MA 3140: Statistical Inference

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An important result

Theorem: If the CRLB for the variance of an UE of $g(\theta)$ is attained, then the class of parametric functions, for which the UEs attain CRLB, is the class of linear functions of $g(\theta)$.

Proof: Let $T(\boldsymbol{X})$ be an UE of $g(\theta)$ such that $Var(T(\boldsymbol{X})) = CRLB$.

Then, $T(\boldsymbol{X})$ and $S_\theta(\boldsymbol{X})$ are linearly related w.p.1, i.e., there exists functions $\alpha(\theta)$ and $\beta(\theta)$ s.t.

$$T(\boldsymbol{X}) + \alpha(\theta)S_\theta(\boldsymbol{X}) = \beta(\theta), \ w.p.1, \ \forall \ \theta \in \Theta.$$

Take expectations,

$$\begin{aligned} E(T(\boldsymbol{X})) + \alpha(\theta)E(S_\theta(\boldsymbol{X})) &= \beta(\theta), \ \forall \ \theta \in \Theta \\ \implies g(\theta) &= \beta(\theta), \ \forall \ \theta \in \Theta, \end{aligned}$$

because $E(S_\theta(\boldsymbol{X})) = 0$.

Proof contd.

Now, let $h(\theta)$ be any other parametric function for which there exists an UE, say $U(\boldsymbol{X})$, for which $Var(U(\boldsymbol{X})) = CRLB$.

Then, $U(\boldsymbol{X})$ and $S_\theta(\boldsymbol{X})$ are linearly related w.p.1, i.e., there exists functions $\alpha^*(\theta)$ and $\beta^*(\theta)$ s.t.

$$U(\boldsymbol{X}) + \alpha^*(\theta) S_\theta(\boldsymbol{X}) = \beta^*(\theta), \; w.p.1, \; \forall \; \theta \in \Theta.$$

Take expectations,

$$\begin{aligned} E(U(\boldsymbol{X})) + \alpha^*(\theta) E(S_\theta(\boldsymbol{X})) &= \beta^*(\theta), \; \forall \; \theta \in \Theta \\ \implies h(\theta) &= \beta^*(\theta), \; \forall \; \theta \in \Theta, \end{aligned}$$

because $E(S_\theta(\boldsymbol{X})) = 0$.

Proof contd.

Thus, we have,

$$\begin{aligned} T(\boldsymbol{X}) + \alpha(\theta) S_\theta(\boldsymbol{X}) &= g(\theta), \ \ w.p.1, \ \forall\ \theta \in \Theta; \\ U(\boldsymbol{X}) + \alpha^*(\theta) S_\theta(\boldsymbol{X}) &= h(\theta), \ \ w.p.1, \ \forall\ \theta \in \Theta. \end{aligned}$$

For a fixed value of $\theta = \theta_0$,

$$\begin{aligned} T(\boldsymbol{X}) + \alpha(\theta_0) S_{\theta_0}(\boldsymbol{X}) &= g(\theta_0), \ \ w.p.1; \\ U(\boldsymbol{X}) + \alpha^*(\theta_0) S_{\theta_0}(\boldsymbol{X}) &= h(\theta_0), \ \ w.p.1. \end{aligned}$$

Proof contd.

Now, by eliminating $S_{\theta_0}(\boldsymbol{X})$, we get

$$\alpha^*(\theta_0)T(\boldsymbol{X}) - \alpha(\theta_0)U(\boldsymbol{X}) = \alpha^*(\theta_0)g(\theta_0) - \alpha(\theta_0)h(\theta_0), \; w.p.1,$$

$$\text{or, } \; aT(\boldsymbol{X}) + bU(\boldsymbol{X}) = c, \; w.p.1,$$

where a, b and c are constants.

Takie expectations,

$$a \; g(\theta) + b \; h(\theta) = c.$$

Thus, g and h are linearly related.

Exponential Family of Distributions

One-parameter Exponential Family

A family of pdfs (or pmfs) is said to be a one-parameter exponential family if it can be expressed as

$$f_\theta(x) = h(x)c(\theta)e^{Q(\theta)T(x)}.$$

Here,

$c(\theta) \geq 0$ and $Q(\theta)$ are real-valued functions of the parameter θ (they cannot depend on x).

$h(x) \geq 0$ and $T(x)$ are real-valued functions of the observation x (they cannot depend on θ).

One parameter Exponential Family contd.

Example 1: $X \sim Bin(n, p)$.

For $x = 0, 1, \ldots, n$ and $0 < p < 1$, the pmf for this family is

$$\begin{aligned} f_p(x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \binom{n}{x} (1-p)^n \left(\frac{p}{1-p}\right)^x \\ &= \binom{n}{x} \ (1-p)^n \ e^{x \log \left(\frac{p}{1-p}\right)} \\ &= h(x) c(p) e^{Q(p) T(x)}, \end{aligned}$$

One parameter Exponential Family contd.

where,

$$h(x) = \begin{cases} \binom{n}{x}, & x = 0, 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

$$c(p) = (1-p)^n,\ 0 < p < 1,$$

$$T(x) = x$$

$$Q(p) = \log \frac{p}{1-p},\ 0 < p < 1.$$

Thus, Binomial distribution (with n known) is a member of the exponential family.

One parameter Exponential Family contd.

Example 2: $X \sim Poisson(\lambda)$.

For $x = 0, 1, \ldots$ and $\lambda > 0$, the pmf for this family is

$$\begin{aligned} f_\lambda(x) &= \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \frac{1}{x!}\ e^{-\lambda}\ e^{x \log \lambda} \end{aligned}$$

Here, $h(x) = \frac{1}{x!}$, $c(\lambda) = e^{-\lambda}$, $T(x) = x$ and $Q(\lambda) = \log \lambda$.

Thus, Poisson distribution is a member of the exponential family.

One parameter Exponential Family contd.

Remark: The set of pdfs given by

$$\begin{aligned} f_\theta(x) &= \frac{1}{\theta} e^{\{1-\frac{x}{\theta}\}},\ 0 < \theta \le x < \infty, \\ &= h(x)c(\theta)e^{Q(\theta)T(x)},\ 0 < \theta \le x < \infty, \end{aligned}$$

where $h(x) = e^1$, $c(\theta) = 1/\theta$, $T(x) = -x$ and $Q(\theta) = 1/\theta$, is not an exponential family.

Reason: As the range involves the parameter, we need to incorporate the range of x into the expression of $f_\theta(x)$ with the use of Indicator function, i.e.,

$$f_\theta(x) = \frac{1}{\theta} e^{\{1-\frac{x}{\theta}\}} I_{[\theta,\infty)}(x).$$

One parameter Exponential Family contd.

Here, the indicator function of a set A, denoted by $I_A(x)$ is the function

$$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Note that $I_{[\theta,\infty)}(x)$ is neither a function of x alone, nor a function of θ alone, and cannot be expressed as an exponential.

Thus, this is not an exponential family.

Multi-parameter Exponential Family

A family of pdfs (or pmfs) is said to be a multi-parameter exponential family if it can be expressed as

$$f_{\boldsymbol{\theta}}(x) = h(x)c(\boldsymbol{\theta})e^{\sum_{i=1}^{k} Q_i(\boldsymbol{\theta})T_i(x)}, \quad \boldsymbol{\theta} \in \mathbb{R}^k.$$

Here,

$c(\boldsymbol{\theta}) \geq 0$ and $Q_1(\boldsymbol{\theta}), \ldots, Q_k(\boldsymbol{\theta})$ are real-valued functions of the vector-valued parameter $\boldsymbol{\theta}$ (they cannot depend on x).

$h(x) \geq 0$ and $T_1(x), \ldots, T_k(x)$ are real-valued functions of the observation x (they cannot depend on $\boldsymbol{\theta}$).

Multi-parameter Exponential Family

Example 1: $X \sim N(\mu, \sigma^2)$, both μ and σ^2 are unknown.

For $\boldsymbol{\theta} = (\mu, \sigma)$, $-\infty < \mu < \infty$, $\sigma > 0$, the pdf for this family is

$$
\begin{aligned}
f_{\mu,\sigma^2}(x) &= \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \\
&= \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{\mu^2}{2\sigma^2}}\; e^{-\frac{x^2}{2\sigma^2}+\frac{\mu x}{\sigma^2}} \\
&= h(x)c(\boldsymbol{\theta})e^{\{Q_1(\boldsymbol{\theta})T_1(x)+Q_2(\boldsymbol{\theta})T_2(x)\}}.
\end{aligned}
$$

Multi-parameter Exponential Family

Here,

$$h(x) = 1, \ \forall \ x;$$

$$c(\boldsymbol{\theta}) = c(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \ e^{-\frac{\mu^2}{2\sigma^2}}, \ -\infty < \mu < \infty, \sigma > 0;$$

$$T_1(x) = x^2; \qquad Q_1(\boldsymbol{\theta}) = -\frac{1}{2\sigma^2}, \ \sigma > 0;$$

$$T_2(x) = x; \qquad Q_2(\boldsymbol{\theta}) = \frac{\mu}{\sigma^2}, \ \sigma > 0.$$

Thus, Normal distribution is a member of two parameter exponential family.

Also, note that the pdf can be written as:

$$f_{\boldsymbol{\theta}}(x) = h(x)c(\boldsymbol{\theta})e^{\{Q_1(\boldsymbol{\theta})T_1(x)+Q_2(\boldsymbol{\theta})T_2(x)\}} I_{(-\infty,\infty)}(x).$$

Exponential Family and CRLB

Consider

$$f_\theta(x) = c(\theta)h(x)e^{Q(\theta)T(x)}$$
$$\log f_\theta(x) = \log c(\theta) + \log h(x) + Q(\theta)T(x)$$
$$\frac{\partial}{\partial\theta}\log f_\theta(x) = \frac{c^{'}(\theta)}{c(\theta)} + T(x)Q^{'}(\theta)$$

Now,

$$S_\theta(\boldsymbol{x}) = \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f_\theta(x_i) = n\frac{c^{'}(\theta)}{c(\theta)} + Q^{'}(\theta)\sum_{i=1}^{n}T(x_i).$$

Here, $W = \frac{1}{n}\sum_{i=1}^{n}T(x_i)$ is linearly related to $S_\theta(\boldsymbol{x})$ with probability 1.
Hence, any linear function of W will be attaining the CRLB for the variance of UE of W.

Determining E(W):

$$\int f_\theta(x)dx = 1 \Longrightarrow \int c(\theta)h(x)e^{Q(\theta)T(x)}dx = 1$$

Differentiating under the integral sign, we get

$$\int c^{'}(\theta)h(x)e^{Q(\theta)T(x)}dx + \int c(\theta)h(x)e^{Q(\theta)T(x)}Q^{'}(\theta)T(x)dx = 0$$

$$\Longrightarrow \frac{c^{'}(\theta)}{c(\theta)} + Q^{'}(\theta)E[T(X)] = 0$$

$$\Longrightarrow E[T(X)] = -\frac{c^{'}(\theta)}{c(\theta)Q^{'}(\theta)}.$$

Exponential Family and CRLB contd.

Example 1: Consider $X \sim Poisson(\lambda)$.

For $x = 0, 1, \ldots$ and $\lambda > 0$, the pmf for this family is

$$f_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{x!}\ e^{-\lambda}\ e^{x \log \lambda}$$

Here, $h(x) = \frac{1}{x!}$, $c(\lambda) = e^{-\lambda}$, $T(x) = x$ and $Q(\lambda) = \log \lambda$.
So, in this case

$$S_\lambda(\boldsymbol{x}) = n\frac{c^{'}(\lambda)}{c(\lambda)} + Q^{'}(\lambda)\sum_{i=1}^{n} T(x_i) = -n\frac{e^{-\lambda}}{e^{-\lambda}} + \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

Therefore, $W = \overline{X}$ and it attains CRLB for variance of UE of λ.

Exponential Family and CRLB contd.

$$E[T(X)] = -\frac{c^{'}(\lambda)}{c(\lambda)\ Q'(\lambda)} = -\frac{-e^{-\lambda}}{e^{-\lambda}\ \frac{1}{\lambda}} = \lambda.$$

Example 2: Let $X \sim \mathit{Geometric}(\theta)$.

For $x = 0, 1, 2, \ldots$ and $0 < \theta < 1$, the pmf for this family is

$$\begin{aligned} f_\theta(x) &= \theta(1-\theta)^x \\ &= \theta\ e^{x \log(1-\theta)} \end{aligned}$$

Here, $h(x) = 1$, $c(\theta) = \theta$, $T(x) = x$ and $Q(\theta) = \log(1-\theta)$.

Exponential Family and CRLB contd.

Consider

$$E[T(X)] = -\frac{c^{'}(\theta)}{c(\theta)Q^{'}(\theta)} = -\frac{1}{\theta(-\frac{1}{1-\theta})} = \frac{1}{\theta} - 1.$$

Here, $W = \overline{X}$. So,

$$E(W) = E(\overline{X}) = \frac{1}{\theta} - 1$$

$V(\overline{X})$ will attain the CRLB for estimating $\frac{1}{\theta} - 1$.

In addition, $\overline{X} - 1$ is UMVUE for $\frac{1}{\theta}$.

Thanks for your patience!