Final Exam - Dec 2020
MA 3140: Statistical Inference

Total time: 3 hours
Total marks: 50

All questions are compulsory.

Q 1. [**3+5 marks**] Let $X_1, X_2, X_3 \overset{i.i.d.}{\sim} Bernoulli(p)$.

(i) Is $T(\boldsymbol{X}) = (X_1 + X_2, X_3)$ a sufficient statistic for p?

(ii) Is it also a minimal sufficient statistic for p?

Q 2. [**4+2+2 marks**] Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, 1)$. Define $W(\boldsymbol{X}) = \frac{1}{2}(X_1 + X_2)$ and $T(\boldsymbol{X}) = X_1$.

(i) Is $\phi(T) = E(W|T)$ unbiased for θ?

(ii) Given that W is an unbiased estimator for θ, compare the variances of $\phi(T)$ and W.

(ii) Is $\phi(T)$ an estimator? Give justification.

Q 3. [**4+4 marks**] Let X is a uniform random sample from $\{1, \ldots, \theta\}$.

(i) When $\theta \in \Omega = \mathbb{N}$, check whether $T(X) = X$ is a complete statistic.

(ii) When $\theta \in \Omega = \mathbb{N} - \{n\}$, check whether $T(X) = X$ a complete statistic.

Q 4. [**5 marks**] Consider the following family of distributions:

$$\mathcal{P} = \{P_\lambda(X = x) : P_\lambda(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}; x = 0, 1, 2, \ldots, \infty; \lambda = 1 \text{ or } 2\}.$$

This is a Poisson family with λ restricted to be 1 or 2. Show that the family $\mathcal{P}$ is not complete.

Q 5. [**5 marks**] Let $X_1, \dots, X_n$ be a random sample from $U(0, \theta)$. Use Basu's Theorem to find

$$E\Big[\frac{X_{(1)}}{X_{(n)}}\Big].$$

Q 6. [**4+3+3 marks**] Let $X_1, \dots, X_n \overset{i.i.d.}{\sim} \textit{Bernoulli}(\theta)$, where $n = 5$. For testing $H_0 : \theta \leq 0.5$ vs $H_1 : \theta > 0.5$, two tests T_1 and T_2 are proposed, where

- T_1 rejects H_0 if, and only if, all success are observed.
- T_2 rejects H_0 if, and only if, 3 or more success are observed.

For T_1 and T_2,

(i) find the power functions.

(ii) compare the maximum probability of making Type I error.

(iii) compare the probability of making Type II error if $\theta = 2/3$.

Q 7. [**6 marks**] A random sample $X_1, \dots, X_n$ is drawn from a Pareto population with pdf

$$f(x; \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I_{[\nu, \infty)}(x), \;\; \theta > 0, \nu > 0.$$

Show that the LRT of

$$H_0 : \theta = 1 \quad \text{vs.} \quad H_a : \theta \neq 1$$

has rejection region of the form $\{\boldsymbol{x} : T(\boldsymbol{x}) \leq c_1 \text{ or } T(\boldsymbol{x}) \geq c_2\}$ where $0 < c_1 < c_2$ and

$$T = \log\left[\frac{\prod_{i=1}^{n} X_i}{(X_{(1)})^n}\right].$$