Assignment - IV

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① $X \sim P(\lambda)$

$H_0 : \lambda = 1$ vs $H_a : \lambda = 4$

$$\phi(x) \begin{cases} 1 & x > 2 \\ 0 & x \leq 2 \end{cases}$$

P(Type I error) = α = Rejecting null when it is true.

$$P_{P=P_0}(\phi(x) = 1) = P_{\lambda=1}(X > 2) = 1 - P_{\lambda=1}(X \leq 2)$$

$$= 1 - P_{\lambda=1}(X=0) - P_{\lambda=1}(X=1) - P_{\lambda=2}(X=2)$$

$$\alpha = 1 - e^{-1}\left[1 + 1 + \frac{1}{2}\right] = 1 - \frac{5}{2e} = \underline{0.0803}$$

β = P(Type II error) = Accepting null when it is false

$$P_{\lambda=4}(X \leq 2) = \sum_{r=0}^{2} \frac{e^{-\lambda}\lambda^r}{r!} = e^{-4}\left[\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!}\right]$$

$$= e^{-4}[4 + 1 + 8]$$

$$\frac{13}{e^4} = \underline{0.2381}$$

(2) $f_X(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}$

$H_0: \sigma = 1$ vs $H_a: \sigma > 1$

$$\phi(x) \begin{cases} 1 & |x| > 1 \\ 0 & |x| \le 1 \end{cases}$$

Size $= \alpha$: P(Type I error)

$\Rightarrow P_{\sigma=1}(|x| > 1)$

$\Rightarrow \frac{1}{2}\left[\int_{-\infty}^{-1} e^{x} dx + \int_{1}^{\infty} e^{-x} dx\right] = \frac{1}{2}\left[2e^{-1}\right] = \frac{1}{e}$

Power: $P_\theta(X \in R)$ $\theta \in \Theta_1$

$\Rightarrow P_{\sigma>1}(|x| > 1)$

$= \frac{1}{2\sigma}\left[\int_{-\infty}^{-1} e^{x/\sigma} dx + \int_{1}^{\infty} e^{-x/\sigma} dx\right]$

$= \frac{1}{2\cancel{\sigma}} \cdot \cancel{\sigma}\left[e^{x/\sigma}\Big|_{-\infty}^{-1} + e^{-x/\sigma}\Big|_{1}^{\infty}\right]$

$= \frac{1}{2}\left[2e^{-1/\sigma}\right] = e^{-1/\sigma}$

$\therefore \sigma > 1 \Rightarrow e^{-1/\sigma} > e^{-1}$ Hence proved. that Power > Size

(2)

(3) $f_\sigma(x) = \dfrac{\sigma}{\pi(\sigma^2+x^2)}$ $\quad H_0: \sigma = 1$ vs $H_a: \sigma = 2$

By NP Lemma Reject H_0 if $\dfrac{f_1(x)}{f_0(x)} \geq k$

$$\frac{f_1(x)}{f_0(x)} = \frac{\dfrac{\sigma_1}{\pi(2^2+x^2)}}{\dfrac{\sigma_0}{\pi(1^2+x^2)}} = \underbrace{2\left[\frac{x^2+1}{x^2+4}\right]}_{Q(x)}$$

$Q'(x) > 0 \ \forall x > 0$ $\Rightarrow$ Increasing function of x for $x>0$

$= \dfrac{12x}{(x^2+4)^2}$ $\quad$ decreasing fn of x for $x<0$

Max is attained at $x = \pm\infty \Rightarrow 2$

Min value $\Rightarrow x = 0 = \dfrac{1}{2}$.

$k \leq \frac{1}{2} \rightarrow$ always reject H_0 & $\alpha = 1$

$k \geq 2 \rightarrow$ always accept H_0 & $\alpha = 0$

$\frac{1}{2} < k < 2 \rightarrow$ Reject H_0 if $Q(x) > k$

$$P_{\sigma=1}\left(2.\frac{x^2+1}{x^2+4} > k\right) = \alpha$$

By solving the size condition we get

Reject H_0 if $|x| > \sqrt{\dfrac{4k-2}{2-k}}$ $\Rightarrow k = \dfrac{4}{3+3\cos(1-\alpha)}$

(3)

(4) $f_\theta(x) = \frac{2}{\theta^2}(\theta - x) \quad 0 < x < \theta$

$H_0 : \theta = 1$ vs $H_a : \theta = 2$

$$\frac{f_1(x)}{f_0(x)} \geq K \Rightarrow \frac{\frac{2}{\theta_1^2}(\theta_1 - x)}{\frac{2}{\theta_0^2}(\theta_0 - x)} \geq K \quad \text{—(1)}$$

$$\Rightarrow \left(\frac{\theta_0}{\theta_1}\right)^2 \left(\frac{\theta_1 - x}{\theta_0 - x}\right) > K, \quad \theta_1 > \theta_0$$

$$\Rightarrow x \geq K'$$

Reject H_0 if $x \geq K'$, where K' is determined by size Condition

$$P_{\theta=1}(x \geq K) = \int_K^1 2(1-x)\,dx$$

$$= 2\left[x - \frac{x^2}{2}\right]_K^1$$

$$\Rightarrow 2\left[1 - \frac{1}{2} - \left[K - \frac{K^2}{2}\right]\right]$$

$$\Rightarrow 1 - 2K + K^2 = (K-1)^2$$

$(K-1)^2 = \alpha$

$\boxed{K' = 1 \pm \sqrt{\alpha}}$

But in (1) $\quad K' = \frac{4K-2}{4K-1} \leq 1 \Rightarrow K' = 1 - \sqrt{\alpha}$

$\therefore$ ~~Accept~~ Reject H_0 when $x \geq 1 - \sqrt{\alpha}$

(5) $X_1, X_2 \ldots X_n \sim f_\theta(x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}, \quad x>0 \quad \theta>0$

$$r(x) = \frac{f(x,\theta_1)}{f(x,\theta_2)}, \quad \theta_1 > \theta_2$$

$$r(x) = \frac{\left[\frac{1}{\Gamma(\theta_1)}\right]^n \prod_{i=1}^{n} x_i^{\theta_1 - 1} \left[\cancel{e^{-\sum x_i}}\right]}{\left[\frac{1}{\Gamma(\theta_2)}\right]^n \cdot \prod_{i=1}^{n} x_i^{\theta_2 - 1} \left[\cancel{e^{-\sum x_i}}\right]}$$

$r(x)$ is increasing function of $\prod_{i=1}^{n} x_i$.

Hence family has MLR in $(\theta, \prod_{i=1}^{n} x_i)$

UMP test for $H_0: \theta \le 3$ vs $H_a: \theta > 3$

We know that $T(X) = \prod_{i=1}^{n} X_i$.

UMP test is given by,

$$\phi(x) = \begin{cases} 1 & \prod_{i=1}^{n} X_i > c \\ 0 & \prod_{i=1}^{n} X_i \le c \end{cases}$$

Where c can be determined by Size Condition,

$$E_{\theta_0}(\phi(x)) = \alpha.$$

⑥ $X_1, X_2, X_3 \ldots X_n \sim Exp(\lambda)$

UMP test $H_0 : \lambda = 1$ vs $H_a : \lambda \neq 1$

$$f(x, \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

we know that exponential distribution belongs to exponential family with $T(x) = \sum_{i=1}^{n} x_i$

$$\therefore \phi(x) \begin{cases} 1 & \sum x_i < c_1 \text{ or } \sum x_i > c_2 \\ 0 & c_1 \leq \sum x_i \leq c_2 \end{cases}$$

$E_{\lambda_0}(\phi(x)) = \alpha \quad (\lambda_0 = 1) \qquad W = \sum X_i \sim gamma(n, 1)$

$E_{\lambda_0}(1 - \phi(x)) = 1 - \alpha$

$P_{\lambda_0}(c_1 \leq \sum x_i \leq c_2) = 1 - \alpha$

$$\int_{c_1}^{c_2} \frac{w^{n-1} e^{-w}}{\Gamma(n)} = 1 - \alpha \quad \text{— ①}$$

$E_{\lambda_0}[T(x)\,\phi(x)] = \alpha E_{\lambda_0}[T(x)]$

$E_{\lambda_0}[W(1 - \phi(x))] = (1-\alpha) E_{\lambda_0}(W)$ [Mean of gamma $= n\lambda$
$= n(1)$
$= n$

$$\int_{c_1}^{c_2} w \cdot g_{n,1}(w)\, dw = (1-\alpha)\, n$$

⑦

$$w\, g_{n,1}\, dw = w.\frac{w^{n-1} e^{-w} \cdot n}{\Gamma(n) \cdot n} = n.\frac{w^{n} e^{-w}}{\Gamma(n+1)}$$

$$\boxed{w\, g_{n,1}(w) = n.\, g_{n+1,1}(w)}$$

$$\Rightarrow \int_{c_1}^{c_2} \cancel{n}.\, g_{n+1,1}(w)\, dw = (1-\alpha)\cancel{n}$$

$$\Rightarrow \int_{c_1}^{c_2} g_{n+1,1}(w)\, dw = 1-\alpha$$

$$\Rightarrow \frac{1}{\Gamma(n+1)} \int_{c_1}^{c_2} w^{n} e^{-w}\, dw = 1-\alpha$$

$$\Rightarrow \frac{1}{\Gamma(n+1)} \Big[w^{n} e^{-w} \Big|_{c_2}^{c_1} + n \underbrace{\int w^{n-1} e^{-w}\, dw}_{①} \Big] = 1-\alpha$$

$$c_1^{n} e^{-c_1} - c_2^{n} e^{-c_2} + n\,\Gamma(n)(1-\alpha) = \Gamma(n+1)\,[1-\alpha]$$

$$\Rightarrow \boxed{c_1^{n} e^{-c_1} = c_2^{n} e^{-c_2}}$$

By solving this equation we would get our c_1, c_2.

(7) $f_\theta(x) = \frac{2}{\theta^2}(\theta - x) \quad 0 < x < \theta$ LRT for $H_0 : \theta = 2$ vs $H_a : \theta \neq 2$.

$$L(\theta, x) = \frac{2}{\theta^2}(\theta - x)$$

$$\log L = -2\log(\theta) + 2\log 2 + \log(\theta - x)$$

$$\frac{\partial}{\partial\theta}\log(L(\theta)) = -\frac{2}{\theta} + \frac{1}{\theta - x} = 0$$

$$\Rightarrow 2x = \theta$$

$$L(\theta_\Omega, x) = \frac{2}{(2x)^2}(2x - x) = \underline{\underline{\frac{1}{2x}}}$$

$$L(\theta_{\Omega_H}, x) = \frac{2}{4}(2 - x) = \frac{2 - x}{2}$$

$$\lambda(x) = \frac{L(\theta_{\Omega_H}, x)}{L(\theta_\Omega, x)} = \frac{\frac{2-x}{2}}{\frac{1}{2x}} = \underline{\underline{x(2 - x)}}$$

According to LRT Reject H_0 if $\lambda(x) \leq c$.

$$x(2 - x) \leq c$$

$$\Rightarrow (x-1)^2 \geq 1 - c$$

$$\left.\begin{array}{l} \text{if } 0 < x \leq 1 \rightarrow x \leq 1 - \sqrt{1-c} \\ 1 \leq x \leq 2 \rightarrow x \geq 1 + \sqrt{1-c} \end{array}\right\} \rightarrow (1)$$

We Know that

$$\sup P_{\theta_0}(\lambda(x) \le c) = \alpha$$

$$\Rightarrow \int_0^{1-\sqrt{1-c}} \frac{1}{2}(2-x)\,dx + \int_{1+\sqrt{1-c}}^{2} \frac{1}{2}(2-x)\,dx = \alpha$$

$$x - \frac{x^2}{4}\Big|_0^{1-\sqrt{1-c}} + x - \frac{x^2}{4}\Big|_{1+\sqrt{1-c}}^{2} = \alpha$$

$$\Rightarrow 1-\sqrt{1-c} = \alpha \quad -(2)$$

Substituting (2) in (1) we got,

$$x \le 1-\sqrt{1-c} \Rightarrow x \le \alpha$$

$$x \geq 1+\sqrt{1-c} \Rightarrow x \geq 2-\alpha$$

$\therefore$ Reject H_0 if $x \le \alpha$ or $x \geq 2-\alpha$ else accept.

(8) $f(x) = \frac{1}{2\sigma} e^{-\left(\frac{|x|}{\sigma}\right)}$

$$L(\sigma, x) = \left(\frac{1}{2\sigma}\right)^n e^{-\sum_{i=1}^{n} |x_i| / \sigma}$$

$H_0: \sigma = 1$ vs $H_a: \sigma \neq 1$

$$\log(L(\sigma, x)) = -n\log(2\sigma) - \frac{\sum |x_i|}{\sigma}$$

$$\frac{\partial \log(L)}{\partial \sigma} = \frac{-n}{2\sigma} \cdot 2 + \frac{\sum |x_i|}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\hat{\sigma} = \sum_{i=1}^{n} |x_i| / n}$$

(10)

(8) $L(\hat{\Omega}) = \dfrac{1}{(2\hat{\sigma})^n} \cdot e^{-\frac{\sum|X_i|}{\sum|X_i|/n}} = \dfrac{1}{(2\hat{\sigma})^n} e^{-n}$

$\hat{L}(\Omega_H) = \dfrac{1}{2^n} e^{-\sum|X_i|}$ [$\sigma = 1$ in Null Hypothesis]

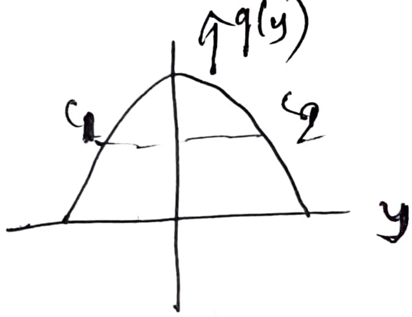
$\lambda(x) = \dfrac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} = \dfrac{\frac{1}{2^n} \cdot e^{-\sum|X_i|}}{\left(\frac{1}{2\hat{\sigma}}\right)^n e^{-n}} = (\hat{\sigma})^n e^{n - \sum|X_i|}$

Let $y = \sum|X_i|$

$\lambda(y) = \dfrac{y^n}{n^n} e^{n-y}$

$g(y) = y^n e^{-y}$, $g'(y) = n y^{n-1} e^{-y} - y^n e^{-y}$

$\Rightarrow e^{-y} \cdot y^{n-1}[n-y] \begin{cases} > 0 & y < n \\ < 0 & y > n \end{cases}$



$\Rightarrow g(y) < 0 \Rightarrow y \not< c_1$ or $\not> c_2$

$\text{Sup}_{\Omega_H}[P(y < c_1) \text{ or } P(y > c_2)] = \alpha$

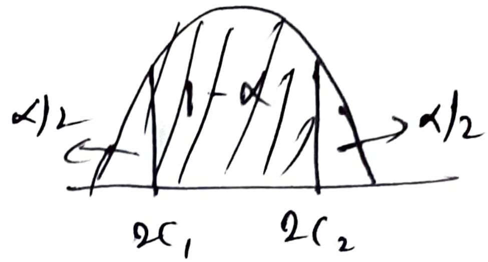
$1 - \alpha = P_{\sigma=1}(c_1 \leq y \leq c_2)$

$1 - \alpha = P_{\sigma=1}\left(\dfrac{2c_1}{\sigma} \leq \dfrac{2y}{\sigma} \leq \dfrac{2c_2}{\sigma}\right)$ $\left[\dfrac{2\sum|X_i|}{\sigma} \sim \chi^2_{2n}\right]$

$\Rightarrow 1 - \alpha = P(2c_1 \leq W \leq c_2)$ $\left[\because \sigma = 1, W = \dfrac{2\sum|X_i|}{\sigma}\right]$

$\Rightarrow 2C_2 = \chi^2_{2n,\alpha/2} \Rightarrow C_2 = \dfrac{\chi^2_{2n,\alpha/2}}{2}$

$2C_1 = \chi^2_{2n,1-\alpha/2} \Rightarrow C_1 = \dfrac{\chi^2_{2n,1-\alpha/2}}{2}$



$\therefore$ Reject H_0 if $y > C_2$ or $y < C_1$

$\Rightarrow \sum|x_i| > \dfrac{\chi^2_{2n,\alpha/2}}{2}$ (or) $\sum|x_i| < \dfrac{\chi^2_{2n,1-\alpha/2}}{2}$ $\left[y = \sum|x_i|\right]$