MA 3140: Statistical Inference

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Example 2

Let $X_1, \ldots, X_n \sim U[0, \theta]$, $\theta > 0$. Find the UMP test for testing

$$H_0 : \theta \leq \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0.$$

Solution: Recall that $\{U[0, \theta] : \theta > 0\}$ has MLR in $(\theta, X_{(n)})$.

Now we apply the result for UMP tests which is given by

$$\text{Reject } H_0 \text{ if } X_{(n)} > c \text{ and Accept } H_0 \text{ if } X_{(n)} \leq c,$$

where c is determined by the size condition

Example 2 contd.

$$P_{\theta=\theta_0}(X_{(n)} > c) = \alpha \implies \int_c^{\theta_0} \frac{ny^{n-1}}{\theta_0^n} dy = \alpha$$

$$\implies \frac{\theta_0^n - c^n}{\theta_0^n} = \alpha$$

$$\implies 1 - \alpha = \left(\frac{c}{\theta_0}\right)^n$$

$$\implies c = \theta_0(1-\alpha)^{1/n}$$

UMP test is: Reject H_0 if $X_{(n)} > \theta_0(1-\alpha)^{1/n}$.

- **Power function:**

$$\beta_\phi^*(\theta) = P_\theta(X_{(n)} > c) = 1 - \left(\frac{c}{\theta}\right)^n \qquad (c = \theta_0(1-\alpha)^{1/n})$$

$$= 1 - \left(\frac{\theta_0}{\theta}\right)^n (1-\alpha), \quad \theta > \theta_0$$

Example 2 contd.

- Consider another test of the form:

$$\phi_2(\boldsymbol{x}) = \begin{cases} 1, & X_n \geq \theta_0 \\ \alpha, & X_n < \theta_0. \end{cases}$$

So,

$$\begin{aligned} E_{\theta_0}\phi_2(\boldsymbol{X}) &= P_{\theta_0}(X_n \geq \theta_0) + \alpha P_{\theta_0}(X_n < \theta_0) \\ &= 0 + \alpha \cdot 1 = \alpha, \end{aligned}$$

when $\theta = \theta_0$, $X_{(n)} \in [0, \theta_0]$.

Thus, ϕ_2 also has size α.

Example 2 contd.

Power of ϕ_2:

$$\begin{aligned}\beta^*_{\phi_2}(\theta) &= P_\theta(X_n \geq \theta_0) + \alpha P_\theta(X_n < \theta)\\ &= \frac{\theta^n - \theta_0^n}{\theta^n} + \alpha \cdot \left(\frac{\theta_0}{\theta}\right)^n = 1 - \left(\frac{\theta_0}{\theta}\right)^n (1-\alpha).\end{aligned}$$

So, $\beta^*_{\phi_1}(\theta) = \beta^*_{\phi_2}(\theta)$, for $\theta > \theta_0$, and hence ϕ_2 is also UMP.

- However, for $\theta < \theta_0$,

$$\beta^*_{\phi_1}(\theta) = 1 - \left(\frac{\theta_0}{\theta}\right)^n (1-\alpha) \leq 1 - (1-\alpha) = \alpha = \beta^*_{\phi_2}(\theta)$$

Thus, ϕ_1 has smaller size for $\theta < \theta_0$, and is a better test than ϕ_2.

Example 2 cont'd

- Now, consider the dual problem:

$$K_0 : \theta \geq \theta_0 \quad \text{vs.} \quad K_1 : \theta < \theta_0$$

UMP test is: Reject K_0 if $X_{(n)} < c$ where c is determined as follows:

$$P_{\theta_0}(X_{(n)} < c) = \alpha \implies \left(\frac{c}{\theta_0}\right)^n = \alpha \implies c = \alpha^{1/n}\theta_0.$$

So the UMP test is Reject K_0 if $X_{(n)} < \theta_0\alpha^{1/n}$.

Example 3

Let $X_1, \ldots, X_n \sim P(\lambda)$, $\lambda > 0$. Find UMP test for testing:

$$H_0 : \lambda \leq \lambda_0 \quad vs. \quad H_1 : \lambda > \lambda_0$$

Solution: The joint pmf of $X_1, \ldots, X_n$ is

$$f(\boldsymbol{x}, \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{\sum x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x_i!}$$

For $\lambda_1 > \lambda_2$,

$$r(\boldsymbol{x}) = \frac{f(\boldsymbol{x}, \lambda_1)}{f(\boldsymbol{x}, \lambda_2)} = \frac{e^{-n\lambda_1} \lambda_1^{\sum x_i}}{e^{-n\lambda_2} \lambda_2^{\sum x_i}} = e^{n(\lambda_2 - \lambda_1)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\sum x_i}$$

Example 3 cont'd

This is an increasing function in $T(\boldsymbol{x}) = \sum x_i$. Hence, MLR in $(\lambda, \sum X_i)$.

So, the UMP test is given by

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sum x_i > c \\ \gamma, & \text{if } \sum x_i = c \\ 0, & \text{if } \sum x_i < c \end{cases} \tag{1}$$

where c and γ are determined by

$$E_{\lambda_0} \phi(\boldsymbol{X}) = \alpha \tag{2}$$

Example 3 cont'd

Note that $Y = \sum X_i \sim P(n\lambda_0)$ under H_0

Condition (2) is simplified as

$$P_{\lambda_0}(Y > c) + \gamma P_{\lambda_0}(Y = c) = \alpha$$

Suppose $\lambda_0 = 1$, $n = 5$, $\alpha = 0.1$. We can use Poisson Distribution Table.

Example 3 cont'd

λ=	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8
x= 0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679	0.3012	0.2466	0.2019	0.1653
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358	0.6626	0.5918	0.5249	0.4628
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197	0.8795	0.8335	0.7834	0.7306
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810	0.9662	0.9463	0.9212	0.8913
4	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963	0.9923	0.9857	0.9763	0.9636
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994	0.9985	0.9968	0.9940	0.9896
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994	0.9987	0.9974
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

λ=	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.5	5.0	5.5
x= 0	0.1353	0.1108	0.0907	0.0743	0.0608	0.0498	0.0408	0.0334	0.0273	0.0224	0.0183	0.0111	0.0067	0.0041
1	0.4060	0.3546	0.3084	0.2674	0.2311	0.1991	0.1712	0.1468	0.1257	0.1074	0.0916	0.0611	0.0404	0.0266
2	0.6767	0.6227	0.5697	0.5184	0.4695	0.4232	0.3799	0.3397	0.3027	0.2689	0.2381	0.1736	0.1247	0.0884
3	0.8571	0.8194	0.7787	0.7360	0.6919	0.6472	0.6025	0.5584	0.5152	0.4735	0.4335	0.3423	0.2650	0.2017
4	0.9473	0.9275	0.9041	0.8774	0.8477	0.8153	0.7806	0.7442	0.7064	0.6678	0.6288	0.5321	0.4405	0.3575
5	0.9834	0.9751	0.9643	0.9510	0.9349	0.9161	0.8946	0.8705	0.8441	0.8156	0.7851	0.7029	0.6160	0.5289
6	0.9955	0.9925	0.9884	0.9828	0.9756	0.9665	0.9554	0.9421	0.9267	0.9091	0.8893	0.8311	0.7622	0.6860
7	0.9989	0.9980	0.9967	0.9947	0.9919	0.9881	0.9832	0.9769	0.9692	0.9599	0.9489	0.9134	0.8666	0.8095
8	0.9998	0.9995	0.9991	0.9985	0.9976	0.9962	0.9943	0.9917	0.9883	0.9840	0.9786	0.9597	0.9319	0.8944
9	1.0000	0.9999	0.9998	0.9996	0.9993	0.9989	0.9982	0.9973	0.9960	0.9942	0.9919	0.9829	0.9682	0.9462
10	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9995	0.9992	0.9987	0.9981	0.9972	0.9933	0.9863	0.9747
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9996	0.9994	0.9991	0.9976	0.9945	0.9890
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9992	0.9980	0.9955
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993	0.9983
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Example 3 cont'd

$\lambda=$		6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	11.0	10.0	12.0	14.0	15.0
$x=$	0	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005	0.0002	0.0005	0.0001	0.0000	0.0000
	2	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028	0.0012	0.0028	0.0005	0.0001	0.0000
	3	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103	0.0049	0.0103	0.0023	0.0005	0.0002
	4	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293	0.0151	0.0293	0.0076	0.0018	0.0009
	5	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671	0.0375	0.0671	0.0203	0.0055	0.0028
	6	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301	0.0786	0.1301	0.0458	0.0142	0.0076
	7	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202	0.1432	0.2202	0.0895	0.0316	0.0180
	8	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328	0.2320	0.3328	0.1550	0.0621	0.0374
	9	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579	0.3405	0.4579	0.2424	0.1094	0.0699
	10	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830	0.4599	0.5830	0.3472	0.1757	0.1185
	11	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968	0.5793	0.6968	0.4616	0.2600	0.1848
	12	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916	0.6887	0.7916	0.5760	0.3585	0.2676
	13	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645	0.7813	0.8645	0.6815	0.4644	0.3632
	14	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165	0.8540	0.9165	0.7720	0.5704	0.4657
	15	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513	0.9074	0.9513	0.8444	0.6694	0.5681
	16	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730	0.9441	0.9730	0.8987	0.7559	0.6641
	17	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857	0.9678	0.9857	0.9370	0.8272	0.7489
	18	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928	0.9823	0.9928	0.9626	0.8826	0.8195
	19	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965	0.9907	0.9965	0.9787	0.9235	0.8752
	20	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984	0.9953	0.9984	0.9884	0.9521	0.9170
	21	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993	0.9977	0.9993	0.9939	0.9712	0.9469
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9990	0.9997	0.9970	0.9833	0.9673
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9995	0.9999	0.9985	0.9907	0.9805
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000	0.9993	0.9950	0.9888
	25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9997	0.9974	0.9938
	26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9967
	27	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9983
	28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9991
	29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996
	30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
	31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Example 4

Let $X \sim f(x, \theta)$ where,

$$f(x,\theta) = \begin{cases} \frac{1}{2}(1+\theta x), & -1 < x < 1, \quad -1 \leq \theta \leq 1 \\ 0, & o/w \end{cases}$$

Find UMP test for testing

$$H_0 : \theta \geq 0 \quad \text{vs.} \quad H_1 : \theta < 0.$$

Solution: For $\theta_1 > \theta_2$,

$$r(x) = \frac{f(x,\theta_1)}{f(x,\theta_2)} = \frac{1+\theta_1 x}{1+\theta_2 x}$$

Since $r(x)$ is increasing in x, we say that the family of densities has MLR in (θ, x).

Example 4 cont'd

So, the UMP test is: Reject H_0 if $X < c$ where c is determined as follows:

$$P_{\theta=0}(X < c) = \alpha \Longrightarrow \int_{-1}^{c} \frac{1}{2} dx = \alpha \Longrightarrow c = 2\alpha - 1$$

Thus, we reject H_0 if $X < 2\alpha - 1$.

- **Power function:** For $\theta < 0$,

$$P_\theta(X < 2\alpha - 1) = \alpha \Longrightarrow \int_{-1}^{2\alpha-1} \frac{1}{2}(1 + \theta x) dx = \alpha[1 + \theta(\alpha - 1)]$$

$$= \begin{cases} \alpha, & \theta = 0 \\ < \alpha, & \theta > 0 \\ > \alpha, & \theta < 0 \end{cases}$$

Theorem 2 on UMP Tests (Two-Tailed Hypothesis)

UMP tests also exists for certain two-sided hypothesis

$$H_0 : \theta \leq \theta_1 \text{ or } \theta \geq \theta_2 \quad (\theta_1 < \theta_2) \quad \text{vs.} \quad H_1 : \theta_1 < \theta < \theta_2.$$

- For the one-parameter exponential family, there exists a UMP test for testing H_0 against H_1 that is of the form

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } c_1 < T(\boldsymbol{x}) < c_2 \quad (c_1 < c_2) \\ \gamma_i, & \text{if } T(\boldsymbol{x}) = c_i,\ i = 1, 2 \\ 0, & \text{if } T(\boldsymbol{x}) < c_1 \text{ or } T(\boldsymbol{x}) > c_2 \end{cases}$$

where c_1, c_2, γ_1 γ_2 are determined by

$$E_{\theta_1}\phi(X) = E_{\theta_2}\phi(X) = \alpha.$$

Example 1

Let $X_1, \ldots, X_n \sim N(\theta, 1)$. Find UMP test for testing

$$H_0 : \theta \leq \theta_1 \text{ or } \theta \geq \theta_2 \quad (\theta_1 < \theta_2) \quad \text{vs.} \quad H_1 : \theta_1 < \theta < \theta_2.$$

Solution: The joint density is

$$\begin{aligned} f(\boldsymbol{x}, \theta) &= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum(x_i - \theta)^2} \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{\sum x_i^2}{2} - \frac{n\theta^2}{2} + \theta \sum x_i} \end{aligned}$$

This belongs to 1-parameter exponential family where $Q(\theta) = \theta$ is a strictly increasing function and $T(\boldsymbol{x}) = \sum X_i$.

Example 1 cont'd

So, by previous theorem, the UMP test is given by

$$\phi(x) = \begin{cases} 1, & \text{if } c_1 < \overline{X} < c_2 \quad (c_1 < c_2) \\ \gamma_i, & \text{if } \overline{X} = c_i, \ i = 1, 2 \\ 0, & \text{if } \overline{X} < c_1 \text{ or } \overline{X} > c_2 \end{cases} \tag{3}$$

where γ_i and c_i, $i = 1, 2$ are determined by

$$E_{\theta_1}\phi(X) = E_{\theta_2}\phi(X) = \alpha \qquad (*)$$

Since $\overline{X} \sim N(\theta, 1/n)$, we may take $\gamma_1 = \gamma_2 = 0$ (wlog).

Example 1 cont'd

Now, $(*)$ gives

$$P_{\theta_i}(c_1 < \overline{X} < c_2) = \alpha, \quad i = 1, 2$$
$$\Longrightarrow P_{\theta_i}(\sqrt{n}(c_1 - \theta_i) < \sqrt{n}(\overline{X} - \theta_i) < \sqrt{n}(c_2 - \theta_i)) = \alpha, \quad i = 1, 2$$
$$\Longrightarrow \Phi(\sqrt{n}(c_2 - \theta_i)) - \Phi(\sqrt{n}(c_1 - \theta_i)) = \alpha, \quad i = 1, 2$$

For given values of θ_1, θ_2, n, α, we can solve the above equations to determine c_1 and c_2.

Example: Let $n = 9$, $\theta_1 = 0$, $\theta_2 = 1$, $\alpha = 0.05$.

Then the above equations reduce to $\Phi(3c_2) - \Phi(3c_1) = 0.05$ and $\Phi(3(c_2 - 1)) - \Phi(3(c_1 - 1)) = 0.05$ which can be solved from the tables of standard normal distributions.

Example 2

Let $X_1, \ldots, X_n$ be a random sample from exponential distribution $f(x, \sigma) = \frac{1}{\sigma} e^{-x/\sigma}$, $x > 0$, $\sigma > 0$. Find UMP test for testing

$$H_0 : \sigma \leq \sigma_1 \text{ or } \sigma \geq \sigma_2 \quad (\sigma_1 < \sigma_2) \quad \text{vs.} \quad H_1 : \sigma_1 < \sigma < \sigma_2.$$

Solution: The joint density is

$$f(\boldsymbol{x}, \sigma) = \frac{1}{\sigma^n} e^{-\frac{\sum(x_i}{\sigma}}$$

This belongs to 1-parameter exponential family where $Q(\sigma) = -\frac{1}{\sigma}$ is increasing in σ and $T(\boldsymbol{x}) = \sum X_i$.

Example 2 cont'd

So, the UMP test: Reject H_0 if $c_1 < \sum X_i < c_2$ where

$$P_{\sigma_j}(c_1 < \sum X_i < c_2) = \alpha, \quad j = 1, 2$$

$$P_{\sigma_j}\left(\frac{2c_1}{\sigma_j} < W < \frac{2c_1}{\sigma_j}\right) = \alpha, \quad j = 1, 2 \quad \left(\because \frac{2\sum X_i}{\sigma_0} \sim \chi^2_{2n}\right)$$

where c_1 and c_2 can be determined from the above equation and using tables of χ^2 distribution for given σ_1, σ_2, n and α.

Suppose $\sigma_1 = 1$, $\sigma_2 = 2$, $n = 5$ and $\alpha = 0.1$.

We may then determine c_1 and c_2 by interpolating from tables of χ^2_{10} tables.

Remark

The UMP test for the dual problem

$$H_0 : \theta_1 \leq \theta \leq \theta_2 \quad \text{vs.} \quad H_1 : \theta < \theta_1 \text{ or } \theta > \theta_2$$

or for

$$H_0^* : \theta = \theta_0 \quad \text{vs.} \quad H_1^* : \theta \neq \theta_0$$

does not exist.

UMP test does not exist

Example 1: Let $X_1, \dots, X_n \sim N(0, \sigma^2)$. Check whether UMP test exists for

$$H_0^* : \sigma = \sigma_0 \quad \text{vs.} \quad H_1^* : \sigma \neq \sigma_0.$$

Solution: Recall that the family of joint pdf's of $\boldsymbol{X} = (X_1, \dots, X_n)$ has an MLR in $T(\boldsymbol{X}) = \sum_{i=1}^n X_i^2$.
So, the UMP test for

$$K_0 : \sigma = \sigma_0 \quad vs. \quad K_1 : \sigma > \sigma_0$$

is given by

$$\phi_1(\boldsymbol{x}) = \begin{cases} 1, & \sum_{i=1}^n X_i^2 > c_1 \\ 0, & \text{otherwise.} \end{cases}$$

UMP test does not exist contd.

Also, the UMP test for

$$K_0 : \sigma = \sigma_0 \quad vs. \quad K_1 : \sigma < \sigma_0$$

is given by

$$\phi_2(\boldsymbol{x}) = \begin{cases} 1, & \sum_{i=1}^{n} X_i^2 < c_2 \\ 0, & \text{otherwise.} \end{cases}$$

If the size is chosen as α, then $c_1 = \sigma_0^2 \chi^2_{n,\alpha}$ and $c_2 = \sigma_0^2 \chi^2_{n,1-\alpha}$.

Clearly, neither ϕ_1 nor ϕ_2 is UMP for $H_0^* : \sigma = \sigma_0$ vs. $H_1^* : \sigma \neq \sigma_0$.

UMP test does not exist contd.

- The power of any test of H_0 for values $\sigma > \sigma_0$ cannot exceed that of ϕ_1; and for values $\sigma < \sigma_0$, it cannot exceed the power of test ϕ_2.

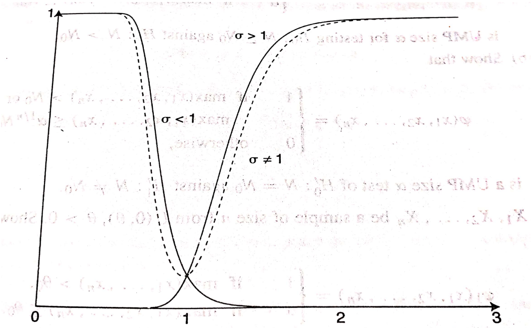

Fig. 1. Power functions of chi-square tests of $H_0: \sigma = \sigma_0$ against H_1.

- Hence, no test of H_0 can be UMP.

UMP test does not exist contd.

Example 2: Let $X_1, \ldots, X_n$ be a random sample from double exponential distribution

$$f(x, \theta) = \frac{1}{2\theta} \exp\left[-\frac{|x|}{\theta}\right], \; x \in \mathbb{R}, \; \theta \in \mathbb{R}.$$

Check whether UMP test exists for testing

$$H_0^* : \theta = \theta_0 \quad \text{vs.} \quad H_1^* : \theta \neq \theta_0.$$

Solution: Note that UMP test for

$$K_0 : \theta \leq \theta_0 \quad vs. \quad K_1 : \theta > \theta_0$$

is given by

$$\phi_1(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \frac{2\sum |X_i|}{\theta_0} \geq \chi^2_{2n,\alpha} \\ 0, & \text{if } \frac{2\sum |X_i|}{\theta_0} < \chi^2_{2n,\alpha} \end{cases}$$

UMP test does not exist contd.

Similarly, the UMP test for

$$L_0 : \theta \geq \theta_0 \quad vs. \quad L_1 : \theta < \theta_0$$

is given by

$$\phi_2(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \frac{2\sum|X_i|}{\theta_0} \leq \chi^2_{2n,1-\alpha} \\ 0, & \text{if } \frac{2\sum|X_i|}{\theta_0} > \chi^2_{2n,1-\alpha} \end{cases}$$

The power of ϕ_1 is less than that of ϕ_2 for $\theta < \theta_0$, whereas the power of ϕ_2 is less than or equal to the power of ϕ_1 for $\theta > \theta_0$.

Thus, no test can be UMP for

$$H_0^* : \theta = \theta_0 \quad \text{vs.} \quad H_1^* : \theta \neq \theta_0.$$

One parameter exponential families

- $\boldsymbol{X} = (X_1, \ldots, X_n)$ has pdf/pmf $f(\boldsymbol{x}, \theta) = c(\theta) e^{\theta T(\boldsymbol{x})} h(\boldsymbol{x})$.

(i) $H_1 : \theta \leq \theta_0 \quad \text{vs} \quad K_1 : \theta > \theta_0$: a UMP test exists.

(ii) $H_2 : \theta \leq \theta_1$ or $\theta \geq \theta_2 \quad \text{vs} \quad K_2 : \theta_1 < \theta < \theta_2$: a UMP test exists.

(iii) $H_3 : \theta_1 \leq \theta \leq \theta_2 \quad \text{vs} \quad K_3 : \theta < \theta_1$ or $\theta > \theta_2$: a UMP test does not exist.

(iv) $H_4 : \theta = \theta_0 \quad \text{vs} \quad K_4 : \theta \neq \theta_0$: a UMP test does not exist.

Thanks for your patience!