Markov Sources Achieve the Feedback Capacity of Finite-State Machine Channels

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Abstract — **The feedback capacity of a finite-state machine channel is achieved by a feedback-dependent Markov source with the same memory length as the channel. The optimal feedback is captured by the conditional probabilities of the channel states given all previous channel outputs, i.e., by the forward coefficients in the BCJR algorithm [1]. We formulate the optimization of the feedback-dependent Markov source distribution as an average-reward-per-stage stochastic control problem, and solve it numerically using dynamic programming algorithms.**

I. Main Results

We consider a finite-state machine channel whose memory length is M_c. The channel input X_t takes its value from a finite alphabet $\mathcal{X}$. The channel output Y_t is induced by the input sequence $X_{t-M_c}^t = [X_{t-M_c}, X_{t-M_c+1}, \ldots, X_t]$ of length M_c+1 and is corrupted by additive white noise N_t. Thus the conditional probability density function of the channel output Y_t satisfies

$$f_{Y_t|X_{-\infty}^t, Y_{-\infty}^{t-1}}(y_t|x_{-\infty}^t, y_{-\infty}^{t-1}) = f_{Y_t|X_{t-M_c}^t}(y_t|x_{t-M_c}^t), \quad (1)$$

and $X_{t-M_c+1}^t = [X_{t-M_c+1}, \ldots, X_t] \in \mathcal{X}^{M_c}$ captures the channel state at time t. Further, we assume the channel is used with noiseless feedback, i.e., the encoder, before sending X_t, knows without error all previous channel outputs Y_1^{t-1}.

For channels used with feedback, Massey [2] showed that the supremum of the directed information rate $I(X \to Y)$ is a feedback capacity upper bound. Tatikonda [3] proved that any directed information rate is achievable by a feedback code, and thus that the feedback capacity is the supremum of the directed information rate $I(X \to Y)$, where the supremum is over the feedback-dependent channel input distribution $\{\Pr(X_t|X_1^{t-1}, Y_1^{t-1}), \text{for } t = 1, 2, \ldots\}$.

The following two theorems reduce the search space.

Theorem 1: The feedback capacity is achieved by a feedback-dependent Markov source, whose memory length is equal to the memory length of the channel.

Theorem 2: The optimal feedback is captured by the vector of conditional probabilities of all possible channel states given all previous channel outputs, i.e., by the forward coefficients of the BCJR algorithm [1].

Thus, the optimal feedback at time t is the vector of conditional probabilities $\underline{\alpha}_{t-1} = \{\Pr(X_{t-M_c}^{t-1} = s|y_1^{t-1}) : s \in \mathcal{X}^{M_c}\}$, and the feedback capacity is

$$C_{fb} = \sup_{\mathbf{P}} I(X \to Y). \quad (2)$$

In (2), the supremum is taken over the stationary feedback-dependent Markov source distribution $\mathbf{P} = \{P(\underline{\alpha}_{t-1}) = \{\Pr(X_t|X_{t-M_c}^{t-1}, \underline{\alpha}_{t-1})\} : \text{all possible } \underline{\alpha}_{t-1}\}$.

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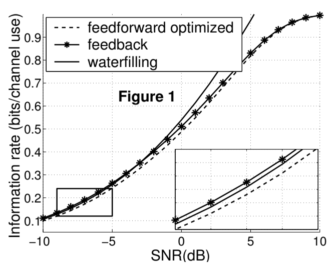
The optimization of the stationary feedback-dependent Markov source distribution $\mathbf{P}$ turns out to be an average-reward-per-stage stochastic control problem [4]. At each stage (time) t, the state is the vector $\underline{\alpha}_{t-1}$, the control (policy) is the feedback-dependent Markov source distribution $P(\underline{\alpha}_{t-1}) = \{\Pr(X_t|X_{t-M_c}^{t-1}, \underline{\alpha}_{t-1})\}$, and the reward is $-\log(f(Y_t|\underline{\alpha}_{t-1})$. Let h^* be the maximum average reward per stage, and $\gamma^*(\underline{\alpha}_{t-1})$ be the optimal relative reward-to-go function for state $\underline{\alpha}_{t-1}$. Then Bellman's equation [4] becomes

$$h^* + \gamma^*(\underline{\alpha}_{t-1}) = \max_{P(\underline{\alpha}_{t-1})} \mathrm{E}\left[-\log f(Y_t|\underline{\alpha}_{t-1}) + \gamma^*(\underline{\alpha}_t)|\underline{\alpha}_{t-1}\right]. \quad (3)$$

We can solve (3) and find an optimal stationary policy $\mathbf{P}$ by using dynamic programming algorithms [4], e.g., value iteration and policy iteration. Then we compute the feedback capacity using the Monte Carlo method in [5].



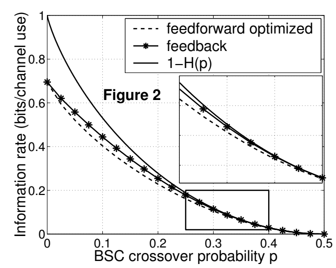


Figure 1 and 2 compare the feedback capacities to the feedforward capacity bounds for the dicode $1-D$ channel and the binary symmetric channel (BSC) with $RLL(0,1)$ input constraint (i.e., no two consecutive 0 inputs are allowed), respectively. For both channels, the gap between the feedback capacity and the feed-forward capacity lower bound (computed by the EM algorithm [6]) is observable. In Figure 1, the feedback capacity exceeds the waterfilling capacity bound at low SNRs, which numerically verifies that feedback increases the capacity of a channel with memory.

References

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