

Again, we treat the integrand as the sum of an infinite geometric series of first term 1 and common ratio xy. We know the series converges since $x, y \in (0, 1)$.

$$\begin{aligned}
I &= \int\limits_{x=0}^{1} \int\limits_{y=0}^{1} \frac{1}{1-xy}\, dy\, dx \\
&= \int\limits_{x=0}^{1} \int\limits_{y=0}^{1} \sum_{n=0}^{\infty} x^n y^n\, dy\, dx \\
&= \int\limits_{x=0}^{1} \left(\sum_{n=0}^{\infty} x^n \cdot \frac{1}{n+1} y^{n+1} \Big|_0^1 \right) dx \\
&= \int\limits_{x=0}^{1} \left(\sum_{n=0}^{\infty} x^n \cdot \frac{1}{n+1} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} x^{n+1} \Big|_0^1 \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

Now we must evaluate the integral I.

First we make a change of variables: let

$$u = \frac{x+y}{2}, v = \frac{y-x}{2} \iff x = u - v, y = u + v.$$

This substitution transforms the unit square into a rotated square in the uv plane with vertices at $(0,0), (\frac{1}{2}, \frac{1}{2}), (1, 0), (\frac{1}{2}, -\frac{1}{2})$.

The change in area dA is given by the determinant of the Jacobian

$$\det J = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

Thus, we have the new integral

$$I = \int\limits_{x=0}^{1} \int\limits_{y=0}^{1} \frac{1}{1-xy}\, dy\, dx = 2 \iint\limits_{R} \frac{1}{1-(u-v)(u+v)}\, dv\, du.$$

We can integrate over the region R in four pieces:

$$I = I_1 + I_1^{'} + I_2 + I_2^{'},$$

where

$$I_1 = 2 \int\limits_{u=0}^{\frac{1}{2}} \int\limits_{v=0}^{u} \frac{1}{1-u^2+v^2}\, dv\, du$$

$$I_1^{'} = 2 \int\limits_{u=0}^{\frac{1}{2}} \int\limits_{v=-u}^{0} \frac{1}{1-u^2+v^2}\, dv\, du$$

$$I_2 = 2 \int\limits_{u=\frac{1}{2}}^{1} \int\limits_{v=0}^{1-u} \frac{1}{1-u^2+v^2}\, dv\, du$$

$$I_2^{'} = 2 \int\limits_{u=\frac{1}{2}}^{1} \int\limits_{v=-1+u}^{0} \frac{1}{1-u^2+v^2}\, dv\, du$$

By making the change of variables $u = u$ and $v = -v$ in $I_1^{'}$ and $I_2^{'}$, we can halve the number of integrals we need to compute (this substitution has $dA = 1$):

$$\begin{aligned} I_1^{'} &= -2 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{-u} \frac{1}{1-(u-v)(u+v)}\, dv\, du \\ &= -2 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{u} \frac{1}{1-(u+v)(u-v)} \cdot -dv\, du \\ &= 2 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{u} \frac{1}{1-(u+v)(u-v)}\, dv\, du \\ &= I_1. \end{aligned}$$

Similarly, it can be shown that $I_2 = I_2^{'}$.
Together, we now have $I = 2I_1 + 2I_2$.

3.1 Evaluating I_1

We have

$$2I_1 = 4 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{u} \frac{1}{1-u^2+v^2}\, dv\, du$$

Note that

$$\int\limits_{0}^{t} \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{t}{a}\right)$$

where a is independent of x.

We can use this to help us evaluate I_1 since u and v are independent of each other. Specifically, let $a^2 = 1 - u^2 \to a = \sqrt{1-u^2}$ (since $0 < u < 1$) and $x^2 = v^2$:

$$\begin{aligned}
2I_1 &= 4 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{u} \frac{1}{1-u^2+v^2}\, dv\, du \\
&= 4 \int\limits_{u=0}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \tan^{-1}\left(\frac{u}{\sqrt{1-u^2}}\right) du
\end{aligned}$$

Let $\sin\theta = u \Leftrightarrow \cos\theta\, d\theta = du$.

$$\begin{aligned}
&= 4 \int\limits_{\theta=0}^{\frac{\pi}{6}} \frac{1}{\cos\theta} \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) (\cos\theta\, d\theta) \\
&= 4 \int\limits_{\theta=0}^{\frac{\pi}{6}} \theta\, d\theta \\
&= 4 \cdot \frac{1}{2} \cdot \frac{\pi^2}{36} \\
&= \frac{\pi^2}{18}.
\end{aligned}$$

3.2 Evaluating I_2

We have

$$2I_2 = 4 \int\limits_{u=0}^{1/2} \int\limits_{v=0}^{u} \frac{1}{1-u^2+v^2}\, dv\, du$$

Again, let $a^2 = 1 - u^2 \to a = \sqrt{1-u^2}$ (since $0 < u < 1$) and $x^2 = v^2$:

$$\begin{aligned}
2I_2 &= 4 \int\limits_{u=\frac{1}{2}}^{1} \int\limits_{v=0}^{1-u} \frac{1}{1-u^2+v^2}\, dv\, du \\
&= 4 \int\limits_{u=\frac{1}{2}}^{1} \frac{1}{\sqrt{1-u^2}} \tan^{-1}\left(\frac{1-u}{\sqrt{1-u^2}}\right) dv\, du
\end{aligned}$$

Let $\cos 2\theta = u \Leftrightarrow -2\sin 2\theta\, d\theta = du$.

$$= 4 \int_{\theta=\frac{\pi}{6}}^{0} \frac{1}{\sin 2\theta} \tan^{-1}\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right)(-2\sin 2\theta \, d\theta)$$
$$= 8 \int_{\theta=0}^{\frac{\pi}{6}} \tan^{-1}\left(\frac{2\sin^2\theta}{\sin 2\theta}\right) d\theta$$
$$= 8 \int_{\theta=0}^{\frac{\pi}{6}} \tan^{-1}\left(\frac{2\sin^2\theta}{2\sin\theta\cos\theta}\right) d\theta$$
$$= 8 \int_{\theta=0}^{\frac{\pi}{6}} \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) d\theta$$
$$= 8 \int_{\theta=0}^{\frac{\pi}{6}} \theta \, d\theta$$
$$= 8 \cdot \frac{1}{2} \cdot \frac{\pi^2}{36}$$
$$= \frac{\pi^2}{9}.$$

Recall $I = 2I_1 + 2I_2$. Therefore,

$$I = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}.$$

To finish, recall $I = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore,

$$I = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4 Corollaries

Two resulting sums from the summation $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ are

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24} \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \tag{4}$$

4.1 Proof of (3)

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2^2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{24}$$

by (2).

4.2 Proof of (4)

Let $S = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. Note that we are summing the reciprocals of all squares congruent to 1 mod 2. Also note that the summation in (3) considered only the reciprocals of squares congruent to 0 mod 2. Therefore, adding the summations from (3) and (4) produces the sum of the reciprocals of squares of all natural numbers:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}.$$

Thus, we may write

$$\begin{aligned} \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\ &= \frac{\pi^2}{24} + S \end{aligned}$$

Therefore,

$$S = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$