HW9
(SOLUTIONS)

1 | a | CH 7, PROB. 4

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y} & 0<y<1,\quad 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$a)\ E[XY] = \int_0^1 \int_0^y xy\frac{1}{y}\,dx\,dy = \int_0^1 \frac{y}{y}\left[\int_0^y x\,dx\right]dy = \int_0^1 \left[\int_0^y x\,dx\right]dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Big|_0^1 = \boxed{\frac{1}{6}}$$

$$b)\ E[X] = \int_0^1\int_0^y x\cdot\frac{1}{y}\,dx\,dy = \int_0^1 \frac{1}{y}\left[\int_0^y x\,dx\right]dy = \int_0^1 \frac{1}{y}\cdot\frac{y^2}{2}\,dy$$

$$= \int_0^1 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^1 = \boxed{\frac{1}{4}}$$

$$c)\ E[Y] = \int_0^1\int_0^y y\cdot\frac{1}{y}\,dx\,dy = \int_0^1\left[\int_0^y dx\right]dy = \int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

b | CH 7, PROB 16

$$X = \begin{cases} Z & \text{if } Z \geq x \\ 0 & \text{otherwise} \end{cases} \qquad \text{AND} \qquad f_Z(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

$$E[X] = E\big[E[X|Z]\big] = E\big[m_X(Z)\big]$$

where $m_X(Z) = E[X|Z]$

So, LET'S FIRST FIND $m_X(Z)$

$$m_X(Z) = E[X|Z] = \begin{cases} 0 & \text{if } Z < x \\ Z & \text{if } Z \geq x \end{cases}$$

LIKEWISE

$$\boxed{m_X(z) = \begin{cases} 0 & \text{if } z < x \\ z & \text{if } z \geq x \end{cases}}$$

Now $E[X] = E\left[E[X|Z]\right]$

$$= E\left[m_X(Z)\right]$$

$$= \int_{-\infty}^{\infty} m_X(z) f_Z(z)\, dz$$

$$= \int_{-\infty}^{x} 0 \cdot f_Z(z)\, dz + \int_{x}^{\infty} z f_Z(z)\, dz$$

$$= \int_{x}^{\infty} z f_Z(z)\, dz$$

$$= \int_{x}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz \qquad \text{sub } u = \frac{z^2}{2}$$

$$= \int_{\frac{x^2}{2}}^{\infty} \frac{e^{-u}}{\sqrt{2\pi}}\, du = -\frac{e^{-u}}{\sqrt{2\pi}}\Bigg|_{\frac{x^2}{2}}^{\infty}$$

$$\boxed{E[X] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$

$\boxed{C}$ CH7, PROB 25

* SINCE $X_1, X_2, \cdots, X_n$ ARE IDENTICALLY DISTRIBUTED,

WE HAVE $\boxed{f_{X_1}(x) = f_{X_2}(x) = \cdots = f_{X_n}(x) = f_X(x)}$ (A)

* NEXT, SINCE $X_1, X_2, \ldots, X_n$ ARE INDEPENDENT, WE HAVE

$$\boxed{f_{X_1, X_2, \cdots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)} \quad (B)$$

* NOW, LET'S FIRST SOLVE THE FOLLOWING INTEGRAL

sub $u = F_X(x) \Rightarrow du = f_X(x)dx$

$$\int_{-\infty}^{r} f_X(x)\left[F_X(x)\right]^i dx = \int_{F_X(-\infty)}^{F_X(r)} u^i du = \int_0^{F_X(r)} u^i du$$

$$= \left.\frac{u^{i+1}}{i+1}\right|_0^{F_X(r)}$$

$$\boxed{\int_{-\infty}^{r} f_X(x)\left[F_X(x)\right]^i dx = \frac{\left[F_X(r)\right]^{i+1}}{i+1}} \quad (C)$$

* NOW, ESTABLISH THE FOLLOWING EQUALITIES

$$\left.\begin{aligned} P(N=n) &= P(X_1 \geq X_2 \geq \cdots \geq X_{n-1} < X_n) \\ P(N>n) &= P(X_1 \geq X_2 \geq \cdots \geq X_{n-1} \geq X_n) \end{aligned}\right\}$$

AND NOW COMBINE THESE TWO ON THE NEXT PAGE

$$P(N \geq n) = P(N = n) + P(N > n)$$

$$= P(X_1 \geq X_2 \geq \cdots \geq X_{n-1} < X_n) + P(X_1 \geq X_2 \geq \cdots \geq X_{n-1} \geq X_n)$$

$$\boxed{P(N \geq n) = P(X_1 \geq X_2 \geq \cdots \geq X_{n-1})} \quad (D)$$

* NEXT, REWRITE (D) AS FOLLOWS

$$P(N \geq n) = P(X_1 \geq X_2 \geq \cdots \geq X_{n-1}) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_{n-4}}\int_{-\infty}^{x_{n-3}}\int_{-\infty}^{x_{n-2}} f_{X_1, X_2, \ldots, X_{n-1}}(x_1, x_2, \ldots x_{n-1})\, dx_{n-1} \cdots dx_1$$

USE (B)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_{n-4}}\int_{-\infty}^{x_{n-3}}\int_{-\infty}^{x_{n-2}} \prod_{i=1}^{n-1} f_X(x_i)\, dx_{n-1}\, dx_{n-2} \cdots dx_1$$

$$P(N \geq n) = \int_{-\infty}^{\infty} f_X(x_1) \int_{-\infty}^{x_1} f_X(x_2) \int_{-\infty}^{x_2} f_X(x_3) \cdots \int_{-\infty}^{x_{n-4}} f_X(x_{n-3}) \int_{-\infty}^{x_{n-3}} f_X(x_{n-2}) \underbrace{\left[\int_{-\infty}^{x_{n-2}} f_X(x_{n-1})\, dx_{n-1}\right]}_{F_X(x_{n-2})} dx_{n-2} \cdots dx_1$$

use (c)

$$= \int_{-\infty}^{\infty} f_X(x_1) \int_{-\infty}^{x_1} f_X(x_2) \int_{-\infty}^{x_2} f_X(x_3) \cdots \int_{-\infty}^{x_{n-4}} f_X(x_{n-3}) \underbrace{\left[\int_{-\infty}^{x_{n-3}} f_X(x_{n-2})\, F_X(x_{n-2})\, dx_{n-2}\right]}_{\frac{[F_X(x_{n-3})]^2}{2}} dx_{n-3} \cdots dx_1$$

use (c)

$$P(N \geq n) = \int_{-\infty}^{\infty} f_X(x_1) \int_{-\infty}^{x_1} f_X(x_2) \int_{-\infty}^{x_2} f_X(x_3) \cdots \underbrace{\left[\int_{-\infty}^{x_{n-4}} f_X(x_{n-3}) \frac{\left[F_X(x_{n-3})\right]^2}{2} dx_{n-3} \right]}_{\frac{\left[F_X(x_{n-4})\right]^3}{3!}} dx_{n-4} \cdots dx_1$$

USE (C)

$\vdots$

$$P(N \geq n) = \int_{-\infty}^{\infty} f_X(x_1) \cdot \frac{\left[F_X(x_1)\right]^{n-2}}{(n-2)!} dx_1$$

SUB $u = F_X(x_1)$

$du = f_X(x_1) dx_1$

$$= \int_{F_X(-\infty)}^{F_X(\infty)} \frac{u^{n-2}}{(n-2)!} du = \int_0^1 \frac{u^{n-2}}{(n-2)!} du = \left. \frac{u^{n-1}}{(n-1)!} \right|_0^1$$

$$\boxed{P(N \geq n) = \frac{1}{(n-1)!}} \quad (E)$$

* NEXT, USING (E), WE GET

$$P(N = n) = P(N \geq n) - P(N > n)$$

$$= P(N \geq n) - P(N \geq n+1)$$

USE (E)

$$= \frac{1}{(n-1)!} - \frac{1}{n!}$$

$$= \frac{n}{n!} - \frac{1}{n!}$$

$$\boxed{P(N = n) = \frac{n-1}{n!}}$$

* FINALLY

$$E[N] = \sum_{n=2}^{\infty} n \cdot P(N=n)$$

$$= \sum_{n=2}^{\infty} n \cdot \frac{n-1}{n!}$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots =$$

$$= \frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \cdots$$

$$= e^1$$

$$= \boxed{e = E[N]}$$

$\boxed{d}$ CH 7, PROBLEM 33

$\boxed{E[X] = 1}$, $\boxed{Var(X) = 5}$

$$E[X^2] - (E[X])^2 = 5$$

$$E[X^2] - 1^2 = 5$$

$$\boxed{E[X^2] = 6}$$

a) $E[(2+X)^2] = E[4 + 4X + X^2] = E[4] + E[4X] + E[X^2]$

$$= 4 + 4E[X] + E[X^2]$$

$$= 4 + 4 \cdot 1 + 6$$

$$= \boxed{14}$$

b) $Var(4+3X) = E[(4+3X)^2] - (E[4+3X])^2$

$= E[4^2 + 2\cdot3\cdot4X + 9X^2] - (E[4]+3E[X])^2$

$= E[4^2] + 2\cdot3\cdot4E[X] + 9E[X^2] - (E[4]+3E[X])^2$

$= 16 + 24E[X] + 9E[X^2] - (4+3E[X])^2$

$= 16 + 24\cdot1 + 9\cdot6 - (4+3\cdot1)^2$

$= 16 + 24 + 54 - 49$

$= \boxed{45}$

<u>Second way</u>: $Var(4+3X) = Var(3X) = 3^2\cdot Var(X) = 3^2\cdot5 = \boxed{45}$

$\boxed{e}$ CH 7, PROB 48

LET $X_1, X_2, X_3, \ldots, X_n, \ldots$ BE RANDOM VARIABLES THAT REPRESENT ROLLS OF DICE

$\{X=i\}$ IS EQUIVALENT TO $\{X_1\neq6, X_2\neq6, \ldots, X_{i-1}\neq6, X_i=6\}$

$\{Y=i\}$ IS EQUIVALENT TO $\{X_1\neq5, X_2\neq5, \ldots, X_{i-1}\neq5, X_i=5\}$

a) $P(X=i) = P(X_1\neq6, X_2\neq6, \ldots, X_{i-1}\neq6, X_i=6) = \left(\frac{5}{6}\right)^{i-1}\cdot\frac{1}{6}$

$E[X] = \sum_{i=1}^{\infty} i\cdot P(X=i) = \sum_{i=1}^{\infty} i\cdot\left(\frac{5}{6}\right)^{i-1}\cdot\frac{1}{6}$

$= \frac{1}{6}\cdot\left[1 + 2\cdot\frac{5}{6} + 3\cdot\left(\frac{5}{6}\right)^2 + 4\cdot\left(\frac{5}{6}\right)^3 + \cdots\right]$

$$E[X] = \frac{1}{6} \cdot \left[1 + 2 \cdot \left(\frac{5}{6}\right) + 3 \cdot \left(\frac{5}{6}\right)^2 + 4 \cdot \left(\frac{5}{6}\right)^3 + \dots \right] \frac{1 - \left(\frac{5}{6}\right)}{1 - \frac{5}{6}}$$

$$= \frac{1}{6} \cdot \left[1 + (2-1)\left(\frac{5}{6}\right) + (3-2)\left(\frac{5}{6}\right)^2 + (4-3)\left(\frac{5}{6}\right)^3 + \dots \right] \cdot \frac{1}{1 - \frac{5}{6}}$$

$$= 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \dots$$

$$= \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^i$$

$$= \frac{1}{1 - \frac{5}{6}} = 6$$

$$\Rightarrow \boxed{E[X] = \frac{1}{6} \sum_{i=1}^{\infty} i \left(\frac{5}{6}\right)^{i-1} = \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^i = 6}$$

b) $P(X = i \mid Y = 1) = P(X_1 \neq 6, X_2 \neq 6, \dots, X_{i-1} \neq 6, X_i = 6 \mid X_1 = 5)$ if $i > 1$

$$= P(X_1 \neq 6 \mid X_1 = 5) \cdot P(X_2 \neq 6) \cdot P(X_3 \neq 6) \cdots P(X_{i-1} \neq 6) \cdot P(X_i = 6)$$

$$= 1 \cdot \left(\frac{5}{6}\right)^{i-2} \cdot \left(\frac{1}{6}\right) \qquad \boxed{\text{if } i = 1}$$

$$P(X = 1 \mid Y = 1) = P(X_1 = 6 \mid X_1 = 5) = 0$$

$$\Rightarrow P(X = i \mid Y = 1) = \begin{cases} 0 & i \leq 1 \\ \frac{1}{6} \left(\frac{5}{6}\right)^{i-2} & i > 1 \end{cases}$$

$$\boxed{E[X \mid Y = 1] = \sum_{i=1}^{\infty} i \cdot P(X = i \mid Y = 1) = \sum_{i=2}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{i-2}}$$

$$E[X|Y=1] = \sum_{i=2}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{i-2}$$

$$= \sum_{i=2}^{\infty} i \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{i-2} + \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{1-2} - \frac{1}{6}\left(\frac{5}{6}\right)^{1-2}$$

$$= \sum_{i=1}^{\infty} i \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{i-2} - \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{1-2}$$

$$= \left(\frac{5}{6}\right)^{-1} \cdot \underbrace{\sum_{i=1}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{i-1}}_{E[X]} - \frac{1}{6}\left(\frac{5}{6}\right)^{1-2}$$

$$= \left(\frac{5}{6}\right)^{-1} \cdot E[X] - \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{1-2}$$

$$= \frac{6}{5} \cdot E[X] - \frac{1}{5} = \frac{6}{5} \cdot 6 - \frac{1}{5} = 7$$

$$\boxed{E[X|Y=1] = 7}$$

c) FIRST TAKE $i \leq 4$

$$P(X=i \mid Y=5) = P(X_1 \neq 6, X_2 \neq 6, \ldots, X_{i-1} \neq 6, X_i = 6 \mid X_1 \neq 5, X_2 \neq 5, X_3 \neq 5, X_4 \neq 5, X_5 = 5)$$

for $i \leq 4$

$$= P(X_1 \neq 6 \mid X_1 \neq 5) \cdots P(X_{i-1} \neq 6 \mid X_{i-1} \neq 5) \cdot P(X_i = 6 \mid X_i \neq 5)$$

for $i \leq 4$

$$(*) \quad \boxed{P(X=i|Y=5) = \left(\frac{4}{5}\right)^{i-1} \cdot \frac{1}{5}} \quad \text{if } i \leq 4$$

NEXT, IF $i = 5$

$$P(X=5|Y=5) = P(X_1 \neq 6, X_2 \neq 6, X_3 \neq 6, X_4 \neq 6, X_5 = 6 \mid X_1 \neq 5, X_2 \neq 5, X_3 \neq 5, X_4 \neq 5, X_5 = 5)$$

$$= P(X_1 \neq 6 | X_1 \neq 5) \cdot P(X_2 \neq 6 | X_2 \neq 5) \cdot P(X_3 \neq 6 | X_3 \neq 5) \cdot P(X_4 \neq 6 | X_4 \neq 5) \cdot \underbrace{P(X_5 = 6 | X_5 = 5)}_{0}$$

$$\boxed{P(X=5|Y=5) = 0} \Rightarrow \boxed{P(X=i|Y=5) = 0 \quad \text{if } i = 5} \; (**)$$

NOW TAKE $i > 5$

$$P(X=i \mid Y=5) = P(X_1 \neq 6, \dots, X_{i-1} \neq 6, X_i = 6 \mid X_1 \neq 5, X_2 \neq 5, \dots X_4 \neq 5, X_5 = 5)$$

$$= \left[\prod_{k=1}^{4} P(X_k \neq 6 \mid X_k \neq 5)\right] \cdot \underbrace{P(X_5 \neq 6 \mid X_5 = 5)}_{1} \left[\prod_{j=5}^{i-1} P(X_j \neq 6)\right] \cdot \underbrace{P(X_i = 6)}_{\frac{1}{6}}$$

(***) $$\boxed{P(X=i \mid Y=5) = \left(\frac{4}{5}\right)^4 \cdot 1 \cdot \left(\frac{5}{6}\right)^{i-6} \cdot \left(\frac{1}{6}\right)} \quad \text{for } i \geq 6$$

$$\boxed{P(X=i \mid Y=5) = \begin{cases} \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{i-1} & 1 \leq i \leq 4 \\ 0 & i = 5 \\ \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6} & i \geq 6 \end{cases}}$$

COMBINE (*), (**) & (***)

$$E[X \mid Y=5] = \sum_{i=1}^{4} i \cdot \frac{1}{5}\left(\frac{4}{5}\right)^{i-1} + \sum_{i=6}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \left(\frac{5}{6}\right)^{i-6}$$

$$= \sum_{i=1}^{4} i \cdot \frac{1}{5}\left(\frac{4}{5}\right)^{i-1} + \underbrace{\sum_{i=6}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6} + \sum_{i=1}^{5} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \left(\frac{5}{6}\right)^{i-6}}_{\text{COMBINING INTO ONE SUM}} - \sum_{i=1}^{5} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6}$$

$$= \underbrace{\sum_{i=1}^{4} i \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{i-1}}_{S_1} + \underbrace{\sum_{i=1}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6}}_{S_2} - \underbrace{\sum_{i=1}^{5} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \left(\frac{5}{6}\right)^{i-6}}_{S_3}$$

$$\boxed{E[X \mid Y=5] = S_1 + S_2 - S_3}$$

NOW COMPUTE SEPARATELY S_1, S_2 & S_3

$$S_1 = \sum_{i=1}^{4} i \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{i-1}$$

$$= 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} \cdot \frac{4}{5} + 3 \cdot \frac{1}{5} \cdot \frac{4^2}{5^2} + 4 \cdot \frac{1}{5} \cdot \frac{4^3}{5^3}$$

$$= \frac{5^3 + 2 \cdot 4 \cdot 5^2 + 3 \cdot 4^2 \cdot 5 + 4 \cdot 4^3}{5^4} = \boxed{\frac{821}{625} = S_1}$$

$$S_2 = \sum_{i=1}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6}$$

$$= \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-5} \underbrace{\sum_{i=1}^{\infty} i \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{i-1}}_{E[X]}$$

$$= \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-5} \cdot E[X] = \frac{4^4}{5^4} \cdot \frac{6^5}{5^5} \cdot E[X] = \frac{4^4}{5^4} \cdot \frac{6^5}{5^5} \cdot 6$$

$$= \frac{4^4 \cdot 6^6}{5^9} = \boxed{\frac{11943936}{1953125} = S_2}$$

$$S_3 = \sum_{i=1}^{5} i \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{i-6}$$

$$= 1 \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-5} + 2 \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \left(\frac{5}{6}\right)^{-4} + 3 \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-3} + 4 \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-2} + 5 \cdot \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{6}\right)^{-1}$$

$$= \frac{1}{6} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{6}{5}\right) \cdot \left[\left(\frac{6}{5}\right)^4 + 2 \cdot \left(\frac{6}{5}\right)^3 + 3 \cdot \left(\frac{6}{5}\right)^2 + 4 \cdot \left(\frac{6}{5}\right) + 5\right]$$

$$= \frac{1}{\cancel{6}} \cdot \frac{4^4 \cdot \cancel{6}}{5^5} \left[\frac{6^4}{5^4} + 2\frac{6^3}{5^3} + 3\frac{6^2}{5^2} + 4\frac{6}{5} + 5\right]$$

$$= \frac{4^4}{5^5} \cdot \frac{6^4 + 2 \cdot 6^3 \cdot 5 + 3 \cdot 6^2 \cdot 5^2 + 4 \cdot 6 \cdot 5^3 + 5 \cdot 5^4}{5^4} = \frac{256}{3125} \cdot \frac{12281}{625} =$$

$$= \boxed{\frac{3143936}{1953125} = S_3}$$

FINALLY,

$$E[X|Y=5] = S_1 + S_2 - S_3$$

$$= \frac{821}{625} + \frac{11943936}{1953125} - \frac{3143936}{1953125}$$

$$= \frac{821}{5^4} + \frac{11943936 - 3143936}{5^9}$$

$$= \frac{821}{5^4} + \frac{8800000}{5^9}$$

$$= \frac{821}{5^4} + \frac{88 \cdot 10^5}{5^9}$$

$$= \frac{821}{5^4} + \frac{88 \cdot \cancel{5^5} \cdot 2^5}{\cancel{5^5} \cdot 5^4} = \frac{821 + 88 \times 2^5}{5^4} = \frac{821 + 88 \cdot 32}{5^4}$$

$$\boxed{E[X|Y=5] = \frac{3637}{625} = 5.8192}$$

$\boxed{f}$ CH 7, PROB 50

FIRST NOTE THAT IF $y \leq 0$ THEN $E[X^2|Y=y]$ IS UNDEFINED !

$\boxed{\text{SO, LET'S COMPUTE } E[X^2|Y=y] \text{ FOR } y>0}$

$$E[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\, dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$$

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$$E[X^2|Y=y] = \frac{\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

$$= \frac{\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\,dx}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$

$$= \frac{\int_0^{\infty} x^2 \cdot \frac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}\,dx}{\int_0^{\infty} \frac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}\,dx} = \frac{\frac{e^{-y}}{y} \cdot \int_0^{\infty} x^2 e^{-\frac{x}{y}}\,dx}{\frac{e^{-y}}{y} \int_0^{\infty} e^{-\frac{x}{y}}\,dx}$$

$$\boxed{E[X^2|Y=y] = \frac{\int_0^{\infty} x^2 e^{-\frac{x}{y}}\,dx}{\int_0^{\infty} e^{-\frac{x}{y}}\,dx}}$$ for $y > 0$

INTEGRAL IN THE NUMERATOR

$$\int_0^{\infty} x^2 e^{-\frac{x}{y}}\,dx = \left[(yx^2 + 2xy^2 - 2y^3)\,e^{-\frac{x}{y}}\right]\Big|_{x=0}^{\infty} = 2y^3$$

INTEGRAL IN THE DENOMINATOR

$$\int_0^{\infty} e^{-\frac{x}{y}}\,dx = -ye^{-\frac{x}{y}}\Big|_{x=0}^{\infty} = y$$

$$\boxed{\frac{\int_0^{\infty} x^2 e^{-\frac{x}{y}}\,dx}{\int_0^{\infty} e^{-\frac{x}{y}}\,dx} = 2y^2}$$

$$\Rightarrow \boxed{E[X^2|Y=y] = \begin{cases} 2y^2 & y > 0 \\ \text{undefined} & y \le 0 \end{cases}}$$

$\boxed{9}$ CH 7, PROB 58

- LET X_i BE THE RANDOM VARIABLE REPRESENTING THE i-TH TOSS

$$P(X_i = 1) = p$$
$$P(X_i = 0) = q = 1-p$$

- LET N BE THE RANDOM VARIABLE REPRESENTING THE TOTAL NUMBER OF TOSSES UNTIL BOTH HEADS & TAILS APPEAR

NOTE $P(N=1)=0$
SO $P(N=i)$ IS NONZERO ONLY FOR $i \geq 2$

- FIRST FIND

$$P(N=i \mid X_1=0) = P(X_2 = X_3 = \cdots = X_{i-1} = 0) \cdot P(X_i = 1)$$
$$= P(X_2=0) \cdot P(X_3=0) \cdots P(X_{i-1}=0) \cdot P(X_i=1)$$

$$\boxed{P(N=i \mid X_1=0) = q^{i-2} \cdot p}$$

- SIMILARLY

$$\boxed{P(N=i \mid X_1=1) = p^{i-2} \cdot q}$$

$$\Rightarrow \quad P(N=i) = P(X_1=0) \cdot P(N=i \mid X_1=0) + P(X_1=1) P(N=i \mid X_1=1)$$
$$= q \cdot q^{i-2} \cdot p + p \cdot p^{i-2} \cdot q$$
$$= p \cdot q^{i-1} + q \cdot p^{i-1}$$

$$\Rightarrow \boxed{P(N=i) = \begin{cases} pq^{i-1} + q p^{i-1} & \text{if } i \geq 2 \\ 0 & \text{otherwise} \end{cases}}$$

a) $$E[N] = \sum_{i=2}^{\infty} i \cdot P(N=i)$$

$$= \sum_{i=2}^{\infty} i \cdot \left(p q^{i-1} + q p^{i-1}\right)$$

$$\boxed{E[N] = p \cdot \left[\sum_{i=2}^{\infty} i \cdot q^{i-1}\right] + q \cdot \left[\sum_{i=2}^{\infty} i \cdot p^{i-1}\right]} \quad (\nabla)$$

NOW COMPUTE

$$\sum_{i=2}^{\infty} i \cdot p^{i-1} = 2p + 3p^2 + 4p^3 + \cdots$$

$$= \left(2p + 3p^2 + 4p^3 + \cdots\right) \frac{(1-p)}{(1-p)}$$

$$= \frac{2p + (3-2)p^2 + (4-3)p^3 + (5-4)p^4 + \cdots}{(1-p)}$$

$$= \frac{p + p + p^2 + p^3 + p^4 + \cdots}{1-p}$$

$$= \frac{p + p \cdot \sum_{i=0}^{\infty} p^i}{1-p} = \frac{p + p \cdot \frac{1}{1-p}}{(1-p)} = \frac{p + p \cdot \frac{1}{q}}{q}$$

$$\boxed{\sum_{i=2}^{\infty} i \cdot p^{i-1} = \frac{p}{q} + \frac{p}{q^2}} \quad (*)$$

SIMILARLY

$$\boxed{\sum_{i=2}^{\infty} i\, q^{i-1} = \frac{q}{p} + \frac{q}{p^2}} \quad (**)$$

NOW SUBSTITUTE (*) & (**) INTO (§)

$$E[N] = p\left[\frac{q}{p} + \frac{q}{p^2}\right] + q\left[\frac{p}{q} + \frac{p}{q^2}\right]$$

$$= q + \frac{q}{p} + p + \frac{p}{q}$$

$$= (q+p) + \frac{q}{p} + \frac{p}{q}$$

$$= 1 + \frac{q}{p} + \frac{p}{q}$$

$$\boxed{E[N] = 1 + \frac{1-p}{p} + \frac{p}{1-p}} \quad \text{if } 0 < p < 1$$

NOTE: if $p=1$ OR $p=0$ then $E[N]=\infty$

b) $P(\text{LAST FLIP LANDS ON HEADS}) = P(\text{FIRST FLIP LANDS ON TAILS})$

$$= P(X_1 = 0)$$

$$= q$$

$$= 1-p \quad \text{if } 0 < p < 1$$

Note:

If $p=0$ then $P(\text{Last flip lands on heads}) = 0$

if $p=1$ then $P(\text{Last flip lands on heads}) = 1$

2

$$f_{X_i}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{X_i}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

a) $F_Y(y) = P(Y \le y)$

$$= P(\min(X_1, X_2, \dots, X_n) \le y)$$

$$= 1 - P(\min(X_1, X_2, \dots, X_n) \ge y)$$

$$= 1 - P(X_1 \ge y) \cdot P(X_2 \ge y) \cdots P(X_n \ge y)$$

$$= 1 - \prod_{i=1}^{n} \left[1 - P(X_i \le y)\right]$$

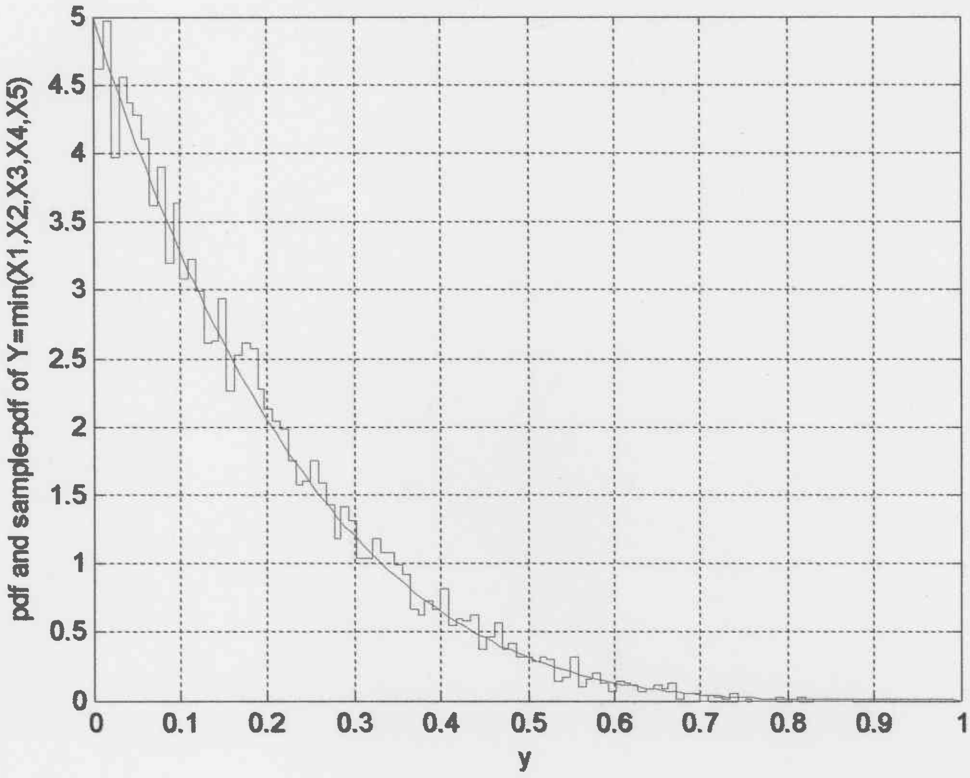
$$\boxed{F_Y(y) = 1 - \left[1 - F_{X_i}(y)\right]^n} \quad \leftarrow \text{for } 0 \le y \le 1$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{\partial \{1 - [1-y]^n\}}{\partial y} = -n \cdot (1-y)^{n-1} \cdot (-1)$$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} n \cdot (1-y)^{n-1} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

MATLAB CONFIRMATION FOR $n=5$

```
% create plots of the pdf and the sample pdf of the random varaible
%           Y=min(X1,X2,X3,X4,X5)
%

n=5;

k=10^4;

X=rand(5,k);
Y=min(X); % creates k random numbers Y=min(X1,X2,X4,X4,X5)

y=0:.01:1;
f_Y=n*(1-y).^(n-1);    % creates the computed pdf

[A,B]=hist(Y,100);
dB=B(2)-B(1);
stairs(B,A/k/dB);  % plots the sample pdf
hold on
plot(y,f_Y)        % plots the computed pdf
hold off

grid

xlabel('y')
ylabel('pdf and sample-pdf of Y=min(X1,X2,X3,X4,X5)')
```



b) $F_Z(z) = P(Z \le z)$

$$= P(\max(X_1, X_2, \dots, X_n) \le z)$$

$$= P(X_1 \le z) \cdot P(X_2 \le z) \cdots P(X_n \le z)$$

$$\boxed{F_Z(z) = F_{X_1}(z)^n} \qquad \text{for } 0 \le z \le 1$$

$$f_Z(z) = \frac{dF_{X_1}(z)}{dz}$$

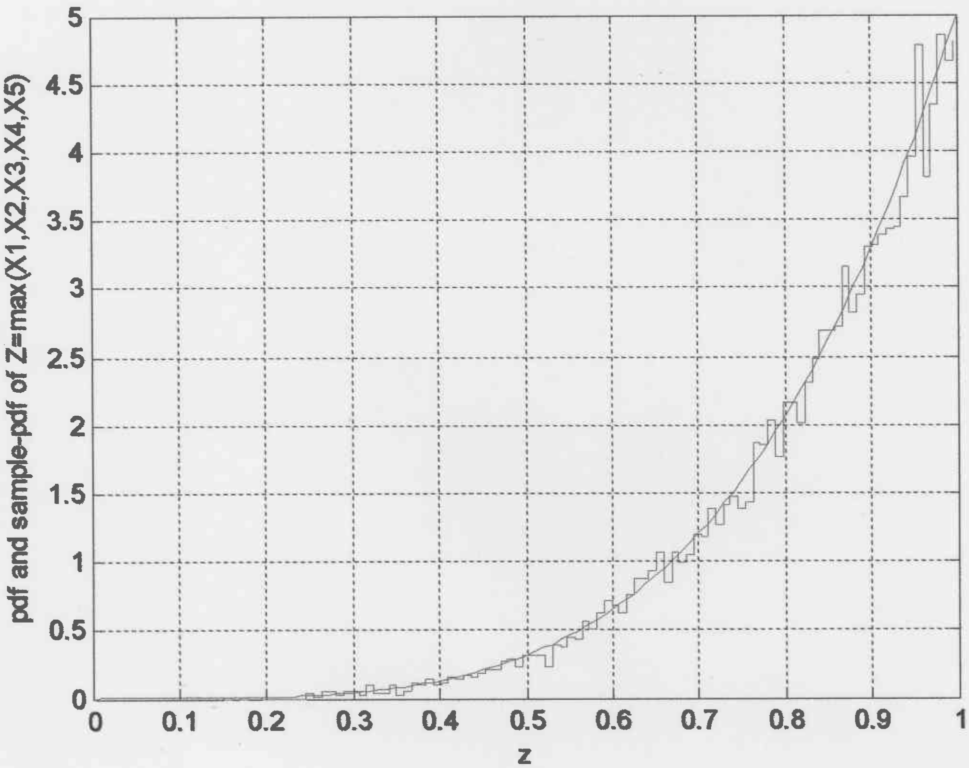
$$= n \cdot \left[F_{X_1}(z)\right]^{n-1} \cdot f_{X_1}(z)$$

$$= n \cdot z^{n-1} \cdot 1$$

$$\boxed{f_Z(z) = \begin{cases} n z^{n-1} & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

MATLAB CONFIRMATION FOR n=5

```
% create plots of the pdf and the sample pdf of the random variable
%            Z=max(X1,X2,X3,X4,X5)
%

n=5;

k=10^4;

X=rand(5,k);
Z=max(X); % creates k random numbers Z=max(X1,X2,X4,X4,X5)

z=0:.01:1;
f_Z=n*z.^(n-1);    % creates the computed pdf

[A,B]=hist(Z,100);
dB=B(2)-B(1);
stairs(B,A/k/dB);  % plots the sample pdf
hold on
plot(z,f_Z)         % plots the computed pdf
hold off

grid

xlabel('z')
ylabel('pdf and sample-pdf of Z=max(X1,X2,X3,X4,X5)')
```



c) ASSUME $0 \le y \le z \le 1$ AND FIND

$$F_{Y,Z}(y,z) = P(Y \le y,\ Z \le z)$$

$$= P(Z \le z) - P(Y > y,\ Z \le z)$$

$$= P(Z \le z) - P(\min(X_1, X_2, \ldots, X_n) > y,\ \max(X_1, X_2, \ldots, X_n) \le z)$$

$$= P(Z \le z) - P(y < X_1 \le z) \cdot P(y < X_2 \le z) \cdots P(y < X_n \le z)$$

$$\boxed{F_{Y,Z}(y,z) = P(Z \le z) - (z-y)^n} \quad \text{for } 0 \le y \le z \le 1$$

$$f_{Y,Z}(y,z) = \frac{\partial^2 F_{Y,Z}(y,z)}{\partial y\, \partial z}$$

$$= n \cdot (n-1) \cdot (z-y)^{n-2} \quad \leftarrow \text{for } 0 \le y \le z \le 1$$

$$\boxed{f_{Y,Z}(y,z) = \begin{cases} n(n-1)(z-y)^{n-2} & \text{if } 0 \le y \le z \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

d) Y & Z ARE <u>NOT INDEPENDENT</u> BECAUSE

$$f_{Y,Z}(y,z) \ne f_Y(y) \cdot f_Z(z)$$

e) $$f_{Y|Z}(y|z) = \frac{f_{Y,Z}(y,z)}{f_Z(z)}$$ ← defined for $0 \le y \le z \le 1$

$$= \frac{n\cdot(n-1)\cdot(z-y)^{n-2}}{n\cdot z^{n-1}}$$

$$\Rightarrow \boxed{f_{Y|Z}(y|z) = \begin{cases} \dfrac{(n-1)(z-y)^{n-2}}{z^{n-1}} & 0 \le y \le z \\ 0 & \text{otherwise} \end{cases}}$$

defined for $0 < z \le 1$

$$E[Y|Z=z] = \int_{-\infty}^{\infty} y\, f_{Y|Z}(y|z)\, dy$$

$$= \int_0^z y \cdot \frac{(n-1)(z-y)^{n-2}}{z^{n-1}}\, dy$$

$$= \int_0^z \left(\frac{y}{z}\right)\cdot(n-1)\cdot\left(1-\frac{y}{z}\right)^{n-2} dy$$

sub: $1 - \frac{y}{z} = u$

$-\frac{dy}{z} = du$

$$= (n-1)\cdot z \cdot \int_0^1 (1-u)\cdot u^{n-2}\, du$$

$$= (n-1)z \cdot \left(\frac{u^{n-1}}{n-1} - \frac{u^n}{n}\right)\Big|_0^1$$

$$= (n-1)\cdot z \cdot \left[\frac{1}{n-1} - \frac{1}{n}\right] = \frac{z}{n}$$

$$\boxed{E[Y|Z=z] = \frac{z}{n}} \quad \Rightarrow \quad \boxed{E[Y|Z] = \frac{Z}{n}}$$

f) SIMILAR TO e), WE GET

$$\boxed{f_{Z|Y}(z|y) = \begin{cases} \dfrac{(n-1)(z-y)^{n-2}}{(1-y)^{n-1}} & y \le z \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

defined for $0 \le y < 1$

$$E[Z|Y=y] = \int_{-\infty}^{\infty} z\, f_{Z|Y}(z|y)\, dz$$

$$= \int_y^1 z \cdot \frac{(n-1)(z-y)^{n-2}}{(1-y)^{n-1}}\, dz$$

$$= (n-1)\int_y^1 \frac{z}{(1-y)} \cdot \left[\frac{z-y}{1-y}\right]^{n-1} dz$$

sub: $u = \dfrac{z-y}{1-y}$

$$= (n-1)\cdot(1-y)\cdot \int_0^1 \left(u + \frac{y}{1-y}\right) \cdot u^{n-2}\, du$$

$$= (n-1)(1-y)\left[\frac{u^n}{n} + \frac{y}{(1-y)}\cdot\frac{u^{n-1}}{(n-1)}\right]\Bigg|_0^1$$

$$= (n-1)(1-y)\cdot\left[\frac{1}{n} + \frac{y}{(1-y)}\cdot\frac{1}{(n-1)}\right]$$

$$\boxed{E[Z|Y=y] = 1 - \frac{1-y}{n}} \implies \boxed{E[Z|Y] = 1 - \frac{1-Y}{n}}$$

g) $$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y \cdot n \cdot (1-y)^{n-1}\,dy$$ sub $1-y=u$

$$= n \cdot \int_0^1 (1-u)u^{n-1}\,du = n \cdot \left[\frac{u^n}{n} - \frac{u^{n+1}}{n+1}\right]\Big|_0^1$$

$$= n\left[\frac{1}{n} - \frac{1}{n+1}\right] = n \cdot \frac{1}{n \cdot (n+1)}$$

$$\boxed{E[Y] = \frac{1}{n+1}}$$

FOR $n=5$, MATLAB GAVE SAMPLE MEAN $0.1655 \approx \frac{1}{6} = \frac{1}{n+1}$

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)\,dz = \int_0^1 z \cdot n \cdot z^{n-1}\,dz$$

$$= n \cdot \frac{z^{n+1}}{(n+1)}\Big|_0^1$$

$$\boxed{E[Z] = \frac{n}{n+1}}$$

For $n=5$, MATLAB GAVE SAMPLE MEAN $0.8343 \approx \frac{5}{6} = \frac{n}{n+1}$

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z)\,dz = \int_0^1 z^2 \cdot n\, z^{n-1}\,dz$$

$$= n \cdot \frac{z^{n+2}}{(n+2)} = \frac{n}{n+2}$$

$$\boxed{E[Z^2] = \frac{n}{n+2}} \Rightarrow \operatorname{Var}(Z) = E[Z^2] - (E[Z])^2$$

$$= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$$

$$\boxed{\operatorname{Var}(Z) = \frac{n}{(n+2)(n+1)^2}}$$

for $n=5$, Matlab gave sample variance $0.0203 \approx \frac{5}{252} = \frac{n}{(n+2)(n+1)^2}$

BY SYMMETRY, WE EXPECT $Var(Y) = Var(Z) = \frac{n}{(n+1)^2(n+2)}$

Let's PROVE IT:

$$E[Y^2] = \int_0^1 y^2 n \cdot (1-y)^{n-1} \qquad \text{sub: } 1-y = u$$

$$= n \int_0^1 (1-u)^2 u^{n-1} du$$

$$= n \int_0^1 (1-2u+u^2) u^{n-1}$$

$$= n \left(\frac{u^n}{n} - \frac{2u^{n+1}}{n+1} + \frac{u^{n+2}}{n+2} \right) \Big|_0^1$$

$$= n \cdot \left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \right] = 1 - \frac{2n}{n+1} + \frac{n}{n+2}$$

$$\boxed{E[Y^2] = \frac{2}{(n+1)(n+2)}}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{2}{(n+1)(n+2)} - \frac{1}{(n+1)^2}$$

$$= \frac{2(n+1) - (n+2)}{(n+1)^2(n+2)}$$

$$\boxed{Var(Y) = \frac{n}{(n+1)^2(n+2)}}$$

for $n=5$, Matlab gave sample variance $0.0200 \approx \frac{5}{252} = \frac{n}{(n+1)^2(n+2)}$

$$E[YZ] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yz\, f_{Y,Z}(y,z)\, dy\, dz$$

$$= \int_0^1 \int_0^z yz \cdot n(n-1)\cdot(z-y)^{n-2}\, dy\, dz$$

$$= n(n-1) \int_0^1 z \left[\int_0^z y(z-y)^{n-2}\, dy\right] dz \qquad \text{sub: } z-y=u$$

$$= n(n-1) \int_0^1 z \left[\int_0^z (z-u)\, u^{n-2}\, du\right] dz$$

$$= n(n-1) \int_0^1 z \left[\frac{z u^{n-1}}{n-1} - \frac{u^n}{n}\right]\Bigg|_0^{z} dz$$

$$= n(n-1) \int_0^1 z \left[\frac{z^n}{n-1} - \frac{z^n}{n}\right] dz$$

$$= \int_0^1 z^{n+1}\, dz$$

$$= \frac{z^{n+2}}{n+2}\Bigg|_0^1$$

$$\boxed{E[YZ] = \frac{1}{n+2}} \Rightarrow Cov(Y,Z) = E[YZ] - E[Y]\cdot E[Z] = \frac{1}{n+2} - \frac{1}{n+1}\cdot\frac{n}{n+1}$$

$$\Rightarrow \boxed{Cov(Y,Z) = \frac{1}{(n+2)(n+1)^2}}$$

Verify by Matlab for $n=5$

```
>> X = rand(5, 10000);
>> Y = min(X);
>> Z = max(X);
>> C = sqrt(var(Z)*var(Y)) * corrcoef(Y,Z);
```

The entry C(1,2) should equal (roughly) $\frac{1}{(n+2)(n+1)^2}$

Try it!