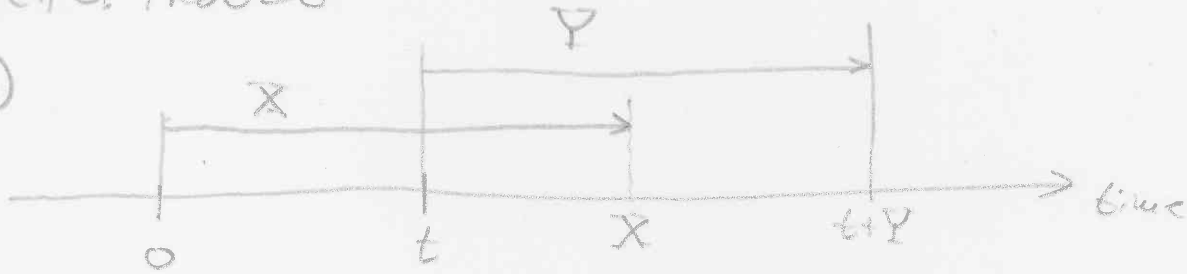
HW8 (SOLUTIONS)

1) $\boxed{a}$ CH6, PROB 28

a)



$0 \leftarrow$ time at which AJ brings car in

$X \leftarrow$ time at which AJ's car is finished

$t \leftarrow$ time at which MJ brings car in

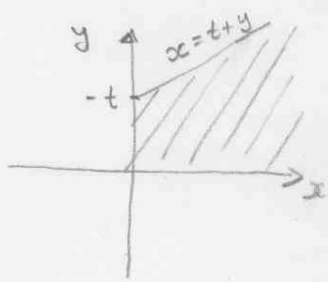
$t+Y \leftarrow$ time at which MJ's car is finished

$$\boxed{\text{FIND } P(X \le t+Y) \quad \text{where } f_{X,Y}(x,y) = e^{-(x+y)}u(x)u(y)}$$

Two cases: 1°) $t<0$

2°) $t \ge 0$

IF $t<0$: $P(X \le t+Y) = \displaystyle\int_0^\infty\left[\int_0^{x-t} e^{-(x+y)}\,dy\right]dx = \int_0^\infty e^{-x}\int_0^{x-t} e^{-y}\,dy\,dx$

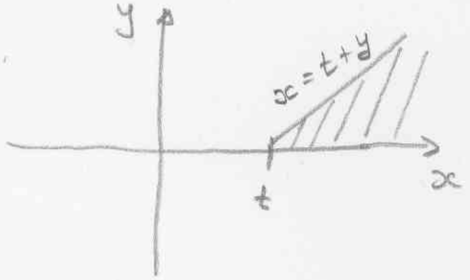


$$= \int_0^\infty e^{-x}\left[\frac{e^{-y}}{(-1)}\Bigg|_0^{x-t}\right]dx = \int_0^\infty e^{-x}\left[1 - e^{-(x-t)}\right]dx$$

$$= \int_0^\infty e^{-x}\,dx - e^{t}\int_0^\infty e^{-2x}\,dx = 1 - \frac{e^t}{2}$$

$$\Rightarrow \boxed{\text{IF } t<0 \text{ then } P(X \le t+Y) = 1 - \frac{e^t}{2}} \quad (*)$$

IF $t \geq 0$: $\quad P(X \leq t+Y) = \int_0^\infty \left[\int_{t+y}^\infty e^{-(x+y)}\, dx \right] dy$



$$= \int_0^\infty e^{-y} \left[\int_{t+y}^\infty e^{-x}\, dx \right] dy$$

$$= \int_0^\infty e^{-y} \left[e^{-(t+y)} \right] dy$$

$$= e^{-t} \int_0^\infty e^{-2y}\, dy = \frac{e^{-t}}{2}$$

$$\boxed{\text{If } t \geq 0 \text{ then } P(X \leq t+Y) = \frac{e^{-t}}{2}} \quad (**)$$

COMBINING (*) & (**), WE GET

$$\boxed{P(X \leq t+Y) = \left[1 - \frac{e^{t}}{2}\right] u(-t) + \frac{e^{-t}}{2} u(t)}$$

b) $P(X+Y \leq 2) = ?$

LET $Z = X+Y$.

SINCE X & Y ARE INDEPENDENT, THE PDF $f_Z(z)$ IS THE CONVOLUTION OF $f_X(z)$ AND $f_Y(z)$

$$f_Z(z) = f_X(z) * f_Y(z) = \begin{cases} 0 & \text{if } z < 0 \\ \int_0^z f_X(t)\, f_Y(z-t)\, dt & \text{if } z \geq 0 \end{cases}$$

NOW FIND $\int_0^z f_X(t) f_Y(z-t)\, dt = \int_0^z e^{-t} \cdot e^{-(z-t)}\, dt = e^{-z} \int_0^z dz = z e^{-z}$

So, $f_Z(z) = z e^{-z} u(z)$

Now $P(X+Y \leq 2) = P(Z \leq 2)$

$$= \int_0^2 f_Z(z)\,dz$$

$$= \int_0^2 z \cdot e^{-z}\,dz$$

$$= \left[-e^{-z} - z e^{-z}\right]\Big|_0^2$$

$$\boxed{P(X+Y \leq 2) = 1 - 3e^{-2}}$$

$\boxed{b}$ CH6, PROB 39

$X = \max[R_1, R_2]$

$Y = \min[R_1, R_2]$

R_1 — roll 1

R_2 — roll 2

R_2 \ R_1	1	2	3	4	5	6
1	$X=1$ $Y=1$	$X=2$ $Y=1$	$X=3$ $Y=1$	$X=4$ $Y=1$	$X=5$ $Y=1$	$X=6$ $Y=1$
2	$X=2$ $Y=1$	$X=2$ $Y=2$	$X=3$ $Y=2$	$X=4$ $Y=2$	$X=5$ $Y=2$	$X=6$ $Y=2$
3	$X=3$ $Y=1$	$X=3$ $Y=2$	$X=3$ $Y=3$	$X=4$ $Y=3$	$X=5$ $Y=3$	$X=6$ $Y=3$
4	$X=4$ $Y=1$	$X=4$ $Y=2$	$X=4$ $Y=3$	$X=4$ $Y=4$	$X=5$ $X=4$	$X=6$ $Y=4$
5	$X=5$ $Y=1$	$X=5$ $Y=2$	$X=5$ $Y=3$	$X=5$ $Y=4$	$X=5$ $Y=5$	$X=6$ $Y=5$
6	$X=6$ $Y=1$	$X=6$ $Y=2$	$X=6$ $Y=3$	$X=6$ $X=4$	$X=6$ $Y=5$	$X=6$ $Y=6$

* THERE ARE EXACTLY $(2i-1)$ OUTCOMES FOR WHICH $X=i$.
* IF $X=i$, ONLY 1 OUTCOME GIVES $Y=i$
* IF $X=i$, THEN 2 OUTCOMES GIVE $Y=j<i$

$$P_{Y|X}(j|i) = \begin{cases} \dfrac{1}{2i-1} & \text{if } j=i \\[2ex] \dfrac{2}{2i-1} & \text{if } j<i \end{cases}$$

FOR X & Y TO BE INDEPENDENT, WE MUST HAVE

$$P_{X,Y}(i,j) = P_X(i)\,P_Y(j)$$

$$P_{Y|X}(j|i)\cdot \cancel{P_X(i)} = \cancel{P_X(i)}\cdot P_Y(j)$$

$$\boxed{P_{Y|X}(j|i) = P_Y(j)}$$

FROM EARLIER, WE HAVE

$$P_{Y|X}(j|i) = \begin{cases} \dfrac{1}{2i-1} & \text{if } j=i \\ \dfrac{2}{2i-1} & \text{if } j<i \end{cases}$$

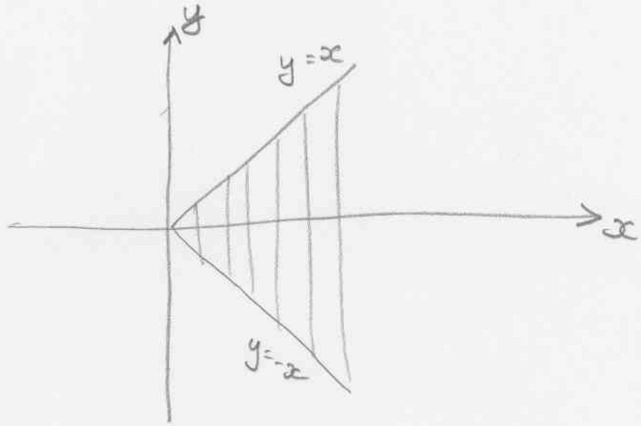
ON THE OTHER HAND,
WE CAN VERIFY

$$P_Y(j) = \frac{13-2j}{36}$$

CLEARLY $\boxed{P_Y(j) \neq P_{Y|X}(j|i)}$

SO X & Y ARE <u>NOT</u> INDEPENDENT

$\boxed{C}$ CH6, PROB 42



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-x}^{x} c(x^2-y^2)e^{-x}u(x)\,dy$$

$$= \left[c\cdot x^2 e^{-x}\int_{-x}^{x} dy - c\cdot e^{-x}\int_{-x}^{x} y^2\,dy\right]u(x)$$

$$= c\cdot\left[2x^3e^{-x} - c\,\frac{2x^3}{3}\,e^{-x}\right]u(x)$$

$$= c\cdot\frac{4}{3}x^3e^{-x}u(x)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \underset{\text{if } x>0}{=} \frac{c(x^2-y^2)e^{-x}}{c\cdot\frac{4}{3}x^3e^{-x}} = \frac{3}{4}\cdot\frac{x^2-y^2}{x^3}$$

$$\boxed{f_{Y|X}(y|x) = \frac{3}{4}\cdot\frac{(x^2-y^2)}{x^3}}$$

$-x \le y \le x$

$$\boxed{f_{Y|X}(y|x) = \begin{cases} \frac{3}{4} \frac{x^2-y^2}{x^3} & -x \le y \le x \\ 0 & \text{otherwise} \end{cases}}$$

$$F_{Y|X}(r|x) = \int_{-\infty}^{r} f_{Y|X}(y|x)\, dy$$

$$= \int_{-x}^{r} \frac{3}{4} \frac{x^2-y^2}{x^3} \cdot dy \qquad \text{if} \quad -x \le r \le x$$

$$= \frac{3}{4} x^{-1} \int_{-x}^{r} dy - \frac{3}{4x^3} \int_{-x}^{r} y^2 dy$$

$$= \frac{3}{4} \frac{r+x}{x} - \frac{\cancel{3}}{4x^3} \frac{r^3+x^3}{\cancel{3}}$$

$$= \frac{3}{4} \frac{r}{x} + \frac{3}{4} - \frac{1}{4} \frac{r^3}{x^3} - \frac{1}{4}$$

$$= \frac{1}{2} + \frac{3}{4} \frac{r}{x} - \frac{1}{4} \frac{r^3}{x^3} \qquad \text{if} \quad -x \le r \le x$$

$$\boxed{F_{Y|X}(y|x) = \begin{cases} \frac{1}{2} + \frac{3}{4} \frac{y}{x} - \frac{1}{4} \frac{y^3}{x^3} & \text{if} \quad -x \le y \le x \\ 0 & \text{otherwise} \end{cases}}$$

$\boxed{d}$ CH6, PROB 52

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X = R\sin\Theta \qquad Y = R\cos\Theta$$

JACOBIAN $J_{X,Y}(r,\theta) = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{bmatrix}$

$$J_{X,Y}(r,\theta) = -r$$

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\sin\theta, r\cos\theta) \cdot |J_{X,Y}(r,\theta)|$$

$$= \frac{1}{\pi} \cdot |-r| = \frac{r}{\pi} \Leftarrow \begin{cases} 0 \le r \le 1 \\ -\pi \le \theta \le \pi \end{cases}$$

$$\boxed{f_{R,\Theta}(r,\theta) = \begin{cases} \frac{r}{\pi} & 0 \le r \le 1,\ -\pi \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases}}$$

$\boxed{e}$ CH6, PROB 53

$$X = R \sin \Theta$$

$$Y = R \cos \Theta$$

because X & Y are independent

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} = \frac{1}{\sqrt{2\pi \cdot 1^2}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi \cdot 1^2}} e^{-\frac{y^2}{2}}$$

$$|J_{X,Y}(r,\theta)| = r \geq 0$$ (see previous problem)

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(x,y) \cdot |J_{X,Y}(r,\theta)|$$

$$= \frac{1}{2\pi} e^{-\frac{r^2 \sin^2\theta + r^2 \cos^2\theta}{2}} \cdot r$$

$$= \frac{r \cdot e^{-r^2/2}}{2\pi} \Leftarrow \begin{cases} 0 \leq r < \infty \\ -\pi \leq \theta \leq \pi \end{cases}$$

$$\boxed{f_{R,\Theta}(r,\theta) = \begin{cases} \dfrac{r e^{-r^2/2}}{2\pi} & 0 \leq r < \infty,\ -\pi \leq \theta \leq \pi \\ 0 & \text{otherwise} \end{cases}}$$

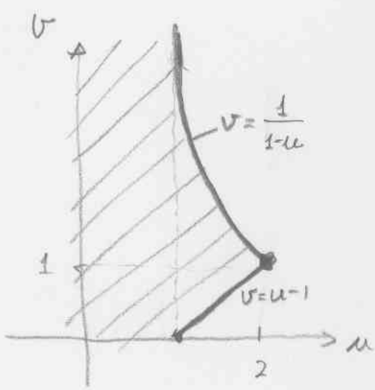
$\boxed{f}$ Chapter 6, Prob 56

a) $U = X + Y$
$V = X/Y$ $\xrightarrow{\text{solve for}}_{X \,\&\, Y}$ $\Rightarrow$ $\boxed{X = \dfrac{U \cdot V}{V+1}}$ $\boxed{Y = \dfrac{U}{V+1}}$

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

JACOBIAN $J_{X,Y}(U,V) = \det \begin{bmatrix} \dfrac{\partial X}{\partial U} & \dfrac{\partial X}{\partial V} \\ \dfrac{\partial Y}{\partial U} & \dfrac{\partial Y}{\partial V} \end{bmatrix}$

$$= \det \begin{bmatrix} \dfrac{V}{V+1} & \dfrac{U}{(V+1)^2} \\ \dfrac{1}{V+1} & -\dfrac{U}{(V+1)^2} \end{bmatrix}$$



$$= \frac{-UV}{(V+1)^3} - \frac{U}{(V+1)^3}$$

$$= -\frac{U}{(V+1)^2}$$

$$|J_{X,Y}(U,V)| = \frac{U}{(V+1)^2}$$

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \cdot |J_{X,Y}(u,v)| = \begin{cases} 1 \cdot \dfrac{u}{(v+1)^2} & \begin{matrix} 0 < v < \infty \\ 0 \le u \le 2 \\ 0 \le u \le v+1 \\ 0 \le u \le \frac{v+1}{v} \end{matrix} \\ 0 & \text{otherwise} \end{cases}}$$

$\left\{ 0 < u \le \min\left[v+1, \frac{v+1}{v}\right] \right.$

b) $U = X \qquad V = \dfrac{X}{Y} \quad \Longrightarrow \quad X = U \qquad Y = \dfrac{U}{V}$

$1 \geq U \geq 0$

$\infty > V \geq U \geq 0$

$$J_{X,Y}(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{dy}{\partial v} \end{bmatrix}$$

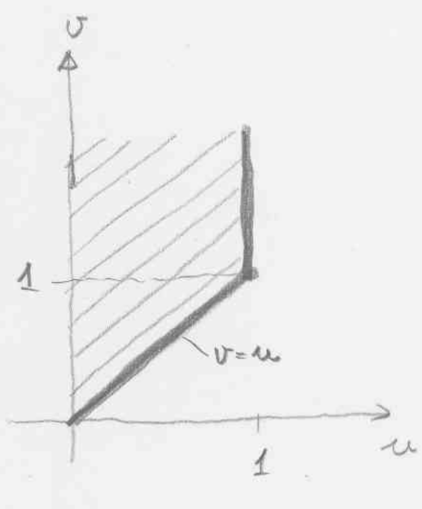
$$= \det \begin{bmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{bmatrix}$$

$$= -\frac{u}{v^2}$$

$$|J_{X,Y}(u,v)| = \frac{u}{v^2}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(u, \frac{u}{v}\right) \cdot |J_{X,Y}(u,v)|$$

$$\boxed{f_{U,V}(u,v) = \begin{cases} 1 \cdot \dfrac{u}{v^2} & \begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq u \leq v < \infty \end{array} \\ 0 & \text{otherwise} \end{cases}}$$



c) $U = X + Y$ $\qquad V = \dfrac{X}{X+Y}$ $\implies$ $X = U \cdot V$ $\qquad Y = U(1-V)$

$0 \le U \le 2$

$0 \le V \le 1$

$0 \le U \cdot V \le 1$

$0 \le U(1-V) \le 1$

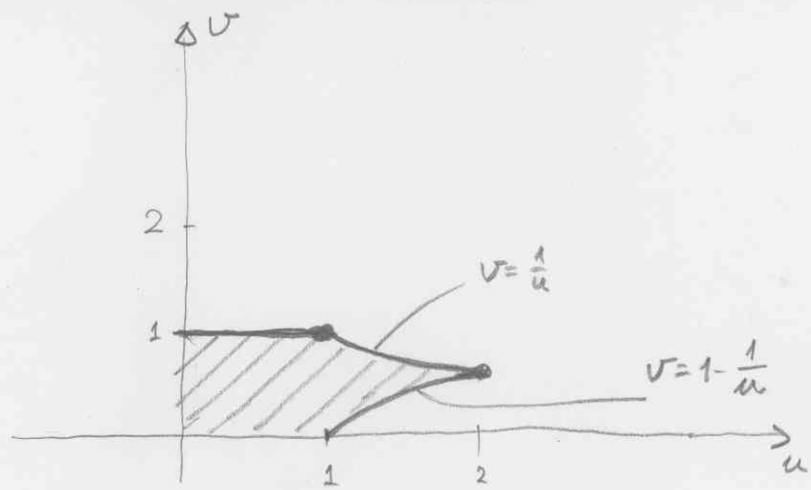
$\Rightarrow 0 \le U \le \min\left[\frac{1}{V}, \frac{1}{1-V}\right]$

$$J_{X,Y}(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \det \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u(1-v) = -u$$

$$\left| J_{X,Y}(u,v) \right| = u$$

$$\boxed{J_{U,V}(u,v) = \begin{cases} u & 0 \le v \le 1, \; 0 \le u \le \min\left[\frac{1}{v}, \frac{1}{1-v}\right] \\ 0 & \text{otherwise} \end{cases}}$$



$\boxed{9}$ CH6, PROB 58

$$f_{X_1,X_2}(x_1,x_2) = \lambda^2 e^{-(x_1+x_2)\lambda}\, u(x_1)\, u(x_2)$$

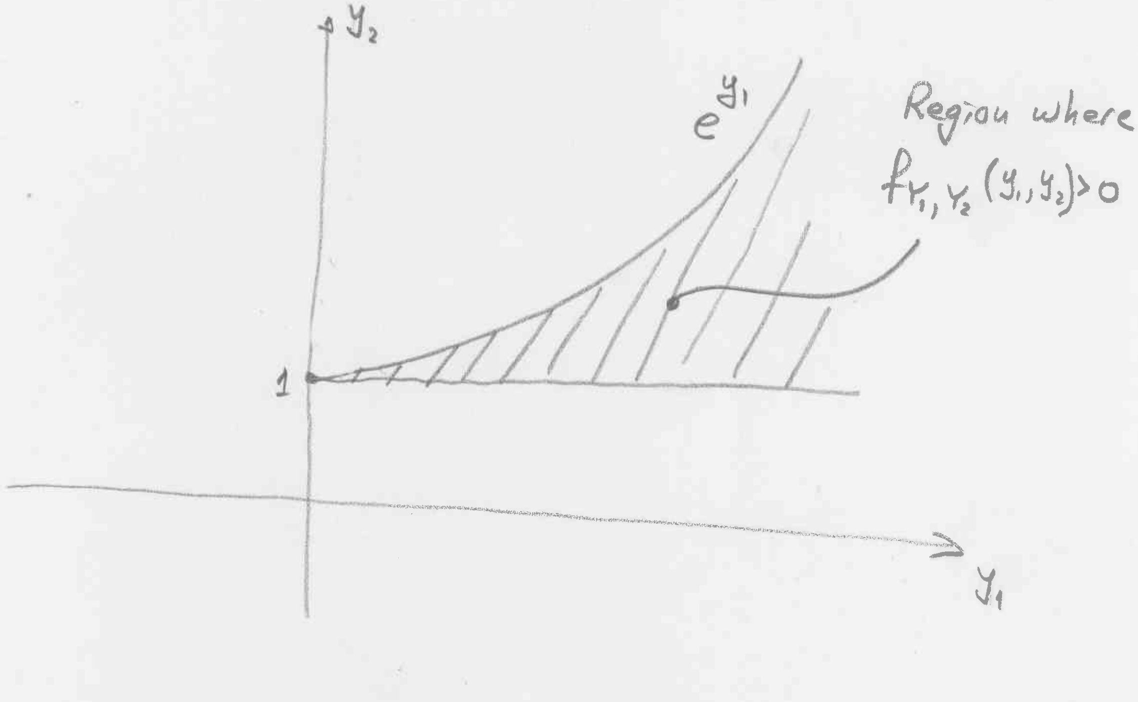
$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= e^{X_1} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X_1 &= \ln Y_2 \\ X_2 &= Y_1 - \ln Y_2 \end{aligned}$$

NOTE $\infty > Y_1 = X_1 + X_2 \geq 0$

$\infty > Y_1 = \ln Y_2 + X_2 \geq 0$

$\Rightarrow Y_1 \geq \ln Y_2$ because $x_2 \geq 0$

$\boxed{e^{Y_1} \geq Y_2}$ Also $\boxed{Y_2 \geq 1}$ because $Y_2 = e^{X_1}$ and $X_1 \geq 0$



JAKOBIAN:

$$J_{X_1,X_2}(Y_1,Y_2) = \det\begin{bmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{bmatrix}$$

$$= \det\begin{bmatrix} 0 & \frac{1}{Y_2} \\ 1 & -\frac{1}{Y_2} \end{bmatrix} = -\frac{1}{Y_2}$$

$$\left|J_{X_1,X_2}(Y_1,Y_2)\right| = \frac{1}{Y_2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(\ln y_2,\ y_1 - \ln y_2)\cdot\frac{1}{y_2}$$

$$= \lambda^2\cdot e^{-(\ln y_2 + y_1 - \ln y_2)\lambda}\cdot u(\ln y_2)\cdot u(y_1 - \ln y_2)\cdot\frac{1}{y_2}$$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \dfrac{\lambda^2 e^{-\lambda y_1}}{y_2} & 1 \le y_2 \le e^{y_1} \\ 0 & \text{otherwise} \end{cases}}$$

$\boxed{2}$ a)

$$\left.\begin{aligned} P_X(1) &= P(X=1) = \frac{6}{9} \\ P_X(2) &= P(X=2) = \frac{3}{9}\cdot\frac{6}{8} \\ P_X(3) &= P(X=3) = \frac{3}{9}\cdot\frac{2}{8}\cdot\frac{6}{7} \\ P_X(4) &= P(X=4) = \frac{3}{9}\cdot\frac{2}{8}\cdot\frac{1}{7}\cdot\frac{6}{6} \end{aligned}\right\} \boxed{\begin{aligned} P_X(i) &= \frac{6\cdot(9-i)!\ 3!}{9!\ (4-i)!} \\ \text{for}\quad & 1 \le i \le 4 \end{aligned}}$$

IF $\boxed{X=i}$ THEN $\boxed{1 \le Y \le 5-i}$

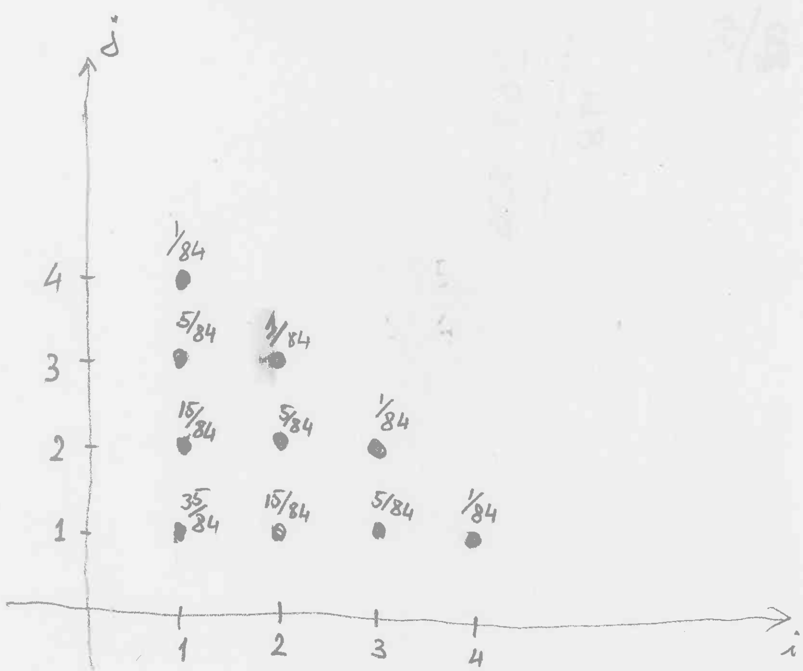
$$\underbrace{\begin{aligned} P_{Y|X}(1|i) &= P(Y=1|X=i) = \frac{5}{9-i} \\ P_{Y|X}(2|i) &= P(Y=2|X=i) = \frac{4-i}{9-i}\cdot\frac{5}{8-i} && \text{if} \quad 2 \le 5-i \\ P_{Y|X}(3|i) &= P(Y=3|X=i) = \frac{4-i}{9-i}\cdot\frac{3-i}{8-i}\cdot\frac{5}{7-i} && \text{if} \quad 3 \le 5-i \\ P_{Y|X}(4|i) &= P(Y=4|X=i) = \frac{4-i}{9-i}\cdot\frac{3-i}{8-i}\cdot\frac{2-i}{7-i}\cdot\frac{5}{6-i} && \text{if} \quad 4 \le 5-i \end{aligned}}$$

$$\boxed{P_{Y|X}(j|i) = P(Y=j|X=i) = \frac{(9-i-j)!}{(9-i)!}\cdot\frac{(4-i)!}{(5-i-j)!}\cdot 5 \qquad \text{if } 1 \le j \le 5-i}$$

$$p_{X,Y}(i,j) = p_X(i) \cdot p_{Y|X}(j|i) =$$

$$= \frac{6 \cdot 5 \cdot 3!}{9!} \cdot \frac{(9-i-j)!}{(5-i-j)!} \quad \text{for} \quad \begin{matrix} 1 \le i \le 4 \\ 1 \le j \le 4 \\ 2 \le i+j \le 5 \end{matrix}$$

$$\Rightarrow \quad p_{X,Y}(i,j) = \begin{cases} \frac{1}{84} & \text{if } i+j=5 \ \& \ 1 \le i \le 4 \ \& \ 1 \le j \le 4 \\ \frac{5}{84} & \text{if } i+j=4 \ \& \ 1 \le i \le 3 \ \& \ 1 \le j \le 3 \\ \frac{15}{84} & \text{if } i+j=3 \ \& \ 1 \le i \le 2 \ \& \ 1 \le j \le 2 \\ \frac{35}{84} & \text{if } i=j=1 \\ 0 & \text{otherwise} \end{cases}$$



b)

$$P_X(1) = P(X=1) = \frac{35}{84} + \frac{15}{84} + \frac{5}{84} + \frac{1}{84} = \frac{56}{84}$$

$$P_X(2) = P(X=2) = \frac{15}{84} + \frac{5}{84} + \frac{1}{84} = \frac{21}{84}$$

$$P_X(3) = P(X=3) = \frac{5}{84} + \frac{1}{84} = \frac{6}{84}$$

$$P_X(4) = P(X=4) = \frac{1}{84}$$

$$P_X(i) = \begin{cases} \frac{56}{84} & i=1 \\ \frac{21}{84} & i=2 \\ \frac{6}{84} & i=3 \\ \frac{1}{84} & i=4 \\ 0 & \text{otherwise} \end{cases}$$

$P_Y(i) = P_X(i)$ because of symmetry

c)

$$E[X] = 1 \cdot \frac{56}{84} + 2 \cdot \frac{21}{84} + 3 \cdot \frac{6}{84} + 4 \cdot \frac{1}{84} =$$

$$E[X] = \frac{120}{84} = \frac{12 \cdot 10}{12 \cdot 7} = \frac{10}{7}$$

$$\boxed{E[X] = E[Y] = \frac{10}{7}}$$

$$E[X^2] = 1^2 \cdot \frac{56}{84} + 2^2 \cdot \frac{21}{84} + 3^2 \frac{6}{84} + 4^2 \frac{1}{84} = \frac{210}{84} = \frac{5}{2}$$

$$\boxed{E[X^2] = E[Y^2] = \frac{50}{21}}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{5}{2} - \left(\frac{10}{7}\right)^2 = \frac{5 \cdot 49 - 2 \cdot 100}{2 \cdot 49} = \frac{45}{98}$$

$$\boxed{Var(X) = Var(Y) = \frac{45}{98}}$$

$$E[X \cdot Y] = 1 \cdot 4 \frac{1}{84} + 1 \cdot 3 \cdot \frac{5}{84} + 2 \cdot 3 \cdot \frac{1}{84}$$

$$+ 1 \cdot 2 \cdot \frac{15}{84} + 2 \cdot 2 \cdot \frac{5}{84} + 3 \cdot 2 \cdot \frac{1}{84}$$

$$+ 1 \cdot 1 \cdot \frac{35}{84} + 2 \cdot 1 \cdot \frac{15}{84} + 3 \cdot 1 \cdot \frac{5}{84} + 4 \cdot 1 \cdot \frac{1}{84}$$

$$E[XY] = \frac{4 + 15 + 6 + 30 + 20 + 6 + 35 + 30 + 15 + 4}{84}$$

$$E[XY] = \frac{165}{84} = \frac{55}{28}$$

$$\boxed{E[XY] = \frac{55}{28}}$$

$$Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$$

$$= \frac{55}{28} - \frac{10}{7} \cdot \frac{10}{7} = -\frac{15}{196}$$

$$\boxed{Cov(X,Y) = -\frac{15}{196}}$$

d) $Cov(X,Y) \neq 0 \Rightarrow$ X & Y ARE CORRELATED $\Rightarrow$ $\boxed{\text{X \& Y ARE NOT INDEPENDENT}}$

e) X & Y ARE CORRELATED BECAUSE $Cov(X,Y) \neq 0$

f) $P_Z(2) = P(Z=2) = P(X+Y=2) = P_{X,Y}(1,1) = \frac{35}{84}$

$P_Z(3) = P(Z=3) = P(X+Y=3) =$

$= P_{X,Y}(1,2) + P_{X,Y}(2,1) = \frac{30}{84}$

$$p_Z(4) = P(X+Y=4) = p_{X,Y}(1,3) + p_{X,Y}(2,2) + p_{X,Y}(3,1)$$

$$p_Z(4) = \frac{5}{84} + \frac{5}{84} + \frac{5}{84} = \frac{15}{84}$$

$$p_Z(5) = 1 - p_Z(2) - p_Z(3) - p_Z(4) = \frac{4}{84}$$

$$p_Z(k) = \begin{cases} \frac{35}{84} & k=2 \\ \frac{30}{84} & k=3 \\ \frac{15}{84} & k=4 \\ \frac{4}{84} & k=5 \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{3}$

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$
$$f_Y(y) = \lambda e^{-\lambda y} u(y)$$
$$\underset{\text{independence}}{\Rightarrow} \boxed{f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)} u(x) u(y)}$$

mgf: $m_X(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx} \cdot \lambda \cdot e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx = \frac{\lambda}{\lambda - t}$

$$m_X(t) = \frac{\lambda}{\lambda - t}$$
$$m_Y(t) = \frac{\lambda}{\lambda - t}$$

$$\Rightarrow m_{X,Y}(t,s) = E\left[e^{tX} e^{sY}\right] = E\left[e^{tX}\right] \cdot E\left[e^{sY}\right]$$

$$\boxed{m_{X,Y}(t,s) = m_X(t) \cdot m_Y(s)}$$

a) $Z = X + Y \Rightarrow f_Z(z) = f_X(z) * f_Y(z)$ ← IF X & Y ARE INDEPENDENT THE PDF OF $X+Y$ IS THE CONVOLUTION OF $f_X(\cdot)$ & $f_Y(\cdot)$

$$f_Z(z) = \lambda e^{-\lambda z} u(z) * \lambda e^{-\lambda z} u(z)$$

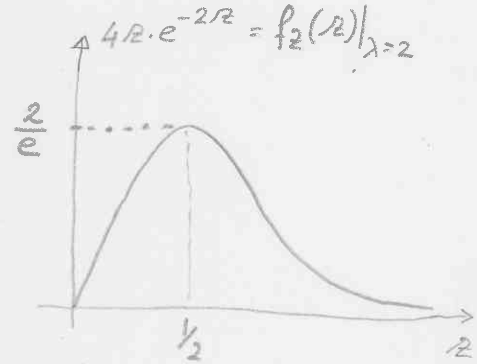
if $z < 0$ then $f_Z(z) = 0$

if $z \geq 0$ then $f_Z(z) = \int_0^z \lambda e^{-\lambda t} \cdot \lambda e^{-\lambda(z-t)}\, dt$

$$= \lambda^2 \cdot e^{-\lambda z} \int_0^z dt$$

$$= \lambda^2 z\, e^{-\lambda z}$$

$$\Rightarrow \boxed{f_Z(z) = \lambda^2 \cdot z \cdot e^{-\lambda z} u(z)}$$



b) $W = X - Y$

$$m_W(t) = E\left[e^{tW}\right] = E\left[e^{t(X-Y)}\right] = \underbrace{E\left[e^{tX}\right]}_{m_X(t)} \cdot \underbrace{E\left[e^{-tY}\right]}_{m_Y(-t)}$$

$$= m_X(t) \cdot m_Y(-t) =$$

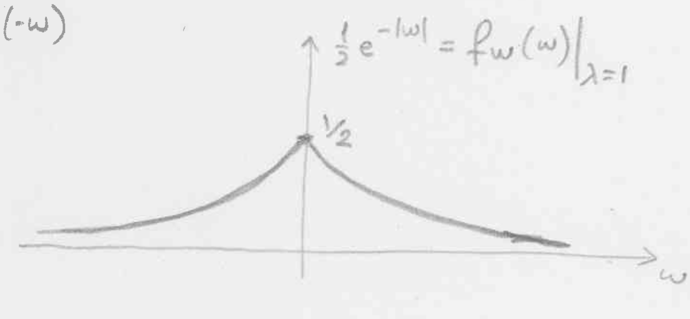
$$= \frac{\lambda}{\lambda - t} \cdot \frac{\lambda}{\lambda + t}$$

$$= \frac{1}{2} \cdot \frac{\lambda}{\lambda - t} + \frac{1}{2} \cdot \frac{\lambda}{\lambda + t}$$

apply inverse Laplace transform

$$f_W(w) = \frac{1}{2} \lambda e^{-\lambda w} u(w) + \frac{1}{2} \lambda e^{\lambda w} u(-w)$$

$$\boxed{f_W(w) = \frac{1}{2} \lambda e^{-\lambda |w|}}$$



c) $V = \min[X, Y]$

$$F_V(v) = P(V \le v)$$

$$= 1 - P(V \ge v)$$

$$= 1 - P[\min(X, Y) \ge v] \quad \leftarrow \text{assume } v \ge 0$$

$$= 1 - P[X \ge v, Y \ge v]$$

$$= 1 - \int_v^\infty \int_v^\infty f_{XY}(x,y)\,dx\,dy$$

$$= 1 - \int_v^\infty \int_v^\infty \lambda^2 e^{-\lambda x} e^{-\lambda y}\,dx\,dy$$

$$= 1 - \left[\int_v^\infty \lambda e^{-\lambda x} dx\right] \cdot \left[\int_v^\infty \lambda e^{-\lambda y} dy\right]$$

$$= 1 - \left[\lambda \cdot \frac{e^{-\lambda x}}{(-\lambda)}\Big|_v^\infty\right] \cdot \left[\lambda \cdot \frac{e^{-\lambda y}}{(-\lambda)}\Big|_v^\infty\right]$$

$$= 1 - e^{-\lambda v} \cdot e^{-\lambda v}$$

$$F_V(v) = \left[1 - e^{-2\lambda v}\right] u(v)$$

$$f_V(v) = \frac{\partial F_V(v)}{\partial v} = 2\lambda e^{-2\lambda v} u(v)$$

$6e^{-6v} = f_V(v)\big|_{\lambda=3}$