Assignment-3

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(1) $f(x) = \dfrac{\beta \alpha^{\beta}}{x^{\beta+1}} \quad x > \alpha, \ \alpha > 0, \ \beta > 2$

(i) α is known

$$f_x(\alpha,\beta) = \prod_{i=1}^{n} \left[\frac{\beta \alpha^{\beta}}{x_i^{\beta+1}} \right] \rightarrow x_i > \alpha$$

$$= \underbrace{\frac{[\beta \alpha^{\beta}]^n}{\prod_{i=1}^{n} x_i^{\beta+1}}}_{g(T(x),\theta)} \underbrace{\left[I_{(\alpha,\infty)} X_{(1)} \right] \left[\prod_{i=2}^{n} I_{(X_1,\infty)} X_i \right]}_{h(x)}$$

$\Rightarrow T(x) = \prod_{i=1}^{n} X_i$ is sufficient statistic.

(ii) β is known

$$f(x,\alpha,\beta) = \prod_{i=1}^{n} \left[\frac{\beta \alpha^{\beta}}{x_i^{\beta+1}} \right] \Rightarrow x_i > \alpha$$

$$= \underbrace{(\alpha^{\beta})^n I_{(\alpha,\infty)} X_{(1)}}_{g(T(x),\theta)} \cdot \underbrace{\frac{\beta^n}{\prod_{i=1}^{n} x_i^{\beta+1}} \left[\prod_{i=2}^{n} I_{(X_1,\infty)} X_i \right]}_{h(x)}$$

$\Rightarrow T(x) = X_{(1)}$ is sufficient statistic.

(iii) α, β are unknown.

$$f(x,\alpha,\beta) = \prod_{i=1}^{n}\left[\frac{\beta\alpha^{\beta}}{x_i^{\beta+1}}\right] \rightarrow X_i > \alpha$$

$$= \underbrace{\frac{(\beta\alpha^{\beta})^n \cdot I_{(\alpha,\infty)}(X_{(1)})}{\prod_{i=1}^{n} x_i^{\beta+1}}}_{g(T(x),\theta)} \cdot \underbrace{\prod_{i=2}^{n} I_{(X_1,\infty)}(X_i)}_{h(x)}$$

where $\boxed{T(X) = \left(\prod_{i=1}^{n} X_i, X_{(1)}\right)}$

Sufficient statistic.

② $\text{Beta}(\lambda,\mu) = \frac{1}{B(\lambda,\mu)} x^{\lambda-1}(1-x)^{\mu-1}$, $\lambda > 0$, $\mu > 0$, $x \in [0,1]$

(i) μ is known,

$$f(x,\lambda,\mu) = \underbrace{\frac{1}{B(\lambda,\mu)} \prod_{i=1}^{n} x_i^{\lambda-1}}_{g(T(x),\theta)} \cdot \underbrace{\prod_{i=1}^{n}(1-x_i)^{\mu-1} \cdot I_{(0,1)}X_{(1)} \cdot \prod_{i=2}^{n} I_{(X_1,1)}X_i}_{h(x)}$$

$\Rightarrow \prod_{i=1}^{n} X_i$ is the sufficient statistic.

(ii) λ is known,

$$f(x,\lambda,\mu) = \underbrace{\frac{1}{[B(\lambda,\mu)]^n} \prod_{i=1}^{n}(1-X_i)^{\mu-1}}_{g(T(x),\theta)} \cdot \prod_{i=1}^{n} X_i^{\lambda-1} I_{(0,1)} X_{(1)} \cdot \prod_{i=2} I_{(X_1,1)} X_i$$

$\Rightarrow \prod_{i=1}^{n}(1-X_i)$ is sufficient.

(iii) μ, λ are unknown.

$$f(x,\mu,\lambda) = \underbrace{\frac{1}{[B(\lambda\,\mu)]^n} \prod_{i=1}^{n} x_i^{\lambda-1} \prod_{i=1}^{n} (1-x_i)^{\mu-1}}_{g(T(x),\,\theta)} \cdot \underbrace{I_{(0,1)}(x_{(1)}) \prod_{i=2}^{n} I_{(x_{(1)},1)}(x_i)}_{h(x)}$$

$\Rightarrow \left(\prod_{i=1}^{n} x_i, \prod_{i=1}^{n} (1-x_i) \right)$ is the Sufficient statistic.

(3) $f(x) = \dfrac{\theta}{(1+x)^{1+\theta}}$ $x>0,\ \theta>0$ MSS ?

$$\frac{f(x,\theta)}{f(y,\theta)} = \frac{\cancel{\theta^n}}{\prod_{i=1}^{n}(1+x_i)^{1+\theta}} \times \frac{\prod_{i=1}^{n}(1+y_i)^{1+\theta}}{\cancel{\theta^n}} = \prod_{i=1}^{n} \left(\frac{1+y_i}{1+x_i}\right)^{1+\theta}$$

This is independent of θ iff

$$\prod_{i=1}^{n}(1+x_i) = \prod_{i=1}^{n}(1+y_i)$$

$\Rightarrow T(X) = \prod_{i=1}^{n}(1+x_i)$ is MSS.

(4) $f(x) = p(1-p)^{x-1}$ $x = 1,2,\ldots$ $0<p<1$

$$\frac{f(x,p)}{f(y,p)} = \frac{\prod_{i=1}^{n}(1-p)^{x_i-1}}{\prod_{i=1}^{n}(1-p)^{y_i-1}} = \frac{(1-p)^{\sum_{i} x_i - n}}{(1-p)^{\sum_{i=1}^{n} y_i - n}}$$

$\Rightarrow$ Independent of p iff $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$

$\boxed{T(X) = \sum_{i=1}^{n} X_i}$ is MSS (3)

(5) $f(x) = e^{\mu - x}$ $x > \mu$, $\mu \in \mathbb{R}$

Completeness $E_\theta[g(T(x))] = 0 \Rightarrow P\{g(T(x)) = 0) = 1$.

$F_{X_1}(x) = P(X_1 \le x)$

$= 1 - P(X_1 > x)$

$= 1 - (P(X_1 > x))^n$

$= 1 - [1 - F_{X_1}(x)]^n$

$F_{X_i}(x) = \int_\mu^x e^{\mu - x} dx$

$= 1 - e^{\mu - x}$

$F_{X_1}(x) = 1 - e^{n(\mu - x)}$

$\boxed{f_{X_1}(x) = n e^{n(\mu - x)}}$, $x > \mu$

$E_\mu[g(T(x))] = 0$

$\int_\mu^\infty g(T(x)) \cdot n e^{n(\mu - x)} dx$

$\Rightarrow n e^{n\mu} \int_\mu^\infty g(T(x)) . e^{-nx} dx = 0$

$\Rightarrow \int_\mu^\infty g(T(x)) . e^{-nx} dx = 0 \ \forall \mu$

$\Rightarrow \int_\infty^\theta g(T(x)) e^{-nx} dx = -\int_\theta^\infty g(T(x)) e^{-nx} dx = 0 \ \forall \mu$

Taking derivative on both sides wrt μ.

$g(\mu) . e^{-n\mu} = 0 \ \forall \mu \Rightarrow g(\mu) = 0 \ \forall \mu$ (4)

Thus $g(T(x)) = g(X_{(0)}) = 0$ With Probability 1 and hence $S = X_1$ is Complete.

(6) $X_1, X_2, X_3 - X_n$ r.s $N(\theta, \theta^2)$

$$N(\theta,\theta^2) = \frac{1}{\sqrt{2\pi}\,\theta} e^{-\frac{(x-\theta)^2}{2\theta^2}}$$

$$f(x,\theta,\theta^2) = \left(\frac{1}{\theta\sqrt{2\pi}}\right)^n \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}\left(\frac{x_i-\theta}{\theta}\right)^2}$$

$$\frac{f(x,\theta,\theta^2)}{f(y,\theta,\theta^2)} = \frac{\cancel{\left(\frac{1}{\theta\sqrt{2\pi}}\right)^n} e^{-\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2\theta^2}}}{\cancel{\left(\frac{1}{\theta\sqrt{2\pi}}\right)^n} e^{-\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2\theta^2}}}$$

$$= \frac{e^{-\sum_{i=1}^{n}\frac{x_i^2}{2\theta^2} + \frac{2\sum x_i\theta}{2\theta^2} - \frac{n\theta^2}{2\theta^2}}}{e^{-\sum_{i=1}^{n}\frac{y_i^2}{2\theta^2} + \frac{2\sum y_i\theta}{2\theta^2} - \frac{n\theta^2}{2\theta^2}}}$$

$$= \frac{e^{-\sum_{i=1}^{n}\frac{x_i^2}{2\theta^2} + \frac{\sum x_i}{\theta}}}{e^{-\sum_{i=1}^{n}\frac{y_i^2}{2\theta^2} + \frac{\sum y_i}{\theta}}}$$

$\Rightarrow$ is independent of θ iff $\sum x_i^2 = \sum y_i^2$ & $\sum x_i = \sum y_i$

$\therefore (\sum x_i, \sum x_i^2)$ is MSS

As $(\bar{x}, s^2)$ is a one-one function of $(\sum x_i, \sum x_i^2)$
$\hookrightarrow$ is also MSS.

Lets take $g(T(x)) = \frac{n+1}{n} S^2 - \bar{X}^2$

We know,

$E[S^2] = \sigma^2 = \underline{\underline{\theta^2}}$

$E[\bar{X}^2] \Rightarrow \bar{X} \sim N\left(\theta, \frac{\theta^2}{n}\right)$

$\Rightarrow E[\bar{X}^2] = Var(\bar{X}) + (E[\bar{X}])^2$

$= \underline{\underline{\frac{\theta^2}{n} + \theta^2}}$

$E[g(T(x))] = \frac{n+1}{n} \cdot E[S^2] - E[\bar{X}^2]$

$= \frac{n+1}{n} \cdot \theta^2 - \theta^2\left[1 + \frac{1}{n}\right]$

$= \underline{\underline{0}}$

$\Rightarrow E[g(T(x))] = 0$ even though g is not identically 0. Hence $(\bar{X}, S^2)$ is not Complete.

⑦ $N(0, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma}\left(e^{-x^2/2\sigma^2}\right)$

(i) $E_\sigma[g(T)] = 0$ for $T = X$

$$\int_{-\infty}^{\infty} g(t) \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}\, dt = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} g(t) \cdot e^{-t^2/2\sigma^2}\, dt = 0$$

$\Downarrow$

Holds true for any odd function such as t, t^3, t^5 etc.

$\therefore$ X is not Complete statistic.

(ii) $T = X^2$ $\rightarrow$ Square of standard normal follows Chi-square with 1 degree of freedom.

PDF is $\frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2}$ $\rightarrow y = x^2$ where x is normal.

$E_\sigma[g(T)] = 0$

$$\Rightarrow \int_{-\infty}^{\infty} g(t) \cdot \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-t/2}\, dt = 0$$

$\Rightarrow$ $t = p^2 \Rightarrow 2p\,dp = dt$

$$\int_{-\infty}^{\infty} \underbrace{g(p^2)}_{\text{even function}} \cdot e^{-p^2/2}\, dp = 0 \quad \text{—(1)}$$

(1) is zero iff $g(p^2) = 0$ $\therefore$ X^2 is complete statistic.

(8) $P(\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ UMVUE of $g(\lambda) = P(X_1 \leq 1) = (1+\lambda)e^{-\lambda}$

We know that $T(X) = \sum_{i=1}^{n} X_i$ is a c.s.s.

(a) Solving by direct method.

$$E[\delta(T)] = g(\lambda)$$

$$T = \sum X_i = P(n\lambda)$$

$$\Rightarrow \sum_{t=0}^{\infty} \delta(t) \frac{e^{-n\lambda}(n\lambda)^t}{t!} = (1+\lambda)e^{-\lambda}$$

$$\Rightarrow \sum_{t=0}^{\infty} \frac{\delta(t)(n\lambda)^t}{t!} = (1+\lambda)e^{\lambda(n-1)}$$

Writing Taylor expansion of e^{x},

$$\sum_{t=0}^{\infty} \frac{\delta(t)(n\lambda)^t}{t!} = (1+\lambda)\sum_{t=0}^{\infty}\frac{[(n-1)\lambda]^t}{t!} = \sum_{t=0}^{\infty}\frac{((n-1)\lambda)^t}{t!} + \sum_{t=0}^{\infty}\frac{(n-1)^t\lambda^{t+1}}{t!}$$

Comparing λ^t coefficients we get,

$$\frac{\delta(t)n^t}{\cancel{t!}} = \frac{(n-1)^t}{\cancel{t!}} + \frac{(n-1)^{t-1}}{\cancel{(t-1)!}} \cdot t$$

$$\Rightarrow \delta(t) = \left(\frac{n-1}{n}\right)^t \left[1 + \frac{t}{n-1}\right]$$

$$\Rightarrow \delta(T) = \left(1-\frac{1}{n}\right)^T \left[1 + \frac{T}{n-1}\right]$$ is UMVUE of $g(\lambda)$.

(7)

(9) $f(x|\theta) = e^{\mu - x} \quad \mu > x, \ \mu \in R$

We know that $X_{(1)}$ is Sufficient for exponential distribution and from Q.5 it is also a CSS.

$$T(X) = X_{(1)} \text{ is CSS}$$

$$f_X(x_{(1)}) = n e^{n(\mu - x)}$$

$$E[X_{(1)}] = \int_{\mu}^{\infty} x . n e^{n(\mu - x)} = n . \left[x . \frac{e^{n(\mu-x)}}{-n} \Big|_{\mu}^{\infty} - \int_{\mu}^{\infty} 1 . \frac{e^{n(\mu-x)}}{-n} dx \right]$$

$$= n \left[\frac{\mu}{n} + \frac{1}{n^2} \underbrace{\int_{\mu}^{\infty} n e^{n(\mu-x)} dx}_{1 \text{ as it is pdf.}} \right]$$

$$= \boxed{\mu + \frac{1}{n}}$$

$$E[X_{(1)}^2] = \int_{\mu}^{\infty} x^2 n e^{n(\mu-x)} dx = n \left[x^2 . \frac{e^{n(\mu-x)}}{-n} \Big|_{\mu}^{\infty} - \int_{\mu}^{\infty} 2x . \frac{e^{n(\mu-x)}}{-n} dx \right]$$

$$= n \left[\frac{\mu^2}{n} + \frac{2}{n^2} \underbrace{\int_{\mu}^{\infty} x . n e^{n(\mu-x)} dx}_{E[X_{(1)}]} \right]$$

$$= n \left[\frac{\mu^2}{n} + \frac{2}{n^2} \left(\mu + \frac{1}{n} \right) \right] = \boxed{\mu^2 + \frac{2}{n} \left(\mu + \frac{1}{n} \right)}$$

UMVUE of μ

$$E\left[X_{(1)} - \frac{1}{n} \right] = \mu \Rightarrow \delta(T) = X_{(1)} - \frac{1}{n} \text{ is UMVUE of } \mu.$$

UMVUE of μ^2

$$\delta(T) = T^2 - \frac{2T}{n} \Rightarrow T(X) = X_{(1)}^2 - \frac{2X_{(1)}}{n}$$

$$E[\delta(T)] = \mu^2 + \frac{2}{n}\left(\mu + \frac{1}{n}\right) - \frac{2}{n}\left(\mu + \frac{1}{n}\right) = \underline{\underline{\mu^2}}$$

$\therefore T^2 - \frac{2T}{n}$ is UMVUE of μ^2

(10) $N(\mu, \sigma^2)$

- Noise ratio - μ/σ

We know that $\bar{X}$ is a CSS, $E[\bar{X}] = \mu$

$\therefore$ $\bar{X}$ is UMVUE for μ.

Let $g(\sigma) = \sigma^r$, since normal is exponential family,

$$S^2 = \sum(X_i - \bar{X})^2 \text{ is a CSS also}$$

$$Y = \frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

We know that $K_{n-1,r} S_X^2$ is UMVUE of σ^2

where $$K_{n-1,r} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^{r/2}\,\Gamma\left(\frac{n-1+r}{2}\right)}$$

$g(\mu, \sigma) = \mu/\sigma$,

$\bar{X} \to$ UMVUE for μ

$K_{n-1,-1}$ is UMVUE for $1/\sigma$

We know that $\bar{X}$, S^2 are independent.

$$E\left[K_{n-1,-1} \cdot \frac{1}{S_X}\right] = 1/\sigma$$

$$\Rightarrow E\left[K_{n-1,-1} \cdot \frac{\bar{X}}{S_X}\right] = \mu/\sigma$$

$\therefore$ $K_{n-1,-1} \dfrac{\bar{X}}{S_X}$ is UMVUE of μ/σ

- Quantile : $\bar{X} + b . K_{n-1,-1}$ is UMVUE for $\mu + b\sigma$.

(8)