Problem 5

This question requests an algorithm to find the best/optimal rigid body transformation between two point sets, $\mathcal{P} = \{p_1, p_2, \dots p_n\}$ and $\mathcal{Q} = \{q_1, q_2, \dots q_n\}$ which can be formalized as[1][2]:

$$F = (\mathbf{R}, \mathbf{t}) = \underset{R,t}{\operatorname{argmin}} \sum_{i=1}^{n} ||(Rp_i + t) - q_i||^2 \tag{1}$$

We can find the optimal translation between the two points by taking the derivative w.r.t t, equating the derivative to 0, and finding t.

$$\begin{aligned}
\frac{\partial F}{\partial t} &= \sum_{i=1}^{n} 2((Rp_i + t) - q_i) \\
&= 2R\sum_{i=1}^{n} p_i + 2t \cdot n + 2\sum_{i=1}^{n} q_i \\
&\implies t = \frac{1}{n}\sum_{i=1}^{n} q_i - R\frac{1}{n}\sum_{i=1}^{n} p_i \\
t &= \overline{q} - R \cdot \overline{p}
\end{aligned}$$

where $\overline{p} = \frac{1}{n}\sum_{i=1}^{n} p_i$ and $\overline{q} = \frac{1}{n}\sum_{i=1}^{n} q_i$

Replacing the value of t in equation 1, we now obtain the minimization as:

$$\begin{aligned}
R &= \underset{R}{\operatorname{argmin}} \sum_{i=1}^{n} ||R(p_i - \overline{p}) - (q_i - \overline{q})||^2 \\
&= \underset{R}{\operatorname{argmin}} \sum_{i=1}^{n} ||Rx_i - y_i||^2
\end{aligned} \tag{2}$$

Now, we aim to find the optimal rotation, we can show from [1] after simplifying the terms that minimizing R can be shown as maximizing the trace of the diagonal matrix $tr(Y^\top RX)$ We compute the SVD decomposition of the covariance matrix $\mathbf{XY}^\top$ to obtain the orthogonal decomposition, i.e.,

$$\mathbf{XY}^\top = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^\top \tag{3}$$

We know that for matrices, $tr(AB) = tr(BA)$, therefore we can write

$$tr(\mathbf{Y}^\top\mathbf{RX}) = tr(\mathbf{RXY}^\top)$$

We need to now maximize $tr(\mathbf{RXY}^\top) = tr(\mathbf{RS}) = tr(\mathbf{RU}\cdot\mathbf{\Sigma}\cdot\mathbf{V}^\top) = tr(\mathbf{\Sigma V}^\top\mathbf{RU})$. Note that $\mathbf{V}^\top, \mathbf{R}, \mathbf{U}$ are orthogonal matrices, which implies that the columns of each of these matrices are required to be orthonormal. Therefore, $\mathbf{M} = \mathbf{V}^\top\mathbf{RU}$ also has orthonormal matrices, further now since we need to find argmax($tr(\mathbf{\Sigma M})$), we note that, we require M to be an identity matrix.

Therefore, we can conclude that

$$\mathbf{M} = \mathbf{I} = \mathbf{V}^\top\mathbf{RU} \implies \mathbf{R} = \mathbf{V}\cdot\mathbf{U}^\top \tag{4}$$

It may be possible that the $det(R) = -1$ which means that the Rotation could also contain reflections. Any reflection matrix can be constructed by inverting the sign of a row of the rotation matrix [1], it follows that we are optimizing for the convex hull of the points $(\pm 1, \ldots, \pm 1)$, with an odd number of -1's. Expressing this as a linear function we have:

$$tr(\Sigma M) = \sigma_1 m_{11} + \sigma_2 m_{12} + ... + \sigma_d m_{dd} \tag{5}$$

$$tr(\Sigma M) = \sigma_1 + \sigma_2 + \sigma_3 + \cdots - \sigma_d \tag{6}$$

This linear function attains its maxima at its vertices, since our domain is a convex polyhedron. Therefore we have 6. Finally we have a formula for $\mathbf{R}$ with reflection as follows:

$$\mathbf{R} = \mathbf{V}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}\mathbf{U}^\top \tag{7}$$

and now the general formula for $\mathbf{R}$ is:

$$\mathbf{R} = \mathbf{V}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & det(VU^\top) \end{pmatrix}\mathbf{U}^\top \tag{8}$$

The algorithm is given as below

- Compute the centroid of the point sets through the formula as below

$$\overline{p} = \frac{1}{N}\sum_{i=1}^{N} p_i, \qquad \overline{q} = \frac{1}{N}\sum_{i=1}^{N} q_i$$

- Compute the centered vectors as:

$$x_i = p_i - \overline{p}, \qquad y_i = q_i - \overline{q}$$

- Compute the co-variance matrix between $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$ as
$$\mathbf{H} = \mathbf{x}\mathbf{y}^\top \quad \text{where H forms a 3x3 matrix}$$

- Compute the SVD of $\mathbf{H}$ as $\mathbf{U\Sigma V}^\top$, now the required optimal rotation in the least squared sense for the point set is
$$\mathbf{R} = \mathbf{V}\mathbf{U}^\top$$

- Finally compute the optimal translation for the rigid body as:
$$\mathbf{t} = -\mathbf{R}\overline{\mathbf{p}} + \overline{\mathbf{q}}$$

Code

The solution to this problem is implemented in `q5.py`. The python code generates a random 3d point set, and transforms it with a known random $\mathbf{R}$ and $\mathbf{t}$ to generate Q. These two point sets are then used as the data for the algorithm. Example 3d point sets, transformed by R and t and the estimated R and t in Figure 1

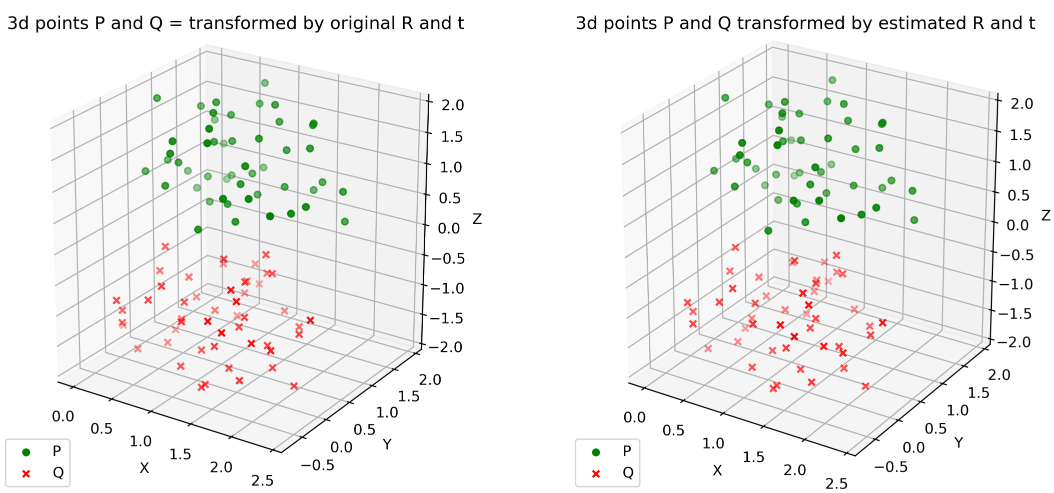

Figure 1: 3D point set P and transformed point set Q ([left] transformed by original R and t, and [right] estimated R and t)

Thought

The above problem can also be thought of in a different way:

We know that SVD decomposes the matrices as orthogonal matrices. In the problem given we are trying to find the least squares closest orthogonal matrix $\mathbf{R}$ which approximates $\mathbf{A}$ an over-constrained system of point transformations from $\mathcal{P}$ to $\mathcal{Q}$.

Intuitively to obtain a rigid body transformation we require that there is no shearing of the object in any of the axes/bases, which effectively amounts to $\Sigma = I$, therefore, we could compute as $\mathbf{R} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^\top = \mathbf{U} \cdot \mathbf{I} \cdot \mathbf{V}^\top$.

Further looking at the previous proof, it seems that this solution gives us the inverse transform from Q to P!

Discussion citation

For this problem, I have discussed with *Tarasha Khurana* andrewID: `tkhurana@andrew.cmu.edu`, and *Rohit Jena* andrewID: `rjena@andrew.cmu.edu`

References

[1] Sorkine-Hornung, Olga and Rabinovich, Michael, *Least-squares rigid motion using svd* Computing, 1, 2017

[2] Arun, K. Somani, Thomas S. Huang, and Steven D. Blostein. *Least-squares fitting of two 3-D point sets* IEEE Transactions on pattern analysis and machine intelligence 5 (1987): 698-700.