Final Exam Solutions

1. i) X_1, X_2 and $X_3 \sim$ Bernoulli (p)

$$f_{\underline{X}}(\underline{x}|p) = p^{x_1 + x_2 + x_3} (1 - p)^{3 - x_1 - x_2 - x_3}$$

$$= p^{x_1 + x_2} (1 - p)^{2 - x_1 - x_2} p^{x_3} (1 - p)^{1 - x_3}$$

Here, $h(\underline{x}) = 1$

$$g(t_1, t_2) = p^{t_1} (1 - p)^{2 - t_1} p^{t_2} (1 - p)^{1 - t_2}$$

$$\therefore f_X(\underline{x}|p) = g(x_1 + x_2, x_3|p) \cdot h(x)$$

Hence T is a sufficient statistic for p.

ii) Let
$$A(\underline{x}) = x_1 + x_2$$
 and $B(\underline{x}) = x_3$

$$f_{\underline{X}}(\underline{x}|p) = p^{A(\underline{x})} (1-p)^{2-A(\underline{x})} p^{B(\underline{x})} (1-p)^{B(\underline{x})}$$

$$= p^{A(\underline{x})+B(\underline{x})} (1-p)^{3-A(\underline{x})-B(\underline{x})}$$

$$\frac{f_{\underline{X}}(\underline{x}|p)}{f_{\underline{Y}}(\underline{y}|p)} = \frac{p^{A(\underline{x})+B(\underline{x})} (1-p)^{3-A(\underline{x})-B(\underline{x})}}{p^{A(\underline{y})+B(\underline{x})} (1-p)^{3-A(\underline{y})-B(\underline{y})}}$$

$$= \left(\frac{p}{1-p}\right)^{A(\underline{x})+B(\underline{x})-A(\underline{y})-B(\underline{y})}$$

The ratio above is constant as a function of p if (but not only if) $A(\underline{x}) = B(\underline{y})$ and $B(\underline{x}) = B(\underline{y})$ because $A(\underline{x}) + B(\underline{x}) = A(\underline{y}) + B(\underline{y})$, even though $A(\underline{x}) \neq A(\underline{y})$ and $B(\underline{x}) \neq B(\underline{y})$, the above ratio is still constant.

Therefore, T is not a MSS for p.

2. i)

$$\phi(T) = E(W|T) = E\left(\frac{X_1 + X_2}{2}|X_1\right)$$

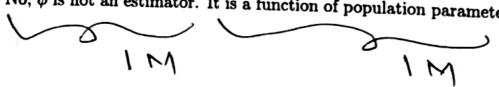
$$= \frac{1}{2}E(X_1|X_1) + \frac{1}{2}E(X_2|X_1)$$

$$= \frac{X_1}{2} + \frac{\theta}{2}$$

$$\Rightarrow E(\phi(T)) = \theta.$$

ii)
$$Var(\phi(T)) = \frac{1}{4} < Var(W) = \frac{1}{2}$$
.

iii) No, ϕ is not an estimator. It is a function of population parameter θ



3. i)
$$X \sim \text{Uniform}(\theta)$$
, $T(X) = X$

$$E_{\theta}(g(T)) = E_{\theta}(g(X)) = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = 0$$

$$\iff \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0$$

$$\iff \sum_{x=1}^{\theta} g(x) = 0$$



For
$$\theta \in \mathbb{N}$$
,

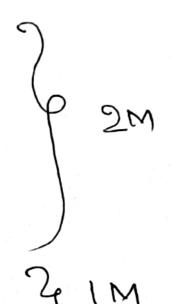
If
$$\theta = 1$$
, $\sum_{\substack{x=1 \ \theta}}^{\theta} g(x) = g(1) = 0$

If
$$\theta = 2$$
, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = 0$

If
$$\theta = k$$
, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) + ... + g(k) = 0$

$$\therefore g(x) = 0 \ \forall \, x \in \mathbb{N}$$

Thus, X is a complete statistic for $\theta \in \Omega = \mathbb{N}$



ii) Define

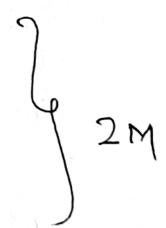
$$g(x) = \left\{ egin{array}{ll} 1 & & if \ x=n, \\ -1 & & if \ x=n+1, \\ 0 & & otherwise. \end{array}
ight.$$

$$E_{\theta}\left(g(T)\right) = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) \Longrightarrow$$

$$E_{ heta}\left(g(T)
ight) = \left\{egin{array}{ll} rac{1}{ heta} & if & heta = n, \ \ 0 & if & heta
eq n. \end{array}
ight.$$

:. when
$$\theta \in \Omega = \mathbb{N} - \{n\}$$
, $g(x) \neq 0$ but $E(g(T)) = 0$.

Thus, T(X) = X is not a complete statistic.



4.

$$P(X = x) = \frac{\lambda^{x}e^{-\lambda}}{x!}, x = 0, 1, 2, \dots \infty, \ \lambda = 1 \text{ or } 2.$$

$$E_{\lambda}(g(T)) = \sum_{t=0}^{\infty} g(t) \frac{\lambda^{t}e^{-\lambda}}{t!} = 0$$

$$E_{\lambda=1}(g(T)) = \sum_{t=0}^{\infty} g(t) \frac{e^{-1}}{t!} = 0$$
and
$$E_{\lambda=2}(g(T)) = \sum_{t=0}^{\infty} g(t) \frac{2^{t}e^{-2}}{t!} = 0$$

$$(1) \implies \sum_{t=0}^{\infty} \frac{g(t)}{t!} = 0$$

$$(3)$$

Define

$$g(t) = \begin{cases} 2, & t = 0, 2 \\ -3, & t = 1, \\ 0, & otherwise. \end{cases}$$

(4)

From (3),

$$\sum_{t=0}^{\infty} \frac{g(t)}{t!} = \frac{g(0)}{0!} + \frac{g(1)}{1!} + \frac{g(2)}{2!} = 2 - 3 + 1 = 0$$

From (4),

$$\sum_{t=0}^{\infty} \frac{g(t)2^t}{t!} = 2 - 6 + 4 = 0$$

Thus, the family is not complete.

5.
$$X_1, X_2, ..., X_n \sim U(0, \theta)$$

We know that $X_{(n)}$ is CSS. Now, $E\left(X_{(1)}\right) = E\left[\frac{X_{(1)}}{X_{(n)}} \cdot X_{(n)}\right]$

$$\Rightarrow E\left(X_{(1)}\right) = E\left[\frac{X_{(1)}}{X_{(n)}}\right] E\left(X_{(n)}\right)$$

$$\therefore E\left[\frac{X_{(1)}}{X_{(n)}}\right] = \frac{E\left(X_{(1)}\right)}{E\left(X_{(n)}\right)}$$

 $(2) \implies \sum_{t=0}^{\infty} \frac{2^t g(t)}{t!} = 0$

This implies if we can show that $\frac{X_{(1)}}{X_{(n)}}$ is ancillary, we can apply Basu's Theorem. Here,

$$f_X(x) = \frac{1}{\theta}, 0 < x < \theta.$$

Let
$$Y = \frac{X}{\theta}$$
. Then $\frac{dx}{dy} = \theta$ and $Y \sim U(0,1)$

$$\therefore \frac{X_{(1)}}{X_{(n)}} = \frac{\theta Y_{(1)}}{\theta Y_{(n)}} = \frac{Y_{(1)}}{Y_{(n)}} \text{ is an ancillary statistic and } E\left[\frac{X_{(1)}}{X_{(n)}}\right] = \frac{E\left[Y_{(1)}\right]}{E\left[Y_{(n)}\right]}$$
Now, since $Y \sim U(0, 1)$

$$f_{Y(y)} = 1, \ 0 < y < 1, \ and$$

$$f_{Y(1)}(y) = n \left[1 - F(y) \right]^{n-1} f(y)$$

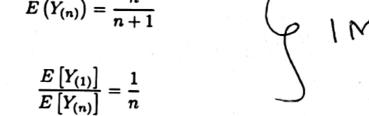
$$= n(1-y)^{n-1}$$

$$\therefore E(Y(1)) = \frac{1}{n+1}$$

Similarly,
$$f_{Y_{(n)}}(y) = n [F(y)]^{n-1} f(y) = n y^{n-1}$$
 and

$$E\left(Y_{(n)}\right)=\frac{n}{n+1}$$

Thus,



i) T_1 rejects H_0 if, and only if, all all success are observed i.e. 6.

$$R = \{\underline{x} : \underline{x} = (1, 1, 1, 1, 1)\}$$
$$= \left\{\underline{x} : \sum_{i=1}^{5} x_i = 5\right\}$$

T₂ rejects if, and only if, 3 or more success are observed i.e.,

$$R = \left\{ \underline{x} : \sum_{i=1}^{5} x_i \ge 5 \right\}$$

$$\beta_1(\theta) = P\left(Reject \ H_0|\theta\right) = P\left(X \in R|\theta\right)$$
$$= P\left(\sum_{i=1}^5 x_i = 5|\theta\right)$$
$$= \theta^5$$





$$\beta_2(\theta) = P\left(\sum_{i=1}^5 x_i \ge 3|\theta\right)$$

$$= {5 \choose 3}\theta^3(1-\theta)^2 + {5 \choose 4}\theta^4(1-\theta) + \theta^5$$

$$= \theta^3(6\theta^2 - 15\theta + 10)$$

ii) When
$$\theta \in \Omega_0 = (0, 0.5]$$
, the power of $\beta(\theta)$ is Type 1 error.
$$\max \beta_1(\theta) = \max \theta^5 = (0.5)^5 = \frac{1}{32} \approx 0.031$$

For T_2 , we need to find max of $\beta_2(\theta)$ for $\Omega_0 = (0, 0.5]$.

$$\beta_2'(\theta) = 30\theta^2(\theta - 1)^2 > 0$$

 $\beta_2(\theta)$ is increasing in $\theta \in (0,1)$. Thus, max of Type 1 error is $\beta_2(0.5) = 0.5.$

iii)
$$(1 - \beta_1(\theta))|_{\frac{2}{3}} = (1 - \theta^5)|_{\frac{2}{3}} \approx 0.868 \qquad \qquad | M$$

$$(1 - \beta_2(\theta))|_{\frac{2}{3}} = 1 - \theta^3(6\theta^2 - 15\theta + 10)|_{\frac{2}{3}} \approx 0.21 \qquad \qquad \boxed{2} M$$

$$\Omega_0 = \{\theta = 1, v > 0\}$$

$$\Omega = \{\theta > 0, v > 0\}$$

 $logL(\theta, v|\underline{x}) = nlog\theta + n\theta logv - (1+\theta)log\prod x_i, \ v < x_{(1)}.$

For any value of θ , this is an increasing function of v for $v < x_{(1)}$.

$$\hat{v} = X_{(1)}$$

 $\frac{\partial log L(\theta, x_{(1)}|\underline{x})}{\partial \theta} = \frac{n}{\theta} + n log x_{(1)} - log \prod_{i=1}^{n} x_i = 0 \text{ and solving for } \theta \text{ yielding,}$

$$\hat{\theta} = \frac{n}{\log\left(\prod x_i/x_{(1)}^n\right)} = \frac{n}{T}$$

Under H_0 ,

$$\hat{\theta} = 1$$

$$\hat{v} = X_{(1)}$$

7.

and Under H_1 ,

$$\hat{\theta} = \frac{n}{T}$$

$$\hat{v} = X_{(1)}$$

So likelihood ratio is

$$\lambda(\underline{x}) = \frac{x_{(1)}^n / (\prod x_i)^2}{(nT)^2 x_{(1)}^{n^2} / (\prod x_i)^{n/T+1}}$$

$$= \left(\frac{T}{n}\right)^n \frac{e^{-T}}{(e^{-T})^{n/T}}$$

$$= \left(\frac{T}{n}\right)^n e^{-T+n}$$

$$\frac{\partial log(\lambda(\underline{x}))}{\partial T} = \frac{n}{T} - 1.$$

Hence, $\lambda(\underline{x})$ is increasing if $T \leq n$ and decreasing if $T \geq n$. Thus, $T \leq c \iff T \leq c_1$ or $T \geq c_2$ for appropriately chosen constants c_1 and c_2 . \geq