MA 3140: Statistical Inference

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Example 4 : Normal UMVUE

Case 1: σ **is known and** $g(\mu) = \mu$

We know that $\overline{X}$ is a CSS.

Also, since $E(\overline{X}) = \mu$, we conclude that $\overline{X}$ is the UMVUE of μ.

Case 2: σ **is known and** $g(\mu) = \mu^2$

Since $Var(\overline{X}) = \sigma^2/n$, we get $E(\overline{X}^2 - \sigma^2/n) = \mu^2$.

Thus, $\overline{X}^2 - \frac{\sigma^2}{n}$ is the UMVUE of μ^2.

Example 4: Normal UMVUE

Case 3: σ is known and $g(\mu) = \mu^3$

Since the odd-ordered central moments of a symmetric distribution is 0, we have

$$\begin{aligned} E(\overline{X} - \mu)^3 &= 0 \\ \Longrightarrow E(\overline{X})^3 - \mu^3 - 3\mu E(\overline{X})^2 + 3\mu^2 E(\overline{X}) &= 0 \\ \Longrightarrow E(\overline{X})^3 - \mu^3 - 3\mu\left(\frac{\sigma^2}{n} + \mu^2\right) + 3\mu^3 &= 0 \\ \Longrightarrow E\left(\overline{X}^3 - 3\frac{\sigma^2}{n}\overline{X}\right) &= \mu^3 \end{aligned}$$

Thus, we conclude that $\overline{X}^3 - 3\frac{\sigma^2}{n}\overline{X}$ is the UMVUE of μ^3.

Example 4: Normal UMVUE

Case 4: μ **is known and** $g(\sigma) = \sigma^r$

Since it is a one-parameter exponential family, we know that $S^2 = \sum (X_i - \mu)^2$ is a CSS.

Also, $Y = S^2/\sigma^2 \sim \chi^2_n$

$$\begin{aligned} E[Y^{\frac{r}{2}}] &= \int y^{\frac{r}{2}} \ \frac{1}{\Gamma \frac{n}{2} \ 2^{\frac{n}{2}}} \ y^{\frac{n}{2}-1} e^{-\frac{y}{2}} dy \\ &= \frac{1}{\Gamma(\frac{n}{2}) \ 2^{\frac{n}{2}}} \int y^{\frac{r+n}{2}-1} e^{-\frac{y}{2}} dy \\ &= \frac{1}{\Gamma(\frac{n}{2}) \ 2^{\frac{n}{2}}} \Gamma\left(\frac{n+r}{2}\right) \ 2^{\frac{n+r}{2}} = \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} \ 2^{\frac{r}{2}} \end{aligned}$$

Example 4: Normal UMVUE

$$E\left[\frac{S^2}{\sigma^2}\right]^{\frac{r}{2}} = E\left[\frac{S}{\sigma}\right]^{r} = \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}\, 2^{\frac{r}{2}}$$

$$\Longrightarrow E\left[\frac{\Gamma(\frac{n}{2})S^r}{2^{\frac{r}{2}}\Gamma(\frac{n+r}{2})}\right] = \sigma^r$$

Thus, when μ is known, $K_{n,r}\ S^r$ is the UMVUE of σ^r, where

$$K_{n,r} = \frac{\Gamma(\frac{n}{2})}{2^{\frac{r}{2}}\Gamma(\frac{n+r}{2})}.$$

Note that for $r = 2$, $K_{n,r} = 1/n$, and hence $E(S^2) = n\sigma^2$.

Example 4: Normal UMVUE

Case 5: μ is unknown and $g(\sigma) = \sigma^r$

We know that $S_X^2 = \sum(X_i - \overline{X})^2$ is a CSS.

Also, $Y = S_X^2/\sigma^2 \sim \chi^2_{n-1}$

Thus, when μ is unknown, $K_{n-1,r}\ S_X^r$ is the UMVUE of σ^r.

Example 4: Normal UMVUE

Case 5: μ and σ^2 unknown and $g(\mu, \sigma) = \frac{\mu}{\sigma}$

Now, $\overline{X}$ is the UMVUE for μ and $K_{n-1,-1} \ \frac{1}{S_X}$ is the UMVUE of $\frac{1}{\sigma}$.

$$E\left[K_{n-1,-1} \ \frac{1}{S_X}\right] = \frac{1}{\sigma}$$

$$E\left[K_{n-1,-1} \ \frac{\overline{X}}{S_X}\right] = \frac{\mu}{\sigma} \qquad (\because \overline{X} \text{ and } S^2 \text{ are independent})$$

Thus, $K_{n-1,-1} \ \frac{\overline{X}}{S_X}$ is the UMVUE of $\frac{\mu}{\sigma}$.

Example 4: Normal UMVUE

Case 6: Estimating the critical value

Within the framework of the Normal one-sample problem, we are often interested in

$$p = P(X_1 \leq w).$$

X_i: performances of past candidates on an entrance examination

w: the cutoff value

$X \geq w$: passing performance with probability $1 - p$.

This is the problem of estimating w above for a given value of p.

Example 4: Normal UMVUE

Solving the equation

$$p = P(X_1 \le w) = \Phi\left(\frac{w - \mu}{\sigma}\right)$$

for w shows that

$$w = g(\mu, \sigma) = \mu + \sigma \Phi^{-1}(p).$$

It follows that the UMVUE of w is

$$\overline{X} + K_{n-1,1}\ S\ \Phi^{-1}(p).$$

Example 4: Normal UMVUE

Case 7: Estimating p for a given value of w.

For $\sigma = 1$, we have $p = P(X_1 \leq w) = \Phi(w - \mu)$.

An UE δ of p is

$$\delta = \begin{cases} 1, & X_1 \leq w \\ 0, & \text{o/w} \end{cases}$$

We also know that $\overline{X}$ is a CSS.

Example 4: Normal UMVUE

The UMVUE of p is

$$
\begin{aligned}
E[\delta|\overline{X}] &= P[X_1 \le w|\overline{X}] \\
&= P[X_1 - \overline{X} \le w - \overline{x} \mid \overline{x}] \\
&= P[X_1 - \overline{X} \le w - \overline{x}] \qquad (\because X_1 - \overline{X} \text{ is ancillary}) \\
&= \Phi\left[\sqrt{\frac{n}{n-1}}(w - \overline{x})\right] \qquad (\because X_1 - \overline{X} \sim N(0, (n-1)/n))
\end{aligned}
$$

Thus, $\Phi\left[\sqrt{\frac{n}{n-1}}(w - \overline{x})\right]$ is the UMVUE of p.

Example 5: Two Sample Normal Problem

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independently distributed according to normal distributions $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively.

Case 1: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ **are unknown**

The joint density

$$\frac{1}{(\sqrt{2\pi})^{m+n}\sigma_X^m \sigma_Y^n} \exp\left[-\frac{1}{2\sigma_X^2}\sum(x_i-\mu_X)^2-\frac{1}{2\sigma_Y^2}\sum(y_j-\mu_Y)^2\right]$$

constitutes an exponential family for which

$$\overline{X},\ \overline{Y},\ S_X^2 = \sum(X_i - \overline{X})^2,\ S_Y^2 = \sum(Y_j - \overline{Y})^2$$

are sufficient and complete.

Example 5: Two Sample Normal Problem

Thus, the UMVU estimators of μ_X and σ_X^r are $\overline{X}$ and $K_{n-1,r}\ S_X^r$.

The UMVU estimators of μ_Y and σ_Y^r are therefore $\overline{Y}$ and $K_{n-1,r}\ S_Y^r$.

Thus, the UMVUE of $\mu_Y - \mu_X$ is $\overline{Y} - \overline{X}$;

and the UMVUE of $\frac{\sigma_Y^r}{\sigma_X^r}$ is $\frac{K_{n-1,r}\ S_Y^r}{K_{n-1,-r}\ S_X^{-r}}$.

Example 5: Two Sample Normal Problem

Case 2: $\sigma_X = \sigma_Y = \sigma$ **are unknown.**

Then, $\overline{X},\ \overline{Y},\ S^2 = \sum(X_i - \overline{X})^2 + \sum(Y_j - \overline{Y})^2$ are sufficient and complete.

The natural unbiased estimators of μ_X, μ_Y, σ^r, $\mu_Y - \mu_X$, and $(\mu_Y - \mu_X)/\sigma$ are all UMVU.

Example 5: Two Sample Normal Problem

Case 3: $\mu_X = \mu_Y$ **but** $\sigma_X \neq \sigma_Y$**.**

Then, $T = (\overline{X},\ \overline{Y},\ S_X^2 = \sum(X_i - \overline{X})^2,\ S_Y^2 = \sum(Y_j - \overline{Y})^2)$ are minimal sufficient.

But are they complete??

NO since $E(\overline{Y} - \overline{X}) = 0$ but $\overline{Y} \neq \overline{X}$ w.p. 1.

Hypothesis Testing

Hypothesis

- Hypothesis: An assertion about the parameters of the population.

  Examples:

  A manufacturer of 10-volt batteries claims that their batteries lasts for N hours.

  In coin-tossing experiment, one frequently assumes that the coin is fair.

  How to check the truth of these assertions?

- Hypothesis Testing: Tests whether a claim (hypothesis) that has been formulated is correct or not.

Null vs. Alternative Hypothesis

Null Hypothesis (denoted by H_0): a condition that is doubted; first tentative specification about the probability model.

$$H_0 : p = .75 \qquad H_0 : \mu_1 = \mu_2 \qquad H_0 : \sigma_1^2 = \sigma_2^2$$

Alternative Hypothesis (denoted by H_1): a condition that we believe in.

$$H_1 : p > .75 \qquad H_1 : \mu_1 \neq \mu_2 \qquad H_1 : \sigma_1^2 < \sigma_2^2$$

Null vs. Alternative Hypothesis cont'd

- In general, $\boldsymbol{X} = (X_1, \dots, X_n)$ and $\boldsymbol{X} \sim F_{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \Theta$.

  Consider $\Theta_0 \subset \Theta$ and $\Theta_1 \subset \Theta$ such that $\Theta_0 \cap \Theta_1 = \phi$.

  Then, the problem of testing of hypothesis is:

$$H_0 : \boldsymbol{\theta} \in \Theta_0 \quad \text{vs.} \quad H_1 : \boldsymbol{\theta} \in \Theta_1$$

- In the previous slide, $\Theta_0 = \{0.75\}$ and $\Theta_1 = (0.75, 1)$.

Remark

- Null and Alternative Hypothesis divide parameter space into disjoint regions.

  However, it is not necessary that they will exhaust the parameter space.

- In case of Binomial distribution, $\Theta = (0, 1)$ but

$$\Theta_0 \cup \Theta_1 = \{0.75\} \cup (0.75, 1) \neq (0, 1) = \Theta.$$

Simple vs. Composite Hypothesis

- The hypothesis which completely specifies a probability model is known as simple hypothesis otherwise, it is referred to as composite hypothesis.

- **Example:** Let $X \sim N(\mu, \sigma^2)$, μ and σ^2 are unknown, $\Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$.

  $H_0 : \mu \leq \mu_0, \sigma^2 > 0$, where μ_0 is known, is a composite H_0.
  $H_1 : \mu > \mu_0, \sigma^2 > 0$, is a composite H_1.

  $H_0 : \mu = \mu_0, \sigma^2 > 0$, is a ??? H_0.

  $H_0 : \mu = 0, \sigma^2 = \sigma_0^2$, where σ_0^2 is known, is a ??? H_0.

Test of Statistical Hypothesis

- A procedure to decide whether to reject or fail to reject the null hypothesis.

  This is done on the basis of the observations that we take from the given population.

Example: Consider a die. Suppose we are interested in p, i.e., probability of occurrence of a six.

$$H_0 : p = 1/6 \qquad vs. \qquad H_1 : p \neq 1/6$$

Toss the die say $n = 60$ times. Let X denote the number of 6's.

If $X = 9, 10, 11$, we fail to reject (accept) H_0

$X \neq \{9, 10, 11\}$, we reject H_0.

Non-randomized test procedure

Based on the sample X, we decide to accept or reject H_0.

$$\mathcal{X} : A \cup R,$$

where $\mathcal{X}$ is the sample space, A is acceptance region for H_0 and R is a rejection region/ critical region for H_0.

If $X \in A$, we accept H_0, and if $X \in R$, we reject H_0.

Examples: Non-randomized test procedure

- Let $X \sim P(\lambda)$.

$$H_0 : \lambda \leq 1 \qquad vs. \qquad H_1 : \lambda > 1$$

If $X = 0$ or 1, then accept H_0 otherwise reject H_0.

$$A = \{0, 1\}, \quad R = \{2, 3, \ldots\}.$$

- Let $X_1, \ldots, X_n \sim N(\mu, 1)$.

$$H_0 : \mu = -1 \qquad vs. \qquad H_1 : \mu = 1$$

If $\overline{X} \leq 0$ then accept H_0, and if $\overline{X} > 0$, reject H_0.

$$A = (-\infty, 0], \quad R = (0, \infty).$$

Type I and Type II errors

Decision	Reality	
	H_0 is true	H_0 is false
Reject H_0	Type I error	Correct
Fail to Reject H_0	Correct	Type II error

- If we reject H_0 when it is true, we commit Type I error and $P(\text{Type I error}) = \alpha$.
- If we fail to reject H_0 when it is false, we commit Type II error and $P(\text{Type II error}) = \beta$.

Example on α and β

- Let $X_1, \ldots, X_n \sim N(\mu, 1)$.

$$H_0 : \mu = -1/2 \qquad vs. \qquad H_1 : \mu = 1/2$$

If $\overline{X} \leq 0$ then accept H_0, and if $\overline{X} > 0$, reject H_0.

$$A = (-\infty, 0], \quad R = (0, \infty).$$

$$\begin{aligned}
\alpha = P(\text{Type I error}) &= P(\text{Rejecting } H_0 \text{ when it is true}) \\
&= P_{\{\mu=-1/2\}}(\overline{X} > 0) \\
&= P\left(\sqrt{n}\left(\overline{X} + \frac{1}{2}\right) > \frac{\sqrt{n}}{2}\right) \quad (\because \overline{X} \sim N(\mu, 1/n)) \\
&= P\left(Z > \frac{\sqrt{n}}{2}\right)
\end{aligned}$$

Example on α and β cont'd

For $n = 16$,

$$\alpha = P(Z > 2) = 0.0228$$

- Let us now calculate β.

$$\begin{aligned}
\beta = P(\text{Type II error}) &= P(\text{Accepting } H_0 \text{ when it is false}) \\
&= P_{\{\mu=1/2\}}(\overline{X} \le 0) \\
&= P\left(\sqrt{n}\left(\overline{X} - \frac{1}{2}\right) \le -\frac{\sqrt{n}}{2}\right) \\
&= \Phi(-2) = 0.0228
\end{aligned}$$

In this case, α and β are equal.

Example on α and β cont'd

In ideal test procedure, both α and β should be minimum (zero).

However, simultaneous minimization of both α and β is not possible.

Consider the modified test procedure for the same problem:

If $\overline{X} < -1/4$ then accept H_0, and if $\overline{X} \geq -1/4$, reject H_0.

$$A^* = (-\infty, -1/4), \quad R^* = [-1/4, \infty).$$

Example on α and β cont'd

$$
\begin{aligned}
\alpha^* &= P(\text{Rejecting } H_0 \text{ when it is true}) \\
&= P_{\{\mu=-1/2\}}(\overline{X} \geq -1/4) \\
&= P\Big(\sqrt{n}\Big(\overline{X} + \frac{1}{2}\Big) \geq \sqrt{n}\Big(-\frac{1}{4} + \frac{1}{2}\Big)\Big) \\
&= P\Big(Z \geq \frac{\sqrt{n}}{4}\Big) = \Phi(-1) = 0.1586
\end{aligned}
$$

Thus, $\alpha^* > \alpha$.

- Let us now calculate β^*.

$$
\begin{aligned}
\beta^* = P(\text{Accepting } H_0 \text{ when it is false}) &= P_{\{\mu=1/2\}}(\overline{X} < -1/4) \\
&= P(Z < -3) \\
&= \Phi(-3) = 0.0013
\end{aligned}
$$

Thus, $\beta^* < \beta$.

Remark

- Thus, it can be seen that we have significantly reduced β but increased α.
- To deal with this, we try to fix an upper bound on one error and then find the test procedure for which the second probability is the minimum.
- A standard convention is to fix α.

$$\begin{aligned} \alpha(\theta) &= P_\theta(X \in R), \quad \theta \in \Theta_0 \\ \sup \ \alpha(\theta) &\leq \alpha \end{aligned} \tag{1}$$

where α is known as the size of the test or level of significance.

Remark

Subject to condition (1), find the test procedure for which

$$\beta(\theta) = P_\theta(X \in A), \quad \theta \in \Theta_1 \tag{2}$$

is minimized (over $\theta \in \Theta_1$)

or

$$1 - \beta(\theta) = P_\theta(X \in R), \quad \theta \in \Theta_1 \tag{3}$$

is maximized, where $1 - \beta(\theta)$ is power function of the test.

Thanks for your patience!