MA 3140: Statistical Inference

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Likelihood Ratio Test

Likelihood Ratio Test (LRT)

Let $\boldsymbol{X}$ be a random vector with pdf (pmf) $f(\boldsymbol{x}, \theta)$, $\theta \in \Omega$, and $L(\theta, \boldsymbol{x})$ is the corresponding likelihood function.
The likelihood ratio test statistic for testing

$$H : \theta \in \Omega_H \quad \text{vs.} \quad K : \theta \in \Omega_K \qquad (\Omega_H \cup \Omega_K = \Omega)$$

is

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\Omega_H} L(\theta, \boldsymbol{x})}{\sup_{\Omega} L(\theta, \boldsymbol{x})} = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)}.$$

A likelihood ratio test (LRT) is any test that has a rejection region of the form $\{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$, where c is determined from the size condition

$$\sup_{\theta \in \Omega_H} P_\theta(\lambda(\boldsymbol{x}) \leq c) = \alpha.$$

Example 1

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find LRT for testing

$$H : \mu \leq 0 \quad \text{vs.} \quad K : \mu > 0.$$

Solution: Step 1:

$$\Omega_H = \{(\mu, \sigma^2) : -\infty < \mu < 0, \sigma^2 > 0\}$$

$$\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}.$$

Step 2: The likelihood function is

$$L(\mu, \sigma^2, \boldsymbol{x}) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left[-\frac{1}{2\sigma^2}\sum(x_i - \mu)^2\right]$$

$$\log L = -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum(x_i - \mu)^2$$

Example 1 cont'd

Now,

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum (x_i - \mu) = \frac{n(\overline{x} - \mu)}{\sigma^2}$$

$$\begin{cases} < 0, & \text{if } \mu > \overline{x} \\ > 0, & \text{if } \mu < \overline{x} \end{cases} \tag{1}$$

So, $\log L$ increases for $\mu < \overline{x}$ and $\log L$ decreases for $\mu > \overline{x}$.

Thus, maximum for μ is attained when $\mu = \overline{x}$ (on Ω), i.e.,

$$\hat{\mu}_{\Omega} = \overline{x}$$

Example 1 cont'd

Further,

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 = \frac{n}{2\sigma^4}\left[\frac{\sum(x_i-\mu)^2}{n} - \sigma^2\right] \begin{cases} < 0, & \text{if } \sigma^2 > \frac{\sum(x_i-\mu)^2}{n} \\ > 0, & \text{if } \sigma^2 < \frac{\sum(x_i-\mu)^2}{n} \end{cases} \tag{2}$$

So, $\log L$ increases when $\sigma^2 < \frac{\sum(x_i-\mu)^2}{n}$ and $\log L$ decreases when $\sigma^2 > \frac{\sum(x_i-\mu)^2}{n}$.

Thus, maximum for σ^2 is attained when $\sigma^2 = \frac{\sum(x_i-\mu)^2}{n}$, i.e.,

$$\hat{\sigma}^2_{\Omega} = \frac{\sum(x_i-\mu)^2}{n}$$

Example 1 cont'd

Therefore, when we consider maximization of $L(\mu, \sigma^2, \boldsymbol{x})$ over Ω, we get

$$\hat{\mu}_\Omega = \overline{x} \quad \text{and} \quad \hat{\sigma}^2_\Omega = \frac{1}{n} \sum (x_i - \overline{x})^2;$$

Substituting these values in $L(\mu, \sigma^2, \boldsymbol{x})$, we get

$$\begin{aligned} \hat{L}(\Omega) &= \frac{1}{(2\pi\hat{\sigma}^2_\Omega)^{n/2}} \exp\left[-\frac{1}{2\hat{\sigma}^2_\Omega} \sum (x_i - \overline{x})^2 \right] \\ &= \frac{1}{(2\pi\hat{\sigma}^2_\Omega)^{n/2}} \exp\left[-\frac{n}{2} \right] \end{aligned}$$

Example 1 cont'd

Step 3: In order to evaluate $\hat{L}(\Omega_H)$, we consider maximization of $L(\mu, \sigma^2, \boldsymbol{x})$ over Ω_H, i.e., we need $\hat{\mu}_{\Omega_H}$ and $\hat{\sigma}^2_{\Omega_H}$.

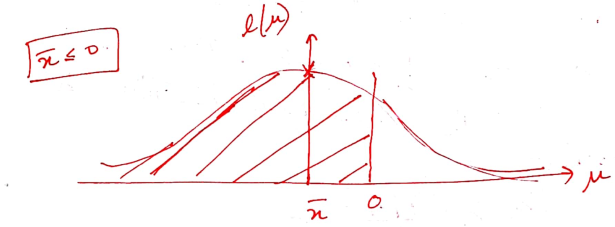
From (6), we have the behaviour of $\log L$ as follows:

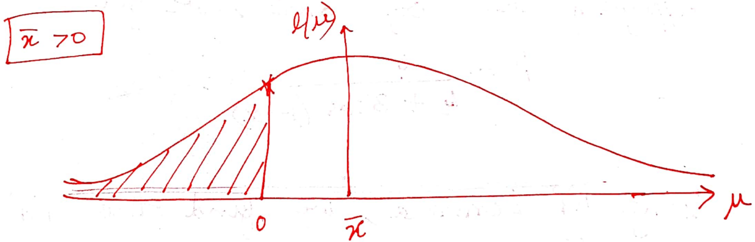
(i) If $\overline{x} \leq 0$, $\hat{\mu}_{\Omega_H} = \overline{x}$.

(ii) If $\overline{x} > 0$, $\hat{\mu}_{\Omega_H} = 0$.

Thus, $\hat{\mu}_{\Omega_H} = \min\{\overline{x}, 0\}$.

Example 1 cont'd





Example 1 cont'd

We also look at maximization wrt σ^2, i.e.,

$$\hat{\sigma}^2_{\Omega_H} = \frac{1}{n} \sum (x_i - \hat{\mu}_{\Omega_H})^2$$

Substituting these values, we get

$$\begin{aligned} \hat{L}(\Omega_H) &= \frac{1}{(2\pi\hat{\sigma}^2_{\Omega_H})^{n/2}} \exp\left[-\frac{1}{2\hat{\sigma}^2_{\Omega_H}} \sum (x_i - \hat{\mu}_{\Omega_H})^2 \right] \\ &= \frac{1}{(2\pi\hat{\sigma}^2_{\Omega_H})^{n/2}} \exp\left[-\frac{n}{2} \right] \end{aligned}$$

Step 4: LRT is: Reject H_0 if

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} < c$$

Example 1 cont'd

Now,

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} < c \iff \left(\frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}}\right)^{n/2} < c$$

$$\iff \frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}} < c_1$$

$$\iff \frac{\frac{1}{n}\sum(x_i - \overline{x})^2}{\frac{1}{n}\sum(x_i - \min\{0, \overline{x}\})^2} < c_1$$

(i) If $\overline{x} \leq 0$, LHS is 1. So, we always accept H and $\alpha = 0$.

(ii) If $\overline{x} > 0$, the test is: Reject H if

Example 1 cont'd

$$
\begin{aligned}
\frac{\sum(x_i-\overline{x})^2}{\sum x_i^2} < c_1 &\iff \frac{\sum(x_i-\overline{x})^2}{\sum(x_i-\overline{x})^2+n\overline{x}^2} < c_1 \\
&\iff \frac{\sum(x_i-\overline{x})^2+n\overline{x}^2}{\sum(x_i-\overline{x})^2} > c_2 \\
&\iff \frac{n\overline{x}^2}{\sum(x_i-\overline{x})^2} > c_3 \\
&\iff \frac{\sqrt{n}\overline{x}}{\sqrt{\sum(x_i-\overline{x})^2}} > c_4 \\
&\iff \frac{\sqrt{n}\overline{x}}{\sqrt{\frac{1}{n-1}\sum(x_i-\overline{x})^2}} > c_5,
\end{aligned}
$$

where c_5 is determined by $\sup\limits_{\mu\le 0} P_\mu\left(\frac{\sqrt{n}\overline{X}}{S} > c_5\right) = \alpha$.

Example 1 cont'd

Recall that $\frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t_{n-1}$ and that

$$P_{\mu}\Big(\frac{\sqrt{n}(\overline{X}-\mu)}{S} > \frac{\sqrt{n}(c_5-\mu)}{S}\Big)$$

is increasing in μ, so it will attain maximum at $\mu = 0$. Therefore, the size condition is

$$P_{\mu=0}\Big(\frac{\sqrt{n}\overline{X}}{S} > c_5\Big) = \alpha, \quad \text{where} \quad c_5 = t_{n-1,\alpha}.$$

Thus, LRT is

$$\text{Reject } H \text{ if } \frac{\sqrt{n}\overline{X}}{S} > t_{n-1,\alpha}$$

$$\text{Accept } H \text{ if } \overline{X} < 0$$

Example 2

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find LRT for testing

$$H : \mu = 0 \quad \text{vs.} \quad K : \mu \neq 0.$$

Solution: Step 1: Here,

$$\Omega_H = \{(\mu, \sigma^2) : \mu = 0, \sigma^2 > 0\}$$

Step 2: We know from previous case that

$$\hat{L}(\Omega) = \frac{1}{(2\pi\hat{\sigma}^2_\Omega)^{n/2}} \exp\left[-\frac{n}{2}\right]$$

where $\hat{\sigma}^2_\Omega = \frac{1}{n}\sum(x_i - \overline{x})^2$.

Example 2 cont'd

Step 3: On Ω_H, $\mu = 0 \Longrightarrow \hat{\mu}_{\Omega_H} = 0$.

This implies $\hat{\sigma}^2_{\Omega_H} = \frac{1}{n}\sum(x_i - \hat{\mu}_{\Omega_H})^2 = \frac{1}{n}\sum x_i^2$.

Substituting these values, we get

$$\hat{L}(\Omega_H) = \frac{1}{(2\pi\hat{\sigma}^2_{\Omega_H})^{n/2}} \exp\left[-\frac{n}{2}\right]$$

Step 4: So, LRT is to Reject H_0 if

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} < c \Longleftrightarrow \left(\frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}}\right)^{n/2} < c \Longleftrightarrow \frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}} < c_1$$

$$\Longleftrightarrow \frac{\sum(x_i - \overline{x})^2}{\sum x_i^2} < c_1$$

Example 2 cont'd

$$
\begin{aligned}
\Longleftrightarrow \frac{\sum x_i^2}{\sum (x_i - \overline{x})^2} > c_2 \Longleftrightarrow & \frac{\sum (x_i - \overline{x})^2 + n\overline{x}^2}{\sum (x_i - \overline{x})^2} > c_2 \\
\Longleftrightarrow & \frac{n\overline{x}^2}{\sum (x_i - \overline{x})^2/(n-1)} > c_3 \\
\Longleftrightarrow & \left|\frac{\sqrt{n}\overline{x}}{S}\right| > c_4 \text{ (taking square roots)}
\end{aligned}
$$

where c_4 is determined by

$$P_{\mu=0}\left(\left|\frac{\sqrt{n}\overline{x}}{S}\right| > c_4\right) = \alpha \Longrightarrow c_4 = t_{n-1,\alpha/2}$$

as $\frac{\sqrt{n}\overline{X}}{S} \sim t_{n-1}$ when $\mu = 0$.

So, LRT is Reject H if $\left|\frac{\sqrt{n}\overline{x}}{S}\right| \geq t_{n-1,\alpha/2}$.

Example 3

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find LRT for testing

$$H : \sigma^2 \leq \sigma_0^2 \quad \text{vs.} \quad K : \sigma^2 > \sigma_0^2.$$

Solution: Step 1: Here,

$$\Omega_H = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 \leq \sigma_0^2\}$$

Step 2: As before,

$$\hat{L}(\Omega) = \frac{1}{(2\pi\hat{\sigma}_\Omega^2)^{n/2}} \exp\left[-\frac{n}{2}\right]$$

where $\hat{\sigma}_\Omega^2 = \frac{1}{n}\sum(x_i - \overline{x})^2$.

Step 3: Over Ω_H,

(i) $\frac{1}{n}\sum(x_i-\overline{x})^2 \le \sigma_0^2$. Then, $\hat{\sigma}^2_{\Omega_H} = \frac{1}{n}\sum(x_i-\overline{x})^2$.

(ii) $\frac{1}{n}\sum(x_i-\overline{x})^2 > \sigma_0^2$. Then, $\hat{\sigma}^2_{\Omega_H} = \sigma_0^2$.

So, $\hat{\sigma}^2_{\Omega_H} = \min\{\sigma_0^2, \frac{1}{n}\sum(x_i-\overline{x})^2\} = \min\{\sigma_0^2, \hat{\sigma}^2_{\Omega}\}$.

Substituting these values, we get

$$\begin{aligned}\hat{L}(\Omega_H) &= \frac{1}{(2\pi\hat{\sigma}^2_{\Omega_H})^{n/2}} \exp\left[-\frac{1}{2\hat{\sigma}^2_{\Omega_H}}\sum(x_i-\overline{x})^2\right] \\ &= \frac{1}{(2\pi\hat{\sigma}^2_{\Omega_H})^{n/2}} \exp\left[-\frac{n\hat{\sigma}^2_{\Omega}}{2\hat{\sigma}^2_{\Omega_H}}\right].\end{aligned}$$

Example 3 cont'd

Step 4: Thus,

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} = \left(\frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}}\right)^{n/2} \exp\left[\frac{n}{2}\left\{1 - \frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}}\right\}\right]$$

When $\hat{\sigma}^2_{\Omega} \leq \sigma^2_0$, $\lambda(\boldsymbol{x}) = 1$. So, we always accept H $(\alpha = 0)$.
When $\hat{\sigma}^2_{\Omega} > \sigma^2_0$,

$$\lambda(\boldsymbol{x}) = y^{\frac{n}{2}} e^{\frac{n}{2}(1-y)} = g(y), \quad y > 1,$$

where $y = \frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_H}} = \frac{\hat{\sigma}^2_{\Omega}}{\sigma^2_0}$.

$$g^{'}(y) = \frac{n}{2} y^{n/2-1} e^{\frac{n}{2}(1-y)}(1-y) < 0$$

Thus, $g(y)$ is decreasing in y.

Example 3 cont'd

So,

$$g(y) < c \iff y > c_2 \iff \frac{1}{n\sigma_0^2}\sum(x_i - \overline{x})^2 > c_2$$
$$\iff \frac{1}{\sigma_0^2}\sum(x_i - \overline{x})^2 > c_3$$

where c_3 is determined by

$$\sup_{\sigma^2 \le \sigma_0^2} P_{\sigma^2}\left(\frac{1}{\sigma_0^2}\sum(x_i - \overline{x})^2 > c_3\right) = \alpha$$

Recall that $W = \frac{1}{\sigma_0^2}\sum(x_i - \overline{x})^2 \sim \chi^2_{n-1}$ under σ_0^2.

So,

$$P_{\sigma^2}\left(\frac{\sum(X_i - \overline{X})^2}{\sigma^2} > \frac{c_3\sigma_0^2}{\sigma^2}\right)$$

is increasing in σ^2.

Example 3 cont'd

Thus, it will attain a maximum at $\sigma^2 = \sigma_0^2$ (for $\sigma^2 \leq \sigma_0^2$).
So the size condition is

$$P_{\sigma_0^2}\Big(\frac{1}{\sigma_0^2}\sum(X_i - \overline{X})^2 > c_3\Big) = \alpha$$

but $\frac{1}{\sigma_0^2}\sum(X_i - \overline{X})^2 \sim \chi^2_{n-1}$ when $\sigma^2 = \sigma_0^2$.
Thus, $c_3 = \chi^2_{n-1,\alpha}$.
So LRT is:

$$\text{Reject } H \text{ if } \frac{1}{\sigma_0^2}\sum(X_i - \overline{X})^2 > \chi^2_{n-1,\alpha}$$

$$\text{Accept } H \text{ if } \frac{1}{n}\sum(X_i - \overline{X})^2 < \sigma_0^2.$$

Example 4

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find LRT for testing

$$H : \sigma^2 = \sigma_0^2 \quad \text{vs.} \quad K : \sigma^2 \neq \sigma_0^2.$$

Solution: Step 1: Here,

$$\Omega_H = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 = \sigma_0^2\}$$

Step 2: As before,

$$\hat{L}(\Omega) = \frac{1}{(2\pi\hat{\sigma}_\Omega^2)^{n/2}} \exp\left[-\frac{n}{2}\right]$$

where $\hat{\sigma}_\Omega^2 = \frac{1}{n}\sum(x_i - \overline{x})^2$.
Step 3: For Ω_H, $\hat{\mu}_{\Omega_H} = \overline{x}$ and $\hat{\sigma}^2_{\Omega_H} = \sigma_0^2$.

$$\hat{L}(\Omega_H) = \frac{1}{(2\pi\hat{\sigma}_0^2)^{n/2}} \exp\left[-\frac{1}{2\sigma_0^2} n\hat{\sigma}_\Omega^2\right]$$

Example 4 cont'd

Step 4: LRT is

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} = y^{n/2} e^{\frac{n}{2}(1-y)} = g(y)$$

where $y = \frac{\hat{\sigma}^2_\Omega}{\sigma^2_0}$.

Now,

$$g^{'}(y) = \frac{n}{2} y^{\frac{n}{2}-1} e^{n/2(1-y)}(1-y) \quad \begin{aligned} &> 0, \text{ if } y < 1 \\ &< 0, \text{ if } y > 1. \end{aligned}$$

So, $g(y)$ attains max at $y = 1$.

$$g^{''}(y) = \frac{n}{2} y^{n/2-2} e^{n/2(1-y)} \left[\frac{n}{2}(1-y)^2 - 1\right] \begin{aligned} &> 0, \text{ if } y < \sqrt{\frac{2}{n}} - 1 \\ &< 0, \text{ if } y > \sqrt{\frac{2}{n}} + 1. \end{aligned}$$

Example 4 cont'd

So, the LRT is to Reject H if

$$\begin{aligned}\lambda(\boldsymbol{x}) < c \iff g(y) < c \\ \iff y < c_1 \text{ or } y > c_2 \\ \iff \frac{\sum(X_i - \overline{X})^2}{\sigma_0^2} < c_1^* \quad \text{or} \quad \frac{\sum(X_i - \overline{X})^2}{\sigma_0^2} > c_2^*,\end{aligned}$$

where c_1^* and c_2^* are determined by

$$P_{\sigma_0^2}\left(c_1^* \leq \frac{\sum(X_i - \overline{X})^2}{\sigma_0^2} < c_2^*\right) = 1 - \alpha$$

where $\frac{\sum(X_i - \overline{X})^2}{\sigma_0^2} \sim \chi^2_{n-1}$.

So these are to be determined from tables of χ^2_{n-1} distribution.

As a convention, one can take $c_1^* = \chi^2_{n-1,1-\alpha/2}$ and $c_1^* = \chi^2_{n-1,\alpha/2}$.

Example 5

Let $X \sim Bin(n, p)$. Find LRT for testing

$$H : p \leq p_0 \quad \text{vs.} \quad K : p > p_0.$$

Solution: Step 1: Here,

$$\Omega = [0, 1] \quad \text{and} \quad \Omega_H = [0, p_0]$$

Step 2:

$$L(p, x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\log L(p, x) = \log \binom{n}{x} + x \log p + (n-x) \log(1-p)$$

$$\frac{d \log L}{dp} = \frac{x}{p} - \frac{n-x}{1-p}$$

Example 5 cont'd

$$\frac{d\log L}{dp} = \frac{x - np}{p(1-p)} = \begin{cases} < 0, & \text{if } p > x/n \\ > 0, & \text{if } p < x/n \end{cases}$$

Thus, $\hat{p}_\Omega = x/n$.

Substituting this, we get

$$\hat{L}(\Omega) = \binom{n}{x}\left(\frac{x}{n}\right)^x\left(1 - \frac{x}{n}\right)^{n-x}.$$

Step 3: On Ω_H, $p \le p_0$.

(i) If $\frac{x}{n} \le p_0$, $\hat{p}_{\Omega_H} = \frac{x}{n}$.

(ii) If $\frac{x}{n} > p_0$, $\hat{p}_{\Omega_H} = p_0$.

Thus, $\hat{p}_{\Omega_H} = \min\{p_0, \frac{x}{n}\}$.

Example 5 cont'd

$$\hat{L}(\Omega_H) = \begin{cases} \binom{n}{x}\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}, & \text{if } x/n < p_0 \\ \binom{n}{x} p_0^x (1-p_0)^{n-x}, & \text{if } x/n > p_0 \end{cases}$$

Step 4: For $x/n < p_0$,

$$\lambda(\boldsymbol{x}) = \frac{\hat{L}(\Omega_H)}{\hat{L}(\Omega)} = 1$$

Thus, we always accept H.

For $\frac{x}{n} > p_0$,

$$\lambda(x) = \frac{p_0^x(1-p_0)^{n-x}}{(x/n)^x(1-x/n)^{n-x}}$$

LRT is: Reject H if $\lambda(x) < c$.

Example 5 cont'd

Consider

$$\lambda^* = \log \lambda(x)$$
$$= x \log p_0 + (n-x)\log(1-p_0) - x\log\frac{x}{n} - (n-x)\log(1-\frac{x}{n}).$$

Now,

$$\frac{d\lambda^*(x)}{dx}$$
$$= \log p_0 - \log(1-p_0) - \log\frac{x}{n} - x\frac{n}{x}\frac{1}{n} + \log(1-\frac{x}{n}) + \frac{n-x}{n-x}\, n\, \frac{1}{n}$$
$$= \log\left[\frac{(n-x)p_0}{x(1-p_0)}\right] < 0$$

since $\frac{x}{n} > p_0$ and $n - x < n(1-p_0)$.
Thus, λ^* or λ is a decreasing function of x.

Example 5 cont'd

LRT is to Reject H if $x > c_1$ where c_1 is to be determined by the size condition

$$\sup_{p \leq p_0} P_p(X > c_1) = \alpha$$

where $P_p(X > c_1)$ is increasing in p and hence, supremum is attained at p_0.

Therefore,

$$P_{p_0}(X > c_1) = \alpha$$

where $X \sim Bin(n, p_0)$.

- Similarly, one can find LRT for testing
  $H : p = p_0 \quad \text{vs.} \quad K : p \neq p_0$

Example 6

Let $X_1, \ldots, X_n$ be a random sample from Exponential Distribution with location parameter, i.e., the pdf is: $e^{\theta - x}, x > \theta$. Find LRT for

$$H : \theta \leq \theta_0 \quad \text{vs.} \quad K : \theta > \theta_0.$$

Solution: Step 1: Here,

$$\Omega_H = \{\theta : \theta \leq \theta_0\}$$

Step 2:

$$L(\theta, \boldsymbol{x}) = e^{n\theta - \sum x_i} = \begin{cases} e^{n(\theta - \bar{x})}, & \text{if } x_{(1)} > \theta \\ 0, & \text{o/w} \end{cases}$$

which is increasing in θ.
Note that $\hat{\theta}_\Omega = X_{(1)}$.

Example 6 cont'd

Therefore,

$$\hat{L}(\Omega) = e^{n(x_{(1)} - \overline{x})}$$

Step 3: For Ω_H,

$$\hat{\theta}_{\Omega_H} = \begin{cases} x_{(1)}, & \text{if } x_{(1)} \leq \theta_0 \\ \theta_0, & \text{if } x_{(1)} > \theta_0 \end{cases}$$

Substituting $\hat{\theta}_{\Omega_H}$, we get

$$\hat{L}(\Omega_H) = \begin{cases} e^{n(x_{(1)} - \overline{x})}, & \text{if } x_{(1)} \leq \theta_0 \\ e^{n(\theta_0 - \overline{x})}, & \text{if } x_{(1)} > \theta_0 \end{cases}$$

Example 6 cont'd

Step 4: For $x_{(1)} \leq \theta_0$, we have $\lambda(\boldsymbol{x}) = 1$. Therefore, we accept H.

For $x_{(1)} > \theta_0$, we have

$$\lambda(\boldsymbol{x}) = e^{n(\theta_0 - \overline{x})} < c \iff x_{(1)} > c$$

where c is determined by

$$P_{\theta_0}(X_{(1)} > c) = \alpha \implies e^{n(\theta_0 - c)} = \alpha.$$

This implies,

$$c = \theta_0 - \frac{\log \alpha}{n}$$

Therefore, LRT is: Reject H if $X_{(1)} > \theta_0 - \frac{\log \alpha}{n}$.

Sufficiency and LRT

Theorem: For testing $H : \theta \in \Omega_H$ vs. $K : \theta \in \Omega_K$, LRT is a function of every sufficient statistic.

Proof:

By Factorization Theorem, we can write

$$L(\theta, \boldsymbol{x}) = g(T(\boldsymbol{x}), \theta) h(\boldsymbol{x})$$

where T is a sufficient statistic.

$$\hat{L}(\Omega) = h(\boldsymbol{x}) \sup_{\theta \in \Omega} g(T(\boldsymbol{x}), \theta) \quad \text{and} \quad \hat{L}(\Omega_H) = h(\boldsymbol{x}) \sup_{\theta \in \Omega_H} g(T(\boldsymbol{x}), \theta).$$

So,

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\theta \in \Omega_H} g(T(\boldsymbol{x}), \theta)}{\sup_{\theta \in \Omega} g(T(\boldsymbol{x}), \theta)}$$

which depends on T.

Thanks for your patience!