## Assignment - 1 Solutions

1. (i). $f(x|\theta) = \begin{cases} \theta(1-\theta)^{x-1} & for\ x = 1,\ 2\ ,\ ... \\ \\ 0 & otherwise. \end{cases}$

   If $g(\theta) = \theta$, then $MLE(g(\theta)) = MME(g(\theta)) = \dfrac{1}{\overline{x}}$. The $MLE$ of $P_\theta(X_1 \geq 4)$ is $(1 - \dfrac{1}{\overline{X}})^3$ since $P_\theta(X_1 \geq 4) = (1-\theta)^3$ and the $MLE$ of $\theta$ is $\dfrac{1}{\overline{x}}$. From the given data the $MLE$ of $P_\theta(X_1 \geq 4) = \left(\dfrac{24}{29}\right)^3$.

   (ii). $f(x|\theta) = \begin{cases} \frac{1}{2\theta} & -\theta \leq x \leq \theta, \\ \\ 0 & otherwise. \end{cases}$

   Calculating MME

   $\mu_1^{'} = 0$ and $\mu_2^{'} = \dfrac{\theta^2}{3} \implies \hat{\theta}_{MME} = \sqrt{\dfrac{3\sum_{i=1}^n x_i^2}{n}}$.

   Calculating MLE

   $L(\theta, \underline{x}) = \dfrac{1}{(2\theta)^n},\ -\theta \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)} \leq \theta$. Maximum of $L$ is attained when $\theta$ takes it's infimum. So $\hat{\theta}_{MLE} = Max\left(-X_{(1)}, X_{(n)}\right)$.

   (iii). $f(x|\theta) = \begin{cases} \dfrac{1}{\sigma} exp\left(-\dfrac{x-\mu}{\sigma}\right) & if\ x > \mu, \\ \\ 0 & otherwise. \end{cases}$

   Calculating MME

   $E(X) = \mu + \sigma$ and $E(X^2) = (\mu+\sigma)^2 + \sigma^2$, implies

   $$\hat{\mu}_{MME} = \overline{x} - \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2}$$

   $$\hat{\sigma}_{MME} = \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2}.$$

   Calculating MLE

$$L^*_{\underline{x}}(\mu,\sigma) = ln\ \ L_{\underline{x}}(\mu,\sigma) \quad = \begin{cases} -n\ ln\sigma - \dfrac{1}{\sigma}\displaystyle\sum_{i=1}^{n}(x_i-\mu) & if\ \ x_{(1)} > \mu, \\ \\ 0 & otherwise, \end{cases}$$

where, $x_{(1)} = min\,\{x_1, x_2, ...x_n\}$. Clearly, $L^*_{\underline{x}}(\mu,\sigma) \le L^*_{\underline{x}}(x_{(1)},\sigma)$, $if, \mu \le x_{(1)}$. $L^*_{\underline{x}}(x_{(1)},\sigma) = -n\ ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(x_i - x_{(1)})$ and

$$\frac{\partial L^*_{\underline{x}}(\mu,\sigma)}{\partial\sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - x_{(1)})$$

So $L^*_{\underline{x}}(\mu,\sigma)$ increases if, $\sigma < \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})$ and decreases if, $\sigma > \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})$. Thus

$$L^*_{\underline{x}}(\mu,\sigma) \le L^*_{\underline{x}}(x_{(1)},\sigma) \le L^*_{\underline{x}}\left(x_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})\right)$$

$\implies$ The MLE of $(\mu,\sigma)$ is $\left(x_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})\right)$

(iv). $f(x|\theta) = \begin{cases} \dfrac{\beta x^{\beta}}{x^{\beta+1}} & if\ \ x > \alpha, \alpha > 0, \beta > 2, \\ \\ 0 & otherwise. \end{cases}$

Calculating MME

$$\hat{\alpha}_{MME} = \frac{\overline{x}\sqrt{\sum_{i=1}^{n} x_i^2}}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2} + \sqrt{\sum_{i=1}^{n} x_i^2}}$$

$$\hat{\beta}_{MME} = 1 + \frac{\sqrt{\sum_{i=1}^{n} x_i^2}}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}$$

Calculating MLE

$$L(\alpha,\beta,\underline{x}) = \frac{\beta^n \alpha^{n\beta}}{\left(\prod_{i=1}^{n} x_i\right)^{\beta+1}}, \quad x_{(1)} > \alpha.$$

$$\hat{\alpha}_{MLE} = x_{(1)}$$

$$\hat{\beta}_{MLE} = \left(\frac{1}{n}\sum_{i=1}^{n} log\left(\frac{x_i}{x_{(1)}}\right)\right)^{-1}$$

(v). The distribution is $N(\mu, \sigma^2)$, $g(\theta) = \dfrac{\mu^2}{\sigma^2}$

<u>Calculating MME</u>

$E(x) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$, implies MME of $(\mu, \sigma^2)$ is $\left(\overline{x}, \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2\right)$

The MME of $g(\theta)$ is $\dfrac{\overline{x}^2}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2}$.

<u>Calculating MLE</u>

The MLE of $(\mu, \sigma^2)$ is $\left(\overline{x}, \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2\right)$ implies the MLE of $g(\theta)$ is $\dfrac{\overline{x}^2}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2}$.

2. The given one is not a proper pmf. **No marks is given to this**

3. Let $X$: The number of items that have failed in less than 100 hours.

$$X \sim Bin(10, \mu)$$

where,

$$\mu = \frac{1}{\theta}\int_0^{100} e^{-x/\theta}dx = 1 - e^{100/\theta}$$

$$\implies \theta = \frac{-100}{ln(1-\mu)}$$

given, $X = 3$, $\hat{\mu} = 0.3$ is the MLE of $\mu$

$$Thus,\ \hat{\mu}_{MLE} = \frac{-100}{ln(0.7)}$$

4. $X_1, X_2, ..., X_n \sim Bin(1, \theta)$

$$\begin{aligned}
L^*_{\underline{x}}(\theta) &= lnL_{\underline{x}}(\theta) \\
&= ln\left(\prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{1-x_i}\right) \\
&= ln\left(\theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{n-\sum_{i=1}^{n}x_i}\right) \\
&= \sum_{i=1}^{n}x_i\, ln\theta + \left(n - \sum_{i=1}^{n}x_i\right)ln(1-\theta)
\end{aligned}$$

$$\frac{dL^*_{\underline{x}}(\theta)}{d\theta} = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{(n - \sum_{i=1}^n x_i)}{1-\theta}$$

Thus, $L^*_{\underline{x}}(\theta)$ increases if $\theta \leq \overline{x}$ and decrease if $\theta \geq \overline{x}$.

$$\hat{\theta}_{MLE}(\underline{x}) = \begin{cases} \frac{1}{4} & if \;\; 0 \leq \overline{x} \leq \frac{1}{4}, \\ \overline{x} & if \;\; \frac{1}{4} \leq \overline{x} \leq \frac{3}{4}, \\ \frac{3}{4} & if \;\; \frac{3}{4} \leq \overline{x} \leq 1. \end{cases}$$

5. (i).The distribution is $N(\mu, \sigma^2)$, $\mu = \mu_0$ is known.

$$L(\sigma^2, \underline{x}) = \frac{1}{\sigma^n (2n)^{n/2}} e^{-\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2}}$$

$$logL(\sigma^2, \underline{x}) = \frac{n}{2} log\sigma^2 - \frac{n}{2} log 2\pi - \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2}$$

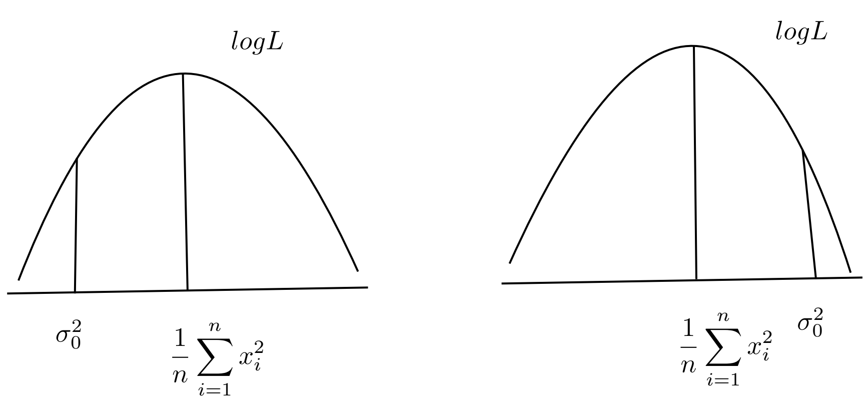
$$\frac{dlogL}{d\sigma^2} = \frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^4}$$

$logL(\sigma^2, \underline{x})$ increases if, $\sigma^2 \leq \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{n}$ and decreases if, $\sigma^2 \geq \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{n}$.

$$Thus, \hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{n}$$

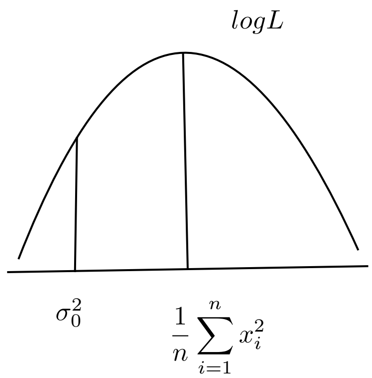
(ii). $\hat{\sigma}^2_{MLE} = \frac{1}{n} \sum_{i=1}^n x_i^2$ if, $\mu_0 = 0$.

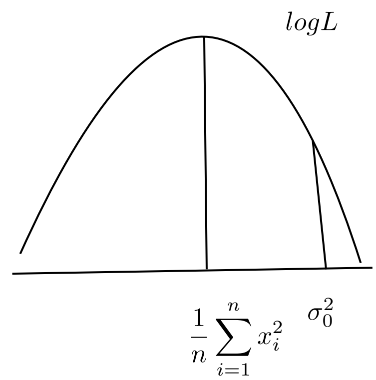
(iii) $\sigma^2 \geq \sigma_0^2$



$$\therefore \hat{\sigma}^2_{MLE} = max\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2, \sigma_0^2\right)$$

(iv) $\sigma^2 \leq \sigma_0^2$




$$\therefore \hat{\sigma}^2_{MLE} = min\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2, \sigma_0^2\right)$$

**Distribution of Marks**

1. (i). 2(MME) + 2(MLE) + 2
   (ii). 2(MME) + 2(MLE)
   (iii). 2(MME) + 2(MLE)
   (iv). 2(MME) + 2(MLE)
   (v). 2(MME) + 2(MLE)
2. 0
3. 5
4. 5
5. (i). 3
   (ii). 1
   (iii). 2
   (iv). 2

**Total - 40**