# Assignment - 2 Solutions

1. (a).

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & if \;\; x > \theta, \\ \\ 0 & otherwise. \end{cases}$$

MLE of $g(\theta) = \theta$ is $\delta_M(\underline{x}) = X_{(1)}$. The density function of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = \begin{cases} e^{-n(x-\theta)} & if \;\; x > \theta, \\ \\ 0 & otherwise. \end{cases}$$

$$E_\theta(X_{(1)}) = \theta + \frac{1}{n}, \;\; for \;\; all \;\; \theta \in (-\infty, \infty)$$
$$E_\theta(X_{(1)} - \frac{1}{n}) = \theta, \;\; for \;\; all \;\; \theta \in (-\infty, \infty)$$

$\implies$ Unbiased estimator of $g(\theta) = \theta$ is $\delta_U(\underline{x}) = X_{(1)} - \dfrac{1}{n}$.
Now, we compare $\delta_U(\underline{x}), \delta_M(\underline{x})$ through MSE,

$$\begin{aligned} MSE_{\delta_M(\underline{x})} - MSE_{\delta_U(\underline{x})} &= E_\theta(X_{(1)} - \theta)^2 - E_\theta(X_{(1)} - \frac{1}{n} - \theta)^2 \\ &= \frac{2}{n} E_\theta(X_{(1)} - \theta) - \frac{1}{n^2} \\ &= \frac{2}{n^2} - \frac{1}{n^2} \\ &> 0. \end{aligned}$$

$\therefore \delta_U(\underline{x})$ is better than $\delta_M(\underline{x})$.

(b). $X \sim Exp(\theta), \;\; g(\theta) = \theta.$
MLE of $g(\theta) = \theta$ is $\delta_M(\underline{x}) = \overline{X} = \dfrac{\sum_{i=1}^n X_i}{n} = \dfrac{T}{n}$.

$$E_\theta(T) = n\theta, \;\; for \;\; all \;\; \theta > 0,$$
$$E_\theta\left(\frac{T}{n}\right) = \theta$$

$\therefore \delta_U(\underline{x}) = \delta_M(\underline{x}) = \overline{X}$.

(c). $X \sim U(0, \theta), \;\; g(\theta) = \theta^r$.

MLE of $\theta$ is $X_{(n)} \implies$ MLE of $g(\theta) = \theta^r$ is $\delta_M(\underline{x}) = X_{(n)}^r$. The density function of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \begin{cases} \dfrac{nx^{n-1}}{\theta^n} & if \;\; 0 < x < \theta, \\ \\ 0 & otherwise. \end{cases}$$

$$\begin{aligned} E_\theta(X_{(n)}^r) &= \frac{n}{n+r}\theta^r \\ \implies \delta_U(\underline{x}) &= \frac{n+r}{n} X_{(n)}^r. \end{aligned}$$

$$\begin{aligned} MSE_{\delta_M(\underline{x})} - MSE_{\delta_U(\underline{x})} &= E_\theta(X_{(n)}^r - \theta^r)^2 - E_\theta\left(\frac{n+r}{n}X_{(n)}^r - \theta^r\right)^2 \\ &= \frac{r^2(n-r)}{n(n+r)(n+2r)}\theta^{2r} \end{aligned}$$

Thus,

$$\begin{aligned} &for \;\; n > r, \;\; \delta_U(\underline{x}) \;\; is \;\; better \;\; than \;\; \delta_M(\underline{x}) \\ &for \;\; n < r, \;\; \delta_M(\underline{x}) \;\; is \;\; better \;\; than \;\; \delta_U(\underline{x}) \\ &for \;\; n = r, \;\; both \;\; hve \;\; same \;\; MSE. \end{aligned}$$

(d). $X \sim N(\theta, 1), \;\; g(\theta) = \theta^2$

MLE of $\theta^2$ is $\delta_M(\underline{x}) = \overline{X}^2$ since the MLE of $\theta$ is $\overline{X}$. $\overline{X} \sim N\left(\theta, \frac{1}{n}\right)$, $E_\theta(\overline{X}^2) = \frac{1}{n} + \theta^2 \implies$ $E_\theta(\overline{X}^2 - \frac{1}{n}) = \theta^2, \;\; for \;\; all \;\; \theta \implies \delta_U(\underline{x}) = \overline{X}^2 - \frac{1}{n}$.

$$\begin{aligned} MSE_{\delta_M(\underline{x})} - MSE_{\delta_U(\underline{x})} &= E_\theta(\overline{X}^2 - \theta^2)^2 - E_\theta\left(\overline{X}^2 - \frac{1}{n} - \theta^2\right)^2 \\ &= \frac{1}{n^2} > 0. \end{aligned}$$

$\therefore \delta_U(\underline{x})$ is better than $\delta_M(\underline{x})$.

2. (a). $X \sim U(-\theta, 2\theta), \;\; g(\theta) = \theta$.

$\mu_1^1 = \frac{3\theta}{2}$ and $E_\theta(T) = E_\theta\left(\frac{2\overline{X}}{3}\right) = \theta$

$T = \frac{2\overline{X}}{3}$ is unbiased and consistent for $\theta$.

(b)&(c). $X \sim N(\mu, \sigma^2) \implies \overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$. Let $s^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \implies W = \dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$.

$$E(\sqrt{W}) = E\left(\frac{\sqrt{n-1}s}{\sigma}\right) = \frac{\sqrt{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \implies E(T_1) = E\left(s.\frac{\sqrt{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right) = \sigma$$

Similarly,

$$E\left(\frac{1}{\sqrt{W}}\right) = E\left(\frac{\sigma}{s\sqrt{n-1}}\right) = \frac{\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n-1}{2}\right)} \implies E(T_2) = E\left(\frac{\sqrt{\frac{2}{n-1}}\Gamma\left(\frac{n-1}{2}\right)}{s \cdot \Gamma\left(\frac{n-2}{2}\right)}\right) = \frac{1}{\sigma}$$

As $\overline{X}$ & $s$ are independent,

$$U_1 = \overline{X}T_2 \ is\ unbiased\ and\ consistent\ for\ \frac{\mu}{\sigma}$$

$$U_2 = \overline{X} + bT_1\ is\ unbiased\ and\ consistent\ for\ \mu + b\sigma$$

(d). $X \sim poisson(\theta),\ g(\theta) = e^{\theta}$.

$\overline{X}$ is an unbiased estimator of $\theta$ and $\overline{X} \xrightarrow{p} \theta$. Using the properties of limits one can prove that $e^{\overline{X}} \xrightarrow{p} e^{\theta}$. Therefore, $e^{\overline{X}}$ is consistent estimator for $e^{\theta}$.

3. $X_1, X_2, ..., X_n \sim U(0, \theta)$

$$MLE\ of\ \theta\ is\ \delta_{MLE}(\underline{x}) = X_{(n)}$$
$$MME\ of\ \theta\ is\ \delta_{MME}(\underline{x}) = 2\overline{X}$$

$E_\theta(X_{(n)}) = \dfrac{n\theta}{n+1}, E_\theta(X_{(n)})^2 = \dfrac{n\theta^2}{n+2}$ $and\ Var_\theta(X_{(n)}) = \dfrac{n\theta^2}{(n+2)(n+1)^2}$

$E_\theta(X_1) = \dfrac{\theta}{2}, E_\theta(X_1^2) = \dfrac{\theta^2}{3}\ and\,Var_\theta(X_1) = \dfrac{\theta^2}{12} \implies MSE_{\delta_{MME}(\underline{x})} = E_\theta\left((2\overline{X} - \theta)^2\right) = \dfrac{\theta^2}{3n}$

$$MSE_{\delta_{MME}(\underline{x})} - Var_\theta(X_{(n)}) = \frac{\theta^2}{3n} - \frac{n\theta^2}{(n+2)(n+1)^2} \geq 0$$

4. $X_1, X_2 \sim exp(\lambda)$, mean $\dfrac{1}{\lambda}$

$$T_1 = \frac{X_1 + X_2}{2}, T_2 = \sqrt{X_1 X_2}$$

$E(X_i) = \dfrac{1}{\lambda} \implies T_1\ is\ unbiased\ for\ \dfrac{1}{\lambda}$. $E(T_2) = E(\sqrt{X_1X_2}) = \left(E(\sqrt{X_1})\right)^2 = \dfrac{\pi}{4\lambda}$ and MSE$(T_2)$ = E$\left(\sqrt{X_1X_2} - \dfrac{1}{\lambda}\right)^2 = \dfrac{2}{\lambda^2}\left(1 - \dfrac{\pi}{4}\right)$. $T_2$ is better than $T_1$ since $Var(T_1) = \dfrac{1}{\lambda^2} > MSE(T_2)$.

5. Given, $T = \alpha T_1 + (1-\alpha)T_2 \implies E(T) = \alpha E(T_1) + (1-\alpha)E(T_2) = \alpha\theta + (1-\alpha)\theta = \theta$, this proves $T$ is an unbiased estimator of $\theta$.

$$Var(T) = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + (\alpha - \alpha^2)\sigma_{12}$$

$Var(T)$ attains minimum at $\alpha = \dfrac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$

6.
$$f(x|\theta) = \frac{x}{\theta} exp\left\{\frac{-x^2}{2\theta}\right\}, \;\; x > 0, \;\; \theta > 0.$$

$$logf = logx - log\theta - \frac{x^2}{2\theta}$$
$$E\left(\frac{\partial logf}{\partial\theta}\right)^2 = E\left(\frac{x^2}{2\theta^2} - \frac{1}{\theta}\right) = \frac{1}{\theta^2}$$

The CRLB bound for the variance of the unbiased estimator of $\theta$ is $\dfrac{\theta^2}{n}$. Also, $T = \dfrac{1}{2n}\displaystyle\sum_{i=1}^{n} X_i^2$ is an unbiased estimator for $\theta$.

$$Var(T) = \frac{\theta^2}{n}, \;\; so\ T\ is\ UMVUE\ for\ \theta.$$

7.
$$f(x|\theta) = \theta(1-x)^{-(\theta+1)}, \;\; x > 0, \;\; \theta > 0$$

$$\frac{\partial logf(x|\theta)}{\partial\theta} = \frac{1}{\theta} - log(1+x)$$
$$E\left(log(1+x)\right) = \frac{1}{\theta} \;\; and \;\; E\left(log(1+x)\right)^2 = \frac{2}{\theta^2}$$
$$E\left(\frac{\partial logf}{\partial\theta}\right)^2 = \frac{1}{\theta^2}.\;\; Thus, \;\; CRLB \;\; is \;\; \frac{\theta^2}{n}$$

$T = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} log(1+x_i)$ is unbiased for $\dfrac{1}{\theta}$ and $Var(T) = \dfrac{\theta^2}{n} \implies T$ is $UMVUE$ of $\dfrac{1}{\theta}$.

8. From the given data, $E(X) = \theta - \dfrac{1}{2}$, So $T = \overline{X} + \dfrac{1}{2}$ is unbiased for $\theta$ and $Var(T) = \dfrac{1+4\theta-4\theta^2}{4n}$.

$$\frac{\partial logf(x|\theta)}{\partial\theta} = \begin{cases} 1/(\theta-1), & x = -1 \\ 0, & x = 0 \\ 1/\theta, & x = 1. \end{cases}$$

$E\left(\dfrac{\partial logf(x|\theta)}{\partial\theta}\right)^2 = \dfrac{1}{2\theta(1-\theta)}$, CRLB is $\dfrac{2\theta(1-\theta)}{n}$. Also, $Var(T) - \dfrac{2\theta(1-\theta)}{n} \geq 0$.

9. (a). (i). <u>$\alpha$ *is known*</u>

$$f(x|\beta) = \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}$$

$$h(x) = \frac{x^{\alpha-1}}{\Gamma\alpha}, c(\beta) = \frac{1}{\beta^\alpha}, w_1(\beta) = \frac{1}{\beta}, t_1(x) = -x.$$

(ii). <u>$\beta$ *is known*</u>

$$f(x|\alpha) = e^{\frac{-x}{\beta}} \frac{1}{\beta^\alpha \Gamma\alpha} exp\left((\alpha-1)logx\right)$$

$$h(x) = e^{\frac{-x}{\beta}}, c(\alpha) = \frac{1}{\beta^\alpha \Gamma\alpha}, w_1(\alpha) = \alpha - 1, t_1(x) = logx$$

(iii). <u>$\alpha$ *and* $\beta$ *are unknown*</u>

$$f(x|\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma\alpha} exp\left((\alpha-1)logx - \frac{x}{\beta}\right)$$

$$h(x) = I_{\{x>0\}}(x), c(\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma\alpha}, w_1(\alpha,\beta) = \alpha - 1, t_1(x) = logx, w_2(\alpha,\beta) = \frac{-1}{\beta}, t_2(x) = x$$

(b). (i). <u>$\alpha$ *is known*</u>

$$h(x) = x^{\alpha-1} I_{[0,1]}(x), c(\beta) = \frac{1}{B(\alpha,\beta)}, w_1(\beta) = \beta - 1, t_1(x) = log(1-x)$$

(ii). <u>$\beta$ *is known*</u>

$$h(x) = (1-x)^{\beta-1} I_{[0,1]}(x), c(\alpha) = \frac{1}{B(\alpha,\beta)}, w_1(\alpha) = \alpha - 1, t_1(x) = logx$$

(iii). <u>$\alpha$ *and* $\beta$ *are unknown*</u>

$$h(x) = I_{[0,1]}(x), c(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}, w_1(\beta) = \beta-1, t_1(x) = log(1-x), w_2(\alpha) = \alpha-1, t_2(x) = logx$$

(c).

$$h(x) = \binom{x-1}{r-1} I_{\{r,r+1,\ldots\}}(x), c(p) = \left(\frac{p}{1-p}\right)^r, w_1(p) = log(1-p), t_1(x) = x.$$

10. (a). <u>*Cauchy*$(1,\theta)$</u>

$$f(x|\theta) = \frac{1}{\pi(1+(x-\theta)^2)} = \frac{1}{\pi} e^{-log(1+(x-\theta)^2)}, \quad -\infty < x < \infty.$$

$$-log(1+(x-\theta)^2) \ \textit{can not be expressed as } w(\theta)t(x).$$

(b). $\underline{Uniform\ (0,\theta)}$

$$f(x|\theta) = \frac{1}{\theta} = e^{-log\theta},\ \ 0 < x < \theta.$$

If $f(x|\theta)$ belongs to exponential family, then from Neyman-Fisher factorization criterion $t(x) = 1$ is a sufficient statistic for all $\theta \in (0, \infty)$. But $t(x) = 1$ is not a sufficient statistic for all $\theta \in (0, \infty)$ and hence $Uniform\ (0, \theta)$ is not exponential family.