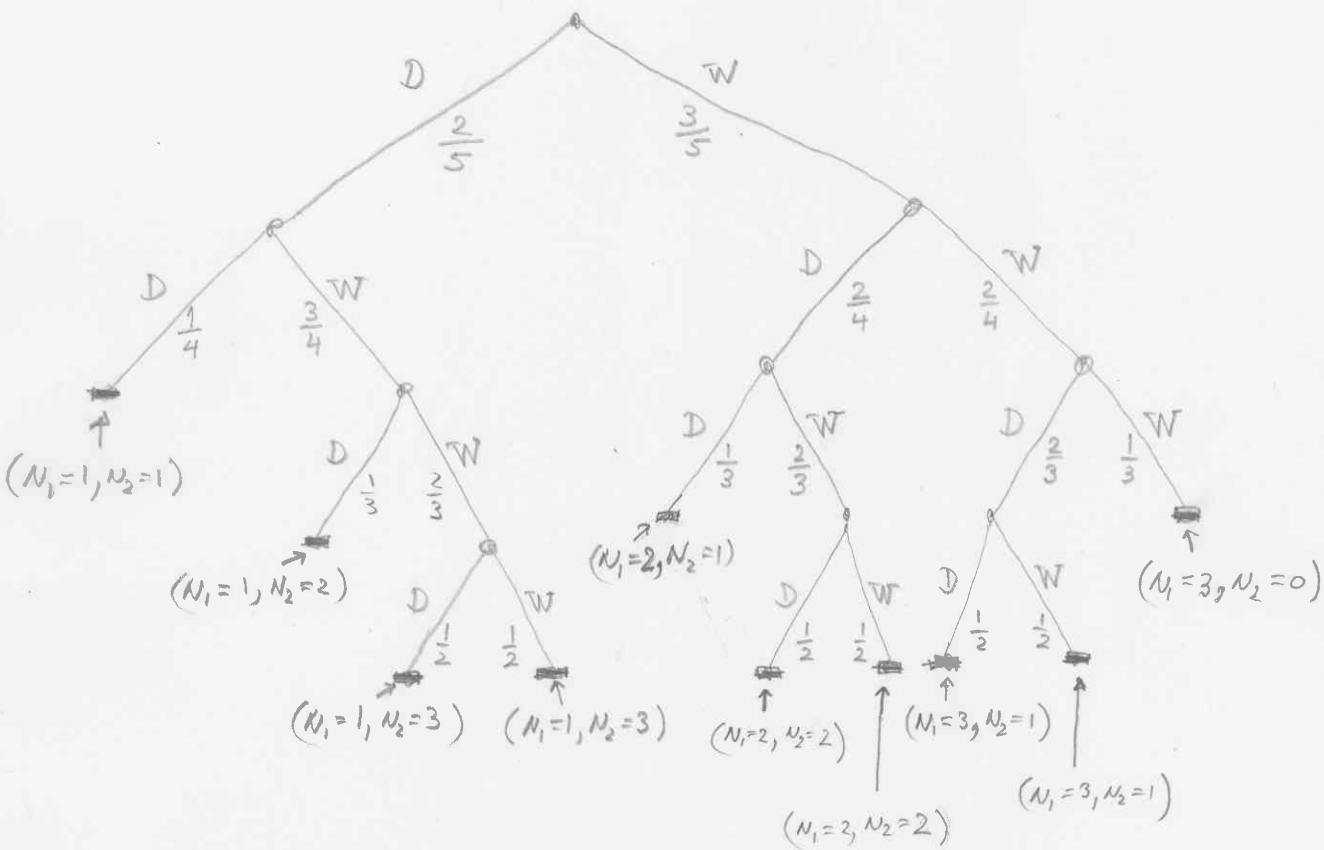
EE 342 HW7
SOLUTIONS

1) $\boxed{a}$ CH 6, PROB 6

LET W DENOTE "DRAWING A WORKING TRANSISTOR"
LET D DENOTE "DRAWING A DEFECTIVE TRANSISTOR"

MAKE A PROBABILITY TREE



$$p_{N_1,N_2}(1,1) = \frac{2}{5}\cdot\frac{1}{4} = 0.1$$

$$p_{N_1,N_2}(1,2) = \frac{2}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} = 0.1$$

$$p_{N_1,N_2}(1,3) = \frac{2}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} = 0.2$$

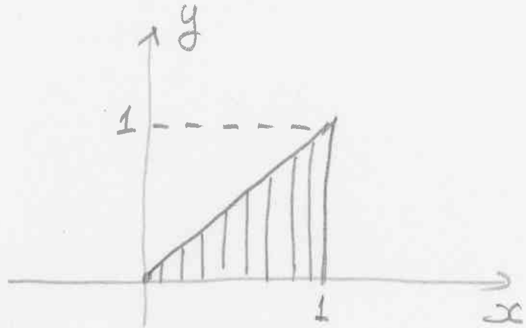
$$p_{N_1,N_2}(2,1) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} = 0.1$$

$$p_{N_1,N_2}(2,2) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} = 0.2$$

$$p_{N_1,N_2}(3,1) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} = 0.2$$

$$p_{N_1,N_2}(3,0) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} = 0.1$$

$\boxed{b}$ CH 6, PROB. 19



FIRST SHOW

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 \int_0^x \frac{1}{x}\,dy\,dx = \int_0^1 \frac{1}{x}\left[\int_0^x dy\right]dx = \int_0^1 \frac{1}{x}\cdot x\,dx = 1$$

(a) $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_y^1 \frac{1}{x}\,dx = \ln x\Big|_y^1 = \begin{cases} \ln \frac{1}{y} & 0<y<1 \\ 0 & \text{othewise} \end{cases}$$

(b) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^x \frac{1}{x}\,dy = \frac{1}{x}\int_0^x dy = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $$E[X] = \int_0^1 x\cdot f_X(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

(d) $$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^1\int_0^x y\cdot f_{X,Y}(x,y)\,dy\,dx = \int_0^1\int_0^x y\cdot\frac{1}{x}\,dy\,dx$$

$$= \int_0^1 \frac{1}{x}\left[\int_0^x y\,dy\right]dx = \int_0^1 \frac{1}{x}\cdot\frac{x^2}{2}\cdot dx = \frac{1}{2}\int_0^1 x\,dx$$

$$= \frac{1}{4}$$

$\boxed{C}$ CH 6, PROB 20

1°)

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

if $x>0$

$$f_X(x) = \int_0^\infty x e^{-(x+y)}\,dy = xe^{-x}\int_0^\infty e^{-y}\,dy = xe^{-x}\left[-e^{-y}\Big|_0^\infty\right] = xe^{-x}$$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} xe^{-x} & x>0 \\ 0 & \text{otherwise} \end{cases}}$$

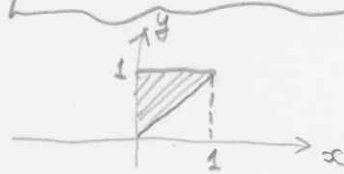
if $y>0$

$$f_Y(y) = \int_0^\infty x e^{-(x+y)}\,dx = e^{-y}\int_0^\infty x e^{-x}\,dx = e^{-y}\left[(-xe^{-x}-e^{-x})\Big|_0^\infty\right] = e^{-y}$$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} e^{-y} & y>0 \\ 0 & \text{otherwise} \end{cases}}$$

Clearly $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$ } So X & Y are <u>independent</u>

2°) $$f_{X,Y}(x,y) = \begin{cases} 2 & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$



for $0<x<1$

$$f_X(x) = \int_x^1 2\,dy = 2\cdot y\Big|_x^1 = 2(1-x)$$

for $0<y<1$

$$f_Y(y) = \int_0^y 2\,dx = 2y$$

$$f_X(x)\cdot f_Y(y) = 2(1-x)\cdot 2y \neq 2 = f_{X,Y}(x,y)$$

$\Rightarrow$ X, Y ARE NOT INDEPENDENT

$\boxed{d}$ CH 6, PROB. 23

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x) & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) $f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 12xy(1-x)\,dy$ for $0<x<1$

$= 12x(1-x)\int_0^1 y\,dy$

$= 12x(1-x)\cdot \frac{1}{2}$

$= 6x(1-x)$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} 6x(1-x) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}}$$

$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = \int_0^1 12xy(1-x)\,dx$ for $0<y<1$

$= 12y\int_0^1 x(1-x)\,dx = 12y\left[\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1\right]$

$= 12\cdot y\left[\frac{1}{2} - \frac{1}{3}\right]$

$= 2y$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 2 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}}$$

Clearly $f_X(x)f_Y(y) = 6x(1-x)\cdot 2y = 12xy(1-x) = f_{X,Y}(x,y)$

$\boxed{\text{So } X \,\&\, Y \text{ are independent}}$

(b) $$E[X] = \int_0^1 x \cdot f_X(x)\,dx = \int_0^1 x \cdot 6x(1-x)\,dx$$

$$= \left(6 \cdot \frac{x^3}{3} - 6\,\frac{x^4}{4}\right)\Big|_0^1$$

$$= \frac{6}{3} - \frac{6}{4} = 6 \cdot \left[\frac{1}{3} - \frac{1}{4}\right] = 6\left[\frac{4-3}{12}\right] = \boxed{\frac{1}{2} = E[X]}$$

(c) $$E[Y] = \int_0^1 y\, f_Y(y)\,dy = \int_0^1 y \cdot 2y\,dy$$

$$= 2 \cdot \frac{y^3}{3}\Big|_0^1 = \boxed{\frac{2}{3} = E[Y]}$$

(d) $$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2 \cdot 6x(1-x)\,dx$$

$$= \left(6 \cdot \frac{x^4}{4} - 6\,\frac{x^5}{5}\right)\Big|_0^1$$

$$= 6\left[\frac{1}{4} - \frac{1}{5}\right] = 6 \cdot \frac{5-4}{20} = \boxed{\frac{3}{10} = E[X^2]}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4}$$

$$\boxed{Var(X) = \frac{2}{40} = \frac{1}{20}}$$

(e) $$E[Y^2] = \int_0^1 y^2 f_Y(y)\,dy = \int_0^1 y^2\, 2y\, dy$$

$$= 2 \cdot \frac{y^4}{4}\Big|_0^1 = \boxed{\frac{1}{2} = E[Y^2]}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9}$$

$$\boxed{Var(Y) = \frac{1}{18}}$$

$\boxed{e}$ CH 6, TH. EX. 18

PROVE: $$P_{X,Y}(i,j) = \frac{P_{X|Y}(i|j)}{\left[\sum_k \frac{P_{X|Y}(k|j)}{P_{Y|X}(j|k)}\right]}$$

START WITH

$$\sum_k \left[\frac{P_{X|Y}(k|j)}{P_{Y|X}(j|k)}\right] = \sum_k \frac{P_{X|Y}(k|j) \cdot P_Y(j)}{P_{Y|X}(j|k) \cdot P_Y(j)}$$

$$= \frac{1}{P_Y(j)} \cdot \left[\sum_k \frac{P_{X|Y}(k|j) \cdot P_Y(j)}{P_{Y|X}(j|i)}\right]$$

$$= \frac{1}{P_Y(j)} \cdot \left[\sum_k \frac{P_{X,Y}(k,j)}{P_{Y|X}(j|i)}\right] = \frac{1}{P_Y(j)} \underbrace{\left[\sum_k P_X(k)\right]}_{1}$$

So $$\boxed{\sum_k \frac{P_{X|Y}(k|j)}{P_{Y|X}(j|k)} = \frac{1}{P_Y(j)}} \quad (*)$$

THEREFORE

$$\frac{p_{X|Y}(i|j)}{\sum_k \frac{p_{X|Y}(k|j)}{p_{Y|X}(j|k)}} \overset{(*)}{\downarrow} = \frac{p_{X|Y}(i|j)}{\left[\frac{1}{p_Y(j)}\right]} = p_{X|Y}(i|j) \cdot p_Y(j)$$

$$= p_{X,Y}(i,j)$$ proof done

2)

a) $m_X(t) = E\left[e^{tX}\right]$

$$= \int_0^\infty e^{-tx} \cdot f_X(x)\,dx$$

$$= \int_0^\infty e^{-tx} \cdot \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} u(x)\,dx$$

$$= \int_0^\infty \frac{(x - t\sigma^2 + t\sigma^2)}{\sigma^2} e^{+\frac{t^2\sigma^2}{2}} \cdot e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx$$

$$= e^{\frac{t^2\sigma^2}{2}}\left[\int_0^\infty \frac{(x-t\sigma^2)}{\sigma^2} e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx + \int_0^\infty t \cdot e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx\right]$$

$$= e^{+t^2\sigma^2/2}\left[\int_{\frac{t^2\sigma^2}{2}}^\infty e^{-z}\,dz + t \cdot \int_0^\infty e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx\right] \quad (!)$$

*) FIRST SOLVE THE INTEGRAL $\int_{\frac{t^2\sigma^2}{2}}^{\infty} e^{-z}\,dz$

$$\boxed{\int_{\frac{t^2\sigma^2}{2}}^{\infty} e^{-z}\,dz = e^{-\frac{t^2\sigma^2}{2}}} \quad (*)$$

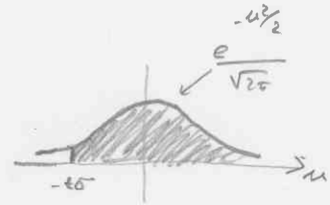
**) NOW SOLVE THE INTEGRAL $\int_{0}^{\infty} e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx$

sub $u = \frac{x - t\sigma^2}{\sigma}$

$$\int_{0}^{\infty} e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx = \int_{-t\sigma}^{\infty} e^{-u^2/2}\cdot\sigma\cdot du$$

$$= \sigma\cdot\int_{-t\sigma}^{\infty} e^{-u^2/2}\,du$$

$$= \sigma\cdot\sqrt{2\pi}\int_{-t\sigma}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$



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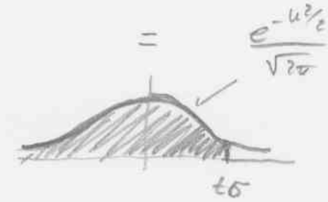


$$= \sigma\cdot\sqrt{2\pi}\int_{-\infty}^{t\sigma} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$\boxed{\int_{0}^{\infty} e^{-\frac{(x-t\sigma^2)^2}{2\sigma^2}}\,dx = \sigma\cdot\sqrt{2\pi}\cdot\Phi(t\sigma)} \quad (**)$$

SUBSTITUTE (*) AND (**) INTO (?) TO GET

$$m_X(t) = e^{+\frac{t^2\sigma^2}{2}}\left[e^{-\frac{t^2\sigma^2}{2}} + t\cdot\sigma\cdot\sqrt{2\pi}\,\Phi(t\sigma)\right]$$

$$\boxed{m_X(t) = 1 + t\cdot\sigma\cdot\sqrt{2\pi}\cdot e^{\frac{t^2\sigma^2}{2}}\cdot\Phi(t\sigma)}$$

b) $m_X = \left.\dfrac{\partial m_X(t)}{\partial t}\right|_{t=0}$

$$= \sigma \cdot \sqrt{2\pi} \cdot \left.\frac{\partial\left[t \cdot e^{t^2\sigma^2/2} \cdot \phi(t\sigma)\right]}{\partial t}\right|_{t=0}$$

$$= \sigma \cdot \sqrt{2\pi}\left[e^{t^2\sigma^2/2} \cdot \phi(t\sigma) \cdot \frac{\partial[t]}{\partial t} + t \cdot \phi(t\sigma)\frac{\partial(e^{t^2\sigma^2/2})}{\partial t} + te^{t^2\sigma^2/2} \cdot \frac{\partial\phi(\sigma t)}{\partial t}\right]_{t=0}$$

$$= \sigma \cdot \sqrt{2\pi}\left[\phi(0) + 0 + 0\right]$$

$$= \sigma \cdot \sqrt{2\pi} \cdot \frac{1}{2} =$$

$$\boxed{m_X = \sigma \cdot \sqrt{\frac{\pi}{2}}}$$

c) $\dfrac{\partial m_X(t)}{\partial t} = \sigma\sqrt{2\pi}\left[e^{\frac{t^2\sigma^2}{2}} \cdot \phi(t\sigma) + t^2\sigma^2 e^{\frac{t^2\sigma^2}{2}}\phi(t\sigma) + te^{t^2\sigma^2/2} \cdot \sigma \cdot \dfrac{e^{-\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}}\right]$

$$= \sigma \cdot \sqrt{2\pi}\, e^{\frac{t^2\sigma^2}{2}}\left[1 + t^2\sigma^2\right]\phi(t\sigma) + t\sigma^2$$

$$\frac{\partial m_X^2(t)}{\partial t^2} = \sigma \cdot \sqrt{2\pi} \cdot \left[t \cdot \sigma^2 e^{\frac{t^2\sigma^2}{2}}(1+t^2\sigma^2)\phi(t\sigma) + e^{t^2\sigma^2/2} \cdot 2t\sigma^2\,\phi(t\sigma) + \right.$$
$$\left. + e^{\frac{t^2\sigma^2}{2}} \cdot \left[1+t^2\sigma^2\right] \cdot \frac{\partial\phi(t\sigma)}{\partial t}\right] + \sigma^2$$

$$\left.\frac{\partial m_X^2(t)}{\partial t^2}\right|_{t=0} = \sigma \cdot \sqrt{2\pi} \cdot \left[0 + 0 + \left.\frac{\partial\phi(t\sigma)}{\partial t}\right|_{t=0}\right] + \sigma^2$$

$$\phi(x) = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du$$

$$\phi'(x) = \frac{\partial \phi(x)}{\partial x} = e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}$$

$$\frac{\partial \phi(t\sigma)}{\partial t} = \frac{\partial \phi(t\sigma)}{\partial (t\sigma)} \cdot \sigma = e^{-\frac{(t\sigma)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sigma$$

$$\boxed{\left.\frac{\partial \phi(t\sigma)}{\partial t}\right|_{t=0} = \frac{\sigma}{\sqrt{2\pi}}}$$

$$\Rightarrow \quad \frac{\partial m_X^2(t)}{\partial t^2} = \sigma \cdot \sqrt{2\pi} \left[\left.\frac{\partial \phi(t\sigma)}{\partial t}\right|_{t=0} \right] + \sigma^2$$

$$= \sigma \cdot \sqrt{2\pi} \left[\frac{1}{\sqrt{2\pi}} \cdot \sigma \right] + \sigma^2$$

$$\boxed{\left.\frac{\partial m_X^2(t)}{\partial t^2}\right|_{t=0} = 2\sigma^2}$$

$$\boxed{E[X^2] = \left.\frac{\partial m_X^2(t)}{\partial t^2}\right|_{t=0} = 2\sigma^2}$$

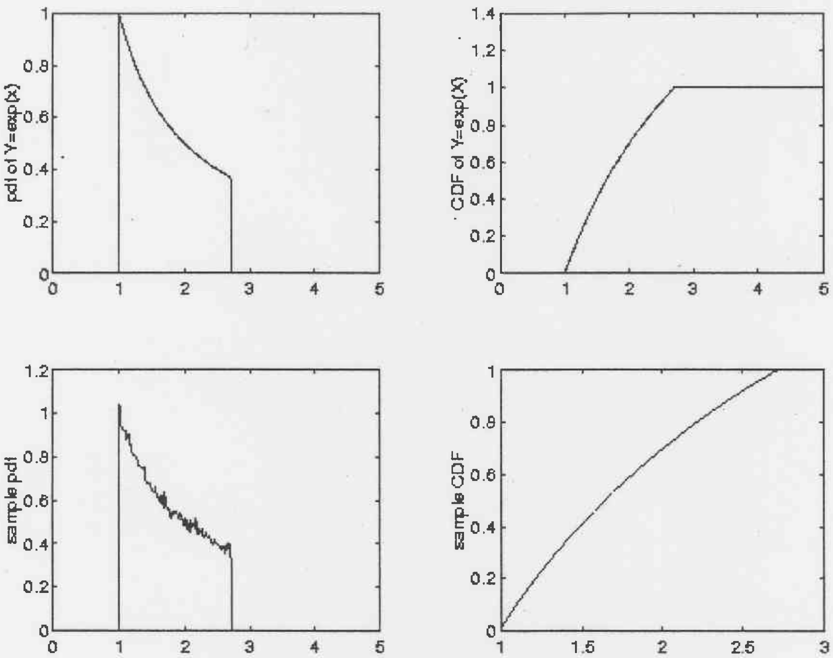
$$Var(X) = E[X^2] - (E[X])^2 = 2\sigma^2 - \left(\sigma \cdot \sqrt{\frac{\pi}{2}}\right)^2$$

$$\boxed{Var(X) = \left(2 - \frac{\pi}{2}\right) \cdot \sigma^2}$$

3) $S_Y = [1,e]$, $F_Y(y) = P(\exp(X) \le y) = P(X \le \log(y)) = \log(y)$ when $1 \le y \le e$.
Then $f_Y(y) = (1/y)\,(u(y-1) - u(y-e))$. $E(Y) = 1-e$, $E(Y^2) = .5\,(e^2 - 1)$, $Var(Y) = .2420$.
Using matlab, to generate pdf `dt=.001; t=dt:dt:5; pdfy=(1<t & t<exp(1))./t;`
`CDF = cumsum(pdfy)*dt;`
To generate random samples `y= exp(rand(1,100000));`
Sample mean is 1.7198 and sample variance is .2427
Sample pdf looks similar to real pdf, but curve is not as smooth. Sample CDF looks very similar to real CDF.

4) $S_Z = [0,1]$, $F_Z(z) = P(\sin(\pi X/2) \le z) = P(X \le (2/\pi)\arcsin(z))$ when $0 \le z \le 1$.
Then $f_Z(z) = (2/\pi)\,(1-z^2)^{-.5}\,(u(z) - u(z-1))$. $E(Z) = 2/\pi$, $E(Z^2) = .5$, $Var(Z) = .0947$.
Matlab simulations similar to part a) except use different function. Sample mean is .6370 and sample variance is .0952.
Sample pdf looks similar to real pdf except near $Z=1$ as here Z goes to ∞. Sample CDF looks very similar to real CDF.

5) $S_W = [0,1]$, $F_W(w) = P((2X-1)2 \le w) = P(.5(1-w^{.5}) \le X \le .5(1+w^{.5}) = w^{.5}$ when $0 \le w \le 1$. The $f_W(w) = (1/2)\,w^{-.5}\,u(w)-u(w-1)$. $E(W) = 1/3$, $E(W^2)=1/5$, $Var(W) = 4/45$.
Matlab simulations similar to part a) except use different function. Sample mean is .3318 and sample variance is .0879. Sample pdf looks similar to real pdf except when $W=0$ as here W goes to ∞. Sample CDF looks very similar to real CDF.


pdf of Y=exp(x)
CDF of Y=exp(X)
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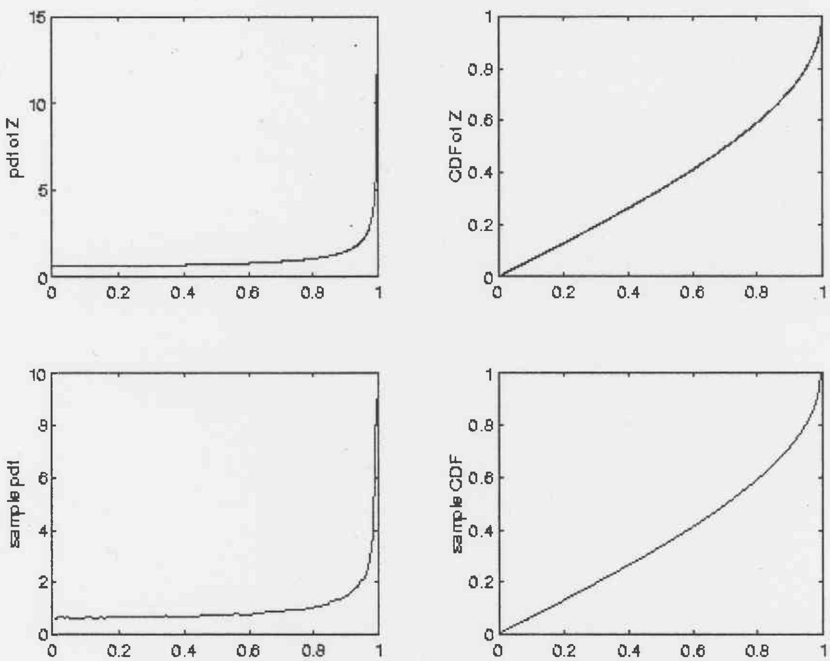
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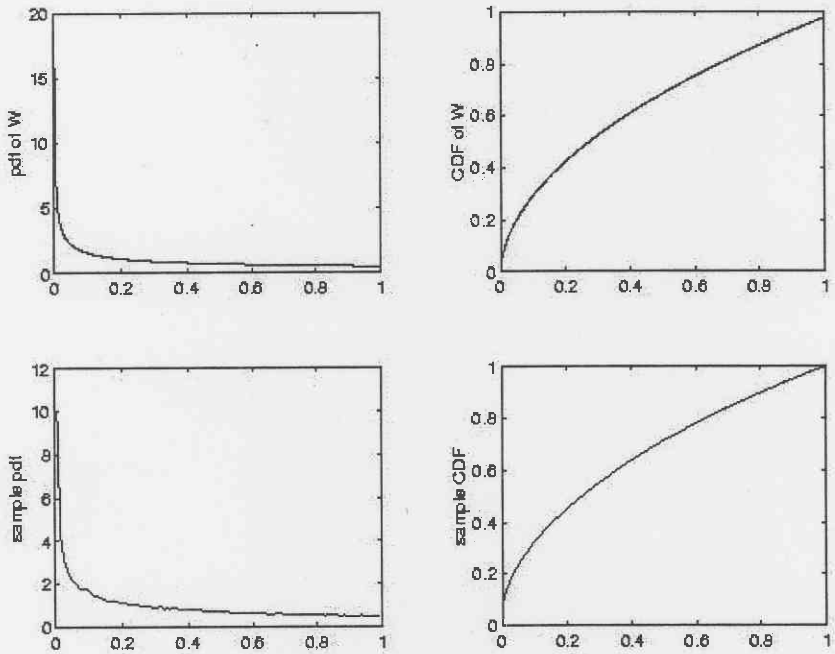
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sample CDF