Final Exam Solutions

1. i) X_1, X_2 and $X_3 \sim$ Bernoulli (p)

$$\begin{aligned} f_{\underline{X}}(\underline{x}|p) &= p^{x_1+x_2+x_3}\,(1-p)^{3-x_1-x_2-x_3} \\ &= p^{x_1+x_2}\,(1-p)^{2-x_1-x_2}\,p^{x_3}\,(1-p)^{1-x_3} \end{aligned}$$

1M

Here, $h(\underline{x}) = 1$

$$\begin{aligned} g(t_1,t_2) &= p^{t_1}\,(1-p)^{2-t_1}\,p^{t_2}\,(1-p)^{1-t_2} \\ \therefore f_{\underline{X}}(\underline{x}|p) &= g(x_1+x_2, x_3|p)\cdot h(x) \end{aligned}$$

2M

Hence T is a sufficient statistic for p.

ii) Let $A(\underline{x}) = x_1 + x_2$ and $B(\underline{x}) = x_3$

$$\begin{aligned} f_{\underline{X}}(\underline{x}|p) &= p^{A(\underline{x})}\,(1-p)^{2-A(\underline{x})}\,p^{B(\underline{x})}\,(1-p)^{B(\underline{x})} \\ &= p^{A(\underline{x})+B(\underline{x})}\,(1-p)^{3-A(\underline{x})-B(\underline{x})} \\ \frac{f_{\underline{X}}(\underline{x}|p)}{f_{\underline{Y}}(\underline{y}|p)} &= \frac{p^{A(\underline{x})+B(\underline{x})}\,(1-p)^{3-A(\underline{x})-B(\underline{x})}}{p^{A(\underline{y})+B(\underline{x})}\,(1-p)^{3-A(\underline{y})-B(\underline{y})}} \\ &= \left(\frac{p}{1-p}\right)^{A(\underline{x})+B(\underline{x})-A(\underline{y})-B(\underline{y})} \end{aligned}$$

2M

The ratio above is constant as a function of p if (but not only if) $A(\underline{x}) = B(\underline{y})$ and $B(\underline{x}) = B(\underline{y})$ because $A(\underline{x}) + B(\underline{x}) = A(\underline{y}) + B(\underline{y})$, even though $A(\underline{x}) \neq A(\underline{y})$ and $B(\underline{x}) \neq B(\underline{y})$, the above ratio is still constant.
Therefore, T is not a MSS for p.

3M

2. i)

$$\begin{aligned} \phi(T) = E(W|T) &= E\left(\frac{X_1+X_2}{2}|X_1\right) \\ &= \frac{1}{2}E(X_1|X_1) + \frac{1}{2}E(X_2|X_1) \\ &= \frac{X_1}{2} + \frac{\theta}{2} \\ \Longrightarrow E(\phi(T)) &= \theta. \end{aligned}$$

4M

ii) $Var\left(\phi(T)\right) = \dfrac{1}{4} < Var(W) = \dfrac{1}{2}$. — 2M

iii) $\underbrace{\text{No, } \phi \text{ is not an estimator.}}_{1M}$ $\underbrace{\text{It is a function of population parameter } \theta}_{1M}$ — 2M

3. i) $X \sim \text{Uniform}(\theta)$, $T(X) = X$

$$E_\theta(g(T)) = E_\theta(g(X)) = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = 0$$

$$\iff \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0$$

$$\iff \sum_{x=1}^{\theta} g(x) = 0$$

1M

For $\theta \in \mathbb{N}$,

If $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$

If $\theta = 2$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = 0$

.

.

.

If $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) + \ldots + g(k) = 0$

2M

$$\therefore g(x) = 0 \;\; \forall x \in \mathbb{N}$$

Thus, X is a complete statistic for $\theta \in \Omega = \mathbb{N}$

1M

ii) Define

$$g(x) = \begin{cases} 1 & if \;\; x = n, \\ -1 & if \;\; x = n+1, \\ 0 & otherwise. \end{cases}$$

2M

$$E_\theta(g(T)) = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) \Longrightarrow$$

$$E_\theta(g(T)) = \begin{cases} \frac{1}{\theta} & if \;\; \theta = n, \\ 0 & if \;\; \theta \neq n. \end{cases}$$

$\therefore$ when $\theta \in \Omega = \mathbb{N} - \{n\}$, $g(x) \neq 0$ but $E(g(T)) = 0$.

Thus, $T(X) = X$ is not a complete statistic.

2M

4.

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!}, x=0,1,2,...\infty,\ \lambda=1\ or\ 2.$$

$$E_\lambda\left(g(T)\right)=\sum_{t=0}^{\infty} g(t)\frac{\lambda^t e^{-\lambda}}{t!}=0$$

$$E_{\lambda=1}\left(g(T)\right)=\sum_{t=0}^{\infty} g(t)\frac{e^{-1}}{t!}=0 \qquad (1)$$

$$and\ E_{\lambda=2}\left(g(T)\right)=\sum_{t=0}^{\infty} g(t)\frac{2^t e^{-2}}{t!}=0 \qquad (2)$$

$$(1)\implies\sum_{t=0}^{\infty}\frac{g(t)}{t!}=0 \qquad (3)$$

$$(2)\implies\sum_{t=0}^{\infty}\frac{2^t g(t)}{t!}=0 \qquad (4)$$

1M

Define

$$g(t)=\begin{cases} 2, & t=0,2 \\ -3, & t=1, \\ 0, & otherwise.\end{cases}$$

2M

From (3),

$$\sum_{t=0}^{\infty}\frac{g(t)}{t!}=\frac{g(0)}{0!}+\frac{g(1)}{1!}+\frac{g(2)}{2!}=2-3+1=0$$

1M

From (4),

$$\sum_{t=0}^{\infty}\frac{g(t)2^t}{t!}=2-6+4=0$$

1M

Thus, the family is not complete.

5. $X_1, X_2, ..., X_n \sim U(0,\theta)$

We know that $X_{(n)}$ is CSS. Now, $E\left(X_{(1)}\right)=E\left[\frac{X_{(1)}}{X_{(n)}}\cdot X_{(n)}\right]$

$\implies E\left(X_{(1)}\right)=E\left[\frac{X_{(1)}}{X_{(n)}}\right]E\left(X_{(n)}\right)$

$$\therefore E\left[\frac{X_{(1)}}{X_{(n)}}\right]=\frac{E\left(X_{(1)}\right)}{E\left(X_{(n)}\right)}$$

1M

This implies if we can show that $\frac{X_{(1)}}{X_{(n)}}$ is ancillary, we can apply Basu's Theorem. Here,

$$f_X(x) = \frac{1}{\theta}, 0 < x < \theta.$$

Let $Y = \frac{X}{\theta}$. Then $\frac{dx}{dy} = \theta$ and $Y \sim U(0,1)$

$\therefore \frac{X_{(1)}}{X_{(n)}} = \frac{\theta Y_{(1)}}{\theta Y_{(n)}} = \frac{Y_{(1)}}{Y_{(n)}}$ is an ancillary statistic and $E\left[\frac{X_{(1)}}{X_{(n)}}\right] = \frac{E\left[Y_{(1)}\right]}{E\left[Y_{(n)}\right]}$

Now, since $Y \sim U(0,1)$

2M

$$\begin{aligned} f_Y(y) &= 1, \ 0 < y < 1, \ and \\ f_{Y_{(1)}}(y) &= n\left[1 - F(y)\right]^{n-1} f(y) \\ &= n(1-y)^{n-1} \end{aligned}$$

$$\therefore E\left(Y_{(1)}\right) = \frac{1}{n+1}$$

1M

Similarly, $f_{Y_{(n)}}(y) = n\left[F(y)\right]^{n-1} f(y) = ny^{n-1}$ and

$$E\left(Y_{(n)}\right) = \frac{n}{n+1}$$

Thus,

$$\frac{E\left[Y_{(1)}\right]}{E\left[Y_{(n)}\right]} = \frac{1}{n}$$

1M

6. i) T_1 rejects H_0 if, and only if, all all success are observed i.e.

$$\begin{aligned} R &= \{\underline{x} : \underline{x} = (1,1,1,1,1)\} \\ &= \left\{\underline{x} : \sum_{i=1}^{5} x_i = 5\right\} \end{aligned}$$

T_2 rejects if, and only if, 3 or more success are observed i.e.,

$$R = \left\{\underline{x} : \sum_{i=1}^{5} x_i \geq 5\right\}$$

1M

$$\begin{aligned} \beta_1(\theta) = P\left(Reject\ H_0|\theta\right) &= P\left(X \in R|\theta\right) \\ &= P\left(\sum_{i=1}^{5} x_i = 5|\theta\right) \\ &= \theta^5 \end{aligned}$$

1M

$$
\begin{aligned}
\beta_2(\theta) &= P\left(\sum_{i=1}^{5} x_i \geq 3|\theta\right) \\
&= \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{4}\theta^4(1-\theta) + \theta^5 \\
&= \theta^3(6\theta^2 - 15\theta + 10)
\end{aligned}
$$

2M

ii) When $\theta \in \Omega_0 = (0, 0.5]$, the power of $\beta(\theta)$ is Type 1 error. 1M

$$max\ \beta_1(\theta) = max\ \theta^5 = (0.5)^5 = \frac{1}{32} \approx 0.031$$

For T_2, we need to find max of $\beta_2(\theta)$ for $\Omega_0 = (0, 0.5]$. 1M

$$\beta_2'(\theta) = 30\theta^2(\theta-1)^2 > 0$$

$\therefore$ $\beta_2(\theta)$ is increasing in $\theta \in (0, 1)$. Thus, max of Type 1 error is $\beta_2(0.5) = 0.5$. 1M

iii)

$$(1-\beta_1(\theta))\,|_{\frac{2}{3}} = (1-\theta^5)\,|_{\frac{2}{3}} \approx 0.868 \quad \text{— 1M}$$

$$(1-\beta_2(\theta))\,|_{\frac{2}{3}} = 1-\theta^3(6\theta^2 - 15\theta + 10)|_{\frac{2}{3}} \approx 0.21 \quad \text{— 2M}$$

7.

$$\Omega_0 = \{\theta = 1, v > 0\}$$
$$\Omega = \{\theta > 0, v > 0\}$$

$$logL(\theta, v|\underline{x}) = nlog\theta + n\theta logv - (1+\theta)log\prod_{i=1}^{n} x_i, \ \ v < x_{(1)}.$$

1M

For any value of θ, this is an increasing function of v for $v < x_{(1)}$.

$$\hat{v} = X_{(1)}$$

1M

$$\frac{\partial logL(\theta, x_{(1)}|\underline{x})}{\partial\theta} = \frac{n}{\theta} + nlogx_{(1)} - log\prod_{i=1}^{n} x_i = 0 \text{ and solving for } \theta \text{ yielding,}$$

$$\hat{\theta} = \frac{n}{log\left(\prod x_i / x_{(1)}^n\right)} = \frac{n}{T}$$

1M

Under H_0,

$$\hat{\theta} = 1$$
$$\hat{v} = X_{(1)}$$

+

and Under H_1,

$$\hat{\theta} = \frac{n}{T}$$
$$\hat{\upsilon} = X_{(1)}$$

1M

So likelihood ratio is

$$\begin{aligned}
\lambda(\underline{x}) &= \frac{x_{(1)}^n / (\prod x_i)^2}{(nT)^2 x_{(1)}^{n^2} / (\prod x_i)^{n/T+1}} \\
&= \left(\frac{T}{n}\right)^n \frac{e^{-T}}{(e^{-T})^{n/T}} \\
&= \left(\frac{T}{n}\right)^n e^{-T+n}
\end{aligned}$$

$$\frac{\partial log(\lambda(\underline{x}))}{\partial T} = \frac{n}{T} - 1.$$

Hence, $\lambda(\underline{x})$ is increasing if $T \leq n$ and decreasing if $T \geq n$. Thus, $T \leq c \iff T \leq c_1$ or $T \geq c_2$ for appropriately chosen constants c_1 and c_2. $\geq$

2