Assignment 2 : MA3140

Q 1. Let $X_1, \ldots, X_n$ be a r.s. from a distribution having pdf (pmf) $f(x|\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the MLE, say $\delta_M(\boldsymbol{X})$, and the UE based on the MLE, say $\delta_U(\boldsymbol{X})$. Also compare the MSEs of $\delta_M(\boldsymbol{X})$ and $\delta_U(\boldsymbol{X})$.

(a) $f(x|\boldsymbol{\theta}) = e^{-(x-\theta)}$, if $x > \theta$, and $= 0$, otherwise; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta$.

(b) $X_1 \sim Exp(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta$.

(c) $X_1 \sim U(0, \theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

(d) $X_1 \sim N(\theta, 1)$; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta^2$.

Q 2. Let $X_1, \ldots, X_n$ be a r.s. from a distribution having pdf (pmf) $f(x|\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. Find CE in each of the following situations.

(a) $X_1 \sim U(-\theta, 2\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta$.

(b) $X_1 \sim N(\mu, \sigma^2)$; $\boldsymbol{\theta} = (\mu, \sigma^2)$, $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\boldsymbol{\theta}) = \frac{\mu}{\sigma}$.

(c) Same as (b) with $g(\boldsymbol{\theta}) = \mu + b\sigma$, where b is any given real number.

(d) $X_1 \sim Poisson(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = e^{\theta}$.

Q 3. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$ distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Of the two estimators, the M.M.E. and the M.L.E, of θ, which one would you prefer with respect to the criterion of the bias?

Q 4. Let X_1, X_2 be a r.s. from an Exponential population with mean $1/\lambda$. Define $T_1 = \frac{X_1+X_2}{2}$ and $T_2 = \sqrt{X_1 X_2}$. Show that T_1 is unbiased and T_2 is biased. Further, prove that $MSE(T_2) \leq Var(T_1)$.

Q 5. Let T_1 and T_2 be UEs of θ with respective variances σ_1^2 and σ_2^2, and $cov(T_1, T_2) = \sigma_{12}$ (assumed to be known). Consider $T = \alpha T_1 + (1-\alpha) T_2$, $0 \leq \alpha \leq 1$. Show that T is unbiased and find the value of α for which $Var(T)$ is minimized.

Q 6. Let X_1, X_2 be a r.s. from a population with density function $f(x) = \frac{x}{\theta} \exp\{-\frac{x^2}{2\theta}\}$, $x > 0$, $\theta > 0$. Find the CRLB for the variance of an UE of θ. Hence, derive a UMVUE for θ.

Q 7. Let X_1, X_2 be a r.s. from a population with density function $f(x) = \theta(1+x)^{-(1+\theta)}$, $x > 0$, $\theta > 0$. Find the CRLB for the variance of an UE of $1/\theta$. Hence, derive a UMVUE for $1/\theta$.

Q 8. Let X_1, X_2 be a r.s. from a discrete population with pmf

$$P(X=-1) = \frac{1-\theta}{2}, \; P(X=0) = \frac{1}{2}, \; P(X=1) = \frac{\theta}{2}, \; 0 < \theta < 1.$$

Find the CRLB for the variance of an UE of θ. Show that the variance of an UE $\overline{X} + \frac{1}{2}$ is more than or equal to this bound.

Q 9. Show that each of the following families is an exponential family.

(a) Gamma Family with either parameter α or β known or both unknown.

(b) Beta Family with either parameter α or β known or both unknown.

(c) Negative Binomial family with r known, $0 < p < 1$.

Q 10. Check whether the following family of distributions of X is a one-parameter exponential family.

(a) $X \sim Cauchy(1, \theta)$.

(b) $X \sim Uniform[0, \theta]$, $\theta \in (0, \infty)$.