Q2 $\phi(y, w_k^T x + b_k, \sigma_k^2) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{\sigma_k^2}} \times$
$\exp\left(-\left\{\frac{(y_i - w_k^T x_i - b_k)^2}{2\sigma_k^2}\right\}\right)$

where $y \in \mathbb{R}$ & $w_k^T x + b_k \in \mathbb{R}$.

Let $w_k^T x + b_k = \mu_k$.

$\therefore \phi(y, \mu_k, \sigma_k^2) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma_k^2} \times \exp\left(\frac{-(y-\mu_k)^2}{2\sigma_k^2}\right)$

$\therefore f(y \mid x; \theta) = \sum_{k=1}^{K} \epsilon_k \phi(y, \mu_k, \sigma_k^2)$

& $\epsilon_k \geqslant 0$ & $\sum \epsilon_k = 1$

Let $S \in \{1, \ldots K\}$ be a discrete RV such that

$P_r\{S = k\} = \epsilon_k$

$\therefore P_r(y \in A) = \sum_{k=1}^{K} \left(\int_A \phi(y; \mu_k, \sigma_k^2) \times \epsilon_k \, dy \right.$

Let $\underline{y} = (y_1, \ldots y_n)$ $\therefore$ the likelihood is
& $\underline{x} = (x_1, \ldots x_n)$

~~$L(\theta; x)$~~ $\prod_{i=1}^{n} f(y_i \mid x_i; \theta)$

$L(\theta, \underline{y} \mid \underline{x}) = \prod_{i=1}^{n} \sum_{k=1}^{K} \epsilon_k \phi(y_i; \mu_k; \sigma_k^2)$

1

Scanned by CamScanner

∴ log likelihood is.

$$l(\theta|\underline{x}) = \sum_{i=1}^{n} \log\left(\sum_{k=1}^{K} \epsilon_k \, \phi(y_i ; \mu_k, \sigma_k^2)\right)$$

Let $\underline{s} = (s_1 \ldots s_n)$

& $\underline{z} = (\underline{y}, \underline{x}, \underline{s})$. Also, let

$$\Delta_{ik} = \begin{cases} 1 & \text{if } s_i = k \\ 0 & \text{if } s_i \neq k \end{cases}.$$

∴ The complete data log likelihood is,

$$\log l(\theta ; \underline{y} \mid \underline{x}, \underline{s}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \Delta_{ik} \left[\log \epsilon_k + \log \phi(y_i ; \mu_k, \sigma_k^2)\right]$$

Now, looking at E-step, we get.

$$Q(\theta, \theta^{(j)}) = E_{\underline{s}|\underline{y},\underline{x}}\left[l(\theta ; \underline{y} \mid \underline{x}, \underline{s}) \mid \underline{y} \mid \underline{x} ; \theta^{(j)} \right]$$

$$\therefore Q(\theta, \theta^{(j)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma_{i,k}^{(j)} \left[\log \epsilon_k + \log \phi(y_i, \mu_k, \sigma_k^2)\right]$$

$$\& \; \gamma_{i,k}^{(j)} = \frac{\epsilon_k^{(j)} \cdot \phi(y_i, \mu_k^{(j)}, \sigma_k^{(j)})}{\sum_{l=1}^{K} \epsilon_l^{(j)} \, \phi(y_i, \mu_l^{(j)}, \sigma_k^{(j)})}$$

$\therefore$ In terms of $\theta, \theta^{(j)}$ & the data for $Q(\theta, \theta^{(j)})$ ...

$$\log l(\theta; \underline{y} \mid \underline{x}, \underline{s}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \Delta_{i,k} \Big[\log \epsilon_K + \log \phi(y_i, w_k^T x_i + b_k, \sigma_k^2)\Big]$$

& $$Q(\theta, \theta^{(j)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma_{(i,k)}^{(j)} \Big[\log \epsilon_K + \log \phi(y_i, w_k^T x_i + b_k, \sigma_k^2)\Big]$$

& $$\gamma_{(i,k)}^{(j)} = \frac{\epsilon_k^{(j)} \phi(y_i, w_k^T x_i + b_k, \sigma_k^2)}{\sum_{l=1}^{K} \epsilon_l^{(j)} \phi(y_i, w_k^T x_i + b_k, \sigma_l^2)}$$

M step:

$$\theta^{(j+1)} = \arg\max_\theta Q(\theta, \theta^{(j)})$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma_{i,k}^{(j)} \Big[\log \phi(y_i, w_k^T x_i + b_k, \sigma_k^2) + \log \epsilon_K\Big]$$

$$\log \phi(y_i, w_k^T x_i + b_k, \sigma_k^2) =$$

$$\log\left(\frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{\sigma_k^2}} \times \exp\left(-\left\{\frac{(y_i - w_k^T x_i - b_k)^2}{2\sigma_k^2}\right\}\right)\right)$$

$$= \underbrace{\log\left(\frac{1}{\sqrt{2\pi}}\right)}_{C} - \frac{1}{2}\log(\sigma_k^2) - \left\{\frac{(y_i - \omega_k^T x_i - b_k)^2}{2\sigma_k^2}\right\}$$

$$= \frac{-1}{2}\log(\sigma_k^2) - \frac{(y_i - \omega_k^T x_i - b_k)^2}{2\sigma_k^2}$$

$$\therefore Q(\theta, \theta^{(j)}) = \sum_{i=1}^{n}\sum_{k=1}^{K} \gamma_{i,k}^{(j)}\left[\log \epsilon_k - \frac{1}{2}\log(\sigma_k^2) - \frac{(y_i - \omega_k^T x_i - b_k)^2}{2\sigma_k^2}\right]$$

$$\cancel{\therefore \frac{\partial Q(\theta, \theta^{(j)})}{\partial b_k} = \sum_{i=1}^{n} \gamma_{i,k}^{(j)} \times \frac{1}{\sigma_k^2} \times (y_i - \omega_k^T x_i - b_k) = 0}$$

$$\cancel{\text{or } \sum y_i - \omega_k^T \sum x_i = b_k.}$$

$$\cancel{\text{or } b_k = \bar{y} - \omega_k^T \bar{x}}$$

$$\cancel{\frac{\partial Q(\theta, \theta^{(j)})}{\partial \omega_k}} =$$

$$\frac{\partial Q(\theta, \theta^{(j)})}{\partial b_k} = \sum_{i=1}^{n} \gamma_{i,k}^{(j)}\left[\frac{1}{\sigma_k^2} \times (y_i - \omega_k^T x_i - b_k)\right] = 0$$

$$\Rightarrow b_k^{(j+1)} = \frac{\sum \gamma_{i,k}^{(j)} y_i - \omega_k^{T(j+1)} \sum \gamma_{i,k}^{(j)} x_i}{\sum \gamma_{i,k}^{(j)}}$$

$$\bar{x}_{\gamma^{(j)}} = \frac{\sum \gamma_{i,k}^{(j)} x_i}{\sum \gamma_{i,k}^{(j)}}$$

$$\bar{y}_\gamma = \frac{\sum \gamma_{i,k}^{(j)} y_i}{\sum \gamma_{i,k}^{(j)}}$$

Similarly,

$$\frac{\partial Q(\theta,\theta^{(j)})}{\partial \omega_k} = \frac{\partial}{\partial \omega_k}\left(\sum_{i=1}^{n} \frac{\gamma_{i,k}^{(j)}}{2\sigma_k^2}\left[-\left(y_i - \omega_k^T x_i - \left(\frac{\bar{y}_\gamma - \omega_k^T \bar{x}_\gamma}{\gamma_{i,k}^{(j)}}\right)\right)^2\right.\right.$$

$$= \frac{\partial}{\partial \omega_k}\left(\sum_{i=1}^{n} -\frac{\gamma_{i,k}^{(j)}}{2\sigma_k^2}\left[\frac{\gamma_{i,k}^{(j)} y_i - \bar{y}_\gamma - \omega_k^T(\gamma_{i,k}^{(j)} x_i - \bar{x}_\gamma)}{\gamma_{i,k}^{(j)}}\right]^2\right)$$

$$\cancel{= + \frac{\gamma_{i,k}^{(j)}}{\sigma_k^2}\left(y_i - \omega_k^T x_i - \frac{\bar{y}_\gamma - \omega_k^T \bar{x}_\gamma}{\gamma_{i,k}^{(j)}}\right) \times (\omega}$$

$$\cancel{= \frac{\partial}{\partial \omega_k}\left(\sum_{i=1}^{n} - \frac{\gamma_{i,k}^{(j)}}{2\sigma_k^2 \gamma_{i,k}^{(j)2}}\left[\gamma_i^{(j)}\right.\right.}$$

Let $(\gamma_{i,k}^{(j)} y_i - \bar{y}_\gamma)/\sqrt{\gamma_{i,k}} = y_{\alpha i}$

& $(\gamma_{i,k}^{(j)} x_i - \bar{x}_\gamma)/\sqrt{\gamma_{i,k}} = x_{\alpha i}$

$$\cancel{= \frac{\partial}{\partial \omega_k}\left(\sum_{i=1}^{n} - \frac{1}{2\sigma_k^2 \gamma_{i,k}^{(j)}}\left[(y_{\alpha i} - \omega_k^T x_{\alpha i})^2\right]\right.}$$

$$= \cancel{(y_{\alpha i} - \omega_k^T x_{\alpha i})}$$

$$= \frac{\partial}{\partial \omega_k}\sum_{i=1}^{n}\left[(y_{\alpha i} - \omega_k^T x_{\alpha i})^2\right]$$

$$\Rightarrow \omega_k = (X_\alpha^T X_\alpha)^{-1} X_\alpha^T Y_\alpha$$

where $X_\alpha = \begin{bmatrix} x_{\alpha 1} \\ \vdots \\ x_{\alpha n} \end{bmatrix}$ $\quad Y_\alpha = \begin{bmatrix} y_{\alpha 1} \\ \vdots \\ y_{\alpha n} \end{bmatrix}$

$$\frac{\partial Q(\theta, \theta^{(j)})}{\partial \sigma_k^2} = \sum_{i=1}^{n}\left(-\frac{1}{2} \times \frac{1}{\sigma_k^2} + \frac{(y_i - \omega_k^T x_i - b_k)^2}{2(\sigma_k^2)^2}\right)\gamma_{i,k}^{(j)} = 0$$

$$\text{or } -\frac{n}{\sigma_k^2} + \frac{\sum \gamma_{i,k}^{(j)} (y_i - \omega_k^T x_i - b_k)^2}{(\sigma_k^2)^2} = 0$$

If $\sigma_k^2 \neq 0$,

$$\Rightarrow \sigma_k^2 \times n = \sum \gamma_{i,k}^{(j)} (y_i - \omega_k^T x_i - b_k)^2$$

$$\text{or } \sigma_k^2 = \frac{1}{n} \sum \gamma_{i,k}^{(j)} (y_i - \omega_k^{T(j+1)} x_i - b_k^{(j+1)})^2$$

Now, for ϵ_k

Let

$$J(\theta, \theta^{(j)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma_{i,k}^{(j)} \log(\epsilon_k) + \beta\left(\sum_{i=1}^{K} \epsilon_k - 1\right)$$

$$\frac{\partial J}{\partial \epsilon_k} = \sum_{i=1}^{n} \frac{\gamma_{i,k}^{(j)}}{\epsilon_k} + \beta = 0$$

$$\therefore \epsilon_k = \frac{\sum \gamma_{i,k}^{(j)}}{-\beta}$$

Now, using $\sum \epsilon_k = 1$, we get

$$-\beta = \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma_{i,k}^{(j)}$$

$$-\beta = n$$

$$\therefore \epsilon_k = \frac{\sum_{i=1}^{n} \gamma_{i,k}^{(j)}}{n}$$

Solving for b_k & ω_k

Let $x_i = [1 \quad x_i]$

$\beta_k = [b_k \quad \omega_k]$

$\therefore$

$$\therefore \frac{\partial Q(\theta, \theta^{(j)})}{\partial \beta_k} = \frac{\partial}{\partial \beta_k}\left(\sum_{i=1}^{k} \frac{\gamma_{i,k}^{(j)}}{\sigma_k^2} (y_i - \beta_k^T x_i)^2\right)$$

$$= \frac{\partial}{\partial \beta_k} \sum_{i=1}^{n} \gamma_{i,k}^{(j)} (y_i - \beta_k^T x_i)^2$$

$$\therefore \beta_k = (X^T W_k^* X)^{-1} X^T W_k^* y$$

where $W_k^* = \begin{bmatrix} \gamma_{1,k}^{(j)} & & & \\ & \gamma_{2,k}^{(j)} & & \\ & & \ddots & \\ & & & \gamma_{n,k}^{(j)} \end{bmatrix}$

d) The # iteration = 10

b = [-0.0192 0.3160]

ω = [0.5328 -1.3234]

V = [3.7 × 10⁻¹¹ 0.3468]

e = [3.2×10⁻⁵ 1]

ω^*

Q1 a) # iterations = 225

The portions which are present at a lot of pixels in the ~~of~~ image & where there is not sudden change in intensity.

~~Comp~~ Compression ratio = 0.0707

R M Abs. error = 0.0232

b) bpp = $\dfrac{k \times m^2 \times 24 + \log_2 k \times N^2/m^2}{N^2}$

Compression ratio = $\dfrac{k \times m^2 \times 24 + \log_2 k \times N^2/m^2}{N^2 \times 24}$

=

Scanned by CamScanner