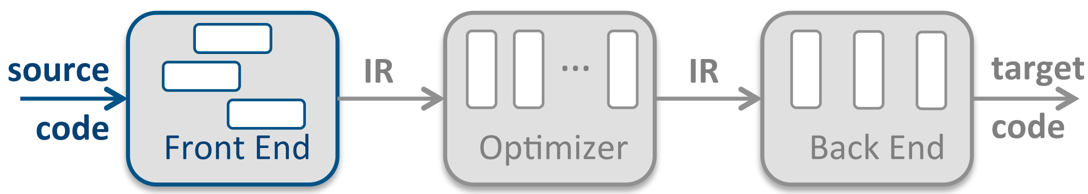
Ignore § 2.4.4 in EaC2e.

Read the replacement section posted on the course web site.

COMP 412
FALL 2017

# Lexical Analysis, III

## Comp 412






**Chapter 2 in EaC2e**

# The Plan for Scanner Construction

**RE → NFA** *(Thompson's construction)* ✔
- Build an **NFA** for each term in the **RE**
- Combine them in patterns that model the operators

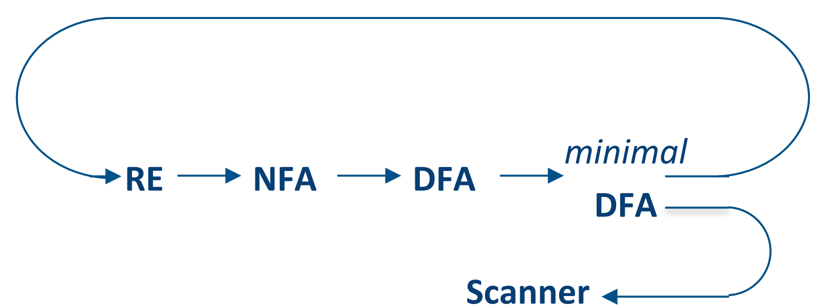
**NFA → DFA** *(Subset construction)* ✔
- Build a **DFA** that simulates the **NFA**

**DFA → Minimal DFA**
- Hopcroft's algorithm
- Brzozowski's algorithm

**Minimal DFA → Scanner**
- See § 2.5 in EaC2e

**DFA → RE**
- All pairs, all paths problem
- Union together paths from $s_0$ to a final state

*The Cycle of Constructions*



# **DFA** Minimization

## The Big Picture

- Discover sets of behaviorally equivalent states in the **DFA**
- Represent each such set with a single new state

Two states $s_i$ and $s_j$ are **behaviorally equivalent** if and only if:

*Recursive definition*

- $\forall\, c \in \Sigma$, transitions from $s_i$ & $s_j$ on $c$ lead to equivalent states
- The set of paths leading from $s_i$ & $s_j$ are equivalent

A **partition** $P$ of a set $S$:

- A collection of subsets of $P$ such that each state $s$ is in exactly one $p_i \in P$
- The algorithm iteratively constructs partitions of the **DFA**'s set of states

We want a partition $P = \{ p_0, p_1, p_2, \ldots p_n \}$ of $D$ that has two properties:

1. If $d_i$ & $d_j \in p_s$ and $c$ takes $d_i \rightarrow d_x$ and $d_j \rightarrow d_y$, then $d_x$ & $d_y \in p_t$, $\forall\, c, i, j, s, t$
2. If $d_i$ & $d_j \in p_s$ and $d_i \in F$ then $d_j \in F$

$D$ is the set of states for the **DFA**: $(D,\Sigma,\delta,s_0,D_A)$

# **DFA** Minimization

**Maximally sized sets ⇒ minimal number of sets**

## Details of the algorithm

- Group states into maximally-sized initial sets, *optimistically* **(property 2)**
- Iteratively subdivide those sets, based on transition graph **(property 1)**
- States that remain grouped together are equivalent

**Initial partition:** $P_0$ has two sets: $\{D_A\}$ & $\{D - D_A\}$ $\qquad D = (D,\Sigma,\delta,s_0,D_A)$

final states — other states

## Property 1 provides the basis for refining, or splitting, the sets

- Assume $s_i$ & $s_j \in p_s$, and $\delta(s_i,\underline{a}) = s_x$, & $\delta(s_j,\underline{a}) = s_y$
- If $s_x$ & $s_y$ are not in the same set $p_t$, then $p_s$ must be split
  - **COROLLARY:** $s_i$ has transition on $\underline{a}$, $s_j$ does not ⇒ $\underline{a}$ splits $p_s$
- A single state in a **DFA** cannot have two transitions on $\underline{a}$
  - Each $p_s$ will become a **DFA** state

# **DFA** Minimization Algorithm (Worklist version)

*Worklist* $\leftarrow \{ D_A , \{ D - D_A \} \}$
*Partition* $\leftarrow \{ D_A , \{ D - D_A \} \}$

*While (Worklist* $\neq \varnothing$*) do*
  *select a set S from Worklist and remove it*
  *for each* $\alpha \in \Sigma$ *do*
    *Image* $\leftarrow \{ x \mid \delta(x, \alpha) \in S \}$
    *for each* $q \in$ *Partition do*
      $p_1 \leftarrow q \cap Image$
      $p_2 \leftarrow q - p_1$
      *if* $p_1 \neq \varnothing$ *and* $p_2 \neq \varnothing$ *then*
        *remove q from Partition*
        *Partition* $\leftarrow$ *Partition* $\cup\, p_1 \cup p_2$
        *if* $q \in$ *Worklist then*
          *remove q from Worklist*
          *Worklist* $\leftarrow$ *Worklist* $\cup\, p_1 \cup p_2$
        *else if* $|p_1| \leq |p_2|$
          *then Worklist* $\leftarrow$ *Worklist* $\cup\, p_1$
          *else Worklist* $\leftarrow$ *Worklist* $\cup\, p_2$

"split q"

adjust Worklist

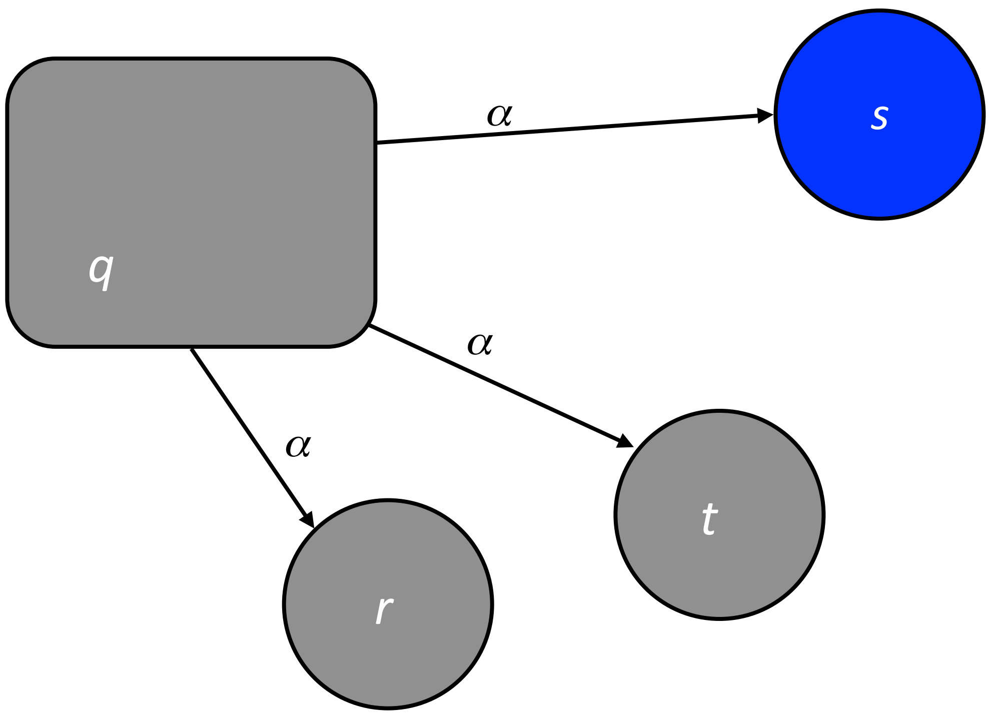
Image is the set of states that have a transition into *S* on $\alpha$: $\delta^{-1}(S,\alpha)$

$p_1$ *is the subset of q that transitions to S on* $\alpha$

$p_2$ *is the rest of q*

# Key Idea: Splitting Q Around Transitions on $\alpha$

## Partitioning Q around S



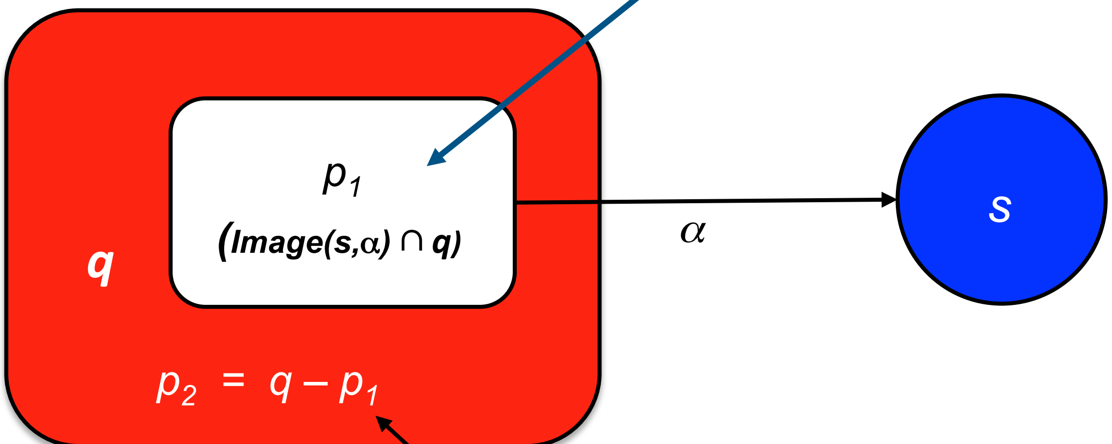

Assume that *q, r, s,* & *t* are sets in the current approximation to the final partition

*q* has transitions on $\alpha$ to *r, s,* & *t,* so it must split around $\alpha$

*As the algorithm considers s and $\alpha$, it will split q.*

# Key Idea: Splitting q around $s$ and $\alpha$

**Find maximal subset of $q$ ($p_1$) that has an $\alpha$-transition into s**




Think of $p_1$ as the image of $s$ into q under the inverse of the transition function:

$p_1 \leftarrow \delta^{-1}(s, \alpha) \cap q$

*$p_2$ must have an $\alpha$-transition to one or more other states in one or more other partitions (e.g., r & s), or states with no $\alpha$-transitions.*

*Otherwise, q does not split!*

# **DFA** Minimization Algorithm (Worklist version)

*Worklist* $\leftarrow \{ D_A, \{D - D_A\}\}$
*Partition* $\leftarrow \{ D_A, \{D - D_A\}\}$

*While (Worklist* $\neq \emptyset$*) do*
  *select a set S from Worklist and remove it*
  *for each* $\alpha \in \Sigma$ *do*
    *Image* $\leftarrow \{ x \mid \delta(x, \alpha) \in S \}$
    *for each* $q \in$ *Partition do*
      $p_1 \leftarrow q \cap Image$
      $p_2 \leftarrow q - p_1$
      *if* $p_1 \neq \emptyset$ *and* $p_2 \neq \emptyset$ *then*
        *remove q from Partition*
        *Partition* $\leftarrow$ *Partition* $\cup\, p_1 \cup p_2$
        *if* $q \in$ *Worklist then*
          *remove q from Worklist*
          *Worklist* $\leftarrow$ *Worklist* $\cup\, p_1 \cup p_2$
        *else if* $|p_1| \leq |p_2|$
          *then Worklist* $\leftarrow$ *Worklist* $\cup\, p_1$
          *else Worklist* $\leftarrow$ *Worklist* $\cup\, p_2$

"split q"

adjust Worklist

Projection is the set of states that have a transition into $S$ on $\alpha$: $\delta^{-1}(S,\alpha)$

$p_1$ *is the subset of q that transitions to S on* $\alpha$

$p_2$ *is the rest of q*

And, as an implementation nit, if we just split $S$ — that is, $S$ was $q$ & it split — we need a new $S$

# **DFA** Minimization Algorithm (Worklist version)

**One last hack …**

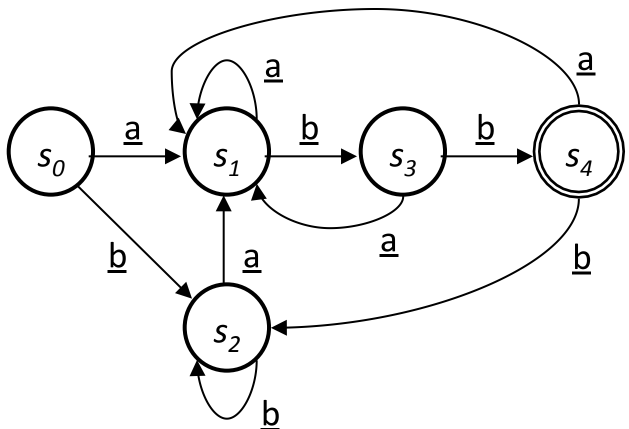
*Worklist* $\leftarrow \{ D_A, \{D - D_A\} \}$
*Partition* $\leftarrow \{ D_A, \{D - D_A\} \}$

*While (Worklist* $\neq \emptyset$*) do*
    *select a set S from Worklist and remove it*
    *for each* $\alpha \in \Sigma$ *do*
        *Image* $\leftarrow \{ x \mid \delta(x, \alpha) \in S \}$
        *for each* $q \in$ *Partition do*
            $p_1 \leftarrow q \cap Image$
            $p_2 \leftarrow q - p_1$
            *if* $p_1 \neq \emptyset$ *and* $p_2 \neq \emptyset$ *then*
                *remove q from Partition*
                *Partition* $\leftarrow$ *Partition* $\cup\, p_1 \cup p_2$
                *if* $q \in$ *Worklist then*
                    *remove q from Worklist*
                    *Worklist* $\leftarrow$ *Worklist* $\cup\, p_1 \cup p_2$
                *else if* $|p_1| \leq |p_2|$
                    *then Worklist* $\leftarrow$ *Worklist* $\cup\, p_1$
                    *else Worklist* $\leftarrow$ *Worklist* $\cup\, p_2$

*If q is a singleton, we can skip the body of the loop because a singleton cannot split.*

# A Detailed Example

### The DFA for ( a | b )* abb

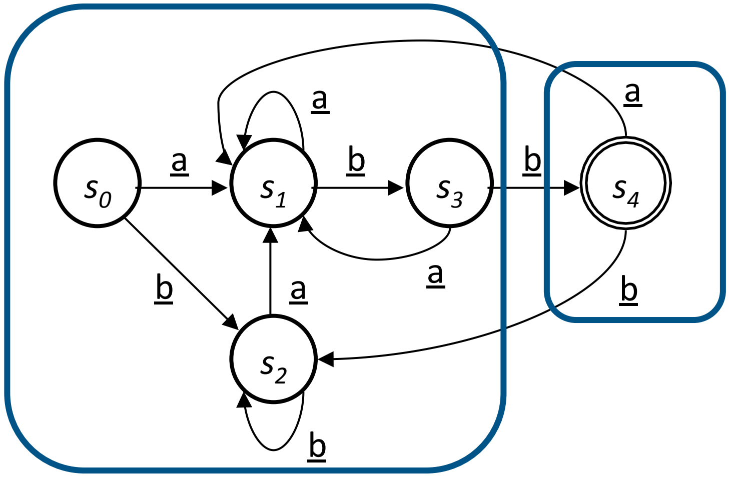



| | Character | |
|---|---|---|
| State | a | b |
| $s_0$ | $s_1$ | $s_2$ |
| $s_1$ | $s_1$ | $s_3$ |
| $s_2$ | $s_1$ | $s_2$ |
| $s_3$ | $s_1$ | $s_4$ |
| $s_4$ | $s_1$ | $s_2$ |

- Deterministic version of **NFA** from last lecture
- Specifically not the minimal **DFA**
- Use same code skeleton as before

# A Detailed Example

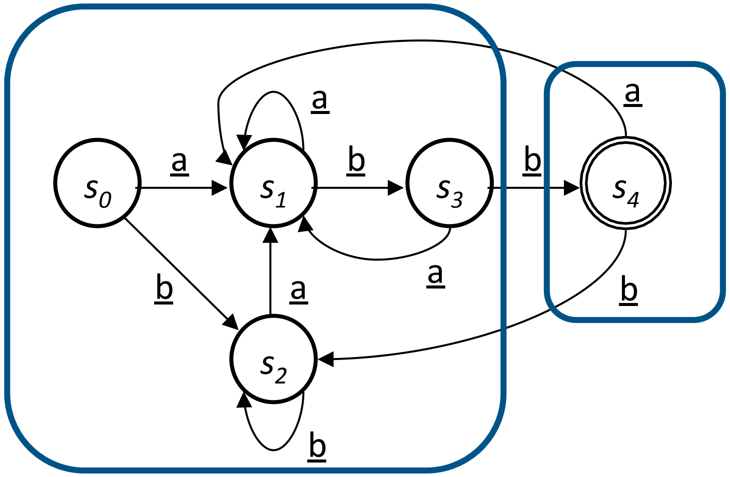
## Splitting a Partition




- The algorithm starts out with { {$s_0$, $s_1$, $s_2$, $s_3$}, { $s_4$ }}
- How does { $s_4$ } split {$s_0$, $s_1$, $s_2$, $s_3$} ?
  - On a, no edges run from {$s_0$, $s_1$, $s_2$, $s_3$} to { $s_4$ }, so nothing splits

# A Detailed Example

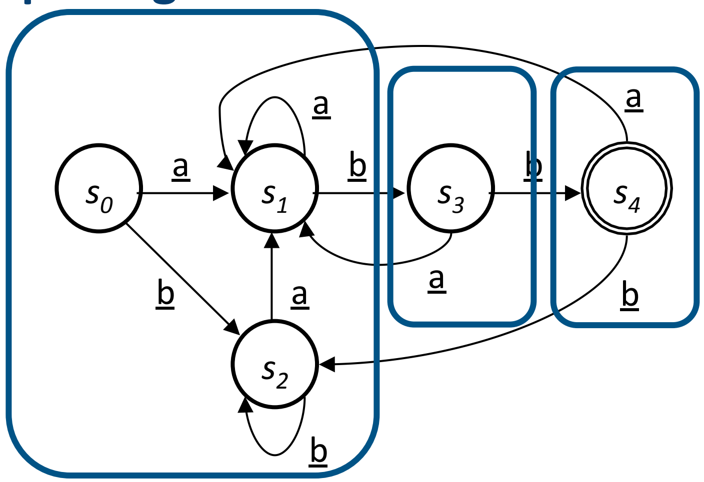
## Splitting a Partition




- The algorithm starts out with { {$s_0$, $s_1$, $s_2$, $s_3$}, { $s_4$ }}
- How does { $s_4$ } split {$s_0$, $s_1$, $s_2$, $s_3$} ?
  - On b, {$s_0$, $s_1$, $s_2$, $s_3$} has edges into both { $s_4$ } and {$s_0$, $s_1$, $s_2$, $s_3$}, so { $s_4$ } splits {$s_0$, $s_1$, $s_2$, $s_3$} into {$s_0$, $s_1$, $s_2$ } and { $s_3$}
    - {$s_0$, $s_1$, $s_2$ } → {$s_0$, $s_1$, $s_2$ } on b
    - { $s_3$ } → { $s_4$ } on b

# A Detailed Example

## Splitting a Partition




- The algorithm starts out with { {$s_0$, s
- How does { $s_4$ } split {$s_0$, $s_1$, $s_2$, $s_3$} ?
  – On b, {$s_0$, $s_1$, $s_2$, $s_3$} has edges into bo
    {$s_0$, $s_1$, $s_2$, $s_3$} into {$s_0$, $s_1$, $s_2$ } and { $s_3$}
    → {$s_0$, $s_1$, $s_2$ } → {$s_0$, $s_1$, $s_2$ } on b
    → { $s_3$ } → { $s_4$ } on b

Now, every state in { $s_3$ } has the same transition on b

- Singleton set ⇒ same transition
- Neither { $s_3$ } nor { $s_4$ } can be split
- { $s_4$ } causes no more splits
- { $s_3$ } will split {$s_0$, $s_1$, $s_2$ } into {$s_0$, $s_1$} and { $s_2$ }

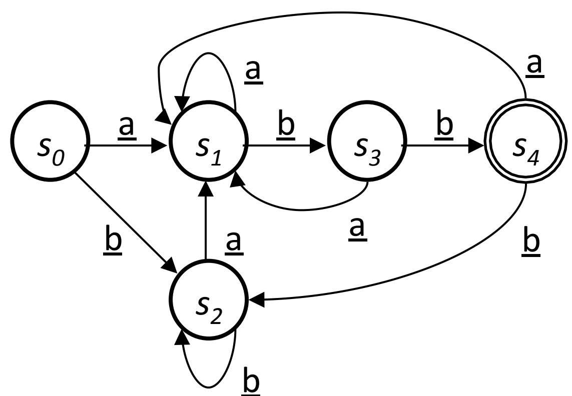
Note that when we split {$s_0$, $s_1$, $s_2$, $s_3$} around { $s_4$ }, we left behind more work — the resulting set, {$s_0$, $s_1$, $s_2$ }, could be split further.

In the algorithm, { $s_3$ } ends up on the worklist, where it will later split {$s_0$, $s_1$, $s_2$ }

# Detailed Example

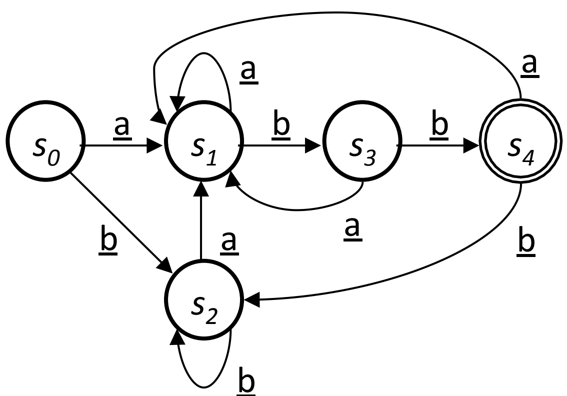
| | *Current Partition* | *Worklist* | *s* | *Split on* a | *Split on* b |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | | | |

Example in this tabular format is for the worklist version of the algorithm.



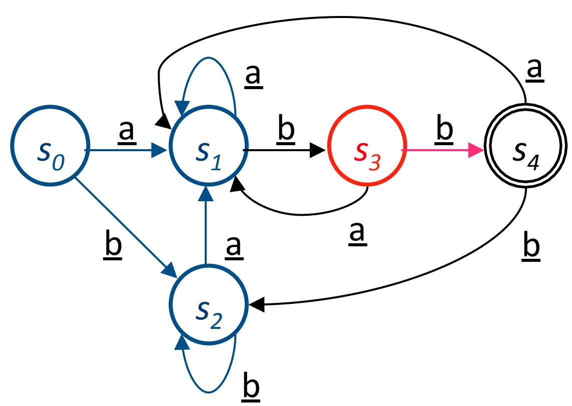
# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on* a | *Split on* b |
|---|---|---|---|---|---|
| *0* | $\{s_4\}$ $\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ $\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | |



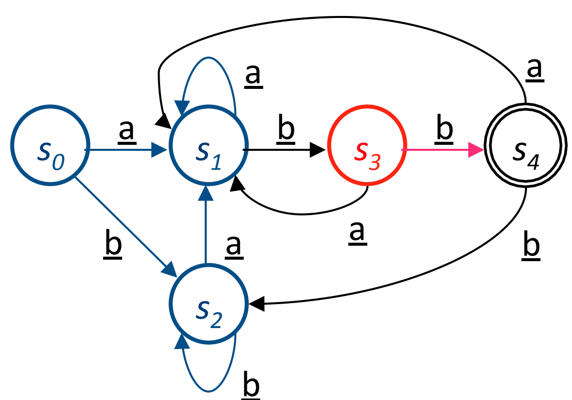
# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on a* | *Split on b* |
|---|---|---|---|---|---|
| *0* | $\{s_4\}$ $\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ $\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}$ $\{s_0,s_1,s_2\}$ |



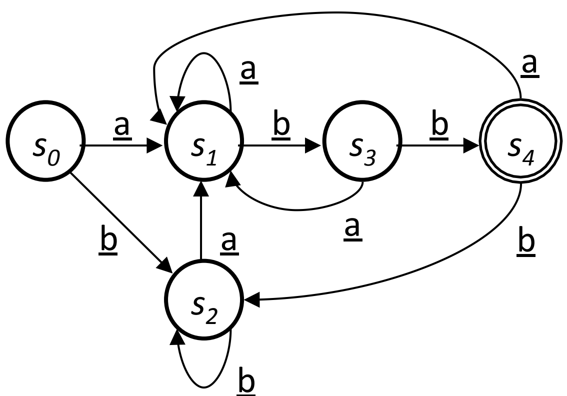
# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on a* | *Split on b* |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\ \{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\ \{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}\ \{s_0,s_1,s_2\}$ | | | |



# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on a* | *Split on b* |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\ \{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\ \{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | |



# Detailed Example

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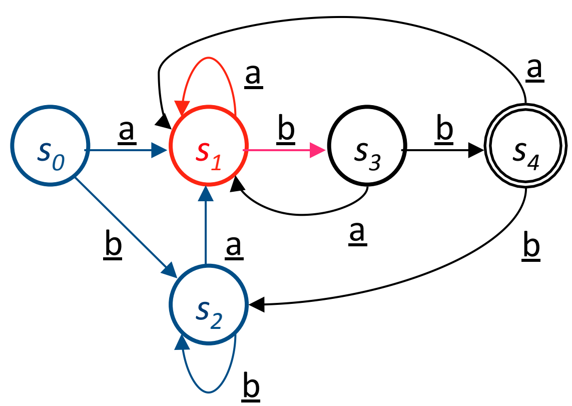
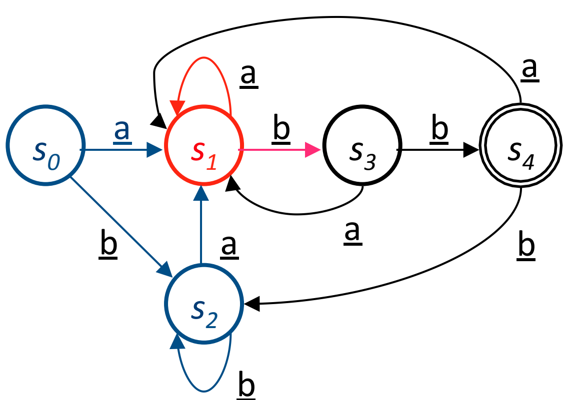
|---|---|---|---|---|---|
| *0* | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\ \{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\ \{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | $\{s_1\}\ \{s_0,s_2\}$ |



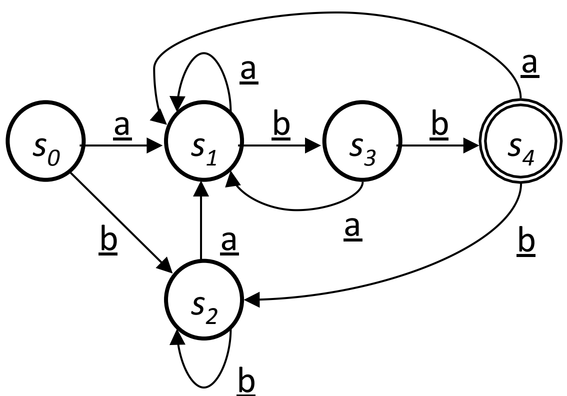
# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on* a | *Split on* b |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\ \{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\ \{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\ \{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}\ \{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | $\{s_1\}\ \{s_0,s_2\}$ |
| *2* | $\{s_4\}\ \{s_3\}\ \{s_1\}\ \{s_0,s_2\}$ | $\{s_1\}\ \{s_0,s_2\}$ | | | |



# Detailed Example

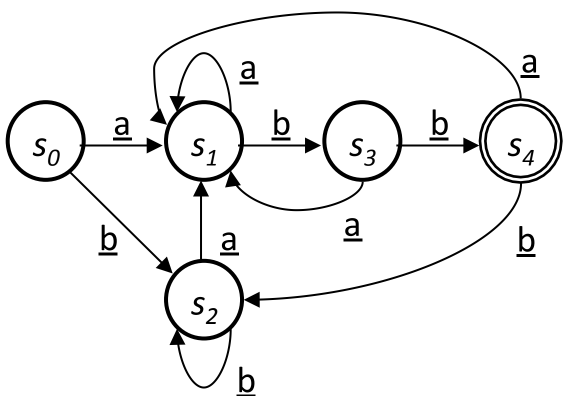
| | *Current Partition* | *Worklist* | *s* | *Split on a* | *Split on b* |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\,\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\,\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\,\{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\,\{s_3\}\,\{s_0,s_1,s_2\}$ | $\{s_3\}\,\{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | $\{s_1\}\,\{s_0,s_2\}$ |
| *2* | $\{s_4\}\,\{s_3\}\,\{s_1\}\,\{s_0,s_2\}$ | $\{s_1\}\,\{s_0,s_2\}$ | $\{s_1\}$ | *none* | *none* |



# Detailed Example

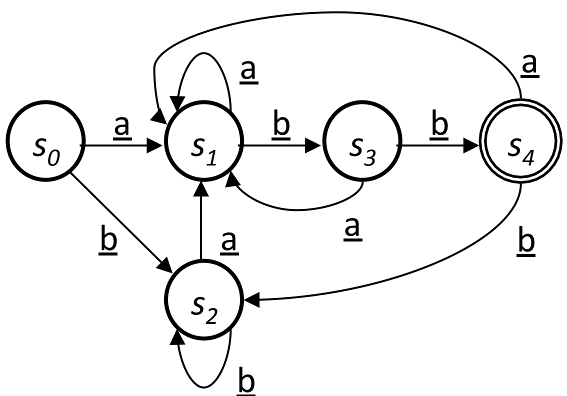
| | *Current Partition* | *Worklist* | *s* | *Split on a* | *Split on b* |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\{s_3\}\{s_0,s_1,s_2\}$ | $\{s_3\}\{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | $\{s_1\}\{s_0,s_2\}$ |
| *2* | $\{s_4\}\{s_3\}\{s_1\}\{s_0,s_2\}$ | $\{s_1\}\{s_0,s_2\}$ | $\{s_1\}$ | *none* | *none* |
| *3* | $\{s_4\}\{s_3\}\{s_1\}\{s_0,s_2\}$ | $\{s_1\}\{s_0,s_2\}$ | $\{s_0,s_2\}$ | *none* | *none* |

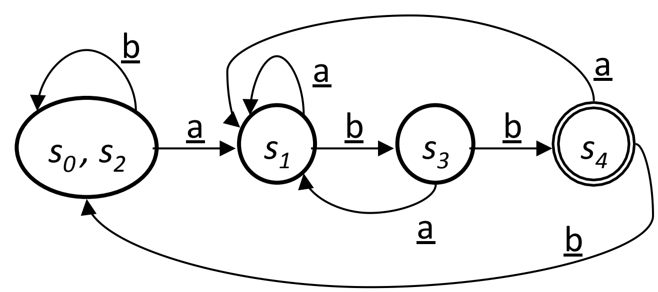
Empty worklist ⇒ done!



# Detailed Example

| | *Current Partition* | *Worklist* | *s* | *Split on* a | *Split on* b |
|---|---|---|---|---|---|
| *0* | $\{s_4\}\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}\{s_0,s_1,s_2,s_3\}$ | $\{s_4\}$ | *none* | $\{s_3\}\{s_0,s_1,s_2\}$ |
| *1* | $\{s_4\}\{s_3\}\{s_0,s_1,s_2\}$ | $\{s_3\}\{s_0,s_1,s_2\}$ | $\{s_3\}$ | *none* | $\{s_1\}\{s_0,s_2\}$ |
| *2* | $\{s_4\}\{s_3\}\{s_1\}\{s_0,s_2\}$ | $\{s_1\}\{s_0,s_2\}$ | $\{s_1\}$ | *none* | *none* |
| *3* | $\{s_4\}\{s_3\}\{s_1\}\{s_0,s_2\}$ | $\{s_1\}\{s_0,s_2\}$ | $\{s_0,s_2\}$ | *none* | *none* |




**20% reduction in number of states**

# **DFA** Minimization Algorithm (Worklist version)

*Worklist* $\leftarrow \{ D_A , \{ D - D_A \} \}$
*Partition* $\leftarrow \{ D_A , \{ D - D_A \} \}$

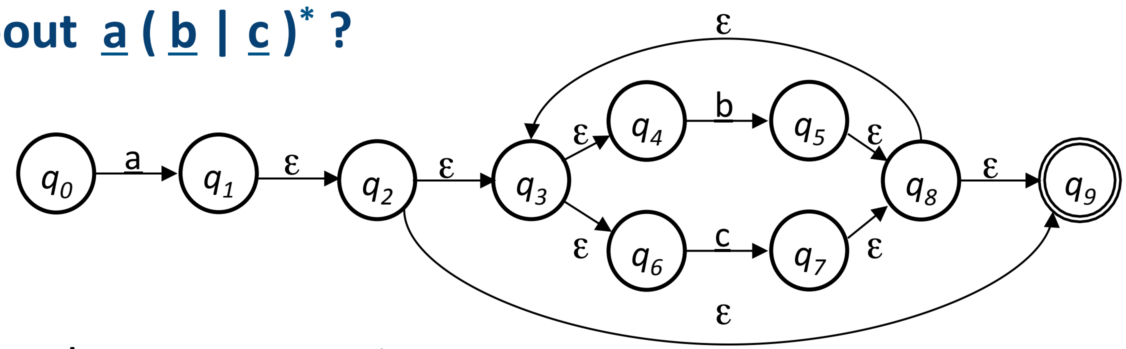
*While (Worklist* $\neq \emptyset$*) do*
  *select a set S from Worklist and remove it*
  *for each* $\alpha \in \Sigma$ *do*
    *Image* $\leftarrow \{ x \mid \delta(x, \alpha) \in S \}$
    *for each* $q \in$ *Partition do*
      $p_1 \leftarrow q \cap Image$
      $p_2 \leftarrow q - p_1$
      *if* $p_1 \neq \emptyset$ *and* $p_2 \neq \emptyset$ *then*
        *remove q from Partition*
        *Partition* $\leftarrow Partition \cup p_1 \cup p_2$
        *if* $q \in$ *Worklist then*
          *remove q from Worklist*
          *Worklist* $\leftarrow Worklist \cup p_1 \cup p_2$
        *else if* $|p_1| \leq |p_2|$
          *then Worklist* $\leftarrow Worklist \cup p_1$
          *else Worklist* $\leftarrow Worklist \cup p_2$

Why does this algorithm halt?

- Fixed-point algorithm
- DFA has finite number of states
- Start with 2 sets in Partition
- Splitting breaks 1 set into 2 smaller ones but never makes a set larger
  → *Monotone behavior*
- Simple, finite limit on |*Partition* |; it cannot be > |*States* |
- Finite # steps, monotone increasing construction ⇒ algorithm halts

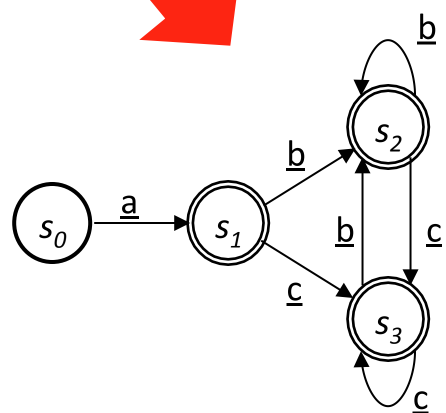
# **DFA** Minimization

## What about $\underline{a}$ ( $\underline{b}$ | $\underline{c}$ )$^*$ ?




First, the subset construction:

| States | | ε-closure(Move(s,*)) | | |
|---|---|---|---|---|
| **DFA** | **NFA** | $\underline{a}$ | $\underline{b}$ | $\underline{c}$ |
| $s_0$ | $q_0$ | $s_1$ | *none* | *none* |
| $s_1$ | $q_1, q_2, q_3, q_4, q_6, q_9$ | *none* | $s_2$ | $s_3$ |
| $s_2$ | $q_5, q_8, q_9, q_3, q_4, q_6$ | *none* | $s_2$ | $s_3$ |
| $s_3$ | $q_7, q_8, q_9, q_3, q_4, q_6$ | *none* | $s_2$ | $s_3$ |

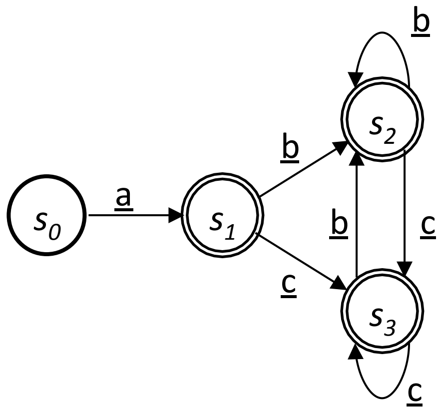



***From last lecture …***

# **DFA** Minimization

## Then, apply the minimization algorithm

| | | Split on | | |
|---|---|---|---|---|
| | *Current Partition* | a | b | c |
| $P_0$ | $\{s_1,s_2,s_3\}$ $\{s_0\}$ | *none* | *none* | *none* |




It splits no states after the initial partition

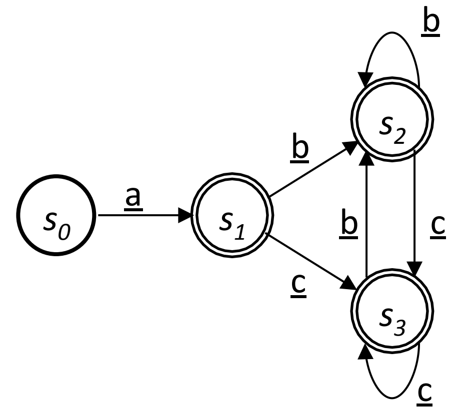
$\Rightarrow$ The minimal **DFA** has two states

- One for $\{s_0\}$
- One for $\{s_1, s_2, s_3\}$

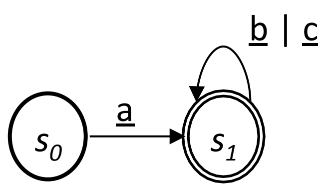
# **DFA** Minimization

## Then, apply the minimization algorithm

| | | *Split on* | | |
|---|---|---|---|---|
| | *Current Partition* | a | b | c |
| $P_0$ | $\{s_1,s_2,s_3\}$ $\{s_0\}$ | *none* | *none* | *none* |




It produces this **DFA**




Earlier, I suggested that a human would design a simpler automaton than Thompson's construction & the subset construction did.

Minimizing that **DFA** produces exactly the **DFA** that I claimed a human would design!

# Abbreviated Register Specification

**Start with a regular expression**

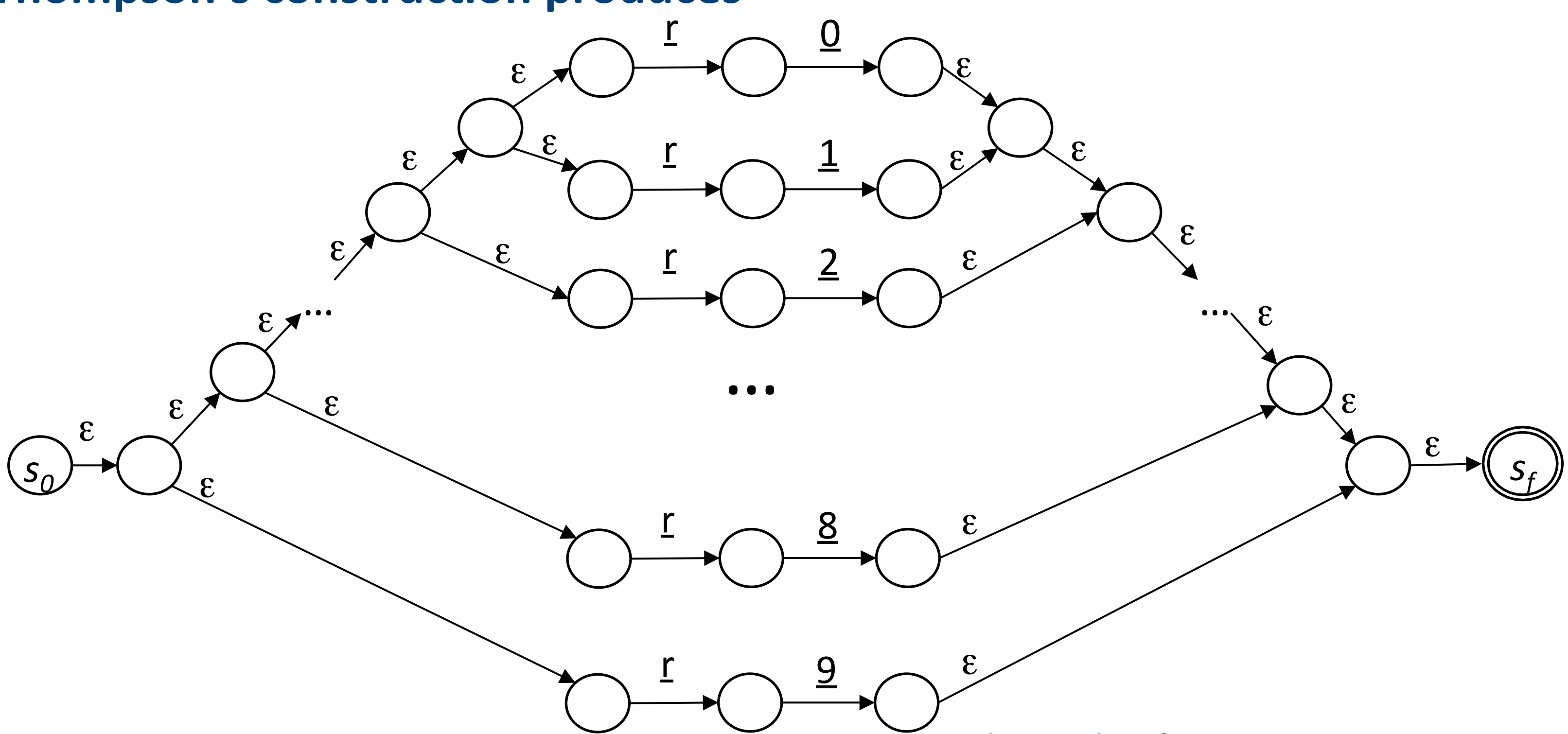
r0 | r1 | r2 | r3 | r4 | r5 | r6 | r7 | r8 | r9

Register names from zero to nine

*The Cycle of Constructions*

# Abbreviated Register Specification

## Thompson's construction produces

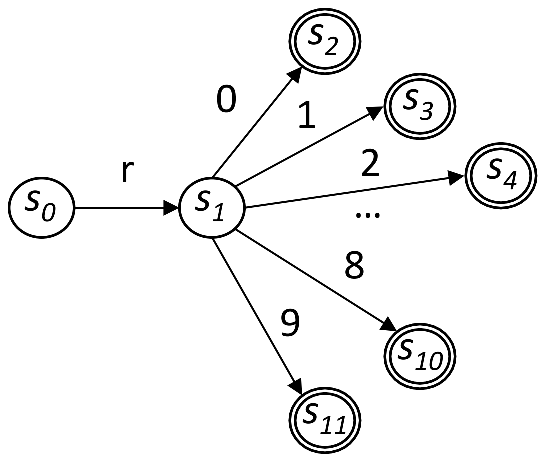



*The Cycle of Constructions*

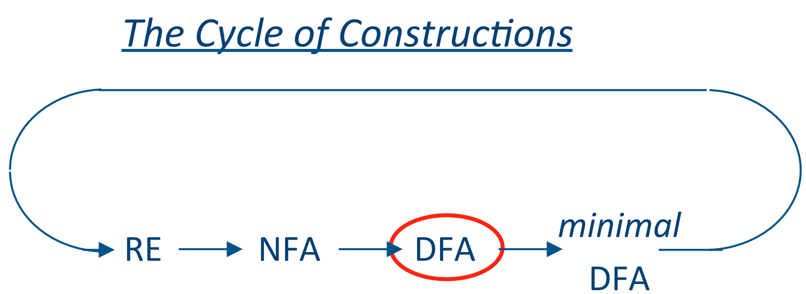

To make the example fit, we have eliminated some of the ε-transitions, e.g., between r and 0

# Abbreviated Register Specification

## Applying the subset construction yields



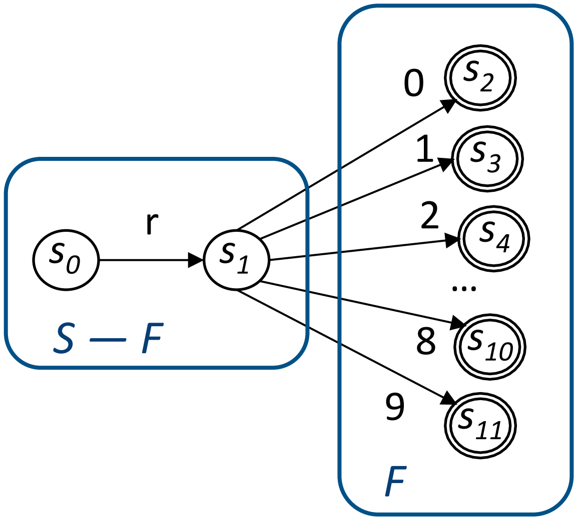

This is a **DFA**, but it has a lot of states …



# Abbreviated Register Specification

## Hopcroft's algorithm

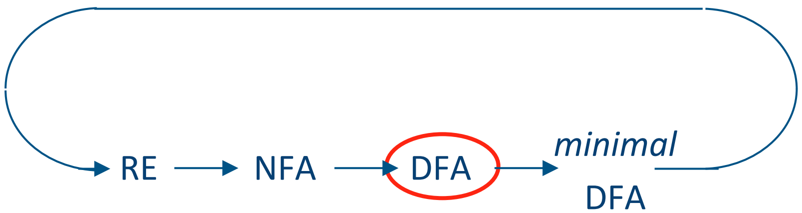
### Initial sets




*F* does not split.

Since no transitions leave it, there are no states to split it.

*The Cycle of Constructions*

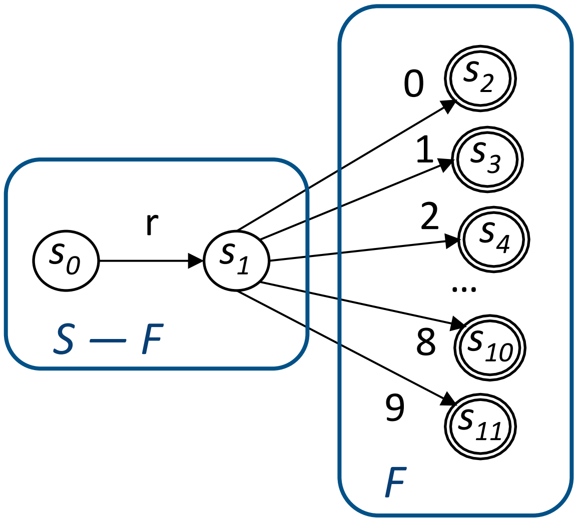



Technically, this edge shows up as 10 transitions, which are combined by construction of the character classifier …

# Abbreviated Register Specification

## Hopcroft's algorithm

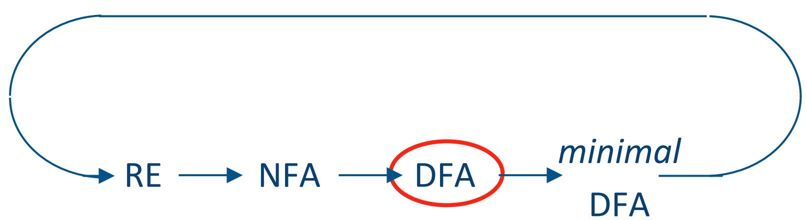
### Initial sets




$\{ S - F \}$ does split

Any character will split it into $\{ s_0 \}$, $\{ s_1 \}$

*The Cycle of Constructions*

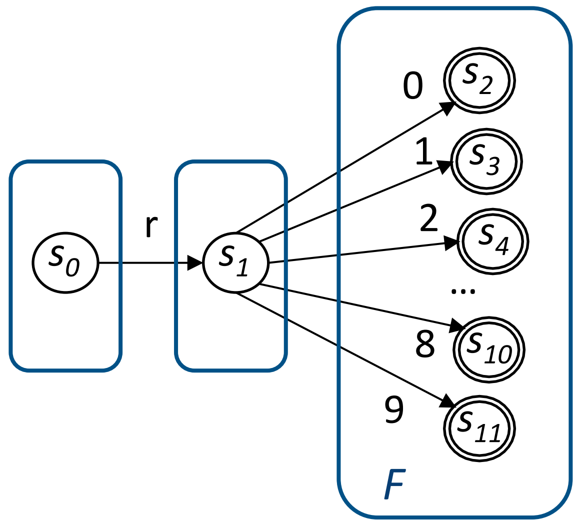



Technically, this edge shows up as 10 transitions, which are combined by construction of the character classifier …

# Abbreviated Register Specification

## Hopcroft's algorithm

### Initial sets




$\{ S - F \}$ does split

Any character will split it into $\{ s_0 \}, \{ s_1 \}$

This partition is the final partition

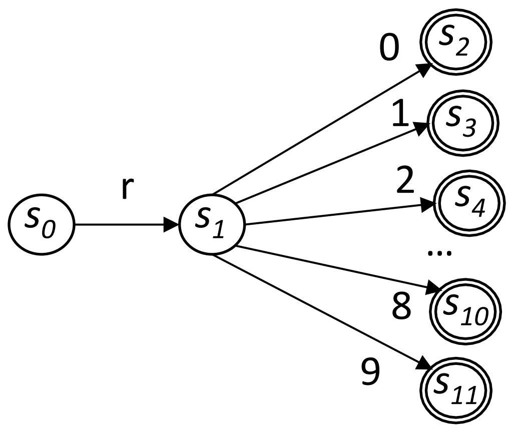
*The Cycle of Constructions*


Technically, this edge shows up as 10 transitions, which are combined by construction of the character classifier …

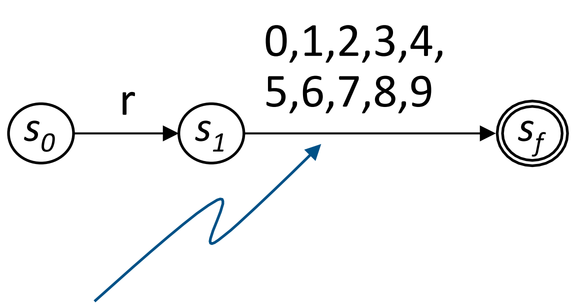
# Abbreviated Register Specification

## Hopcroft's algorithm

### Initial sets




### Becomes, through minimization




Technically, this edge shows up as 10 transitions, which are combined by construction of the character classifier …

**The Critical Takeaway Points:**

- The construction will build a minimal **DFA**
- The size of the **DFA** relates to the language described by the **RE**, not the size of the **RE**
- The result is a **DFA**, so it has **O**(1) cost per character
- The compiler writer can use the "most natural" or "intuitive" **RE**

*The Cycle of Constructions*

# The Plan for Scanner Construction

**RE → NFA** *(Thompson's construction)* ✔
- Build an **NFA** for each term in the **RE**
- Combine them in patterns that model the operators

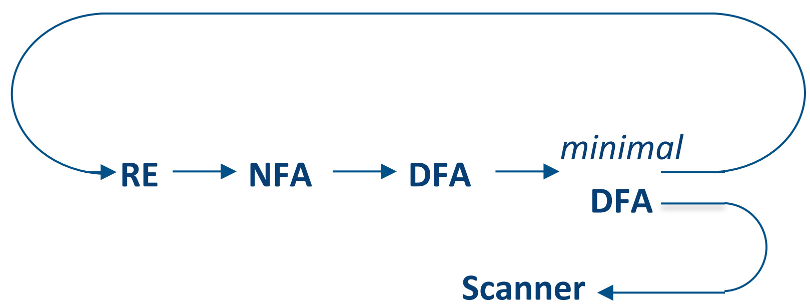
**NFA → DFA** *(Subset construction)* ✔
- Build a **DFA** that simulates the **NFA**

**DFA →** Minimal **DFA**
- Hopcroft's algorithm ✔
- Brzozowski's algorithm

Minimal **DFA** → Scanner
- See § 2.5 in EaC2e

**DFA → RE**
- All pairs, all paths problem
- Union together paths from $s_0$ to a final state

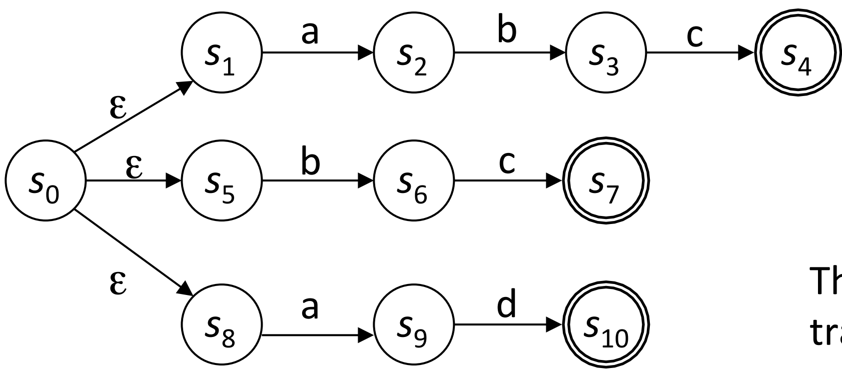
*The Cycle of Constructions*



# Brzozowski's Algorithm for **DFA** Minimization

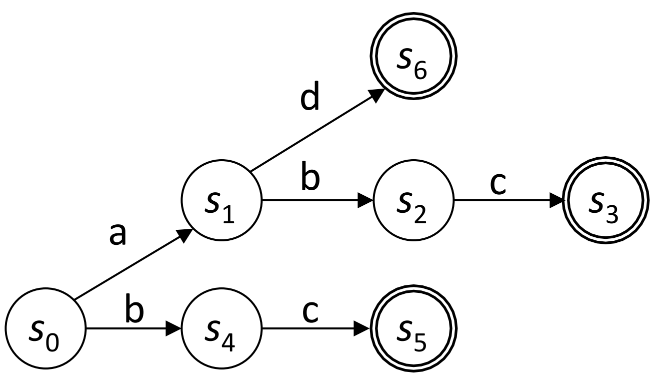
## The Intuition

- The subset construction merges prefixes in the **NFA**




abc | bc | ad

Thompson's construction would leave ε-transitions between each single-character automaton




Subset construction eliminates ε-transitions and merges the paths for a. It leaves duplicate tails, such as bc, intact.

# Brzozowski's Algorithm

## Idea: Use The Subset Construction Twice

- For an **NFA** *N*
  - Let *reverse(N)* be the **NFA** constructed by making initial state final, adding a new start state with an ε-transition to each previously final state, and reversing the other edges
  - Let *subset(N)* be the **DFA** produced by the subset construction on *N*
  - Let *reachable(N)* be *N* after removing any states that are not reachable from the initial state
- Then,

$$reachable(subset(reverse(\ reachable(subset(reverse(N))\ )))$$

is a minimal **DFA** that implements *N*     [Brzozowski, 1962]

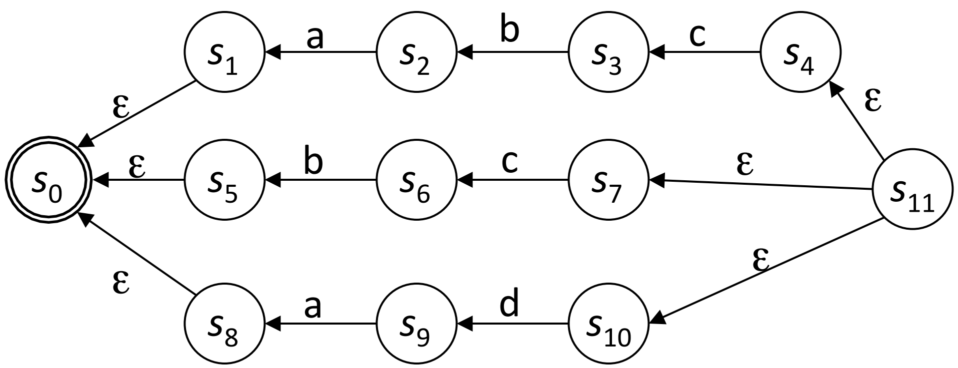
*Not everyone finds this result to be intuitive.*

*Neither algorithm dominates the other.*

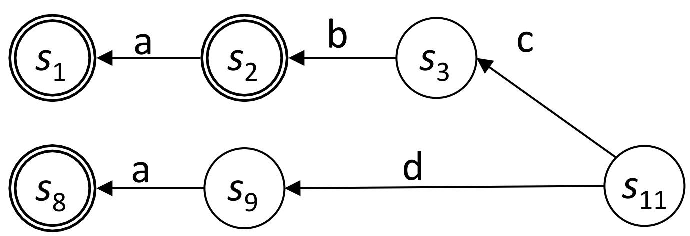
# Brzozowski's Algorithm

## Step 1

- The subset construction on *reverse(NFA)* merges suffixes in original NFA




**Reversed NFA**

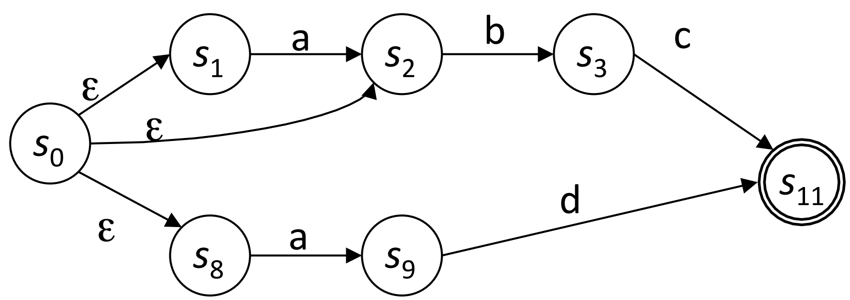



**subset(reverse(NFA))**

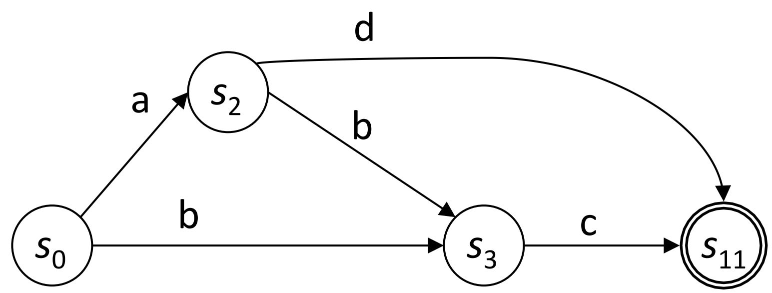
# Brzozowski's Algorithm

## Step 2

- Reverse it again & use subset to merge prefixes …



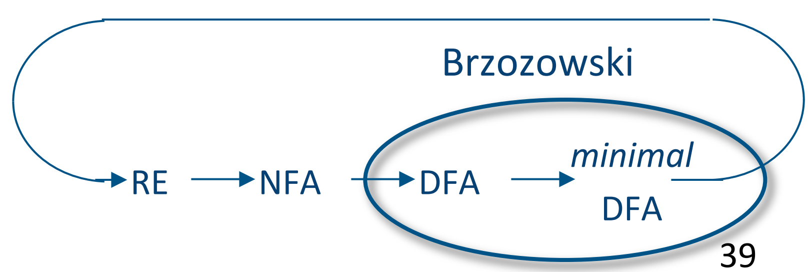

**Reverse it, again**




Minimal DFA

**And subset it, again**

*The Cycle of Constructions*



# Abbreviated Register Specification

## Start with a regular expression

r0 | r1 | r2 | r3 | r4 | r5 | r6 | r7 | r8 | r9

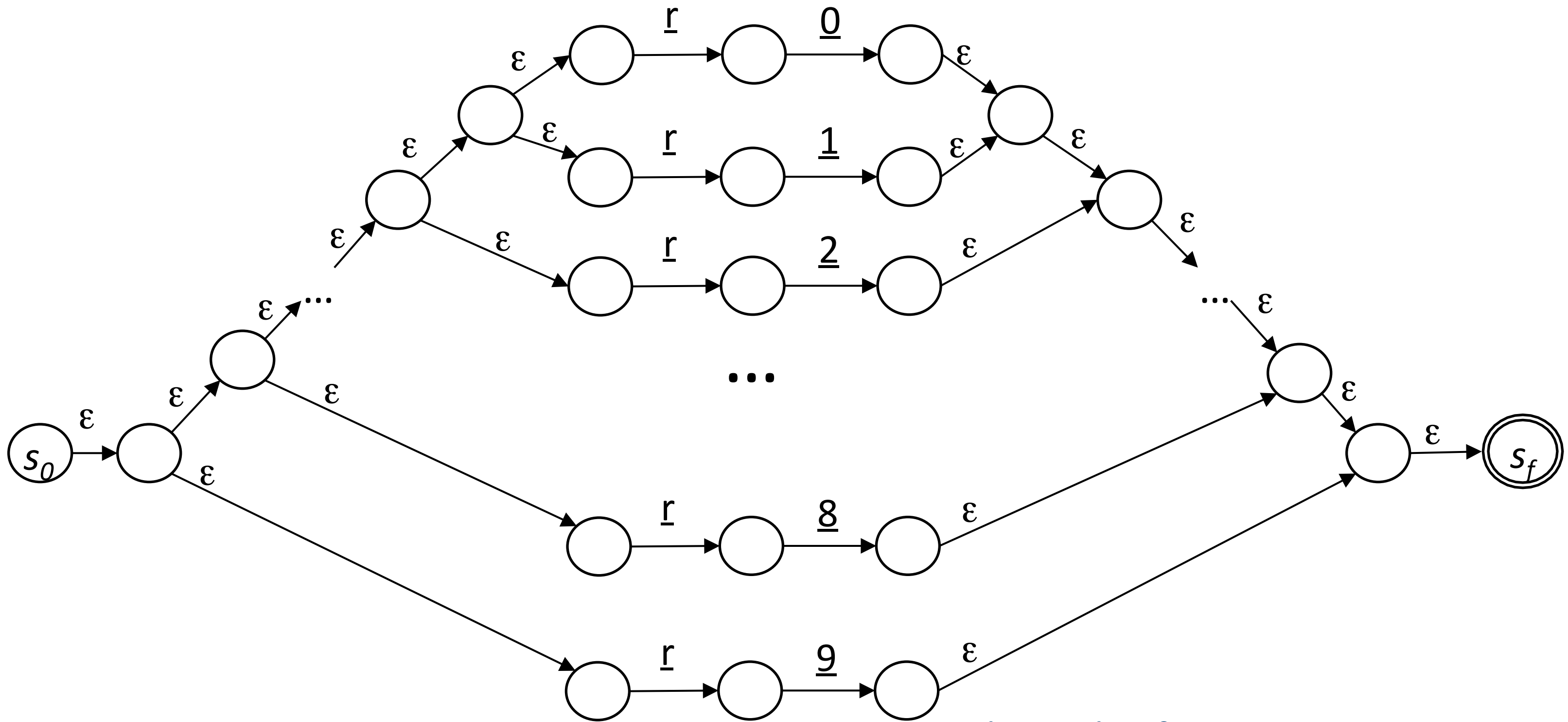
Register names from zero to nine

*The Cycle of Constructions*

RE → NFA → DFA → *minimal* DFA

# Abbreviated Register Specification

**Thompson's construction produces something along these lines**

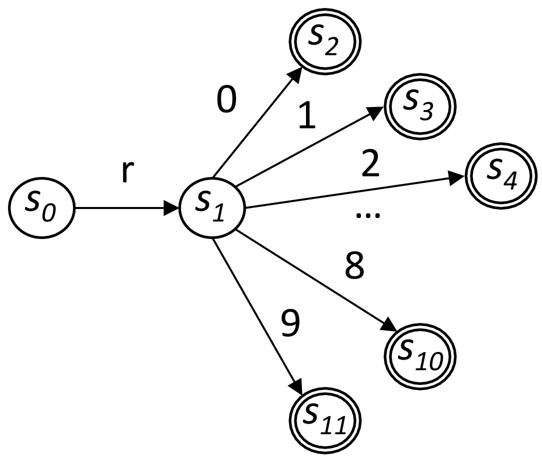



To make the example fit, we have eliminated some of the ε-transitions, e.g., between r and 0

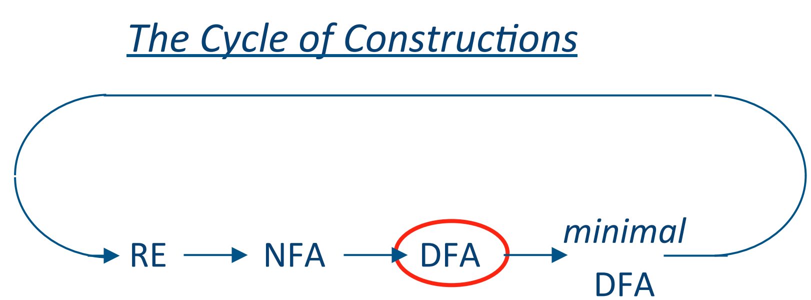
*The Cycle of Constructions*

# Abbreviated Register Specification

## Applying the subset construction yields

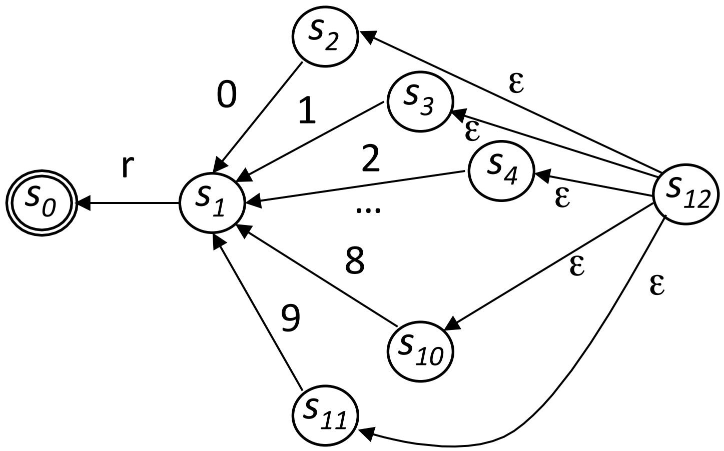



This is a **DFA**, but it has a lot of states …

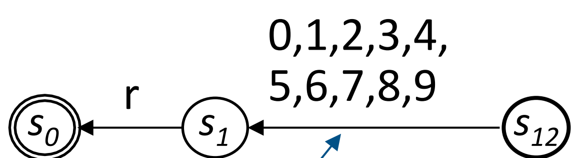


# Abbreviated Register Specification

## Applying Brzozowski's algorithm, step 1




Reversed **NFA**



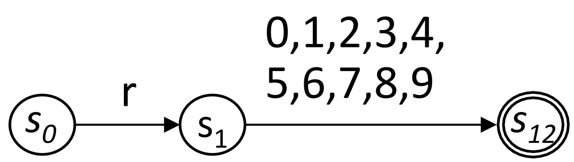

After Subset Construction

Technically, this edge shows up as 10 edges, which need to be combined…

*The Cycle of Constructions*

# Abbreviated Register Specification

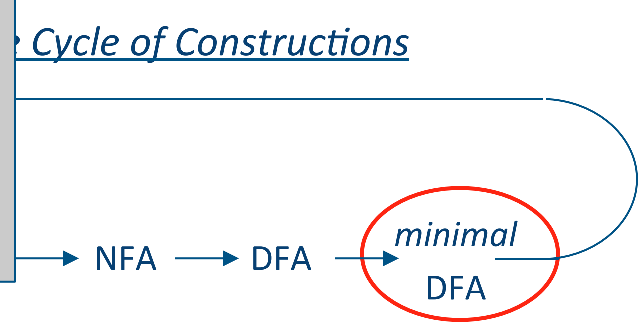
## Brzozowski, step 2 reverses that DFA and subsets it again




A skilled human might build this **DFA**

**The Critical Point:**

- The construction will build a minimal **DFA**
- The size of the **DFA** relates to the language described by the **RE**, not the size of the **RE**
- The result is a **DFA**, so it has **O**(1) cost per character
- The compiler writer can use the "most natural" or "intuitive" **RE**



## One Last Algorithm

# RE Back to DFA

The Wikipedia page on "Kleene's algorithm" is pretty good. It also contains a link to Kleene's 1956 paper. This form of the algorithm is usually attributed to McNaughton and Yamada in 1960.

## Kleene's Construction

*for* $i \leftarrow 0$ *to* $|D| - 1$; *// label each immediate path*
  *for* $j \leftarrow 0$ *to* $|D| - 1$;
    $R^0_{ij} \leftarrow \{ a \mid \delta(d_i, a) = d_j \}$;
    *if* $(i = j)$ *then*
      $R^0_{ii} = R^0_{ii} \mid \{\varepsilon\}$;

*for* $k \leftarrow 0$ *to* $|D| - 1$; *// label nontrivial paths*
  *for* $i \leftarrow 0$ *to* $|D| - 1$;
    *for* $j \leftarrow 0$ *to* $|D| - 1$;
      $R^k_{ij} \leftarrow R^{k-1}_{ik} (R^{k-1}_{kk})^* R^{k-1}_{kj} \mid R^{k-1}_{ij}$

$L \leftarrow \{\}$ *// union labels of paths from*
*For each final state* $s_i$ *//* $s_0$ *to a final state* $s_i$
  $L \leftarrow L \mid R^{|D|-1}_{0i}$

$R^k_{ij}$ is the set of paths from $i$ to $j$ that include no state higher than $k$

*The Cycle of Constructions*

RE → NFA → DFA → *minimal* DFA

**Adaptation of all points, all paths, low cost algorithm**

# Limits of Regular Languages

## Not all languages are regular

**RL**'s $\subset$ **CFL**'s $\subset$ **CSL**'s

You cannot construct **DFA**'s to recognize these languages

- $L = \{ p^k q^k \}$ *(parenthesis languages)*
- $L = \{ wcw^r \mid w \in \Sigma^* \}$

Neither of these is a regular language *(nor an RE)*

But, this is a little subtle. You <u>can</u> construct **DFA**'s for

- Strings with alternating 0's and 1's

  $( \varepsilon \mid 1 ) ( 01 )^* ( \varepsilon \mid 0 )$
- Strings with and even number of 0's and 1's

**RE**'s can count bounded sets and bounded differences

# Limits of Regular Languages

## Advantages of Regular Expressions

- Simple & powerful notation for specifying patterns
- Automatic construction of fast recognizers
  - **O**(1) cost per input character
- Many kinds of syntax can be specified with REs

## Disadvantages of Regular Expressions

- Many interesting constructs are not regular
  - Balanced parentheses, nested **if-then** and **if-then-else** constructs
- The **DFA** recognizer has no real notion of grammatical structure
  - Gives no help with meaning