Upper Bounds on the Capacities of Non-Controllable Finite-State Channels Using Dynamic Programming Methods

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***Abstract*—A non-controllable finite-state channel (FSC) is a finite-state channel in which the user can't control channel states. That is, the channel state of a non-controllable FSC evolves freely according to an uncontrollable probability law. Thus far, good upper bounds on capacities of general non-controllable FSCs remain unknown. Here we develop upper bounds that use delayed feedback and delayed state information, and propose dynamic programming methods to numerically evaluate the bounds.**

I. Introduction

In recent years, several methods to numerically evaluate information rates and bounds on capacities of various finite-state channels (FSCs) were proposed. In particular, [1–4] presented Monte Carlo methods for evaluating information rates of FSCs whose inputs are i.i.d (independent and identically distributed) as well as Markov random processes. A generalization of the well-known Blahut-Arimoto algorithm [5, 6] was proposed to numerically optimize (maximize) information rates for a given Markov processes order. These methods, coupled with recent proofs [7, 8] that Markov processes asymptotically achieve capacities of FSCs, can be utilized to very closely lower bound the capacities. Known upper bounds include the bound in [9], and the feedback capacity [10], which are both relatively loose. The upper bounds can be further numerically tightened by computing *delayed* feedback capacities [10]. However, the delayed feedback capacity bound in [10] is applicable only to controllable FSCs, e.g., ISI channels (see [10]). No similar tool has thus far been proposed for non-controllable FSCs.

In this paper, we develop a sequence of tightening bounds on the capacities of *indecomposable* [11] *non-controllable* FSCs[1]. Non-controllable FSCs [6] are finite-state machine channels in which the user cannot control the channel state. We develop upper bounds on the capacities of non-controllable FSCs using *delayed* feedback and *delayed* channel state information. Through three theorems, we show that the upper bounds can be achieved by finite-order conditional Markov sources, conditioned on *delayed* feedback and *delayed* channel state information. Similar to [10], we formulate the computations of the upper bounds as average reward per stage stochastic control problems [14] and propose dynamic programming methods to numerically evaluate the bounds.

[1]Note that for some special non-controllable FSCs such as the Gilbert-Elliott channel [12] and certain Gilbert-Elliott-like channels [13], the capacity-achieving distributions are known, and the capacities can be evaluated using the tools in [1–4]. For general non-controllable FSCs, however, closely bounding the capacities seems to be the only practical approach.

Structure: The channel model and the channel capacity expression are given in the next section. Upper bounds on the capacities of non-controllable FSCs are developed in Section III. In Section IV, we introduce dynamic programming methods to optimize the channel inputs and thus compute the bounds. Numerical results are presented in Section V, followed by the conclusion in Section VI.

Notation: The t-th member of a sequence of random variables $\{X_1, X_2, \ldots, X_N\}$ is denoted by X_t and its realization is denoted by x_t. A vector of random variables $[X_i, X_{i+1}, \ldots, X_j]$ is shortly denoted by X_i^j and its realization is denoted by x_i^j. By default, we set $X^j \triangleq X_1^j$ and $x^j \triangleq x_1^j$. The expectation $\mathbf{E}[g(X)]$, is also denoted by $\mathbf{E}_X[g(X=x)]$. This is useful notation to denote expectations of information rates, e.g., $\mathbf{E}_X[\,I(Y;Z|X=x)] \triangleq \sum_x [\Pr(x)\, I(Y;Z|X=x)]$.

II. Channel Model and Capacity

Let $t \in \mathbb{Z}$ denote discrete time. Let S_t denote the state of the non-controllable FSC at time t, and its realization be s_t, which is drawn from a finite alphabet $\mathcal{S}$, i.e., $s_t \in \mathcal{S}$ and $|\mathcal{S}| < \infty$. Variables X_t and Y_t denote the channel input and output at time t, whose realizations are x_t and y_t, respectively. The input alphabet $\mathcal{X}$ is finite, i.e., $x_t \in \mathcal{X}$ and $|\mathcal{X}| < \infty$, and the output alphabet $\mathcal{Y}$ could be finite or continuous. The non-controllable FSC satisfies the following probability assumption

$$\Pr\big(y_t, s_t \big| x^t, s^{t-1}, y^{t-1}\big) = \Pr(y_t|x_t, s_t)\Pr(s_t|s_{t-1})\,. \tag{1}$$

Example (The RLL$(1,\infty)$-GE Channel): The channel input is a binary run-length-limited (RLL) sequence satisfying the RLL$(1,\infty)$ constraint, i.e., there are no consecutive ones in the sequence. The channel is a Gilbert-Elliott (GE) channel (see Fig. 1) with two states, a "good" state and a "bad" state. Denote the state alphabet by $\mathcal{S} \triangleq \{g,b\}$. Transition probabilities between channel states are $p(b|g) \triangleq \Pr(S_t = b|S_{t-1} = g)$ and $p(g|b) \triangleq \Pr(S_t = g|S_{t-1} = b)$. When the channel state is $S_t = g$, the channel acts as a binary symmetric channel (BSC) with

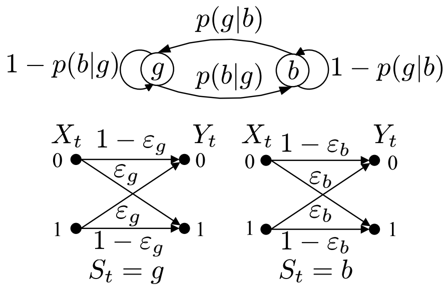

Fig. 1. The Gilbert-Elliott channel.

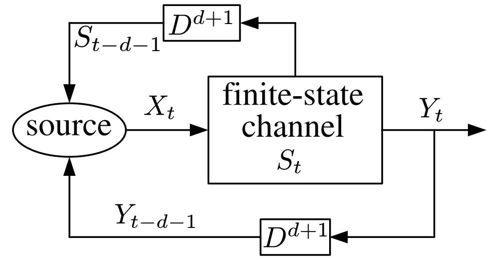

Fig. 2. A non-controllable FSC model with d-delayed FB and d-delayed SI.

cross-over probability ε_g. When the channel is in the bad state $S_t = b$, the channel is a BSC with cross-over probability ε_b. ❑

In [15], Massey introduced the directed information between the channel input and output as

$$I\left(X^N \to Y^N\right) \triangleq \sum_{t=1}^{N} I\left(X^t; Y_t \middle| Y^{t-1}\right)$$

and showed that the mutual information equals the directed information if the channel is used without feedback. For simplicity, we denote $\mathcal{I}(X \to Y)$ as the directed information *rate*[2] between the channel input and output hereafter, that is,

$$\mathcal{I}(X \to Y) \triangleq \lim_{N\to\infty} \frac{1}{N} I\left(X^N \to Y^N\right). \tag{2}$$

The capacity of an indecomposable FSC can be expressed as

$$C = \sup_{\{\Pr(x_t|x^{t-1})\}} \mathcal{I}(X \to Y)$$

where $\left\{\Pr\left(x_t \middle| x^{t-1}\right)\right\}$ indicates that the supremum is taken over all stationary channel input processes.

III. Upper Bound on the Capacity

Definition 1: Let the following be available at time t at the source (see Fig. 2): (1) the channel output sequence Y^{t-d-1}, referred to as the ***d*-delayed feedback**, and (2) the channel state sequence S^{t-d-1}, referred to as the ***d*-delayed state information**. Let the channel input X_t depend on the past channel inputs X^{t-1}, on the d-delayed feedback Y^{t-d-1}, and on the d-delayed state information S^{t-d-1} according to the conditional probability law $\Pr\left(x_t \middle| x^{t-1}, s^{t-d-1}, y^{t-d-1}\right)$. We shall denote by $\mathcal{P}(d)$ the set of all possible input processes $\{X_t\}$ satisfying such a law, i.e., $\mathcal{P}(d) \triangleq \left\{\Pr\left(x_t \middle| x^{t-1}, s^{t-d-1}, y^{t-d-1}\right)\right\}$. Equivalently, $\mathcal{P}(d)$ represents the set of all stationary sources (channel inputs) used with d-delayed feedback (FB) and d-delayed state information (SI). ❑

Definition 2: Define $\mathcal{I}_d(X,S \to Y)$ as the information rate[3]

$$\mathcal{I}_d(X, S \to Y) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} I\left(X^t, S^{t-d}; Y_t \middle| Y^{t-1}\right). \tag{3}$$

❑

[2] We note that the limits in (2) may not exist for all channels and all sources. In that case, the "lim inf" should be used instead of "lim". However, in this paper, we consider only indecomposable channels and stationary channel inputs, in which the limits in (2) do exist.

[3] In Section IV, we will see that maximizing $\mathcal{I}_d(X,S \to Y)$ is considered to be a stochastic control problem, which implies that the limits in (3) do exist for those distributions that maximize $\mathcal{I}_d(X,S \to Y)$, see [14], [16].

Definition 3: Define $\mathcal{I}^*_{FB,SI}(d)$ as the supremum

$$\mathcal{I}^*_{FB,SI}(d) \triangleq \sup_{\mathcal{P}(d)} \mathcal{I}_d(X, S \to Y).$$

❑

Obviously, $\mathcal{I}^*_{FB,SI}(d)$ is an upper bound on the capacity of non-controllable FSCs, i.e., for any non-negative integer d,

$$C \le \mathcal{I}^*_{FB,SI}(d). \tag{4}$$

Moreover, we have a nested sequence of upper bounds

$$C \le \cdots \le \mathcal{I}^*_{FB,SI}(d) \le \cdots \le \mathcal{I}^*_{FB,SI}(1) \le \mathcal{I}^*_{FB,SI}(0).$$

We next introduce theorems that will facilitate the computation of the bound in (4).

Theorem 1: $\mathcal{I}_d(X, S \to Y)$ in (3) can be simplified as

$$\mathcal{I}_d(X,S \to Y) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} I\left(X^t_{t-d+1}, S_{t-d}; Y_t \middle| Y^{t-1}\right). \tag{5}$$

❑

Proof: This is a result of the channel assumption in (1) that implies $\Pr\left(y_t \middle| x^t, s^{t-d}, y^{t-1}\right) = \Pr\left(y_t \middle| x^t_{t-d+1}, s_{t-d}, y^{t-1}\right)$. ■

Definition 4: A source sequence $\{X_t\}$ used with d-delayed FB and d-delayed SI is said to be **an *m*-th order conditional Markov source**, conditioned on the d-delayed FB and d-delayed SI, if the conditional probability function satisfies

$$\Pr\left(x_t \middle| x^{t-1}, s^{t-d-1}, y^{t-d-1}\right) = \Pr\left(x_t \middle| x^{t-1}_{t-m}, s_{t-d-1}, y^{t-d-1}\right).$$

Denote by $\mathcal{P}_m(d)$ the set of all such sources parameterized by the probability law $\Pr\left(x_t \middle| x^{t-1}_{t-m}, s_{t-d-1}, y^{t-d-1}\right)$; that is,

$$\mathcal{P}_m(d) \triangleq \left\{\Pr\left(x_t \middle| x^{t-1}_{t-m}, s_{t-d-1}, y^{t-d-1}\right)\right\}.$$

❑

Clearly, $\mathcal{P}_0(d) \subset \mathcal{P}_1(d) \subset \cdots \subset \mathcal{P}_m(d) \subset \cdots \subset \mathcal{P}(d)$.

Theorem 2: For any integer $d \ge 0$, the upper bound $\mathcal{I}^*_{FB,SI}(d)$ is achieved by a d-th order conditional Markov source, conditioned on d-delayed FB and d-delayed SI, i.e.,

$$\mathcal{I}^*_{FB,SI}(d) = \sup_{\mathcal{P}_d(d)} \mathcal{I}_d(X, S \to Y).$$

❑

Proof: See Appendix A. ■

By Theorem 2, the upper bound $\mathcal{I}^*_{FB,SI}(d)$ can be achieved by finite d-th order Makov sources, which reduce the complexity of computations. Although [7, 8] proved that Makov processes asymptotically achieve capacities of FSCs, finite order Makov sources are considered in practice, which induce lower bounds only.

Definition 5: Denote $\alpha_t(\ell) \triangleq \Pr\left(\left(X^t_{t-d+1}, S_{t-d}\right) = \ell \middle| y^{t-d}\right)$, where $\ell \in \{0, 1, \ldots, M-1\}$ for $M \triangleq |\mathcal{X}|^d \times |\mathcal{S}|$. Let $\underline{A}_t$

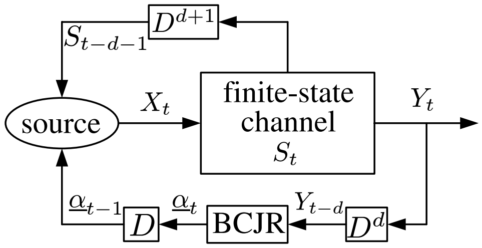

Fig. 3. A non-controllable FSC whose source is in the set $\mathcal{P}'_m(d)$.

be a random vector of causal posterior probabilities, whose realization is $\underline{\alpha}_t{=}[\alpha_t(0),\alpha_t(1),\ldots,\alpha_t(M{-}1)]$. ❑

We can use the forward recursion of the BCJR algorithm [17] to compute all values of $\alpha_t(\ell)$, if $\underline{\alpha}_{t-1}$, y_{t-d} and transition probabilities $\{\Pr(x_t|x_{t-d}^{t-1},s_{t-d-1},y^{t-d-1})\}$ are given. We shortly denote this forward recursion as

$$\underline{\alpha}_t = F_{BCJR}\big(\underline{\alpha}_{t-1},\{\Pr(x_t|x_{t-d}^{t-1},s_{t-d-1},y^{t-d-1})\},y_{t-d}\big). \quad (6)$$

Definition 6: Let $\mathcal{P}'_m(d)$ denote the set of all m-th order conditional Markov sources, conditioned on d-delayed FB and d-delayed SI, for which the following equation holds

$$\Pr\big(x_t|x_{t-m}^{t-1},s_{t-d-1},y^{t-d-1}\big) = \Pr\big(x_t|x_{t-m}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big).$$

Shortly, we denote the set $\mathcal{P}'_m(d)$ as

$$\mathcal{P}'_m(d) \triangleq \left\{\Pr\big(x_t|x_{t-m}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\right\}. \qquad ❑$$

Clearly, $\mathcal{P}'_m(d)\subset\mathcal{P}_m(d)$. Fig. 3 depicts a non-controllable FSC model, whose source belongs to the set $\mathcal{P}'_m(d)$.

Theorem 3: If a d-th order conditional Markov source achieves the upper bound $\mathcal{I}^*_{FB,SI}(d)$, then the conditional probabilities of the source $\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},y_{t-d-1}\big)$ satisfy

$$\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},y_{t-d-1}\big) = \Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)$$

for all t. In other words, the upper bound $\mathcal{I}^*_{FB,SI}(d)$ can be taken over the set of sources $\mathcal{P}'_d(d)$, i.e,

$$\mathcal{I}^*_{FB,SI}(d) = \sup_{\mathcal{P}'_d(d)} \mathcal{I}_d(X,S\to Y). \quad (7)$$

❑

Proof: See Appendix B. ■

IV. Source Optimization

A. Stochastic Control Formulation

As shortened notation, we use $I(X;y)\triangleq H(X)-H(X|y)$. Then by Theorem 3, we can rewrite $\mathcal{I}_d(X,S\to Y)$ in (5) as

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}_{Y^N}\left[\sum_{t=1}^{N} I\big(X_{t-d+1}^t,S_{t-d};Y_t{=}y_t|Y^{t-1}{=}y^{t-1}\big)\right]$$
$$=\lim_{N\to\infty}\frac{1}{N}\mathbf{E}_{Y^N}\left[\sum_{t=1}^{N} I\big(X_{t-d+1}^t,S_{t-d};y_t|\underline{\alpha}_{t-1},y_{t-d}^{t-1}\big)\right].$$

Then the optimization problem (7) is an average reward per stage stochastic control problem [14]. For this stochastic control system, the *state variable* is the random vector $\underline{A}_{t-1}$ whose realization is $\underline{\alpha}_{t-1}$. The *policy* or *control* is the collection of all conditional probabilities $\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)$ for all realizations x_{t-d}^t, s_{t-d-1} and $\underline{\alpha}_{t-1}$, shortly denoted by $\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\}$. The *system disturbance* is Y_{t-d}^t. The *reward function* at stage t is

$$\begin{aligned}\phi\left(\underline{\alpha}_{t-1},\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\},y_{t-d}^t\right)&\\ \triangleq I\big(X_{t-d+1}^t,S_{t-d};y_t|\underline{\alpha}_{t-1},y_{t-d}^{t-1}\big)&\\ = I\big(X_{t-d+1}^t,S_{t-d};y_t|y^{t-d-1},y_{t-d}^{t-1}\big)&.\end{aligned}$$

Theorem 4: In the stochastic control problem (7), the state variable process $\underline{A}_t$ is a Markov process. ❑

Proof: By Theorem 3, the input process X_t is a source in the set $\mathcal{P}'_d(d)$ (see Definition 6). Similar to (6), we have

$$\underline{\alpha}_t = F_{BCJR}\left(\underline{\alpha}_{t-1},\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\},y_{t-d}\right) \quad (8)$$

which implies that $\underline{\alpha}_t$ depends recursively on $\underline{\alpha}_{t-1}$, on transition probabilities $\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\}$, and on y_{t-d}. The probability measure function $\Pr\big(y_{t-d}|y^{t-d-1}\big)$ satisfies

$$\begin{aligned}\Pr\big(y_{t-d}|y^{t-d-1}\big) = \sum_{x_{t-d}^t,s_{t-d-1}} &\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\\ &\times \alpha_{t-1}\big(x_{t-d}^{t-1},s_{t-d-1}\big)\Pr\big(y_{t-d}|x_{t-d},s_{t-d-1}\big). \quad (9)\end{aligned}$$

Moreover, using $\underline{\alpha}_{t-1}$ as defined in Definition 5, we have

$$\Pr\big(y_{t-d}|y^{t-d-1}\big) = \Pr\big(y_{t-d}|\underline{\alpha}^{t-1},y^{t-d-1}\big). \quad (10)$$

Equations (9) and (10) reveal that if $\underline{A}_{t-1}=\underline{\alpha}_{t-1}$ is given, Y_{t-d} is independent of $\underline{A}_0^{t-2}$ and Y^{t-d-1}. Hence, from (8) we have that $\underline{A}_t$ is independent of $\underline{A}_0^{t-2}$ if $\underline{A}_{t-1}$ is given; that is, $\underline{A}_t$ is a Markov process. ■

Let λ be the *maximum average reward* and $\mathcal{J}(\underline{\alpha})$ be the *optimal relative reward-to-go function* (or *return function*) for the state $\underline{A}=\underline{\alpha}$. Denote $\Phi\big(\underline{\alpha}_{t-1},\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\}\big)\triangleq \mathbf{E}_{Y_{t-d}^t}\big[I\big(X_{t-d+1}^t,S_{t-d};y_t|\underline{\alpha}_{t-1},y_{t-d}^{t-1}\big)\big]$. Then Bellman's equation [14] for the stochastic control problem (7) is

$$\begin{aligned}\lambda+\mathcal{J}(\underline{\alpha}_{t-1}) = \max_{\mathcal{P}'_d(d)}\Big\{&\mathbf{E}_{Y_{t-d}}\left[\mathcal{J}(\underline{\alpha}_t)\right]\\ &+\Phi\big(\underline{\alpha}_{t-1},\{\Pr\big(x_t|x_{t-d}^{t-1},s_{t-d-1},\underline{\alpha}_{t-1}\big)\}\big)\Big\} \quad (11)\end{aligned}$$

where $\underline{\alpha}_t$ is obtained by (8). Since the state process $\underline{A}_t$ is a Markov process, there exists at least one optimal stationary policy [14] in the set $\mathcal{P}'_d(d)$, denoted by

$$\{\Pr(j|i,\underline{\alpha})\}\triangleq\left\{\Pr\big(X_t{=}j|(X_{t-d}^{t-1},S_{t-d-1}){=}i,\underline{A}_{t-1}{=}\underline{\alpha}\big)\right\}$$

that solves Bellman's equation (11). Moreover, there exist efficient dynamic programming algorithms (e.g., value iteration and policy iteration [14]) to solve Bellman's equation (11) and then find the optimal stationary policy $\{\Pr(j|i,\underline{\alpha})\}$ that achieves the supremum $\mathcal{I}^*_{FB,SI}(d)$. Thus far, similar to [1–4], we can use the Monte Carlo method to evaluate the supremum $\mathcal{I}^*_{FB,SI}(d)$. Next, we show how to obtain the optimal policy using value iteration.

B. Value Iteration Algorithm

Define a family of reward-to-go functions $\{J_k(\underline{\alpha})|k=0,1,\ldots\}$ which are subject to the following conditions:

1) For any state vector $\underline{\alpha}$, the *terminal reward function* [14] is $J_0(\underline{\alpha})=0$.

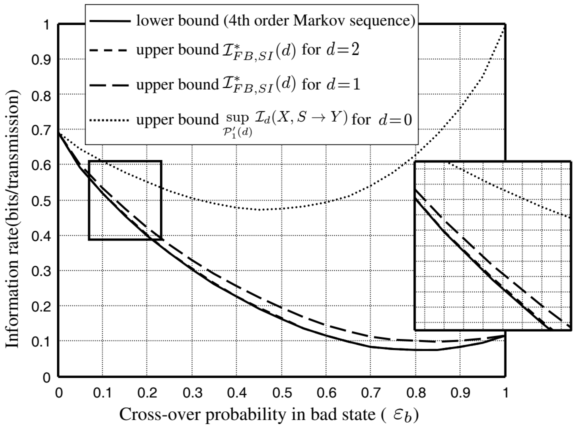

Fig. 4. Bounds on the capacity of the GE channel with RLL inputs.

2) At stage k, the reward-to-go functions $J_k(\underline{\alpha})$ are recursively generated by

$$J_k(\underline{\alpha})=\max_{\{\Pr(j|i,\underline{\alpha})\}}\left\{\mathbf{E}_{Y_{t-d}}[J_{k-1}(\underline{\alpha}')]+\Phi(\underline{\alpha},\{\Pr(j|i,\underline{\alpha})\})\right\}$$

where $\underline{\alpha}'=F_{BCJR}\left(\underline{\alpha},\{\Pr(j|i,\underline{\alpha})\},y_{t-d}\right)$, computed by the BCJR algorithm. Moreover, an optimal policy is obtained at stage k as

$$\{\Pr(j|i,\underline{\alpha})\}^* = \arg\max_{\{\Pr(j|i,\underline{\alpha})\}}\left\{\mathbf{E}_{Y_{t-d}}[J_{k-1}(\underline{\alpha}')]+\Phi(\underline{\alpha},\{\Pr(j|i,\underline{\alpha})\})\right\}.$$

In general, it is hard to find the optimal stationary policy in closed form using value iteration. The following is a quantized numerical approximation of the value iteration algorithm.

Algorithm *(Quantized Value Iteration Algorithm)*

1) **Initialization:**
   - Choose a large positive integer n.
   - Choose a finite quantizer $\hat{\underline{\alpha}}=\mathcal{Q}(\underline{\alpha})$.
   - Initialize the terminal reward $J_0(\hat{\underline{\alpha}})=0$ for all $\hat{\underline{\alpha}}$.

2) **Recursions:** For $k=1,2,\ldots,n$, for any $\hat{\underline{\alpha}}$, compute the k-stage reward-to-go functions as

$$J_k(\hat{\underline{\alpha}})=\max_{\{\Pr(j|i,\hat{\underline{\alpha}})\}}\left\{\Phi(\hat{\underline{\alpha}},\{\Pr(j|i,\hat{\underline{\alpha}})\})+E_{Y_{t-d}}\left[J_{k-1}(\hat{\underline{\alpha}}')\right]\right\}$$

where $\hat{\underline{\alpha}}'=\mathcal{Q}\left(F_{BCJR}\left(\hat{\underline{\alpha}},\{\Pr(j|i,\hat{\underline{\alpha}})\},y_{t-d}\right)\right)$.

3) **Optimized source:** For any $\hat{\underline{\alpha}}=\mathcal{Q}(\underline{\alpha})$, the optimized distribution is taken as

$$\{\Pr(j|i,\hat{\underline{\alpha}})\}^* = \arg\max_{\{\Pr(j|i,\hat{\underline{\alpha}})\}}\left\{\Phi(\hat{\underline{\alpha}},\{\Pr(j|i,\hat{\underline{\alpha}})\})+E_{Y_{t-d}}\left[J_n(\hat{\underline{\alpha}}')\right]\right\}$$

in which, similar to the step of Recursions, $\hat{\underline{\alpha}}'$ is a quantization of the vector obtained by the BCJR algorithm.

V. Numerical results

Because of the limit of pages, we only consider the channel given in Section II, i.e., the RLL$(1,\infty)$-GE channel whose inputs satisfy the RLL$(1,\infty)$ constraint. We chose this channel because it was already used in a prior publication [6]. For this example, we chose the transition probabilities between the channel states as $p(b|g)=p(g|b)=0.3$, the cross-over probability in the "good" state as $\varepsilon_g=0.001$ and the cross-over probability in the "bad" state, ε_b, to be a variable varying from 0 to 1. Fig. 4 shows upper bounds $\mathcal{I}^*_{FB,SI}(d)$ obtained by using methods of this paper and a lower bound computed using the technique presented in [5, 6].

VI. Conclusion

We developed a sequence of tightening upper bounds on the capacities of non-controllable FSCs. The upper bounds $\mathcal{I}^*_{FB,SI}(d)$ can be achieved by conditional Markov sources $\mathcal{P}'_m(d)$, conditioned on d-delayed feedback and d-delayed state information. The computation of the upper bound $\mathcal{I}^*_{FB,SI}(d)$ is formulated as an average reward per stage stochastic control problem [14]. A value iteration algorithm [14] is used to optimize the conditional Markov sources. However, if the number d (the order of Makov sources or the delay of FB and SI) is large, the trellis is so complex that the quantized method is no longer numerically feasible. For the RLL$(1,\infty)$-GE channel, at most 6-state trellis (i.e., $d=2$) is explored.

Appendix A: Proof of Theorem 2

Proof: Let $\mathcal{P}_1$ be an arbitrary source in $\mathcal{P}(d)$, i.e., $\mathcal{P}_1\in\mathcal{P}(d)$. Due to the assumption in (1), the source $\mathcal{P}_1$ induces the joint probability

$$\begin{aligned}&\Pr^{(\mathcal{P}_1)}(x^t_{t-d+1},s_{t-d},y^t)\\ &=\sum_{x^{t-d},s^{t-d-1}}\Pr^{(\mathcal{P}_1)}(x^{d+1},s_1,y_1)\prod_{\tau=d+2}^{t-1}\Pr^{(\mathcal{P}_1)}\left(x_\tau\middle|x^{\tau-1},s^{\tau-d-1},y^{\tau-d-1}\right)\\ &\quad\times\Pr(s_{\tau-d},y_{\tau-d}|s_{\tau-d-1},x_{\tau-d})\Pr\left(y^t_{t-d+1}\middle|x^t_{t-d+1},s_{t-d}\right)\end{aligned} \quad (12)$$

and the conditional probability

$$\Pr^{(\mathcal{P}_1)}\left(x_t\middle|x^{t-1}_{t-d},s_{t-d-1},y^{t-d-1}\right)=\frac{\sum_{x^{t-d-1},s^{t-d-2}}\Pr^{(\mathcal{P}_1)}(x^t,s^{t-d-1},y^{t-d-1})}{\sum_{x^{t-d-1},s^{t-d-2}}\Pr^{(\mathcal{P}_1)}(x^{t-1},s^{t-d-1},y^{t-d-1})}.$$

Use probabilities $\left\{\Pr^{(\mathcal{P}_1)}\left(x_t\middle|x^{t-1}_{t-d},s_{t-d-1},y^{t-d-1}\right)\middle|t=0,1,\ldots\right\}$ and $\Pr^{(\mathcal{P}_1)}(x^{d+1},s_1,y_1)$ to construct a new source $\mathcal{P}_2$ satisfying

$$\Pr^{(\mathcal{P}_2)}\left(x_t\middle|x^{t-1},s^{t-d-1},y^{t-d-1}\right)\triangleq\Pr^{(\mathcal{P}_1)}\left(x_t\middle|x^{t-1}_{t-d},s_{t-d-1},y^{t-d-1}\right)$$

and $\Pr^{(\mathcal{P}_2)}(x^{d+1},s_1,y_1)\triangleq\Pr^{(\mathcal{P}_1)}(x^{d+1},s_1,y_1)$. The source $\mathcal{P}_2$ satisfies $\mathcal{P}_2\in\mathcal{P}_d(d)\subset\mathcal{P}(d)$, and induces the joint probability

$$\begin{aligned}&\Pr^{(\mathcal{P}_2)}(x^t_{t-d+1},s_{t-d},y^t)\\ &\overset{(a)}{=}\sum_{x^{t-d},s^{t-d-1}}\Pr^{(\mathcal{P}_2)}(x^{d+1},s_1,y_1)\Pr\left(y^t_{t-d+1}\middle|x^t_{t-d+1},s_{t-d}\right)\\ &\quad\times\prod_{\tau=d+2}^{t}\Pr^{(\mathcal{P}_2)}\left(x_\tau\middle|x^{\tau-1},s^{\tau-d-1},y^{\tau-d-1}\right)\Pr(s_{\tau-d},y_{\tau-d}|x_{\tau-d},s_{\tau-d-1})\\ &\overset{(b)}{=}\sum_{x^{t-d},s^{t-d-1}}\Pr^{(\mathcal{P}_1)}(x^{d+1},s_1,y_1)\Pr\left(y^t_{t-d+1}\middle|x^t_{t-d+1},s_{t-d}\right)\\ &\quad\times\prod_{\tau=d+2}^{t}\frac{\Pr^{(\mathcal{P}_1)}(x^\tau_{\tau-d},s^{\tau-d}_{\tau-d-1},y^{\tau-d})}{\Pr^{(\mathcal{P}_1)}(x^{\tau-1}_{\tau-v},s_{\tau-d-1},y^{\tau-d-1})}\\ &=\sum_{x^{t-d},s^{t-d-1}}\Pr^{(\mathcal{P}_1)}(x^t,s^{t-d},y^{t-d})\Pr\left(y^t_{t-d+1}\middle|x^t_{t-d+1},s_{t-d}\right)\end{aligned} \quad (13)$$

where equality (a) follows from (1) and equality (b) follows from the construction of the source $\mathcal{P}_2$. Equalities in (12) and (13) mean that the source $\mathcal{P}_2 \in \mathcal{P}_d(d) \subset \mathcal{P}(d)$ induces the same information rate $I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| Y^{t-1}\big)$ as the source $\mathcal{P}_1 \in \mathcal{P}(d)$ does. Since $\mathcal{P}_1$ is chosen from $\mathcal{P}(d)$ arbitrarily, the supremum $\mathcal{I}_{FB,SI}^*(d)$ can be taken over the set of conditional Markov sources $\mathcal{P}_d(d)$ instead of the set $\mathcal{P}(d)$. ■

Appendix B: Proof of Theorem 3

Proof: First, we show that Bellman's principle of optimality [14] holds. For any integer $T \in [1, N]$, we have

$$\begin{aligned}&\sum_{t=1}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| Y^{t-1}\big)\\ &\quad= \sum_{t=1}^{T-1} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| Y^{t-1}\big) + \int_{y^{T-d-1}} \Pr(y^{T-d-1})\\ &\qquad \times \sum_{t=T}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| y^{T-d-1}, Y_{T-d}^{t-1}\big)\, dy^{T-d-1}. \qquad (14)\end{aligned}$$

Given optimal policies $\big\{\Pr\big(x_t \big| x_{t-d}^{t-1}, s_{t-d-1}, y^{t-d-1}\big) \big| 1 \le t < T\big\}$, the term $\sum_{t=1}^{T-1} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| Y^{t-1}\big)$ in (14) can be computed using the probability measure function

$$\begin{aligned}&\Pr\left(x^{T-1}, s^{T-d-1}, y^{T-1}\right) = \Pr(x^{d+1}, s_1, y_1)\, \Pr\big(y_{T-d}^{T-1} \big| x_{T-d}^{T-1}, s_{T-d}\big)\\ &\times \prod_{\tau=d+2}^{T-1} \Pr\big(x_\tau \big| x_{\tau-d}^{\tau-1}, s_{\tau-d-1}, y^{\tau-d-1}\big) \Pr(s_{\tau-d}, y_{\tau-d} | x_{\tau-d}, s_{\tau-d-1})\end{aligned}$$

which is independent of the policies after time T, i.e., $\left\{\Pr\left(x_t \big| x_{t-d}^{t-1}, s_{t-d-1}, y^{t-d-1}\right) \big| T \le t \le N\right\}$. Thus the policies after time T are optimal if and only if they maximize the last term of (14), which proves Bellman's optimal principle [14].

Next, we show that if after time T we utilize policies

$$\left\{\Pr\big(x_t \big| x_{t-d}^{t-1}, s_{t-d-1}, \underline{\alpha}_{T-1}, y_{T-d}^{t-d-1}\big) \big| T \le t \le N\right\}$$

instead of the general policies

$$\left\{\Pr\big(x_t \big| x_{t-d}^{t-1}, s_{t-d-1}, y^{T-d-1}, y_{T-d}^{t-d-1}\big) \big| T \le t \le N\right\}$$

we can still maximize the last term in (14). To show this, suppose two different vectors y^{T-d-1} and $\tilde{y}^{T-d-1}$ induce the same posterior probability vectors $\underline{\alpha}_{T-1}$ and $\underline{\tilde{\alpha}}_{T-1}$. If policies after time T for y^{T-d-1} and $\tilde{y}^{T-d-1}$ are identical, then

$$\Pr\big(x_{T-d}^N, s_{T-d}^{N-d}, y_{T-d}^N \big| y^{T-d-1}\big) = \Pr\big(x_{T-d}^N, s_{T-d}^{N-d}, y_{T-d}^N \big| \tilde{y}^{T-d-1}\big).$$

It follows that

$$\begin{aligned}&\sum_{t=T}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| y^{T-d-1}, Y_{T-d}^{t-1}\big)\\ &\qquad = \sum_{t=T}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| \tilde{y}^{T-d-1}, Y_{T-d}^{t-1}\big). \qquad (15)\end{aligned}$$

Since the polices for y^{T-d-1} and $\tilde{y}^{T-d-1}$ have the same vector $\underline{\alpha}_{T-1} = \underline{\tilde{\alpha}}_{T-1}$, both sides of (15) are equal to

$$\begin{aligned}&\sum_{t=T}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| \underline{\alpha}_{T-1}, Y_{T-d}^{t-1}\big)\\ &\qquad = \sum_{t=T}^{N} I\big(X_{t-d+1}^t, S_{t-d}; Y_t \big| \underline{\tilde{\alpha}}_{T-1}, Y_{T-d}^{t-1}\big).\end{aligned}$$

Therefore, policies $\big\{\Pr\big(x_t \big| x_{t-d}^{t-1}, s_{t-d-1}, \underline{\alpha}_{T-1}, y_{T-d}^{t-d-1}\big) \big| T \le t \le N\big\}$ maximize the last term in (14). Moreover, such policies are optimal not only for the vector y^{T-d-1} but also for any other different vector $\tilde{y}^{T-d-1}$, as long as $\underline{\alpha}_{T-1} = \underline{\tilde{\alpha}}_{T-1}$.

Since T is chosen arbitrarily, the optimal source in the set $\mathcal{P}_d'(d)$ achieves the same supremum $\mathcal{I}_{FB,SI}^*(d)$ as the optimal source in the set $\mathcal{P}_d(d)$ does. ■

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