Problem 1

For this problem we are trying to minimize the surface area of revolution , which can be given as follows:

$$\mathcal{L} = \int_{x_0}^{x_1} 2\pi y\sqrt{1+(y')^2}dx$$
$$\mathcal{L} = 2\pi\int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx$$
$$\mathcal{L} = \int_{x_0}^{x_1} F(x,y,y')dx$$

Here the integrand of the above equation is the functional that needs to be minimized, therefore we have from the Euler-Lagrange equation, the following:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Now we calculate the following:

$$\frac{\partial F}{\partial y} = \sqrt{1+(y')^2}$$
$$\frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{1+(y')^2}}$$

Here we observe that the $\frac{\partial F}{\partial x} = 0$, therefore we use the *Beltrami Identity* [1]. We observe that:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$
$$y'\frac{\partial F}{\partial y} = \frac{dF}{dx} - \left(\frac{\partial F}{\partial x} + y''\frac{\partial F}{\partial y'}\right) \tag{1}$$

Further by multiplying both sides of the *Euler Lagrange* equation with y' we have the equation:

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{2}$$

Substituting equation 2 in equation 1 we have:

$$y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{dF}{dx} - \left(\frac{\partial F}{\partial x} + y''\frac{\partial F}{\partial y'}\right)$$
$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \left(y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = 0$$

The terms inside the bracket can be simplified by observing that it is an integration by parts:

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = 0$$
$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

We observe that $\frac{\partial F}{\partial x} = 0$ since there is no dependence of F on x. Now on integrating the above equation we obtain:

$$F - y'\frac{\partial F}{\partial y'} = c \tag{3}$$

where c is a constant of integration. The equation 3 is also called the *Beltrami identity* and is a special case of *Euler Lagrange* equation. Using the above in our problem we now have:

$$\begin{aligned}
y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} &= c \\
y(1+(y')^2) - y(y')^2 &= c\sqrt{1+(y')^2} \\
y &= c\sqrt{1+(y')^2} \\
(y')^2 &= \frac{(y^2-c^2)}{c^2} \\
\implies (\frac{dx}{dy})^2 &= \frac{c^2}{(y^2-c^2)} \\
\implies x &= \int \frac{c}{\sqrt{y^2-c^2}}dy
\end{aligned}$$

Now we get:

$$x = c\log(\sqrt{y^2-c^2} + y) + b$$

On simplifying the equation further we obtain that

$$y = c\cosh\left(\frac{x-b}{c}\right)$$

From the endpoint conditions we can obtain the values for b and c integration constants:

$$y_0 = c\cosh\left(\frac{x_0-b}{c}\right)$$
$$y_1 = c\cosh\left(\frac{x_1-b}{c}\right)$$

This essentially forms a *Catenoid* when the curve is C^2, but when this condition is not required, the equation can be as follows:

$$y = \begin{cases} y_0, & x_0 \\ 0, & x_0 < x < x_1 \\ y_1, & x_1 \end{cases}$$

This surface is effectively two circular rings at the end points, as shown below:

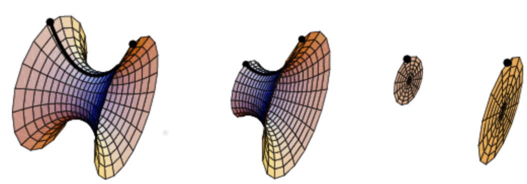

Figure 1: Image of catenoid: Right-most image, shows the end condition when curve is not C^2, (source: Wolfram Alpha)

When the endpoints $(x_0, y_0), (x_1, y_1)$ are widely spaced, then the curve tends towards not being C^2 and vice-versa.

Problem 2

We know that to find the shortest curve between two points A and B on the surface of a sphere, we need to minimize the surface integral given as below:

$$\mathcal{L} = \int_A^B dS$$

Here since the surface is that of a sphere in cartesian coordinates we have for dS:

$$dS = \sqrt{dx^2 + dy^2 + dz^2}$$

Converting the above into spherical coordinates where $x = \rho\cos(\theta)\cos(\phi)$, $y = \rho\sin(\theta)sin(\phi)$, $z = \rho\cos(\phi)$ we have the following:

$$\begin{aligned} dx &= \frac{\partial x}{\partial R}dR + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi &&= 0 + R\sin(\theta)\sin(\phi)d\theta + R\cos(\theta)sin(\phi)d\phi \\ dy &= \frac{\partial y}{\partial R}dR + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi &&= 0 + R\cos(\theta)\sin(\phi)d\theta + R\sin(\theta)\cos(\phi)d\phi \\ dz &= \frac{\partial z}{\partial R}dR + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi &&= -R\sin(\phi)d\phi \end{aligned}$$

Where R is the radius of the sphere. Finally after simplification of the surface integral we obtain the following:

$$dS = R\sqrt{(\sin^2\phi\left(\frac{d\theta}{d\phi}\right)^2 + 1)}d\phi$$
$$\mathcal{L} = \int_{\phi_A}^{\phi_B} dS = \int_{\phi_A}^{\phi_B} F d\phi \tag{4}$$

Here we represent $\theta' = \frac{d\theta}{d\phi}$, then to minimise the above functional we apply the *Euler-Lagrange* equation which states:

$$\frac{\partial F}{\partial \theta} - \frac{d}{d\phi}\left(\frac{\partial F}{\partial \theta'}\right) = 0$$

Since the first term in the *Euler Lagrange* equation does not have any dependency to θ, we have:

$$0 - \frac{d}{d\phi}\left(\frac{\partial F}{\partial \theta'}\right) = 0$$

Now integrating the equation w.r.t ϕ

$$\implies \frac{\partial F}{\partial \theta'} = k \qquad \text{where k is an integration constant} \tag{5}$$

We will now solve the partial differential on the LHS of equation 5 by substituting from equation 4, as follows:

$$\frac{\theta' \sin^2(\phi)}{\sqrt{\sin^2(\phi)(\theta')^2 + 1}} = k$$

On squaring and cross-multiplying on both sides

$$\implies k^2 + k^2(\sin^2(\phi))(\theta')^2 = (\theta')^2 \sin^4(\phi)$$
$$\implies \theta' = \frac{k}{\sin(\phi)\sqrt{\sin^2(\phi) - k^2}}$$

Now on integrating on both sides we obtain:

$$\theta = \int \frac{k}{\sin(\phi)\sqrt{\sin^2(\phi) - k^2}} d\phi$$

From the given integral in the handout, we have:

$$\theta = -\arcsin\left(\frac{\cot(\phi)}{\sqrt{\frac{1}{k^2} + 1}}\right) + \theta_0$$
$$\theta = -\arcsin\left(w\cot(\phi)\right) + \theta_0 \tag{6}$$

where θ_0 and $w = \frac{1}{\sqrt{\frac{1}{k^2}+1}}$ are integration constants which can be found from the initial conditions. Now we need to show that equation 6 is the equation of an arc of the great circle of the sphere.

We know that the equation of the plane passing through the origin (center of the sphere) is given as:

$$Ax + By + Cz = 0$$

Since the equation of the great circle is the intersection of the above plane and the surface of the sphere, we can substitute the spherical coordinate values for x, y and z respectively to obtain the following:

$$\begin{aligned} AR\cos(\theta)\cos(\phi) + BR\sin(\theta)\sin(\phi) + CR\cos(\phi) &= 0 \\ A\cos(\theta)\cos(\phi) + B\sin(\theta)\sin(\phi) + C\cos(\phi) &= 0 \\ A\cos(\theta) + B\sin(\theta) &= -C\cot(\phi) \\ \sqrt{A^2+B^2}\sin(\theta+\theta_0) &= -C\cot(\phi) \qquad (7) \\ & \qquad (8) \end{aligned}$$

Since equation 6 and 7 have the same form, we observe that the shortest curve between two points on a sphere is indeed an arc of the great circle

Problem 3

We are given the equation from lecture that:

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0$$

Also we know that for the brachistochrone problem we have:

$$F = \sqrt{\frac{1 + (y')^2}{y_0 - y}}$$
$$F_{y'} = \frac{y'}{\sqrt{(y_0 - y)(1 + (y')^2)}}$$

Substituting the above two equations in the given equation from lecture we get:

$$y' - \frac{g_y}{g_x + g_y y'}(1 + (y')^2) = 0$$
$$g_x y' + g_y (y')^2 = g_y + g_y (y')^2$$
$$y' = \frac{g_y}{g_x}$$

Now if we take the full derivative of $g(x, y) = 0$ w.r.t to the parametric term t, we get:

$$\frac{dg(x,y)}{dt} = g_x \frac{dx}{dt} + g_y \frac{dy}{dt} = 0$$
$$\implies \frac{g_y}{g_x} = -\frac{dx}{dy}$$

This shows that $\frac{g_y}{g_x}$ is perpendicular to the slope of the curve $g(x, y)$ and hence we see that at the endpoint, the optimizing curve $y(x)$ intersects the iso-contour of $g(x, y) = 0$ orthogonally.

Problem 4

(a) For any system we have from Lagrangian dynamics the following:

$$\mathcal{L} = \mathbf{T} - \mathbf{V}$$

Where $\mathcal{L}$ is the lagrangian and $\mathbf{T}$ and $\mathbf{V}$ are the kinetic and potential energy of the system respectively. In the problem that we are given, since the effect of gravity is not considered, the potential energy of the system is 0.

For this problem we will consider the generalized coordinates as θ_1 and θ_2 and the generalized forces as τ_1 and τ_2

The Kinetic energy of the system in cartesian coordinates can be given as:

$$\mathbf{T} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_{21}^2 + \frac{1}{2}m_2v_{22}^2$$

Here $v_1 = l_1\dot{\theta_1}$ in terms of the general coordinates. To obtain the velocities of the other mass m_2 we have the following: For the mass m_2 above m_1 in the diagram:

$$x_1 = l_1\cos(\theta_1) + l_2\cos(\theta_1 + \theta_2)$$
$$y_1 = l_1\sin(\theta_1) + l_2\sin(\theta_1 + \theta_2)$$

Similarly for the mass m_2 below m_1 in the diagram:

$$x_2 = l_1\cos(\theta_1) - l_2\cos(\theta_1 + \theta_2)$$
$$y_2 = l_1\sin(\theta_1) - l_2\sin(\theta_1 + \theta_2)$$

We differentiate the above two positions to obtain the velocity components in the cartesian coordinates as:

$$\dot{x_1} = -l_1\sin(\theta_1)\dot{\theta_1} - l_2\sin(\theta_1 + \theta_2)(\dot{\theta_1} + \dot{\theta_2})$$
$$\dot{y_1} = l_1\cos(\theta_1)\dot{\theta_1} + l_2\cos(\theta_1 + \theta_2)(\dot{\theta_1} + \dot{\theta_2})$$

Similarly for second mass m_2 we have:

$$\dot{x_2} = -l_1\sin(\theta_1)\dot{\theta_1} + l_2\sin(\theta_1 + \theta_2)(\dot{\theta_1} + \dot{\theta_2})$$
$$\dot{y_2} = l_1\cos(\theta_1)\dot{\theta_1} - l_2\cos(\theta_1 + \theta_2)(\dot{\theta_1} + \dot{\theta_2})$$

Now we have the velocities as:

$$v_{21}^2 = \dot{x_1}^2 + \dot{y_1}^2 = l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1} + \dot{\theta_2})^2 + 2l_1l_2\dot{\theta_1}(\dot{\theta_1} + \dot{\theta_2})\cos(\theta_2)$$
$$v_{22}^2 = \dot{x_2}^2 + \dot{y_2}^2 = l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1} + \dot{\theta_2})^2 - 2l_1l_2\dot{\theta_1}(\dot{\theta_1} + \dot{\theta_2})\cos(\theta_2)$$

Now we have the Lagrangian for the system as follows:

$$\mathcal{L} = \mathbf{T} = (\frac{1}{2}m_1 + m_2)l_1^2\dot{\theta_1}^2 + m_2l_2^2(\dot{\theta_1} + \dot{\theta_2})^2$$

We have from the Principle of Stationary action that:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) - \tau_1 \tag{9}$$
$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) - \tau_2 \tag{10}$$

Since there is no potential energy term due to the lack of gravity the L.H.S of equations 9 and 10 are 0, since the Lagrangian $\mathcal{L}$ does not have any term explicitly related to θ_1 and θ_2. Now we can write down the other terms:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}} = (m_1 + 2m_2)l_1^2\dot{\theta_1} + 2m_2l_2^2(\dot{\theta_1} + \dot{\theta_2})$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) = (m_1 + 2m_2)l_1^2\ddot{\theta_1} + 2m_2l_2^2(\ddot{\theta_1} + \ddot{\theta_2})$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}} = 2m_2l_2^2(\dot{\theta_1} + \dot{\theta_2})$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) = 2m_2l_2^2(\ddot{\theta_1} + \ddot{\theta_2})$$

Substituting in equation 9 and 10 we have:

$$\tau_1 = (m_1 + 2m_2)l_1^2\ddot{\theta_1} + 2m_2l_2^2(\ddot{\theta_1} + \ddot{\theta_2}) \tag{11}$$
$$\tau_2 = 2m_2l_2^2(\ddot{\theta_1} + \ddot{\theta_2}) \tag{12}$$
$$\tag{13}$$

or writing in matrix form:

$$\begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} = \begin{bmatrix} m_1l_1^2 + 2m_2l_1^2 + 2m_2l_2^2 & 2m_2l_2^2 \\ 2m_2l_2^2 & 2m_2l_2^2 \end{bmatrix} \begin{bmatrix}\ddot{\theta_1} \\ \ddot{\theta_2}\end{bmatrix}$$

(b) From part (a) we have the equation for τ_1:

$$\tau_1 = (m_1 l_1^2 + 2m_2 l_1^2 + 2m_2 l_2^2)\ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_2$$

When we set $\ddot{\theta}_2 = 0$ we have:

$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\ddot{\theta}_1$$

We can understand this equation in the following way. We know that

$$\tau = \frac{d}{dt}(L)$$

where L denotes the angular momentum. In the given problem, the angular momentum of mass m_1 can be given as:

$$L_1 = m_1 \omega l_1^2 = m_1 \dot{\theta}_1 l_1^2$$

Similarly the angular momentum L_2 for mass m_2 can be given as:

$$L_2 = m_2 \omega r_1^2 + m_2 \omega r_2^2$$

where r_1 and r_2 are the distances of the masses from the ground joint rotating at $\dot{\theta}_1$:

$$r_1^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos(\theta_2)$$
$$r_2^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\theta_2)$$

Now finally we can write τ_1 as follows:

$$\tau_1 = \frac{d}{dt}\left(m_1 \dot{\theta}_1 l_1^2 + m_2 \dot{\theta}_1 2(l_1^2 + l_2^2)\right)$$
$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\ddot{\theta}_2$$

Which is the same equation that we obtained initially.

References

[1] Beltrami Identity in Calculus of Variations
`https://en.wikipedia.org/wiki/Beltrami_identity`