1 $\delta_M(x) \to$ MLE ; $\delta_U(x) \to$ UE !

Assignment-II

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(a) $f(x|\theta) = e^{-(x-\theta)} \quad \forall x > \theta \ ; \ g(\theta) = \theta$

$\delta_M(x)$:

$$L(\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} \quad \forall \theta < x_i$$

$$L(\theta) = e^{-\sum_{i=1}^{n}(x_i-\theta)} \quad \forall \theta < x_i.$$

$$\log(L(\theta)) = -\sum_{i=1}^{n}(x_i-\theta) \quad \forall \theta < x_i \Rightarrow n\theta - \sum_{i=1}^{n} x_i$$

$L(\theta)$ is maximum when θ is maximum, hence,

$$\boxed{\therefore \text{MLE is } X_{(1)}}$$

$\delta_U(x)$:

$$E[X_{(1)}] = \int_{\theta}^{\infty} x f_{X_{(1)}}(x)\,dx$$

$$F_{X_{(1)}}(x) = P(X_1 \le x)$$
$$= 1 - P(X_{(1)} > x)$$
$$= 1 - P(X_1 > x, X_2 > x \ldots X_n > x)$$
$$= 1 - [P(X_1 > x)]^n$$
$$= 1 - [1 - F_{X_1}(x)]^n$$
$$= 1 - [1 - [1 - e^{\mu - x}]]^n$$

$$F_{X_i}(x) = \int_{\mu}^{x} e^{\mu - t}\,dt$$
$$= -e^{\mu - t}\Big|_{\mu}^{x}$$
$$\boxed{F_{X_i}(x) = 1 - e^{\mu - x}}$$

$$F_{X_{(1)}}(x) = 1 - e^{n(\mu - x)}, \quad x > \mu$$

$$f_{X_{(1)}}(x) = n e^{n(\mu - x)}, \quad x > \mu$$

$$E[X_1] = \int_{\mu}^{\infty} x f_{X_{(1)}}(x)\, dx$$

$$= \int_{\mu}^{\infty} x\, n\, e^{n(\mu - x)}\, dx \Rightarrow n \int_{\mu}^{\infty} x e^{n(\mu-x)}\, dx$$

$$\Rightarrow n \left[x \int_{\mu}^{\infty} e^{n(\mu-x)} dx + \frac{1}{n} \int_{\mu}^{\infty} e^{n(\mu-x)} \right]$$

$$\Rightarrow n \left[\frac{\mu}{n} + \frac{1}{n^2} \right] = \mu + \frac{1}{n}$$

$$E[X_1] = \mu + \frac{1}{n}$$

$$\boxed{\delta_U(x) = X_1 - \frac{1}{n}}$$

MSE of $\delta_M(x)$:

$$MSE = \text{Variance} + \text{Bias}^2.$$

$$V(X_1) = E[X_1^2] - (E[X])^2$$

$$E[X_1^2] = \int_{\mu}^{\infty} x^2 f_{X_{(1)}}(x)\, dx = \int_{\mu}^{\infty} x^2\, n\, e^{n(\mu-x)}\, dx$$

$$= \frac{x^2 \cdot e^{n(\mu-x)}}{-n} \Big|_{\mu}^{\infty} + \frac{2}{n} \int_{\mu}^{\infty} n x e^{n(\mu-x)}\, dx$$

$$= \frac{\mu^2}{n} + \frac{2}{n}\left[\frac{\mu}{n} + \frac{1}{n^2}\right] = \left(\mu + \frac{1}{n}\right)^2 + \frac{1}{n^2}$$

$$V(X_1) = \cancel{\left(\mu + \frac{1}{n}\right)^2} + \frac{1}{n^2} - \cancel{\left(\mu + \frac{1}{n}\right)^2} = \frac{1}{n^2}$$

$$\text{Bias}(X_1) = E[X_1] - \mu = \cancel{\mu} + \frac{1}{n} - \cancel{\mu} = \frac{1}{n}$$

$$\therefore MSE \text{ of } \delta_M(x) = \frac{1}{n^2} + \left(\frac{1}{n}\right)^2 = 2/n^2$$

$\Rightarrow$ MSE of $\delta_v(x)$:

$$MSE_2 = E\left[\left(x_1 - \frac{1}{n}\right) - \theta\right]^2 = V\left(x_1 - \frac{1}{n}\right) = V(x_1) = \frac{1}{n^2}$$

$\therefore MSE_2 < MSE_1$

$$\boxed{MSE\ \delta_v(x) < MSE\ \delta_M(x)}$$

(b) $X_1 \sim \exp(\theta)$; $\theta \in (0, \infty)$; $g(\theta) = \theta$

$$f(x|\theta) = \theta e^{-x\theta}$$

$$L(\theta) = \prod_{i=1}^{n} \theta e^{-x_i\theta}$$

$$\log(L(\theta)) = n\log\theta - \theta\sum_{i=1}^{n} x_i$$

$$\frac{d}{d\theta}(\log(L(\theta))) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \theta = \frac{1}{\left(\frac{\sum x_i}{n}\right)} = \frac{1}{\bar{x}}$$

$$\boxed{\delta_M(x) = \frac{1}{\bar{x}}}$$

We know that if X is exponential, $\sum_{i=1}^{n} x_i \sim \Gamma(n, \theta)$

$$\Downarrow$$

PDF: $\frac{\theta^n}{\Gamma(n)} x^{n-1} e^{-x\theta}$ $\quad x \in (0, \infty)$

$$E\left[\frac{n}{\sum x_i}\right] = n E\left[\frac{1}{\sum x_i}\right]$$

If $X \sim \Gamma(n, \theta)$, $E\left[\frac{1}{x}\right] = \frac{\theta}{n-1}$

Proof: $E[x^{-1}] = \int_0^\infty \frac{1}{x} \cdot \frac{\theta(\theta x)^{n-1}}{\Gamma(n)} e^{-\theta x}\, dx$

$$= \frac{\theta\,\Gamma(n-1)}{\Gamma(n)} \int_0^\infty \underbrace{\frac{\theta(\theta x)^{n-2} e^{-\theta x}}{\Gamma(n-1)}}_{\Gamma(n-1,\theta) \to PDF} dx$$

$$= \frac{\theta}{n-1}$$

$$\Rightarrow E\left[\frac{n-1}{\sum x_i}\right] = \theta \Rightarrow \boxed{\delta_U(X) = \frac{n-1}{\sum x_i}}$$

$\rightarrow$ MSE of $\delta_M(X)$: $V\left(\frac{1}{\bar{x}}\right) + Bias^2\left(\frac{1}{\bar{x}}\right)$ $\quad Y \sim \sum x_i$

$$\Rightarrow V\left(\frac{n}{\sum x_i}\right) + Bias^2\left(\frac{n}{\sum x_i}\right)$$

$$\Rightarrow n^2 V\left(\frac{1}{y}\right) + Bias^2\left[n\left(\frac{1}{y}\right)\right]$$

$$\boxed{V\left(\frac{1}{y}\right) = \frac{\theta^2}{(n-1)^2(n-2)}}$$

$$\Rightarrow \frac{n^2\theta^2}{(n-1)^2(n-2)} + \left(n \cdot \frac{\theta}{n-1} - \theta\right)^2 = \frac{\theta^2}{(n-1)^2}\left[\frac{n^2}{n-2} + 1\right]$$

$$= \frac{\theta^2}{(n-1)^2} \frac{(n-1)(n+2)}{(n-2)} = \frac{\theta^2}{n-2}\left[\frac{n+2}{n-1}\right]$$

$\rightarrow$ MSE of $\delta_U(x)$: $V\left(\frac{n-1}{\sum x_i}\right) + bias\left(\frac{n-1}{\sum x_i}\right)^2$ $\quad \hookrightarrow 0$

$$\Rightarrow \frac{\cancel{(n-1)^2}\,\theta^2}{\cancel{(n-1)^2}(n-2)} \Rightarrow \frac{\theta^2}{n-2}$$

$$\therefore \; MSE(\delta_U(x)) < MSE(\delta_M(x))$$

(c) $X_1 \sim U(0,\theta)$ $g(\theta) = \theta^r$

$U(0,\theta) \begin{cases} \frac{1}{\theta} & 0 < x_i < \theta \\ 0 & o.w \end{cases}$

$L(\theta) = \theta^{-n}$

$\Rightarrow$ $L(\theta)$ is max. at $\theta = X_{(n)}$

$\therefore$ $\underline{\underline{\delta_M(g(\theta))}} = (X_{(n)})^r$.

$\underline{\underline{f_{X_{(n)}}(x)}} \to ?$

$F_{X_{(n)}}(x) = P(X_n \le x)$

$= P(X_1 \le x, \dots X_n \le x)$

$= P[X_1 \le x]^n \Rightarrow \begin{cases} 0 & x < 0 \\ (x/\theta)^n & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$

$\Rightarrow f_{X_{(n)}}(x) \begin{cases} \frac{nx^{n-1}}{\theta^n} & \theta \ge x,\ x \ge 0 \\ 0 & ow \end{cases}$

$\Rightarrow E[X_{(n)}^r] = \int x^r f_{X_{(n)}}(x)\, dx$

$= \int_0^\theta x^r \frac{nx^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \int_0^\theta x^{r+n-1}\, dx$

$= \frac{n}{\theta^n} \left. \frac{x^{r+n}}{r+n} \right|_0^\theta$

$\Rightarrow E[X_{(n)}^r] = \frac{n}{n+r} \theta^r$

$\Rightarrow E\left[\left(\frac{n+r}{n}\right) X_{(n)}^r\right] = \theta^r$

$\therefore$ $\boxed{\delta_U(x) = \left(\frac{n+r}{n}\right) X_{(n)}^r}$

$\rightarrow$ MSE$(\delta_M(x))$:

$$E[(X_{(n)}^r - \theta^r)^2] = V(X_n^r) + \text{bias}^2(X_n^r)$$

$$\Rightarrow V(X_n^r) = E[(X_n^r)^2] - (E[X_n^r])^2$$

$$E[X_n^{2r}] = \int_0^\theta x^{2r} f_{X_n}(x)\,dx = \int_0^\theta x^{2r} \cdot \frac{n x^{n-1}}{\theta^n}\,dx$$

$$= \frac{n}{\theta^n} \int_0^\theta x^{n+2r-1} = \frac{n\theta^{2r}}{n+2r}$$

$$V(X_{(n)}^r) \Rightarrow \frac{n\theta^{2r}}{n+2r} - \left(\frac{n}{n+r}\right)^2 \theta^{2r} = \frac{n r^2 \theta^{2r}}{(n+r)^2 (n+2r)}$$

$$\text{Bias}(X_n^r) = (E[X_n^r] - \theta^r)^2$$

$$= \frac{n\theta^r}{n+r} - \theta^r = \frac{-r\theta^r}{n+r}$$

$$\text{MSE}(\delta_M(x)) = \frac{n r^2 \theta^{2r}}{(n+r)^2(n+2r)} + \frac{r^2 \theta^{2r}}{(n+r)^2} = \frac{2r^2\theta^{2r}}{(r+n)(n+2r)}$$

$\rightarrow$ MSE$(\delta_V(x))$: $E\left[\left(\left(\frac{n+r}{n}\right) X_n^r - \theta^r\right)^2\right] = V\left(\frac{n+r}{n} X_n^r\right)$

$$= \left(\frac{n+r}{n}\right)^2 V(X_n^r) = \left(\frac{\cancel{n+r}}{n}\right)^2 \cdot \frac{n r^2 \theta^{2r}}{\cancel{(n+r)^2}(n+2r)} = \frac{r^2\theta^{2r}}{n(n+2r)}$$

$$MSE(\delta_M(x)) = MSE(\delta_V(x)) \cdot \frac{2}{n(n+r)}$$

$$MSE_1 - MSE_2 = \frac{r^2\theta^{2r}}{(n+2r)} \times \frac{(n-r)}{n(n+r)} \Rightarrow$$

$MSE(\delta_U(x)) < MSE(\delta_M(x)) \rightarrow$ if $n > r$

$MSE(\delta_U(x)) = MSE(\delta_M(x)) \rightarrow$ if $n = r$

$MSE(\delta_U(x)) = MSE(\delta_M(x)) \rightarrow$ if $n < r$

(d) $X \sim N(\theta, 1)$; $g(\theta) = \theta^2$

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}$$

$$\log(L(\theta)) = \log\left(\frac{1}{(2\pi)^{1/2}}\right) - \frac{\sum_{i=1}^{n}(X_i - \theta)^2}{2}$$

$$\frac{d}{d\theta}(\log(L(\theta))) = \sum_{i=1}^{n} \frac{2(X_i - \theta)}{2} = 0$$

$$\Rightarrow \sum X_i = n\theta \Rightarrow \boxed{\hat{\theta} = \bar{X}}$$

$$\boxed{\delta_M(x) = \bar{x}} \Rightarrow \delta_M(g(\theta)) = \bar{x}^2$$

$$E[\bar{x}^2] = E[\sum_i X_i^2 + 2\sum_{i \neq j} X_i X_j]$$

$$= E[\sum_i X_i^2] + 2\sum_{i \neq j} E[X_i X_j]$$

$$\sum_i E[X_i^2] + 2\sum_{i \neq j} E[X_i]E[X_j] \leftarrow X_i, X_j \text{ are independent}$$

$$= n E[X^2] + 2\frac{(n)(n-1)}{2}\left(E(X)\right)^2$$

$$= n\theta^2 + n + n(n-1)\theta^2$$

$$\Rightarrow E[\bar{x}^2] = \frac{1}{n^2}\left(n + n^2\theta^2\right) = \underline{\underline{\theta^2 + \frac{1}{n}}}$$

$$E\left[x^2 - \frac{1}{n}\right] = \theta^2$$

$$\therefore x^2 - \frac{1}{n} \text{ is UE of } \theta^2$$

$\underline{MSE(\delta_M(x))}$: $E[(\bar{x}^2 - \theta^2)^2] = Var(\bar{x}^4) + Bias^2(\bar{x}^2)$

$$\text{MSE of } \delta_V(x) = E\left[x^2 - \frac{1}{n} - \theta^2\right]^2 = Var\left(x^2 - \frac{1}{n}\right) + Bias^2\left(x^2 - \frac{1}{n}\right)$$

These both are independent

$$= Var(\bar{x}^4) + \underbrace{Bias^2\left(x^2 - \frac{1}{n}\right)}_{0 \text{ (UE)}}$$

$$= Var(\bar{x}^4)$$

$$Bias(\bar{x}^2) = E[\bar{x}^2] - \theta^2$$

$$= \theta^2 + \frac{1}{n} - \theta^2$$

$$= \underline{\underline{\frac{1}{n}}}$$

$$MSE(\delta_M(x)) = Var(\bar{x}^4) + Bias^2(\bar{x}^2)$$

$$= Var(\bar{x}^4) + \frac{1}{n^2}$$

$$MSE(\delta_V(x)) = Var(\bar{x}^4)$$

$$\therefore MSE(\delta_{M(x)}) \geq MSE(\delta_V(x))$$

(2)

(a) $X_1 \sim U(-\theta, 2\theta)$; $g(\theta) = \theta$

$$f(x|\theta) \begin{cases} 1/3\theta & -\theta < x_1 < \dots < x_n < 2\theta \\ 0 & \text{else} \end{cases}$$

$L(\theta) = \left(\frac{1}{3\theta}\right)^n \Rightarrow$ MLE of θ is $X_{(n)}$

$E[X] = \frac{2\theta - \theta}{2} = \frac{\theta}{2}$

$\Rightarrow E[\bar{x}] = \mu$, We know that sample mean $\bar{x}$ is Consistent for Population mean,

$$P(|\bar{X} - \mu| > \varepsilon) \leq \frac{Var(\bar{x})}{\varepsilon^2}$$

$$\leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\leq \frac{(b-a)^2}{12n\varepsilon^2} \to 0 \text{ as } n \to \infty$$

$\therefore \bar{X}$ is CE for $\theta/2$

$\Rightarrow 2\bar{X}$ is CE for θ.

(d) $X_1 \sim Poisson(\theta)$; $g(\theta) = e^\theta$

$\bar{X}$ is Consistent for θ, $\to$ by WLLN.

As $g(\theta)$ is Continous function,

$e^{\bar{X}}$ is Consistent for e^θ.

(b) $X_1 \sim N(\mu, \sigma^2)$; $g(\theta) = \mu/\sigma$

$\bar{X}$ is consistent for μ

$(MLE_{\sigma^2}) \Rightarrow \prod_{i=1}^{n} f(x|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma^n} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$

$\log(L(\sigma^2)) = \sum -\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2 - n\log\sigma$

$= \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$

$= \frac{-n}{2\sigma^2} + \frac{1}{2}\left[\sum_{i=1}^{n}(x_i-\mu)^2\right]\frac{1}{(\sigma^2)^2}$

$= \frac{1}{2\sigma^2}\left[\frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)^2\right] - n\right]$

$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu)^2}$ (as $\sigma \neq 0$)

$\left(\frac{n-1}{n}\right)s^2 \to$ unbiased for σ^2

$P\left(\left|\frac{n-1}{n}s^2 - \sigma^2\right| > \varepsilon\right) \le \frac{Var\left(\frac{n-1}{n}s^2\right)}{\varepsilon^2}$

$\le \left(\frac{n-1}{n}\right)^2 \frac{Var(s^2)}{\varepsilon^2}$

$\le \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{(n-1)} \cdot \frac{1}{\varepsilon^2} \to \left[\frac{n-1}{\sigma^2}s^2 \sim \chi^2_{n-1}\right]$

$\le \frac{2\sigma^4}{\left(\frac{n^2}{n-1}\right)\varepsilon^2} \longrightarrow 0$ as $n \to \infty \Rightarrow \left(\frac{n-1}{n}\right)s^2$ is CE of σ^2

$g(\theta) = \mu/\sigma \Rightarrow CE = \dfrac{\bar{x}}{\sqrt{\frac{n-1}{n}s^2}} \Rightarrow \underline{\underline{\sqrt{\frac{n}{n-1}} \cdot \dfrac{\bar{x}}{s}}}$

ⓒ $g(\theta) = \mu + b\sigma \Rightarrow CE = \bar{x} + b\left(\sqrt{\frac{n-1}{n}}\, s\right)$

as $g(\theta)$ is Conthhous

③ $X_i \sim U(0,\theta) \quad \theta \in \Theta = (0,\infty)$

<u>MME:</u> $M_1 = E[X] = \theta/2$

$m_1 = \dfrac{\sum x_i}{n}$

$M_1 = m_1$

$\Rightarrow \boxed{\hat{\theta} = 2\bar{x}}$

<u>MLE:</u>

$\ln(L(\theta)) = -n\ln\theta$

$\boxed{\hat{\theta} = X_{(n)}}$

<u>Bias of MLE Estimator:</u>

$E[X_{(n)}] = \int_0^\theta x\, f_x(x_{(n)})$

$F_{X_{(n)}}(x) = P(X_{n} \le x)$

$= P(X_1 \le x, X_2 \le x, \ldots X_n \le x)$

$= [P[X_{(n)} \le x]]^n \begin{cases} 0 & x < 0 \\ \left(\frac{x}{\theta}\right)^n & 0 < x \le \theta \\ 0 & \text{else} \end{cases}$

$\underline{\underline{f_{x_n}(x) = \dfrac{n x^{n-1}}{\theta^n}}}$

$E[X_n] - \theta \Rightarrow$ Bias of MLE estimator

$$E[X_n] = \int_0^\theta x_{(n)}\, n \cdot \frac{x^{n-1}}{\theta^n} = \left(\frac{n}{n+1}\right) \cdot \frac{1}{\theta^n}\, x^{n+1} \Big|_0^\theta$$

$$= \frac{n\theta}{n+1}$$

$$\text{Bias} = \frac{-\theta}{n+1} \quad (\text{underbiased})$$

Bias of MME:

$$2E[\bar{x}] - \theta$$

$$= 2\left[\frac{\theta}{2}\right] - \theta$$

$$= 0$$

$\therefore$ As Bias of MME is 0, it is better estimator compared to MLE.

(4) $X_1, X_2 \sim \exp(\lambda)$

$$T_1 = \frac{X_1 + X_2}{2} \; ; \; T_2 = \sqrt{X_1 X_2}$$

$$E[T_1] = \frac{1}{2} E[X_1 + X_2] = E[X]$$ [As X_1, X_2 are independent and indenticially distributed and by applying linearity]

As $E[X] = \frac{1}{\lambda}$

$\therefore \frac{X_1 + X_2}{2}$ is unbiased estimator for mean.

$$\therefore \ MSE(T_2) = Var(\sqrt{x_1 x_2}) + Bias^2(\sqrt{x_1 x_2})$$

$$= \frac{1}{\lambda^2} - \frac{\pi^2}{\lambda^2 . 16} + \frac{1}{\lambda^2}\left[\frac{\pi}{4} - 1\right]^2$$

$$= \frac{2}{\lambda^2} - \frac{\pi}{2\lambda^2}$$

$$MSE(T_2) = \frac{4-\pi}{2\lambda^2}, \quad 4-\pi < 1$$

$$\therefore \ MSE(T_2) \leq Var(T_1)$$

(5) T_1, T_2 are UE of θ

$\downarrow \quad \downarrow$

$\sigma_1^2 \quad \sigma_2^2$

$$Cov(T_1 T_2) = \sigma_{12}$$

$$T = \alpha T_1 + (1-\alpha) T_2 \qquad 0 \leq \alpha \leq 1$$

$E[T] = \alpha E[T_1] + (1-\alpha) E[T_2]$ [Linearity of expectation]

$= \alpha\theta + (1-\alpha)\theta$ [$T_1 T_2$ are Unbiased]

$E[T] = \theta$, Hence T is also Unbiased estimator of θ.

$$Var(aA + bB) = a^2 Var(A) + b^2 Var(B) + 2ab\, Cov(A, B)$$

$$\Rightarrow Var(T) = \alpha^2 Var(T_1) + (1-\alpha)^2 Var(T_2) + 2\alpha(1-\alpha)\, Cov(T_1, T_2)$$

$$\Rightarrow \quad = \alpha^2(\sigma_1^2) + (1-\alpha)^2(\sigma_2^2) + 2\alpha(1-\alpha)\, Cov(T_1 T_2)$$

$$Var(T) = \alpha^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) + 2\alpha(\sigma_{12} - \sigma_2^2) + \sigma_2^2$$

$T_2 \rightarrow \sqrt{X_1 X_2}$

$E[\sqrt{X_1 X_2}] = E[\sqrt{X_1}]E[\sqrt{X_2}]$ (iids)

$E[\sqrt{X}] = \int_0^\infty \sqrt{x}\, \lambda e^{-\lambda x} dx \Rightarrow \sqrt{\lambda x} = t$

$\frac{2t dt}{\lambda} = dx$

$\Rightarrow \int_0^\infty \frac{t}{\sqrt{\lambda}} \cdot \cancel{\lambda} e^{-t^2} \cdot \frac{2t dt}{\cancel{\lambda}}$

$\Rightarrow \frac{2}{\sqrt{\lambda}} \int_0^\infty t^2 e^{-t^2} dt$

We know that,

$\int_{-\infty}^{\infty} e^{-\alpha x^2} = \left(\frac{\pi}{\alpha}\right)^{1/2}$

Differentiating wrt 'α';

$\boxed{\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} = \frac{\pi^{1/2}}{2\alpha^{3/2}}}$

$\Downarrow$

$\int_0^\infty x^2 e^{-\alpha x^2} = \frac{\pi^{1/2}}{4\alpha^{3/2}}$

$\Rightarrow \frac{2}{\sqrt{\lambda}} \cdot \frac{\pi^{1/2}}{4} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$

$E[\sqrt{X_1 X_2}] = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \cdot \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = \pi/4\lambda$ → Biased estimator.

$Var(T_1) = \frac{\sigma^2}{n} = \frac{1}{\lambda^2} \cdot \frac{1}{2} = \frac{1}{2\lambda^2}$

$Var(\sqrt{X_1 X_2}) = E[X_1 X_2] - (E[\sqrt{X_1 X_2}])^2$

$= \frac{1}{\lambda^2} - \frac{\pi^2}{16\lambda^2} = \frac{1}{\lambda^2}\left[1 - \frac{\pi^2}{16}\right]$

$$\frac{d}{d\alpha}(Var(T)) = 2\alpha(\sigma_1^2+\sigma_2^2-2\sigma_{12}) + (\sigma_{12}-\sigma_2^2) = 0$$

$$\Rightarrow \alpha = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2+\sigma_2^2-2\sigma_{12}}$$

⑥ $f(x) = \frac{x}{\theta}\exp\left(-\frac{x^2}{2\theta}\right) \quad x>0,\ \theta>0$

UMVUE for $g(\theta) = \theta$.

$\psi'(\theta) = 1$

$$\log f_\theta(x) = \log x - \log\theta - \frac{x^2}{2\theta}$$

$$\frac{\partial}{\partial\theta}(\log f_\theta(x)) = -\frac{1}{\theta} + \frac{x^2}{2\theta^2} \Rightarrow \text{is in form}$$

$$T(x) - \psi(\theta) = K(\theta)\frac{\partial}{\partial\theta}\log f_\theta(x)$$

hence CRLB can be attained.

$$I(\theta) = E\left[\frac{x^2}{2\theta^2}-\frac{1}{\theta}\right]^2 = \frac{1}{4\theta^4}E[x^2-2\theta] \longrightarrow (1)$$

$$E[x^2] = \int_0^\infty x^2 \cdot \frac{x}{\theta}\cdot e^{-\frac{x^2}{2\theta}}\,dx$$

Let $x^2 = t$

$2x\,dx = dt$

$$\int_0^\infty t\cdot\frac{1}{\theta}e^{-t/2\theta}\cdot\frac{dt}{2} = \frac{1}{2\theta}\int_0^\infty te^{-t/2\theta}\,dt$$

$$\Rightarrow t\int_0^\infty e^{-t/2\theta}dt + \int(-2\theta)e^{-t/2\theta}dt$$

$$= \frac{4\theta^2}{2\theta} = 2\theta \qquad \therefore E[x^2] = 2\theta$$

$\Rightarrow I(\theta) = \frac{1}{4\theta^4} Var(X^2)$ (from ①)

$Var(X^2) = E[X^4] - (E[X^2])^2$

$\Rightarrow E[X^4] = \int_0^\infty x^4 \cdot \frac{x}{\theta} e^{-x^2/2\theta} dx$

$x^2 = t \rightarrow 2x dx = dt$

$\Rightarrow \frac{1}{2\theta} \int_0^\infty x^5 e^{-t/2\theta} dt$

$\Rightarrow \frac{1}{2\theta} \int_0^\infty t^2 e^{-t/2\theta} dt$

$= 2 \cdot \int_0^\infty t e^{-t/2\theta} dt = 2 \cdot \left[\frac{t e^{-t/2\theta}}{-1/2\theta} \Big|_0^\infty + 2\theta \int_0^\infty e^{-t/2\theta} dt \right]$

$= 2 \cdot 2\theta \cdot 2\theta = 8\theta^2$

$Var(X^2) = 8\theta^2 - (2\theta)^2 = 4\theta^2$

$I_1(\theta) = \frac{1}{\theta^2} \Rightarrow I(\theta) = 2/\theta^2 \Rightarrow CRLB = \theta^2/2$

Let $T(X) = \frac{X_1^2 + X_2^2}{4}$

$E[T(X)] = \frac{1}{4} \times 2 \times E[X^2] = \theta$

$Var(T(X)) = \frac{1}{16} \cdot 2 \cdot var[X^2] = \frac{1}{8} \times 4\theta^2 = \frac{\theta^2}{2} = CRLB$

$\therefore$ UMVUE is $\frac{X_1^2 + X_2^2}{4}$

(7) $X_i \sim f(x) = \theta(1+x)^{-(1+\theta)}$ $\quad x>0, \theta>0$

UE of $1/\theta$, UMVUE of $1/\theta$.

$\psi(\theta) = 1/\theta \Rightarrow \psi'(\theta) = \dfrac{-1}{\theta^2}$

$\log f_\theta(x) = \log\theta - (1+\theta)\log(1+x)$

$\dfrac{d}{d\theta}\log f_\theta(x) = \dfrac{1}{\theta} - \log(1+x)$

$E\left[\dfrac{\partial}{\partial\theta}\log f_\theta(x)\right]^2 = E\left[\left(\dfrac{1}{\theta} - \log(1+x)\right)^2\right]$

$= E\left[\left(\log(1+x) - \dfrac{1}{\theta}\right)^2\right]$

$= Var(\log(1+x)) + Bias^2(\log(1+x))$

$\Downarrow$

$E[\log^2(1+x)] - E[\log(1+x)]^2$

$\rightarrow E[\log(1+x)]: \quad \displaystyle\int_0^\infty \log(1+x)\,\theta\,(1+x)^{-(1+\theta)}\,dx$

$\Rightarrow 1+x = e^{t/\theta}$

$dx = e^{t/\theta}\cdot\dfrac{dt}{\theta}$

$\Rightarrow \displaystyle\int_0^\infty \frac{t}{\theta}\cdot\theta\cdot e^{-\frac{t(1+\theta)}{\theta}}\cdot e^{t/\theta}\,\frac{dt}{\theta}$

$\Rightarrow \dfrac{1}{\theta}\displaystyle\int_0^\infty t e^{-t}\,dt$

$= \underline{\underline{1/\theta}}$

$\rightarrow E[\log^2(1+x)] = \int_0^\infty \log^2(1+x) \cdot \theta \cdot (1+x)^{-(1+\theta)} dx$

$\rightarrow 1+x = e^{t/\theta}, \quad dx = e^{t/\theta}\frac{dt}{\theta}$

$= \int_0^\infty \frac{t^2}{\theta^2} \cdot \cancel{\theta} \cdot e^{-\frac{t(1+\theta)}{\theta}} \cdot e^{t/\theta} \frac{dt}{\cancel{\theta}}$

$= \frac{1}{\theta^2} \int_0^\infty t^2 e^{-t} dt$

$= \frac{1}{\theta^2}\left[t^2 e^{-t} \Big|_0^\infty + 2\int t e^{-t} dt \right]$

$= \frac{2}{\theta^2}$

$\therefore Var(\log(1+x)) = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$

$\Rightarrow Bias^2[\log(1+x)] = E[\log(1+x)] - 1/\theta = 0$

$I_1(\theta) = Var(\log(1+x)) + Bias^2(\log(1+x))$

$= \frac{2}{\theta^2} - \frac{1}{\theta^2} + 0 = \frac{1}{\theta^2}$

$I(\theta) = \frac{2}{\theta^2} \Rightarrow CRLB = \frac{(\psi'(\theta))^2}{I(\theta)} = \frac{\left(\frac{-1}{\theta^2}\right)^2}{2/\theta^2} = \frac{1}{2\theta^2}$

Consider $T(x) = \frac{\log(1+X_1) + \log(1+x_2)}{2}$

$E[T(X)] = \frac{1}{2} \cdot 2 \cdot E[\log(1+x_1)] = \frac{1}{\theta}$

$$Var(T(x)) = \frac{1}{4} \cdot 2 \cdot Var(\log(1+X_1)) = \frac{1}{2}\cdot\frac{1}{\theta^2} = \frac{1}{2\theta^2} = \text{CRLB}$$

$$\therefore \frac{1}{2}\left[\log(1+X_1) + \log(1+X_2)\right] \rightarrow \text{UMVUE of } \frac{1}{\theta}.$$

(9) (a) $\Gamma(\alpha,\beta) = \frac{x^{\beta-1} e^{-x/\alpha}}{\alpha^{\beta}\Gamma(\beta)} I_x$

Exponential family:

$$h(x)\, c(\theta)\, e^{Q(\theta)T(x)}$$

→ α is unknown

$$\frac{I_x}{\Gamma(\beta)} \cdot \frac{x^{\beta-1} e^{-x/\alpha}}{\alpha^{\beta}} = \frac{I_x}{\Gamma(\beta)} \cdot x^{\beta-1} e^{-x/\alpha - \beta\log\alpha}$$

$$= \underbrace{\frac{x^{\beta-1}}{\Gamma(\beta)}}_{h(x)} \cdot \underbrace{e^{-x/\alpha}}_{c(\alpha)} \cdot e^{-\beta\log\alpha}$$

$T(x) = -\beta$

$Q(\alpha) = \log\alpha$

$\therefore$ Hence an exponential family.

→ β is unknown

$$\Rightarrow e^{-x/\alpha} \cdot \frac{x^{\beta-1}}{\alpha^{\beta}\Gamma(\beta)}$$

$$\Rightarrow e^{-x/\alpha} \cdot e^{\beta-1(\log x) - \beta\log\alpha - \log(\Gamma(\beta))}$$

$$\Rightarrow \underbrace{e^{-x/\alpha}}_{h(x)} \cdot \underbrace{e^{-\beta\log\alpha - \log\Gamma(\beta)}}_{C(\beta)} \cdot e^{(\beta-1)\log x}$$

$T(x) = \log x$

$Q(\beta) = \beta - 1$

→ α, β both are unknown,

$$= \frac{e^{-x/\alpha} \cdot x^{\beta-1}}{\alpha^{\beta}\Gamma(\beta)} = e^{-x/\alpha + (\beta-1)\log x - \beta\log\alpha - \log\Gamma(\beta)}$$

$$= \underbrace{1}_{h(x)} \cdot \underbrace{e^{-\beta\log\alpha - \log\Gamma(\beta)}}_{c(\theta)} \cdot e^{\overbrace{-x/\alpha + (\beta-1)\log x}}$$

$\theta = (\alpha, \beta)$

$T_1(x) = \cancel{x}$

$Q_1(\alpha,\beta) = -1/\alpha$

$T_2(x) = \log x$

$Q_2(\alpha,\beta) = \beta - 1$

∴ Hence its an exponential family.

(b) Beta family:

$$\beta(\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

→ α is unknown:

$$(1-x)^{\beta-1} \cdot \frac{x^{\alpha-1}}{B(\alpha,\beta)}$$

$$= \underbrace{(1-x)^{\beta-1}}_{h(x)} \underbrace{e^{-\log B(\alpha,\beta)}}_{c(\alpha)} \cdot e^{\overbrace{(\alpha-1)\log x}}$$

$T(x) = \log x \quad Q(\alpha) = \alpha - 1$

→ β is unknown:

$$x^{\alpha-1} \cdot e^{(\beta-1)\log(1-x) - \log B(\alpha,\beta)}$$

$$\Rightarrow \underbrace{x^{\alpha-1}}_{h(x)} \cdot \underbrace{e^{-\log B(\alpha,\beta)}}_{c(\beta)} \cdot e^{\overbrace{(\beta-1)\log(1-x)}}$$

$T(x) = \log(1-x)$

$Q(\beta) = \beta - 1$

$\rightarrow \alpha, \beta$ are unknown.

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

$$= e^{(\alpha-1)\log x + (\beta-1)\log(1-x) - \log B(\alpha,\beta)}$$

$$= \underbrace{1}_{h(x)} . \underbrace{e^{-\log B(\alpha,\beta)}}_{C(\alpha,\beta)} . e^{(\alpha-1)\log x + (\beta-1)\log(1-x)}$$

$\downarrow$

$T_1(x) = \log x \quad T_2(x) = \log(1-x)$

$Q_1(\alpha,\beta) = \alpha - 1 \quad Q_2(\alpha,\beta) = \beta - 1$

Exponential family $\rightarrow C(\theta)\, h(x)\, e^{\sum_i Q_i(\theta) T_i(x)}$

(c) Negative binomial. With r known $0<p<1$

$$f(k;r,p) = \binom{k+r-1}{k} p^r (1-p)^k$$

$$= \underbrace{\binom{k+r-1}{r}}_{h(k)} \underbrace{e^{r\log p}}_{c(p)} e^{\underbrace{k\log(1-p)}}$$

$Q(p) = \log(1-p)$

$T(p) = k$

$\therefore$ Hence an exponential family.

(10) (a) $X \sim \text{Cauchy}(1, \theta)$

$$f(x_0, \delta) = \frac{1}{\pi\delta}\left[\frac{\delta^2}{(x-x_0)^2+\delta^2}\right]$$

$$f(1,\theta) = \frac{1}{\pi\theta\left[1+\left(\frac{x-1}{\theta}\right)^2\right]}$$

$$= \frac{1}{\pi\theta} \cdot e^{-\log\left[1+\left(\frac{x-1}{\theta}\right)^2\right]},$$

Let $-\log\left[1+\left(\frac{x-1}{\theta}\right)^2\right] = Q(\theta)\,T(x)$

Let $Q(1) = k$ & $\theta = 1$

& $T(x) = -\frac{\log\left[1+(x-1)^2\right]}{k}$

$$\left.Q(\theta) = \frac{-\log\left[1+\left(\frac{x-1}{\theta}\right)^2\right]k}{-\log\left[1+(x-1)^2\right]}\right\} \rightarrow Q \text{ is also a fn of } x.$$

$\therefore$ Cauchy$(1,\theta)$ is not one Parameter exponential family.

(b) $U(0,\theta)$: $f(x) = \frac{x}{\theta}$ $0 < x \le \theta$

$$x \cdot e^{-\log\theta} \cdot I_{(0,\theta)}(x)$$

As x depends on θ,

$U(0,\theta)$ is not an exponential.

$$= \frac{\left[1+(1-2\theta)^2\right]\cdot\frac{1}{2} - (1-2\theta)^2}{4\theta^2(1-\theta)^2} = \frac{\frac{1}{2}\left[1+(1-2\theta)^2 - 2(1-2\theta)^2\right]}{4\theta^2(1-\theta)^2}$$

$$= \frac{\frac{1}{2}\left[1-(1-2\theta)^2\right]}{4\theta^2(1-\theta)^2} = \frac{\frac{1}{2}\left[4\theta - 4\theta^2\right]}{4\theta^2(1-\theta)^2} = \underline{\underline{\frac{1}{2\theta(1-\theta)}}}$$

$$I(\theta) = n\, I_1(\theta) = \frac{1}{\theta(1-\theta)} \quad [n=2]$$

$$\psi(\theta) = \theta,\ \psi'(\theta) = 1,\ CRLB = \frac{1}{1/\theta(1-\theta)} = \underline{\underline{\theta(1-\theta)}}$$

$\therefore$ CRLB for Variance of UE of θ is $\theta(1-\theta)$.

Given $\bar{x} + \frac{1}{2} = T$

$$E[T] = E\left[\bar{x} + \tfrac{1}{2}\right] \text{ — } (1)$$

$$\rightarrow E[\bar{x}] = \frac{-1(1-\theta)}{2} + 0\left(\frac{1}{2}\right) + \frac{\theta}{2} = \theta - \frac{1}{2}$$

$$\rightarrow E[x^2] = \frac{1-\theta}{2} + 0\left(\frac{1}{2}\right) + 1\cdot\left(\frac{\theta}{2}\right) = \underline{\underline{\frac{1}{2}}}$$

$$\Rightarrow E[T] = \theta - \frac{1}{2} + \frac{1}{2} = \underline{\underline{\theta}}$$

$\therefore$ T is an Unbiased estimator for θ.

$$\rightarrow Var(T) = E[T^2] - \underbrace{(E[T])^2}_{\rightarrow\, 0 \text{ as it is an UE.}}$$

$$Var\left(\bar{x} + \tfrac{1}{2}\right) = Var(\bar{x}) = \frac{1}{4} Var(x_1 + x_2) = \frac{1}{2} Var(x_1)$$

$$= \frac{1}{2}\left(E[x_1^2] - (E[x_1])^2\right) = \frac{1}{2}\left[\frac{1}{2} - \left(\theta - \frac{1}{2}\right)^2\right]$$

(8) $P(X=-1) = \frac{1-\theta}{2}$, $P(X=0) = \frac{1}{2}$, $P(X=1) = \frac{\theta}{2}$

Pmf $f(x) = \frac{1}{2}\theta^{\frac{x(x+1)}{2}}(1-\theta)^{\frac{x(x-1)}{2}}$

$$\log(f(x)) = \log\frac{1}{2} + \frac{x(x+1)}{2}\log\theta + \frac{x(x-1)}{2}\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}(\log f(x)) = \frac{x(x+1)}{2\theta} - \frac{x(x-1)}{2(1-\theta)}$$

$$= \frac{x}{2}\left[\frac{x+1}{\theta} - \frac{x-1}{1-\theta}\right]$$

$$= \frac{x}{2}\left[\frac{x - x\theta + 1 - \theta - \theta x + \theta}{\theta(1-\theta)}\right]$$

$$= \frac{x}{2} \times \frac{x - 2\theta x + 1}{\theta(1-\theta)} = \frac{x^2(1-2\theta) + x}{2\theta(1-\theta)}$$

$$I_1(\theta) = E\left[\frac{\partial}{\partial\theta}\log f_\theta(x)\right]^2$$

$$= E\left[\frac{x^2(1-2\theta)+x}{2\theta(1-\theta)}\right]^2 = \frac{1}{4\theta^2(1-\theta)^2}E\left[x^2(1-2\theta)+x\right]^2$$

$$= \frac{1}{4\theta^2(1-\theta)^2} \cdot E\left[x^4(1-2\theta)^2 + x^2 + 2x^3(1-2\theta)\right]$$

↳ getting individual expectations

$$= \frac{1}{4\theta^2(1-\theta)^2}\left[0.0 + (1-2\theta)^2 + 1 + 2(1-2\theta)\right]\frac{\theta}{2} + \left[(1-2\theta)^2 + 1 - 2(1-2\theta)\right]\left(\frac{1-\theta}{2}\right)$$

$$= \frac{\left[(1-2\theta)^2+1\right]\left[\frac{\theta}{2}+\frac{1}{2}-\frac{\theta}{2}\right] + \frac{2(1-2\theta)}{2}\left[\theta - 1 + \theta\right]}{4\theta^2(1-\theta)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \theta^2 - \frac{1}{4} + \theta\right]$$

8 $$= \frac{1}{2}\left[\frac{1}{4} + \theta(1-\theta)\right]$$

$$Var(T) = \frac{1}{8} + \frac{1}{2}\theta(1-\theta)$$

Considering $Var(T) - CRLB = \frac{1}{8} + \frac{1}{2}\theta(1-\theta) - \theta(1-\theta)$

$$= \underbrace{\frac{1}{8} - \frac{1}{2}\theta(1-\theta)}_{g(\theta)}$$

$$g'(\theta) = -\frac{1}{2}(1-2\theta) = 0 \Rightarrow \underline{\underline{\theta = \frac{1}{2}}}$$

$$\Rightarrow g''(\theta) = \frac{1}{2} > 0$$

$$g\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \underline{\underline{0}}$$

$$Var(T) - CRLB \geq 0$$

$$Var(T) \geq CRLB$$

$\therefore$ Variance of $\bar{x} + \frac{1}{2} \geq CRLB$.