MA 3140: Statistical Inference

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Power function

- Power function is defined as follows:

$$\beta^*(\theta) = P_\theta(X \in R), \quad \theta \in \Theta_1.$$

  Ideally, $\beta^*(\theta)$ is 0 for all $\theta \in \Theta_0$ and 1 for all $\theta \in \Theta_1$.

- **Example:** Let $X \sim Bin(10, p)$.

  Consider

$$H_0 : p = 0.6 \quad vs. \quad H_1 : p > 0.6$$

  Test procedure: Reject H_0 if $X \geq 7$ and accept H_0 if $X \leq 6$.

Power function

- First, calculate α.

$$\begin{aligned}\alpha = P(\text{Type I error}) &= P(\text{Rejecting } H_0 \text{ when it is true})\\ &= P_{p=0.6}(X \geq 7)\\ &= \sum_{k=7}^{10} \binom{10}{k} (0.6)^k (0.4)^{10-k} = 0.382\end{aligned}$$

- Let us now calculate β.

$$\begin{aligned}\beta(p) &= P(\text{Accepting } H_0 \text{ when it is false})\\ &= P_p(X \leq 6), \quad p > 0.6\\ &= \sum_{k=0}^{6} \binom{10}{k} p^k (1-p)^{(10-k)}\end{aligned}$$

Power function

- Finally, calculate $\beta^*(p)$.

$$\beta^*(p) = 1 - \beta(p) = \sum_{k=7}^{10} \binom{10}{k} p^k (1-p)^{(10-k)}, \quad p > 0.6.$$

Using Binomial Distribution Table, we can obtain these values:

p	0.7	0.8	0.9	0.95
$\beta(p)$	0.35	0.121	0.013	0.001
$\beta^*(p)$	0.65	0.879	0.987	0.999

Thus, $\beta^*(p)$ increases in p.

Remark

- The job of the statistician is to fix an appropriate α.
- Usually, $\alpha = 0.05, 0.01, 0.1$.
- $\alpha = 0.05$ indicates that we are willing to take 5% risk of rejecting the null hypothesis when it is true.

Test function

- **Test function**: A function ϕ on the sample space that is 1 if $\boldsymbol{x} \in R$ and 0 if $\boldsymbol{x} \in R^c$.
- Consider $X \sim P_\theta$;

$$H_0 : \theta \in \Theta_0 \qquad vs \qquad H_1 : \theta \in \Theta_1.$$

$$P_\theta(X \in R) = E_\theta \phi(X) = \beta^*_\phi(\theta)$$

- If $\theta \in \Theta_0$, β^*_ϕ denotes the probability of Type I error.
- If $\theta \in \Theta_1$, β^*_ϕ denotes the power of the test.

Test function

So, the problem of finding an optimal test procedure is to find a test function ϕ such that

$$\text{Maximize } \beta_\phi^* = E_\theta \phi(X), \quad \theta \in \Theta_1$$

subject to

$$\beta_\phi^* = E_\theta \phi(X) \leq \alpha, \quad \forall\ \theta \in \Theta_0.$$

Here, the restriction $\beta_\phi^* \leq \alpha$, $\forall\ \theta \in \Theta_0$ means that if H_0 was true, ϕ rejects it with a probability $\leq \alpha$.

Most powerful/ Uniformly most powerful test

- If Θ_1 is a singleton set, we maximize β^*_ϕ with respect to θ.

  This gives a test say ϕ_0, which is referred to as the most powerful (MP) test.

- If Θ_1 is not singleton, then β^*_ϕ has to be maximized over $\theta \in \Theta_1$.

  This gives a uniformly most powerful (UMP) test.

MP tests

Neyman Pearson Lemma

Theorem: Let π_0 and π_1 be populations with known distributions f_0 and f_1, respectively. Then for testing

$$H_0 : f = f_0 \quad \text{against} \quad H_1 : f = f_1,$$

we can define a test ϕ with a constant k such that

$$E_0\phi(X) = \alpha \tag{1}$$

and

$$\phi(x) = \begin{cases} 1, & \text{when } f_1(x) > kf_0(x) \\ 0, & \text{when } f_1(x) < kf_0(x) \end{cases} \tag{2}$$

If ϕ satisfies (1) and (2) for some k then it is MP test for H_0 against H_1 at level α.

Neyman Pearson Lemma

Proof:

To prove that ϕ is the MP test, let us consider ϕ^* as any other test with $E_0\phi^*(X) \leq \alpha$.

Define

$$A_1 = \{x : \phi(x) - \phi^*(x) > 0\}$$

and

$$A_2 = \{x : \phi(x) - \phi^*(x) < 0\}.$$

If $x \in A_1$, then

$$\phi(x) > \phi^*(x) \Longrightarrow \phi(x) > 0 \Longrightarrow f_1(x) > kf_0(x).$$

If $x \in A_2$, then

$$\phi(x) < \phi^*(x) \Longrightarrow \phi(x) < 1 \Longrightarrow f_1(x) < kf_0(x).$$

Neyman Pearson Lemma

So,

$$\Big(\phi(x) - \phi^*(x)\Big)\Big(f_1(x) - kf_0(x)\Big) \geq 0, \ \forall \ x \in A_1 \cup A_2.$$

Also,

$$\int_{A_1 \cup A_2} \Big(\phi(x) - \phi^*(x)\Big)\Big(f_1(x) - kf_0(x)\Big) dx > 0.$$

This implies

$$\int (\phi(x) - \phi^*(x)) f_1(x) dx > k \int (\phi(x) - \phi^*(x)) f_0(x) dx$$

$$\implies \beta^*_{\phi} - \beta^*_{\phi^*} > 0$$

$$\implies \beta^*_{\phi} > \beta^*_{\phi^*}$$

Thus, ϕ is more powerful than ϕ^*; hence, ϕ is the MP test of level α.

Example 1

Let $X_1, \ldots, X_n \sim N(\mu, 1)$. Find the MP test for testing

$$H_0 : \mu = \mu_0 \quad vs. \quad H_0 : \mu = \mu_1$$

Case 1: $\mu_0 < \mu_1$

Solution: The joint density of $X_1, \ldots, X_n$

$$\begin{aligned} f_\mu(\boldsymbol{x}) &= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum(x_i - \mu)^2} \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum(x_i^2 + \mu^2 - 2\mu x_i)} \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum x_i^2 - \frac{n\mu^2}{2} + \mu x_i} \end{aligned}$$

Example 1 contd.

By NP Lemma, the test is

$$\text{Reject } H_0 \text{ when } \frac{f_1(\boldsymbol{x})}{f_0(\boldsymbol{x})} \geq k$$

Thus,

$$\begin{aligned}
\frac{f_{\mu_1}(\boldsymbol{x})}{f_{\mu_0}(\boldsymbol{x})} = e^{\frac{n\mu_0^2}{2} - \frac{n\mu_1^2}{2}} e^{(\mu_1 - \mu_0)n\bar{x}} &\geq k \\
\implies \qquad e^{(\mu_1 - \mu_0)n\bar{x}} &\geq k_1 \\
\implies \quad n(\mu_1 - \mu_0)\bar{x} &\geq k_2 \\
\implies \qquad \bar{x} &\geq k_3,
\end{aligned}$$

where $\overline{X} \sim N(\mu, 1/n)$.

Example 1 contd.

Now,

$$\begin{aligned}\alpha &= P_0(\overline{X} \geq k_3) \qquad (\because \overline{X} \sim N(\mu_0, 1/n) \text{ when } H_0 \text{ is true}) \\ &= P_0(\sqrt{n}(\overline{X} - \mu_0) \geq \sqrt{n}(k_3 - \mu_0)) \\ &= P_0(Z \geq z_\alpha),\end{aligned}$$

where $z_\alpha = \sqrt{n}(k_3 - \mu_0)$ is the upper $100\alpha\%$ in the standard normal distribution.
Thus, the test is

$$\text{Reject } H_0 \text{ if } \sqrt{n}(\overline{X} - \mu_0) \geq z_\alpha$$

$$\text{Accept } H_0 \text{ if } \sqrt{n}(\overline{X} - \mu_0) < z_\alpha$$

This is the MP test of size α.

Example 1 contd.

- Suppose $\mu_0 = 0$, $\mu_1 = 1$, $n = 25$, $\alpha = 0.05$ and $z_{0.05} = 1.645$

(i) Let $\overline{X} = 0.2$.

Then

$$Z = \sqrt{n}(\overline{X} - \mu_0) = 5(0.2 - 0) = 1$$

Since $Z < z_\alpha$, we cannot reject $H_0 : \mu = 0$ at 5%.

(ii) Let $\overline{X} = 0.4$.

Then,

$$Z = \sqrt{n}(\overline{X} - \mu_0) = 5(0.4 - 0) = 2$$

Since $Z > z_\alpha$, we reject $H_0 : \mu = 0$ at 5%.

Example 1 contd.

- **Calculate Power:**

$$
\begin{aligned}
& P_{\mu=\mu_1}(\sqrt{n}(\overline{X} - \mu_0) \geq z_\alpha) \\
& = P_{\mu=\mu_1}(\sqrt{n}(\overline{X} - \mu_1) + \sqrt{n}(\mu_1 - \mu_0) \geq z_\alpha) \\
& = P(Z \geq z_\alpha - \sqrt{n}(\mu_1 - \mu_0)) \\
& = P(Z \geq 1.645 - 5) \\
& = P(Z \geq -3.355) \approx 1
\end{aligned}
$$

Example 1 contd.

Case 2: $\mu_0 > \mu_1$

Solution: In this case, proceeding as before, the rejection region becomes

$$\overline{X} \leq k_3^*$$

This implies

$$\alpha = P_0(\overline{X} \leq k_3^*) = P_0(\sqrt{n}(\overline{X} - \mu_0) \leq \sqrt{n}(k_3^* - \mu_0))$$
$$= P(Z \leq -z_\alpha)$$

Thus, the MP test is

$$\text{Reject } H_0 \text{ if } \sqrt{n}(\overline{X} - \mu_0) \leq -z_\alpha$$

$$\text{Accept } H_0 \text{ otherwise}$$

Example 1 contd.

- Suppose $\mu_0 = 0$, $\mu_1 = -1$, $n = 25$, $\overline{X} = -0.6$, $\alpha = 0.05$ and $-z_{0.05} = -1.645$.

  Then,
  $$Z = \sqrt{n}(\overline{X} - \mu_0) = 5(-0.6 - 0) = -3$$

  Since $Z < -z_\alpha$, we reject $H_0 : \mu = 0$ at 5%.

- **Calculate Power:**

$$\begin{aligned}
&P_{\mu=\mu_1}(\sqrt{n}(\overline{X} - \mu_0) \leq -z_\alpha) \\
&= P_{\mu=\mu_1}(\sqrt{n}(\overline{X} - \mu_1) + \sqrt{n}(\mu_1 - \mu_0) \leq -z_\alpha) \\
&= P(Z \leq -z_\alpha + \sqrt{n}(\mu_0 - \mu_1)) \\
&= P(Z \leq -1.645 + 5) \approx 1
\end{aligned}$$

Example 2

Let $X_1, \ldots, X_n \sim N(0, \sigma^2)$. Find the MP test for testing

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs. \quad H_1 : \sigma^2 = \sigma_1^2$$

Case 1: $\sigma_1^2 > \sigma_0^2$

Solution: The joint density of $X_1, \ldots, X_n$ under H_0 and H_1 are:

$$f_0(\boldsymbol{x}) = \frac{1}{(\sigma_0\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma_0^2}\sum x_i^2}$$

$$f_1(\boldsymbol{x}) = \frac{1}{(\sigma_1\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma_1^2}\sum x_i^2}$$

Example 2 contd.

NP Lemma gives the form of the MP test

$$\text{Reject } H_0 \text{ if } \frac{f_1(\boldsymbol{x})}{f_0(\boldsymbol{x})} \geq k$$

where k is determined by the size condition.
This is equivalent to

$$\left(\frac{\sigma_0\sqrt{2\pi}}{\sigma_1\sqrt{2\pi}}\right)^n \exp\left[\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum x_i^2\right] \geq k$$

Taking log and adjusting the constants, we can write the rejection region as

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum x_i^2 \geq k_1,$$

where k_1 is determined by the size condition.

Example 2 contd.

This implies that Reject H_0 if

$$\sum x_i^2 \geq k_2 \qquad (\because \sigma_1^2 > \sigma_0^2).$$

In order to determine k_2, we employ the size condition, i.e.,

$$\begin{aligned} \alpha = P(\text{Type I error}) &= P(\text{Rejecting } H_0 \text{ when it is true}) \\ &= P_{\sigma^2=\sigma_0^2}\left(\sum X_i^2 \geq k_2\right) \end{aligned}$$

Note that $Y_i = \frac{X_i}{\sigma_0} \sim N(0,1)$ (under H_0).
$Y_1, \ldots, Y_n$ are independent and $\sum Y_i^2 \sim \chi_n^2$.

Example 2 contd.

Now,

$$\alpha = P(\text{Reject } H_0 \text{ when it is true}) = P_{\sigma_0^2}\Big(\frac{\sum X_i^2}{\sigma_0^2} \geq k_3\Big)$$

$$\implies k_3 = \chi^2_{n,\alpha}$$

Thus the MP test for testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$ at level α is

$$\text{Reject } H_0 \text{ if } \frac{\sum X_i^2}{\sigma_0^2} \geq \chi^2_{n,\alpha}$$

otherwise accept H_0.

Example 2 contd.

Case 2: $\sigma_1^2 < \sigma_0^2$

Solution: The test procedure will get modified as follows:

$$\text{Reject } H_0 \text{ if } \sum x_i^2 \leq k_2^*.$$

In order to determine k_2^*, we employ the size condition, i.e.,

$$\alpha = P(\text{Reject } H_0 \text{ when it is true}) = P_{\sigma_0^2}\Big(\frac{\sum X_i^2}{\sigma_0^2} \leq k_3^*\Big)$$

$$\implies k_3^* = \chi^2_{n,1-\alpha}$$

Thus the MP test is

$$\text{Reject } H_0 \text{ if } \frac{\sum X_i^2}{\sigma_0^2} \leq \chi^2_{n,1-\alpha}$$

Thanks for your patience!