On the Sum $\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!}$

Akash Pardeshi

University of Illinois Laboratory High School

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Introduction

In this paper, I will prove the following:

$$\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!} = \frac{i(i+1)...(i+a)(i+a+1)}{(a+2)!}.$$

Prepatory Proofs

Before we can prove this, we must show a few other things, starting with

$$\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!} = \sum_{n=1}^{1} \frac{n(n+1)...(n+a-1)}{a!} + \sum_{n=1}^{2} \frac{n(n+1)...(n+a-1)}{a!} + ...+ \sum_{n=1}^{i-1} \frac{n(n+1)...(n+a-1)}{a!} + \sum_{n=1}^{i} \frac{n(n+1)...(n+a-1)}{a!}. \quad (1)$$

If we write each of the terms on the RHS in closed form we get

$$\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!} = \frac{1(2)...(1+a-1)(1+a)}{(a+1)!} + \frac{2(3)...(2+a-1)(2+a)}{(a+1)!} + ...+ \frac{(i-1)(i)...(i+a-1)(i+a)}{(a+1)!} + \frac{i(i+1)...(i+a-1)(i+a)}{(a+1)!}.$$

The RHS is the written out expansion of the sum on the LHS so our original claim is true. From this, we can also say that

$$\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!} = \sum_{n=1}^{1} \frac{n(n+1)...(n+a-1)}{a!} + \sum_{n=1}^{2} \frac{n(n+1)...(n+a-1)}{a!} + ...+ \sum_{n=1}^{i-1} \frac{n(n+1)...(n+a-1)}{a!} + \sum_{n=1}^{i} \frac{n(n+1)...(n+a-1)}{a!}$$

$$\sum_{n=1}^{i} \frac{n(n+1)...(n+a)}{(a+1)!} = \sum_{n=1}^{i-1} \frac{n(n+1)...(n+a)}{(a+1)!} + \sum_{n=1}^{i} \frac{n(n+1)...(n+a-1)}{a!}. \quad (2)$$

If we apply the expansion from 1 to each term in the expansion, we get

$$\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)+\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}+\sum_{n=1}^{2}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)+...+$$
$$\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}+\sum_{n=1}^{2}\frac{n(n+1)...(n+a-2)}{(a-1)!}+...+\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)+$$
$$\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}+\sum_{n=1}^{2}\frac{n(n+1)...(n+a-2)}{(a-1)!}+...+\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right).$$

We can factor this expression by realizing that all i terms contain the term,

$$\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}$$

$i-1$ terms contain the term,

$$\sum_{n=1}^{2}\frac{n(n+1)...(n+a-2)}{(a-1)!}$$

and only the final term contains the term

$$\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}.$$

Thus, the factored form is

$$i\sum_{n=1}^{1}\frac{n(n+1)...(n+a-2)}{(a-1)!}+(i-1)\sum_{n=1}^{2}\frac{n(n+1)...(n+a-2)}{(a-1)!}+...+1\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}. \quad (3)$$

If we repeat this process, we would get $1\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}$ terms in the first grouping, 2 in the second, and i in the last. Similarly, there would be $1\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}$ in the second grouping, and $i-1$ in the last. If we carry this process on for all terms, we arrive at the factorization

$$\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}=\left(\sum_{n=1}^{i}n\right)\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right)+\left(\sum_{n=1}^{i-1}n\right)\left(\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right)+...+$$
$$\left(\sum_{n=1}^{1}n\right)\left(\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right). \quad (4)$$

With these tools, we can now prove our theorem.

Proof

We start by writing out our sum

$$\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{1(2)...(a-1)(a)(a+1)}{(a+1)!}+\frac{2(3)...(a)(a+1)(a+2)}{(a+1)!}+...+\frac{i(i+1)...(i+a-1)(i+a)(i+a+1)}{(a+1)!}.$$

We can re-write the last two factors in each term with $i+a$ as such

$$\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{1(2)...(a-1)((i+a)-i)((i+a)-(i-1))}{(a+1)!}+\frac{2(3)...(a)((i+a)-(i-1))((i+a)-(i-2))}{(a+1)!}+...+\frac{i(i+1)...(i+a-2)((i+a)-1)(i+a)}{(a+1)!}.$$

We can re-write this as

$$(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=(a-1)!\frac{1(2)...(a-1)}{(a-1)!}((i+a)-i)((i+a)-(i-1))+$$
$$(a-1)!\frac{2(3)...(a)}{(a-1)!}((i+a)-(i-1))((i+a)-(i-2))+...+$$
$$(a-1)!\frac{i(i+1)...(i+a-2)}{(a-1)!}((i+a)-1)(i+a)$$

Using the statement we are trying to prove to simplify

$$(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=(a-1)!\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)-i)((i+a)-(i-1))+$$
$$(a-1)!\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)-(i-1))((i+a)-(i-2))+...+$$
$$(a-1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)-2)((i+a)-1)+$$
$$(a-1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)-1)(i+a).$$

Multiplying the binomials

$$(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!}=(a-1)!\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)^2-i(i+a)-(i-1)(i+a)+(i-1)i)+$$
$$(a-1)!\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)^2-(i+a)(i-1)-(i-a)(i-2)+(i-2)(i-2))+...+$$
$$(a-1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)^2-2(i+a)-1(i+a)+1(2))+$$
$$(a-1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}((i+a)^2-1(i+a)).$$

Combining the first terms of each grouping

$$(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!} = (a-1)!(i+a)^2\left(\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}+\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}+...+\right.$$

$$\left.\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right)-(a-1)!(i+a)\left(i\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}+(i-1)\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}+...+\right.$$

$$\left.2\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}+1\sum_{n=1}^{i}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right)-(a-1)!(i+a)$$

$$\left((i-1)\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}+(i-2)\sum_{n=1}^{2}\frac{n(n+1)...(n+a-3)}{(a-2)!}+...+\right.$$

$$\left.2\sum_{n=1}^{i-2}\frac{n(n+1)...(n+a-3)}{(a-2)!}+1\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right)+(a-1)!\left((i-1)i\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right.$$

$$\left.+(i-2)(i-1)\sum_{n=1}^{i-2}\frac{n(n+1)...(n+a-3)}{(a-2)!}+...+1(2)\sum_{n=1}^{1}\frac{n(n+1)...(n+a-3)}{(a-2)!}\right).$$

By our previous observations in equations 1, 3, and 4, we can say

$$(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a)}{(a+1)!} = (a-1)!(i+a)^2\left(\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)-(a-1)!(i+a)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-1)}{a!}$$

$$-(a-1)!(i+a)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+(a-1)!\cdot 2!\left(\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}\right).$$

By equation 2, we can rewrite the equation as

$$(a+1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}+(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-1)}{a!} = (a-1)!(i+a)^2\left(\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)$$

$$-(a-1)!(i+a)\left(\sum_{n=1}^{i}\frac{n(n+1)...(n+a-1)}{a!}+\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}\right)+(a-1)!\cdot 2!\left(\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}\right).$$

Further expanding the equation

$$(a+1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}+(a+1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!} =$$

$$(a-1)!(i+a)^2\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-(a-1)!(i+a)\left(\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+\right.$$

$$\left.\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}+\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}\right)+(a-1)!\cdot 2!\left(\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}\right).$$

Combining like terms

$$\Big((a+1)!-2!(a-1)!\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}+(a+1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}=$$
$$(a-1)!(i+a)^2\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-(a-1)!(i+a)\left(2\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}\right)$$

$$\Big((a+1)!-2!(a-1)!\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}+(a+1)!\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}+(a+1)!\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}=$$
$$(a-1)!(i+a)^2\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-(a-1)!2(i+a)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}-(a-1)!(i+a)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}\Bigg)$$

Factoring

$$\Big((a+1)!-2!(a-1)!\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big((a-1)!(i+a)^2-(a-1)!(i+a)-(a+1)!\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big((a-1)!2(i+a)+(a+1)!\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}.$$

Factoring an $(a-1)!$ out of all terms

$$(a-1)!(a(a+1)-2!)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$(a-1)!\Big((i+a)^2-(i+a)-a(a+1)\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-(a-1)!\Big(2(i+a)+a(a+1)\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

Dividing by $(a-1)!$

$$(a(a+1)-2!)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big((i+a)^2-(i+a)-a(a+1)\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big(2(i+a)+a(a+1)\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

$$(a^2+a-2)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big(i^2+2ai+a^2-i-a-a^2-a\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big(2i+2a+a^2+a\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big(i^2+2ai-i-2a\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big(2i+3a+a^2\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}.$$

Factoring

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big(i(i-1)+2a(i-1)\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big(2i-2+3a+a^2+2\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$\Big((i-1)(i+2a)\Big)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-\Big(2(i-1)+(a+1)(a+2)\Big)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

Splitting the last term

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=$$
$$(i-1)(i+2a)\sum_{n=1}^{i}\frac{n(n+1)...(n+a-2)}{(a-1)!}-(a+1)(a+2)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}-2(i-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a-1)}{a!}$$

Writing each term in its closed form

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=(i-1)(i+2a)\frac{i(i+1)...(i+a-2)(i+a-1)}{a!}$$
$$-(a+1)(a+2)\frac{(i-1)i...(i+a-2)(i+a-1)}{(a+1)!}-2(i-1)\frac{(i-1)i...(i+a-2)(i+a-1)}{(a+1)!}$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}\Big((i+2a)(a+1)-(a+1)(a+2)-2(i-1)\Big)$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}\Big(ai+2a^2+i+2a-a^2-3a-2-2i+2\Big)$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}\Big(ai+a^2-a-i\Big)$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}\Big(a(i+a)-(i+a)\Big)$$

$$(a+2)(a-1)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}(a-1)(i+a)$$

$$(a+2)\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)}{(a+1)!}(i+a)$$

$$\boxed{\sum_{n=1}^{i-1}\frac{n(n+1)...(n+a)}{(a+1)!}=\frac{(i-1)i...(i+a-1)(i+a)}{(a+2)!}.}$$