1 (i) $f(x/\theta) \begin{cases} \theta(1-\theta)^{x-1} & x=1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$

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MME:

$$M_1 = E[X] = \mu = \frac{1}{\theta} \quad (\mu_{geometric} = 1/p)$$

$$m_1 = \frac{\sum X_i}{n} = \bar{X}$$

$$\Rightarrow m_1 = M_1$$

$$\bar{x} = 1/\theta$$

$\Rightarrow$ MME of θ is $1/\bar{x}$

MLE:

$$L(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1}$$

$$L(\theta) = \theta^n (1-\theta)^{\sum x_i - n}$$

$$\log(L(\theta)) = n\log\theta + \left(\sum x_i - n\right)\log(1-\theta)$$

Differentiating and equating to zero.

$$\frac{\partial(\log(L(\theta)))}{\partial\theta} = \frac{n}{\theta} - \frac{(\sum x_i - n)}{1-\theta} = 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{\sum x_i - n}{1-\theta}$$

$$\theta = n/\sum x_i \Rightarrow \theta = 1/\frac{\sum x_i}{n} \Rightarrow \boxed{\theta = \frac{1}{\bar{X}}}$$

$\therefore$ Both MLE and MME are same

$x_1 = 2,\ x_2 = 7,\ x_3 = 6,\ x_4 = 5,\ x_5 = 9$

$P(X_1 \geq 4)$

$$\bar{x} = \frac{\sum x_i}{5} = \frac{2+7+6+5+9}{5} = \frac{29}{5} = 5.8 \approx \underline{\underline{0.172}}$$

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - \left(\theta(1-\theta)^0 + \theta(1-\theta) + \theta(1-\theta)^2\right)$$

$$= 1 - \left(\frac{1-(1-\theta)^3}{\theta}\right)$$

$$\Rightarrow (1-\theta)^3 = (1-0.172)^3 = \underline{\underline{0.567}}$$

(ii) $U(-\theta, \theta)$

$$f(x/\theta) \begin{cases} 1/2\theta & -\theta \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_1 = E[X_1] = \frac{1}{2\theta} \int_{-\theta}^{\theta} x\,dx = \underline{\underline{0}} \rightarrow$$ 1st Population moment doesn't exist.

$$\hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

Finding second moments,

$$\cancel{M_2 = \int_{-\theta}^{\theta} x^2 f(x)\,dx = \int_{-\theta}^{\theta} x^2}$$

$$M_2 = \frac{1}{2\theta}\int_{-\theta}^{\theta} x^2\,dx \Rightarrow \frac{1}{2\theta}\cdot \frac{x^3}{3}\Big|_{-\theta}^{\theta} = \underline{\underline{\frac{\theta^2}{3}}}$$

$$m_2 = \frac{\sum x_i^2}{n}$$

$$M_2 = m_2 \Rightarrow \frac{\theta^2}{3} = \frac{\sum x_i^2}{n} \rightarrow \boxed{\hat{\theta} = \sqrt{\frac{3\sum x_i^2}{n}}}$$

<u>MLE:</u> $L(\theta) = \frac{1}{(2\theta)^n} \Rightarrow -\theta \le x \le \theta$

$$-\theta \le x_1 \cdots < x_n \le \theta$$

$L(\theta)$ is maximum when θ is minimum, this happens at $x_1 \Rightarrow$ MLE of θ is $\underline{\underline{x_1}}$

$$\frac{d}{d\theta}\ln(L(\theta)) \Rightarrow \frac{d}{d\theta}(-n\ln 2\theta)$$

if we order the statistic

$\Rightarrow -\frac{n}{\theta} \Rightarrow <0$ for $\theta > 0 \Rightarrow$ hence $\underline{x_1}$ is the estimate at which max likelihood is achieved.

(iii) $f(x|\theta) = \begin{cases} \frac{1}{\sigma}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right) & x > \mu \\ 0 & \text{otherwise} \end{cases}$

$\theta = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$

$g(\theta) = (\mu, \sigma)$

$$M_1 = E[x]$$

$$= \int_{\mu}^{\infty} \frac{x}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx$$

$$= \frac{1}{\sigma} \int_{\mu}^{\infty} x e^{-\left(\frac{x-\mu}{\sigma}\right)} dx$$

$$= \frac{x}{\sigma} \int_{\mu}^{\infty} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx - \int_{\mu}^{\infty} 1 . \int e^{-\frac{x-\mu}{\sigma}}$$

$$\Rightarrow \frac{1}{\sigma}\left(\mu\sigma + \sigma^2\right) = \underline{\underline{\mu+\sigma}} \Rightarrow M_1 = \mu + \sigma$$

$$m_1 = \frac{\sum x_i}{n} = \bar{x}$$

$$\Rightarrow \boxed{\bar{x} = \mu + \sigma}$$

$$M_2 = E[x^2]$$

$$\int_{\mu}^{\infty} \frac{x^2}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx$$

$$\Rightarrow \frac{1}{\sigma}\left\{ \frac{x^2 e^{-\left(\frac{x-\mu}{\sigma}\right)}}{-1/\sigma} \right\}_{\mu}^{\infty} + \sigma \int_{\mu}^{\infty} 2x . e^{-\left(\frac{x-\mu}{\sigma}\right)}$$

$$\Rightarrow \frac{1}{\sigma}\left\{ \mu^2\sigma + 2\sigma \underbrace{\int_{\mu}^{\infty} \frac{x . e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\sigma}}_{E[x] = \mu+\sigma} \times \sigma \right\}$$

$$\Rightarrow \frac{1}{\sigma}\left[\mu^2\sigma + 2\sigma^2(\mu+\sigma) \right]$$

$$\Rightarrow M_2 = (\mu+\sigma)^2 + \sigma^2$$

$\Rightarrow M_2 = (\mu+\sigma)^2 + \sigma^2$

$m_2 = \frac{\sum x_i^2}{n}$, substituting $\mu+\sigma = \bar{x}$

$\bar{x}^2 + \sigma^2 = \frac{\sum x_i^2}{n} \Rightarrow \sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2$

$\rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$\sigma^2 = \frac{n-1}{n} s^2 \Rightarrow \boxed{\begin{aligned} \sigma &= s\sqrt{1-1/n} \\ \mu &= \bar{x} - s\sqrt{1-\frac{1}{n}} \end{aligned}}$

MLE:

$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\left(\frac{x_i-\mu}{\sigma}\right)}$

$= \left(\frac{1}{\sigma}\right)^n e^{\left(-\frac{\sum x_i}{\sigma} + \frac{n\mu}{\sigma}\right)}$

$= \left(\frac{1}{\sigma}\right)^n e^{-1/\sigma\left[\sum x_i - \mu\right]}$

$\log(L(\theta)) = n\log(1/\sigma) - \frac{1}{\sigma}\left(\sum x_i - \mu\right)$

$\log(L(\theta,\mu)) = -n\log\sigma - \frac{1}{\sigma}\left(\sum x_i - \mu\right)$

$L(\mu,\sigma)$ is an increasing function of μ

$\therefore \underline{\hat{\mu} = x_1}$

$$\frac{d}{d\sigma}\left(\log(L(\mu,\sigma))\right) = \frac{d}{d\sigma}\left[-n\log\sigma - \frac{1}{\sigma}\left[\sum_{i=1}^{n}(x_i-\mu)\right]\right]$$

$$= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)\right] = 0$$

$$\Rightarrow \frac{n}{\sigma} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)\right]$$

$$\Rightarrow \boxed{\hat{\sigma} = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)\right]}$$

$\therefore$ MLE and MME are different.

(v) $N(\mu,\sigma^2)$ $\theta=(\mu,\sigma^2)$: $g(\theta) = \mu^2/\sigma^2$

(i) MME:

$M_1 = E[X] = \mu$

$m_1 = \frac{\sum x_i}{n} = \bar{x}$

$\Rightarrow \boxed{\bar{x} = \mu}$

$M_2 = E[X^2] = Var(X) + (E[X])^2$

$= \mu^2 + \sigma^2$

$m_2 = \frac{\sum x_i^2}{n}$

$\Rightarrow \sigma^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$

$\sigma^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$

$\boxed{\sigma^2 = \frac{n-1}{n}s^2}$

$$g(\theta) = \frac{\mu^2}{\sigma^2} = \frac{(\bar{x})^2\, n}{(n-1)\, s^2} \Rightarrow \boxed{\frac{\bar{x}^2 \cdot n}{(n-1)s^2}}$$

(ii) MLE:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}$$

$$\log(L(\theta)) = -n\log(\sqrt{2\pi}) - n\log\sigma - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$= -\frac{n}{2}(\log 2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$\frac{\partial \log(L(\theta,\mu,\sigma))}{\partial\sigma} = \frac{-n}{2\sigma^2} + \frac{1}{(\sigma^2)^2}\left[\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$= \frac{1}{2\sigma^2}\left[\frac{1}{\sigma^2}\sum(x_i-\mu)^2 - n\right] = 0$$

$$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$$

$$\frac{\partial}{\partial\mu}\log(L(\theta,\mu,\sigma)) = \frac{2}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum x_i = n\mu$$

$$\underline{\underline{\mu = \bar{x}}}$$

$$g(\theta) = \frac{\mu^2}{\sigma^2} = \frac{\bar{x}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$\therefore$ MLE and MME are different

(2)

x	θ: 1	2	3	Max. Probability
1	1/2	1/4	1/4	1/2
2	3/5	1/5	1/5	3/5
3	1/3	1/2	1/6	1/2
4	1/6	1/6	2/3	2/3

$\therefore$ MLE of θ is $\begin{cases} 1 & x=1 \\ 1 & x=2 \\ 2 & x=3 \\ 3 & x=4 \end{cases}$

(3) Let X be the random Variable denoting no. of items that have failed in <100 hrs.

$X \sim Bin(10, \mu)$

$$\mu = \frac{1}{\theta}\int_0^{100} e^{-x/\theta}\,dx = 1 - e^{-100/\theta}$$

$$\Rightarrow \theta = \frac{-100}{\ln(1-\mu)}, \quad \text{Given } x=3,\ \hat{\mu} = \frac{3}{10}$$

$$\text{MLE of } \theta = \frac{-100}{\ln(0.7)}$$

④ $x_1, x_2, x_3, \ldots x_n \to Bin(1,\theta)$

$$L(\theta) = \prod_{i=1}^{n} \binom{1}{x_i} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \prod_{i=1}^{n} \binom{1}{x_i} \theta^{\Sigma x_i} (1-\theta)^{n-\Sigma x_i}$$

Let $\tilde{C}_x = \prod_{i=1}^{n} \binom{1}{x_i}$

$$\cancel{\frac{d}{d\theta}(L(\theta))} \ \ln(L(\theta)) = \ln \tilde{C}_x + \left(\Sigma x_i\right) \ln\theta + (n - \Sigma x_i) \ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\left(\ln(L(\theta))\right) = \frac{\Sigma x_i}{\theta} - \frac{(n - \Sigma x_i)}{1-\theta}$$

$$\Rightarrow \theta = \frac{\Sigma x_i}{n} = \underline{\underline{\bar{x}}}$$

$L(\theta)$ is $\uparrow$ if $\theta < \bar{x}$ else $\downarrow$ if $\theta \geq \bar{x}$

Case #I: $0 \leq \bar{x} < \frac{1}{4} \Rightarrow$ Maximised at $\theta = \frac{1}{4}$

Case #2: $\frac{1}{4} \leq \bar{x} \leq \frac{3}{4} \Rightarrow$ Maximised at $\theta = \bar{x}$

Case #3: $\bar{x} \geq 3/4 \Rightarrow$ " " $\theta \geq 3/4$

(5) $X_1, X_2, X_3, \dots X_n \rightarrow N(\mu, \sigma^2)$

(i) MLE of σ^2 for normal distribution,

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

$$\ln(L(\theta)) = \ln\left[\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{\sum -\left(\frac{(x_i-\mu)^2}{2\sigma^2}\right)}\right]$$

$$= ~~n\log~~ \; n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \sum - \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$\Rightarrow -n\log(\sqrt{2\pi}) - n\log\sigma - \frac{1}{2\sigma^2}\left(\sum(x_i-\mu)^2\right)$$

$$\Rightarrow \frac{d}{d\sigma^2}\left(\log(L(\theta))\right) = 0 - \frac{n}{2\sigma^2} - \left(\frac{1}{2}\sum(x_i-\mu)^2\right)\left(\frac{-1}{(\sigma^2)^2}\right)$$

$$\Rightarrow \frac{-n}{2\sigma^2} + \frac{1}{(\sigma^2)^2}\left[\frac{1}{2}\sum(x_i-\mu)^2\right] = 0$$

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

$\therefore$ MLE of σ^2 is $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \longrightarrow \mu = \mu_0$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$$

$$\frac{d}{d\mu}\left(\log L(\theta)\right) = \frac{2}{2\sigma^2} \sum (x_i - \mu) = 0$$

$$\Rightarrow \sum x_i = \underline{\underline{n\mu}}$$

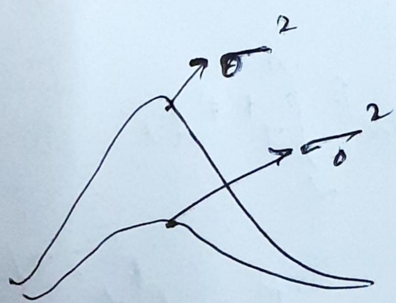
$$\hat{\mu} = \frac{\sum x_i}{n}$$

$$\Rightarrow \boxed{\hat{\mu} = \bar{x}}$$

ii) $\mu_0 = 0$ MLE of $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - 0)^2$

$$= \underline{\underline{\frac{1}{n} \sum x_i^2}}$$

(iii) $\sigma^2 \geq \sigma_0^2$

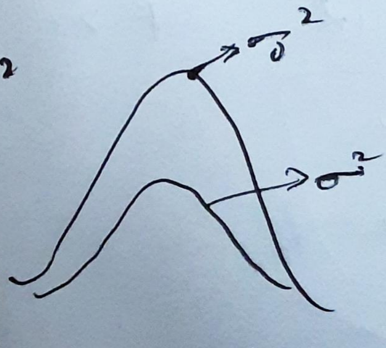


$\sigma^2 \rightarrow$ spread of the curve

Maximum occurs at σ^2

MLE is σ^2

(iv) $\sigma^2 \leq \sigma_0^2$



$\rightarrow$ Max. occurs at σ_0^2, MLE is σ_0^2

(iv) $f(x/\theta) \begin{cases} \dfrac{\beta\alpha^{\beta}}{x^{\beta+1}} & \text{if } x > \alpha \\ 0 & \text{otherwise} \end{cases}$

$$M_1 \Rightarrow E[x] = \int_{\alpha}^{\infty} x f(x)\,dx = \beta\alpha^{\beta}\int_{\alpha}^{\infty} \frac{dx}{x^{\beta}}$$

$$\Rightarrow \beta\alpha^{\beta}\int_{\alpha}^{\infty} x^{-\beta}\,dx = \beta\alpha^{\beta} \left.\frac{x^{-\beta+1}}{1-\beta}\right|_{\alpha}^{\infty}$$

$$= \frac{\beta}{1-\beta}\cdot\alpha^{\beta}\left[-\alpha^{-\beta+1}\right]$$

$$= \frac{\beta}{\beta-1}\cdot\alpha = \frac{\alpha\beta}{\beta-1}$$

$$m_1 = \frac{\sum x_i}{n} = \bar{x} \Rightarrow \bar{x} = \frac{\alpha\beta}{\beta-1} \quad \text{—} (1)$$

$$M_2 \Rightarrow E[x^2] = \int_{\alpha}^{\infty} x^2 \frac{\beta\alpha^{\beta}}{x^{\beta+1}}\,dx$$

$$\Rightarrow \beta\alpha^{\beta}\int_{\alpha}^{\infty}\frac{dx}{x^{\beta-1}} = \beta\alpha^{\beta}\int_{\alpha}^{\infty} x^{1-\beta}\,dx$$

$$= \beta\alpha^{\beta}\left.\frac{x^{2-\beta}}{2-\beta}\right|_{\alpha}^{\infty} = \frac{\beta}{\beta-2}\cdot\alpha^{\beta}\alpha^{2-\beta}$$

$$\boxed{M_2 = \frac{\beta\alpha^2}{\beta-2}}$$

$$m_2 = \frac{\sum x_i^2}{n}$$

$$m_2 = M_2$$

$$\Rightarrow \frac{\beta\alpha^2}{\beta-2} = \frac{\sum x_i^2}{n} \quad \text{--- } (2)$$

$(2) \div (1)^2$

$$\Rightarrow \frac{\frac{\sum x_i^2}{n}}{\bar{x}^2} = \frac{\beta\alpha^2}{(\beta-2)} \frac{(\beta-1)^2}{\alpha^2\beta^2}$$

$$\Rightarrow \frac{\sum x_i^2}{n\bar{x}^2} = \frac{(\beta-1)^2}{\beta(\beta-2)} = 1 + \frac{1}{\beta(\beta-2)}$$

$$\Rightarrow \beta^2 - 2\beta - \frac{n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2} = 0$$

$$\beta = \frac{2 \pm \sqrt{4 + \frac{4n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2}}}{2}$$

Since $\beta > 2$

$$\therefore \beta = \frac{2 + \sqrt{4 + \frac{n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2}}}{2}$$

$$\beta = \frac{2 + 2\sqrt{\dfrac{\Sigma x_i^2 - n\bar{x}^2 + n\bar{x}^2}{\Sigma x_i^2 - n\bar{x}^2}}}{2}$$

$$= 1 + \sqrt{\frac{\Sigma x_i^2}{\Sigma x_i^2 - n\bar{x}^2}}$$

$$= 1 + \sqrt{\frac{\frac{\Sigma x_i^2}{n}}{\frac{1}{n}\Sigma x_i^2 - n\bar{x}^2}} = 1 + \sqrt{\frac{\frac{\Sigma x_i^2}{n}}{\frac{1}{n}\sum_{i=1}^{n}(x_i^2 - \bar{x}^2)}}$$

$$= 1 + \sqrt{\frac{\frac{\Sigma x_i^2}{n}}{\frac{1}{n}\Sigma(x_i - \bar{x})^2}}$$

$$= 1 + \sqrt{\frac{\frac{\Sigma x_i^2}{n}}{\frac{n-1}{n}s^2}} \Rightarrow \boxed{\beta = 1 + \frac{1}{s}\sqrt{\frac{\Sigma x_i^2}{n-1}}}$$

$$\alpha = \left(\frac{\beta - 1}{\beta}\right)\bar{x}$$

$$\Rightarrow \alpha = \frac{\frac{1}{s}\sqrt{\frac{\Sigma x_i^2}{n-1}} \cdot \bar{x}}{1 + \frac{1}{s}\sqrt{\frac{\Sigma x_i^2}{n-1}}}$$

MLE:

$$L(\theta) = \prod_{i=1}^{n} \frac{\beta\alpha^{\beta}}{x_i^{\beta+1}}, \quad x > \alpha$$

$$= (\beta\alpha^{\beta})^{n} \cdot \frac{1}{\prod_{i=1}^{n} x_i^{\beta+1}}$$

$$\log(L(\theta)) = n\log\beta + n\beta\log\alpha - \log\left(\prod_{i=1}^{n} x_i^{\beta+1}\right)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + n\log\alpha - \log\left(\prod_{i=1}^{n} x_i\right) = 0$$

$$\frac{n}{\beta} + n\log\alpha = \log\left(\prod_{i=1}^{n} x_i\right)$$

$$\frac{n}{\beta} = \log\left(\prod_{i=1}^{n} x_i\right) - n\log\alpha$$

$$\frac{n}{\beta} = \log\left(\frac{\prod_{i=1}^{n} x_i}{\alpha^{n}}\right)$$

$$\beta = \frac{n}{\log\left(\prod_{i=1}^{n} \frac{x_i}{\alpha}\right)} = \frac{n}{\sum_{i=1}^{n} \log(x_i/\alpha)}$$

$L(\theta)$ is ↑ wrt α.

$X > \alpha$

$X_1 > \alpha$

$\therefore X_1 - \varepsilon \rightarrow$ MLE of α $\Rightarrow$ MLE & MME are different.

Given data: 3, 5, 2, 3, 4, 1, 4, 3, 3, 3

$$\bar{x} = \frac{3+5+2+3+4+1+4+3+3+3}{10} = \underline{3.1}$$

$$\frac{\sum x_i^2}{n} = 10.7 \Rightarrow \sum x_i^2 = \underline{107}$$

$$s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2 = \frac{1}{9}\left[5(0.01)^2 + (1.9)^2 + (1.1)^2 + (2.1)^2 + 2.(0.9)^2\right]$$

$$= \frac{1}{9}\left[0.05 + 3.61 + 1.21 + 4.41 + 1.62\right]$$

$$= \underline{1.21}$$

$$\beta = 1 + \sqrt{\frac{107}{9 \times 1.21}} = 1 + \sqrt{\frac{107}{10.89}} = 1 + \sqrt{9.82}$$

$$= \underline{4.133}$$

$$\alpha = \frac{(\beta - 1)\bar{x}}{\beta} = \frac{3.133 \times 3.1}{4.133}$$

$$= \underline{2.35}$$

MLE: $\alpha = x_1 - \varepsilon$

Let $\varepsilon = 0 \Rightarrow \underline{x_1 = 1}$

$$\beta = \frac{n}{\sum \log\left(\frac{x_i}{\alpha}\right)} = \frac{10}{\sum_{i=1}^{n} \log(x_i)} = \frac{10}{\log(3^5 \cdot 4^2 \cdot 5 \cdot 2)} = \underline{2.179}$$