MA 3140: Statistical Inference

Dr. Sameen Naqvi
Department of Mathematics, IIT Hyderabad
Email id: sameen@math.iith.ac.in

Method 3: Conditioning

Example 1: Revisiting Poisson Distribution

Let $X_1, \ldots, X_n \sim P(\theta)$. Suppose we are interested in finding the UMVUE for $g(\theta) = e^{-\theta}$.

Solution: We know that $T = \sum X_i \sim P(n\theta)$ is a CSS.

Let

$$\delta(X_1) = \begin{cases} 1, & \text{if } X_1 = 0 \\ 0, & \text{if } X_1 \neq 0 \end{cases}$$

Since $E[\delta(X_1)] = P(X_1 = 0) = e^{-\theta}$, we find that $\delta(X_1)$ is unbiased for $g(\theta)$. Consider

Method 3: Conditioning cont'd

$$
\begin{aligned}
\eta(t) = E[\delta(X_1)|T = t] &= 1.P(X_1 = 0|T = t) + 0.P(X_1 \neq 0|T = t) \\
&= P(X_1 = 0|\sum_{i=1}^{n} X_i = t) \\
&= \frac{P(X_1 = 0, \sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{P(X_1 = 0)P(\sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{e^{-\theta} \; \frac{e^{-(n-1)\theta}[(n-1)\theta]^t}{t!}}{\frac{e^{-n\theta}(n\theta)^t}{t!}} = \left(1 - \frac{1}{n}\right)^t
\end{aligned}
$$

So, $\left(1 - \frac{1}{n}\right)^T$ is UMVUE for $e^{-\theta}$.

Method 3: Conditioning cont'd

Example 2: Hypergeometric Distribution

Let X have Hypergeometric distribution with pmf

$$P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, \ldots, n, x \leq M, n-x \leq N-M,$$

where N is known and M is unknown. Suppose you are interested in estimating M.

Solution: Here, we know that X is sufficient.

To check completeness of X, let $E[g(X)] = 0$.

$$\Longrightarrow \sum_{x=0}^{n} g(x) \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = 0, \quad \forall\ M$$

Method 3: Conditioning cont'd

Now,

when $M = 0 \Longrightarrow g(0) = 0$.

when $M = 1 \Longrightarrow g(0)\binom{N-1}{n} + g(1)\binom{N-1}{n-1} = 0 \Longrightarrow g(1) = 0$.

By induction, it can be shown that X is complete.

$$EX = \frac{n}{N}M \Longrightarrow E\left(\frac{N}{n}X\right) = M$$

Thus, we conclude that $\frac{N}{n}X$ is UMVUE of M.

Method 3: Conditioning cont'd

Example 3: Exponential UMVUE

Let $X_1, \ldots, X_n \sim \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$, $\lambda > 0$.

Suppose we are interested to find the UMVUE of the reliability function $e^{-m/\lambda}$.

Solution: Define

$$\delta(X_1) = \begin{cases} 1, & \text{if } X_1 > m \\ 0, & \text{otherwise} \end{cases}$$

Thus, $E[\delta(X_1)] = e^{-m/\lambda}$.

We also know that $T = \sum X_i$ or $\overline{X}$ is CSS.

Method 3: Conditioning cont'd

Now,

$$
\begin{aligned}
\eta(t) = E[\delta(X_1)|T] &= 1.P[X_1 > m|\overline{X}] + 0 \\
&= P\left[\frac{X_1}{\overline{X}} > \frac{m}{\overline{x}}|\overline{X}\right] \\
&= P\left[\frac{X_1}{\overline{X}} > \frac{m}{\overline{x}}\right] \qquad \left(\because \frac{X_1}{\overline{X}} \text{ is ancillary}\right) \\
&= P\left[\frac{X_1}{\frac{1}{n}\sum X_i} > \frac{m}{\frac{1}{n}\sum x_i}\right] \\
&= P\left[\frac{X_1}{X_1 + \sum_{i=2}^{n} X_i} > \frac{m}{x_1 + \sum_{i=2}^{n} x_i}\right] \\
&= P\left[Z > \frac{m}{n\overline{x}}\right],
\end{aligned}
$$

where $Z = \frac{X_1}{X_1 + \sum_{i=2}^{n} X_i} \sim Be(1, n-1)$.

Method 3: Conditioning cont'd

$$P\left[Z > \frac{m}{n\overline{x}}\right] = \frac{1}{Be(1, n-1)} \int_{\frac{m}{n\overline{x}}}^{1} (1-z)^{n-2}\ z^{1-1} dz$$

$$= \frac{\Gamma n}{\Gamma 1\ \Gamma(n-1)} \int_{\frac{m}{n\overline{x}}}^{1} (1-z)^{n-2} dz$$

$$= \left(1 - \frac{m}{n\overline{x}}\right)^{n-1}$$

So, $\left(1 - \frac{m}{nT}\right)^{n-1}$ is UMVUE for $e^{-m/\lambda}$.

Method 3: Conditioning cont'd

- **Beta Distribution:** An absolutely continuous random variable X is said to follow Beta distribution with shape parameters $a > 0$ and $b > 0$ (written as $X \sim Be(a, b)$) if its pdf is given by

$$f_X(x) = \begin{cases} \frac{x^{a-1}\,(1-x)^{b-1}}{Be(a,b)}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Consider two independent random variables X and Y.
  - If $X \sim Gamma(a, b_1)$ and $Y \sim Gamma(a, b_2)$ then $\frac{X}{X+Y} \sim Be(b_1, b_2)$.
  - If $X \sim Gamma(a_1, b)$ and $Y \sim Gamma(a_2, b)$ then $\frac{X}{X+Y} \sim Be(a_1, a_2)$.

Thanks for your patience!