Mid Sem Solutions

1. $X_1, X_2, ..., X_n \sim U(0,\theta),\ \theta \in \Theta = (0,\infty)\,.\ \ \mathcal{D} = \left\{\delta_\alpha : \delta_c(\underline{X}) = cX_{(n)}, c\ >\ 0\right\}.$

   Fix $\theta \in \Theta$, then for $c \in (0,\infty)$,

$$\begin{aligned}
M_{\delta_c}(\theta) &= E_\theta(cX_{(n)} - \theta)^2 \\
&= c^2 E_\theta(X_{(n)}^2) - 2c\theta E_\theta(X_{(n)}) + \theta^2 \\
\frac{\partial M_{\delta_c}(\theta)}{\partial c} &= 2cE_\theta(X_{(n)}^2) - 2E_\theta(X_{(n)}) \\
\frac{\partial^2 M_{\delta_c}(\theta)}{\partial^2 c} &= 2E_\theta(X_{(n)}^2) > 0
\end{aligned}$$

   Thus, for fixed $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at

$$c = \frac{\theta E_\theta(X_{(n)})}{E_\theta(X_{(n)}^2)}$$

$$\text{We have, } E_\theta(X_{(n)}) = \frac{n\theta}{n+1}, \theta \in \Theta \text{ and } E_\theta(X_{(n)}^2) = \frac{n\theta^2}{n+2}, \theta \in \Theta$$

   Thus, for every $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at $c = \dfrac{n+2}{n+1}$.

   Therefore, among the estimators in class $\mathcal{D}$, $\delta_{\frac{n+2}{n+1}}(\underline{X}) = \dfrac{n+2}{n+1}X_{(n)}$ has the smallest MES for each $\theta \in \Theta$.

2. (i). $X_1, X_2, ..., X_n \sim f$ with mean μ and variance σ^2,

$$\begin{aligned}
E\left(\sum_{i=1}^n a_i X_i\right) &= \sum_{i=1}^n a_i E(X_i) \\
&= \mu \sum_{i=1}^n a_i \\
&= \mu,\ \ if,\ \ \sum_{i=1}^n a_i = 1.
\end{aligned}$$

   (ii).

$$\begin{aligned}
V\left(\sum_{i=1}^n a_i X_i\right) &= \sum_{i=1}^n a_i^2 V(X_i) \\
&= \sigma^2 \sum_{i=1}^n a_i^2
\end{aligned}$$

We need to minimize $\sum_{i=1}^{n} a_i^2$ subject to the constraint $\sum_{i=1}^{n} a_i = 1$. Add and subtract the mean of the a_i, i.e. 1/n, to get

$$\begin{aligned}\sum_{i=1}^{n} a_i^2 &= \sum_{i=1}^{n}\left[\left(a_i - \frac{1}{n}\right) + \frac{1}{n}\right]^2 \\ &= \sum_{i=1}^{n}\left(a_i - \frac{1}{n}\right)^2 + \frac{1}{n} \text{ (The cross term is zero)}\end{aligned}$$

Hence, $\sum_{i=1}^{n} a_i^2$ is minimized by choosing $a_i = \frac{1}{n}$, for all i. Thus, $\sum_{i=1}^{n} \frac{X_i}{n} = \overline{X}$ has the minimum variance among all linear unbiased estimators.

3. (i). $X_1, X_2, ..., X_n \ (n \geq 2) \sim N(0, \sigma^2)$

$$f(x;\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}, \ \ x \in \mathbb{R}, \ \ \sigma > 0.$$

$$\begin{aligned}log f(x;\sigma) &= -log\sigma - \frac{1}{2}log 2\pi - \frac{-x^2}{2\sigma^2} \\ \frac{\partial log f(x;\sigma)}{\partial\sigma} &= \frac{-1}{\sigma} + \frac{x^2}{\sigma^3} \\ &= \frac{1}{\sigma}\left(\frac{x^2}{\sigma^2} - 1\right) \\ \implies E\left(\frac{\partial log f}{\partial\sigma}\right)^2 &= \frac{1}{\sigma^2} E\left(\frac{X^2}{\sigma^2} - 1\right)^2 \\ &= \frac{2}{\sigma^2}\end{aligned}$$

Therefore, the CRLB for the variance of σ is $\frac{\sigma^2}{2n}$.

(ii). $T_1 = \alpha \sum_{i=1}^{n} |X_i|$

$$\begin{aligned}E|X_i| &= \int_{-\infty}^{\infty} |x| \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} dx \\ &= 2\int_{0}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} dx \\ &= \frac{2\sigma}{\sqrt{2\pi}}\end{aligned}$$

$$E(T_1) = \frac{2n\sigma}{\sqrt{2\pi}}\alpha = \sigma \implies \alpha = \frac{1}{n}\sqrt{\frac{\pi}{2}}$$

Thus, $T_1 = \frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n}|X_i|$ is an unbiased estimator of σ. Now,

$$\begin{aligned}
Var(T_1) &= \frac{\pi}{2n}Var(|X_i|) \\
&= \frac{\pi}{2n}\left(E(X_i)^2 - (E|X_i|)^2\right) \\
&= \frac{\pi}{2n}\left(\sigma^2 - \frac{2\sigma^2}{\pi}\right) \\
&= \frac{(\pi-2)}{2n}\sigma^2 > \frac{\sigma^2}{2n}
\end{aligned}$$

So T_1 does not attain CRLB although T_1 is unbiased estimator of σ.

Consider, $T_2 = \beta\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}$

$$\begin{aligned}
\text{We know that, } U &= \frac{\sum_{i=1}^{n} X_i^2}{\sigma^2} \sim \chi_n^2 \\
E\left(U^{1/2}\right) &= \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\
E(T_2) &= \beta\frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\sigma = \sigma, \text{ implies} \\
\beta &= \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}
\end{aligned}$$

Thus, $T_2 = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}$ is also an unbiased estimator of σ. Now,

$$
\begin{aligned}
Var(T_2) &= \frac{1}{2}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^2 Var\left(\sum_{i=1}^{n} X_i^2\right)^{1/2} \\
&= \frac{1}{2}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^2 \left(E\left(\sum_{i=1}^{n} X_i^2\right) - \left\{E\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}\right\}^2\right) \\
&= \frac{1}{2}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^2 \left[n - 2\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right]\sigma^2 \\
&= \left[\frac{n}{2}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^2 - 1\right]\sigma^2 > \frac{\sigma^2}{2n}
\end{aligned}
$$

(iii). It can be shown that $Var(T_2) < Var(T_1)$

$$\therefore T_2 \text{ is a better estimator.}$$