International Symposium on Information Theory and its Applications, ISITA2004
Parma, Italy, October 10–13, 2004

On Optimal Markov Policies for Identifying Markov Channels

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Abstract

We consider the issue of identifying a channel from a finite list of Markov channels. In particular, we consider time-invariant Markov policies of order k, i.e., policies for which the next probing input to the channel depends only on the previous k inputs/outputs of the channel. We show that if the hypothesis test is binary and the Markov channels have order $\ell = 0$ (i.e., the channels are memoryless), then there exists an optimal Markov policy of order $k = 0$ in the sense that no other time-invariant Markov policy of any finite order attains a better exponent to the average probability of error. This parallels a similar result in channel coding for Markov feedback channels. There, it is known that feedback does not increase the capacity of memoryless channels [1] (i.e., the feedback policy that achieves capacity for a Markov channel of order $\ell = 0$ is a Markov policy of order $k = 0$ [2]).

1. INTRODUCTION

In traditional hypothesis testing, we are given a set of hypotheses $\mathcal{H}$. For each hypothesis $h \in \mathcal{H}$, we know the probability law for an observable variable Y, i.e., we know $P_Y^h[y]$. We make n observations $y_1^n = [y_1, y_2, \ldots, y_n]$ and based on these observations, we need to infer the hypothesis $h \in \mathcal{H}$. This is a well known problem with well known solutions in the context of Bayesian and Neyman-Pearson decision making [3]. Furthermore, the type 2 error exponent (for Newman-Pearson) or average error exponent (for Bayesian) detection is well known and may be derived by the method of types [1, 4].

Now, consider a finite set of hypotheses $\mathcal{H}$ such that for each $h \in \mathcal{H}$ we identify a discrete time-invariant Markov channel $P^h_{Y_\ell|X_0^\ell, Y_0^{\ell-1}}[y_\ell|x_0^\ell, y_0^{\ell-1}] > 0$ of order ℓ. We refer to this type of problem as a *finite-hypothesis channel identification* or *channel detection* problem.

In this paper we will consider the following scenario. At time $t > k$, an input to the channel is selected according to a time-invariant distribution $Q_{X_k|X_0^{k-1}, Y_0^{k-1}}[x_t|x_{t-k}^{t-1}, y_{t-k}^{t-1}]$, which we call a Markov policy and shall frequently abbreviate as Q. Then, at some later time n, based on the entire realization of x_1^n and y_1^n, a decision is made as to the nature of the true hypothesis $h \in \mathcal{H}$. The objective is to study the class of Markov policies that optimize some performance measure. The performance measure we choose is the error exponent of the average probability of error for a maximum *a posteriori* (MAP) decision.

This paper is structured as follows. In Section 2, we introduce our notation. Specifically, in Section 2.1, we review the method of Markov types and in Section 2.2 define the error exponent that we study in this paper. In Section 3 we derive our main result, Theorem 4 which states that for discriminating between two Markov channels of order $\ell = 0$, an optimal probing policy of order $k = 0$ exists. This is achieved by means of an intermediary result, Theorem 3. In Theorem 3, the exact error exponent is derived for the class of irreducible Markov policies of order $k \geq 0$. Finally, Section 4 concludes this work.

2. PRELIMINARIES

2.1. Markov Types

Similar to previous work on error exponents, our approach is based on the method of types [4]. Since circular Markov types will be of central importance in this paper, we now develop our notation for them. In particular, if we denote by $U_{X_0,\ldots,X_k} = U_{X_0^k}$ the kth order circular Markov type[1] of a sequence x_1^n, then $U_{X_0^k}$ is a probability mass function (PMF) defined by the relative frequencies,

$$U_{X_0^k}[a_0, \ldots, a_k] = \frac{1}{n}|\{i : x_i^{i+k} = a_0^k, 1 \leq i \leq n\}|, \quad (1)$$

with the cyclic convention that $x_{i+n} = x_i$.

[1]Sometimes Markov types are referred to as higher order

We denote by $\mathcal{U}^n_{X_0^k}$ the set of all kth order circular Markov types for X sequences of length n. Likewise, $T(U_{X_0^k})$ denotes the set of all length n X sequences whose Markov type is $U_{X_0^k}$. Finally, we denote by $\mathcal{F}_{X_0^k}$ the space of all distributions $F_{X_0^k}$ on the tuple $(X_0,\ldots,X_k)$ whose marginalization satisfies $F_{X_0^{k-1}}[x_0^{k-1}] = F_{X_1^k}[x_0^{k-1}]$ with topology induced by the usual topology on $\mathbb{R}^n$. We immediately note that $U_{X_0^k} \in \mathcal{F}_{X_0^k}$.

If we denote the topological interior of a set A by $\mathbf{i}(A)$, then for any $F_{X_0^k} \in \mathbf{i}(\mathcal{F}_{X_0^k})$ ($\mathcal{F}_{X_0^k}$ viewed as a subset of $\mathbb{R}^n$), we have that $F_{X_0^k} > 0$. Hence, the corresponding $F_{X_k|X_0^{k-1}} > 0$ and $F_{X_k|X_0^{k-1}}$ represents an irreducible (equivalently ergodic since the state space is finite [5, 6]) kth order Markov chain. Conversely, for any ergodic Markov chain $F_{X_k|X_0^{k-1}}$ there is a unique invariant distribution $F_{X_0^{k-1}}$. Hence, each interior point of $\mathcal{F}_{X_0^k}$ is in one-to-one correspondence with an ergodic Markov chain $F_{X_k|X_0^{k-1}}$. Furthermore, for any ergodic Markov chain $F_{X_k|X_0^{k-1}}$, we may associate a unique $F_{X_0^k} \in \mathcal{F}_{X_0^k}$ (which need not be in the interior).

If $\Gamma_{X_k|X_0^{k-1}}[x_k|x_0^{k-1}] = 0$ for some x_0^k implies that $U_{X_0^k}[x_0^k] = 0$, we shall denote this by the shorthand notation $U_{X_0^k} \ll \Gamma_{X_k|X_0^{k-1}}$.

Bounds on the number of Markov types have been proven for first order Markov types [7]. As stated in Csiszár [4], these readily generalize to any order. We now state the relevant bounds for arbitrary orders, the proof of which is a simple extension of the first order results in [7].

Theorem 1. *(Davisson, Longo and Sgarro) Let x_1^n be a sequence with kth order Markov type $U_{X_0^k} \in \mathcal{U}^n_{X_0^k}$ and $P[x_1^n]$ a probability mass function on $\mathcal{X}^n$ defined by*

$$P[x_1^n] = \mu_{X_1^k}[x_1^k] \prod_{m=k+1}^{n} \Gamma_{X_k|X_0^{k-1}}[x_m|x_{m-k}^{m-1}]. \tag{2}$$

with $\mu_{X_1^k} > 0$ and $\Gamma_{X_k|X_0^{k-1}} > 0$. Then, for some $\alpha > 0$ and $\beta > 0$ (which depend on $\mu_{X_1^k}$ and $\Gamma_{X_k|X_0^{k-1}}$ respectively), the following hold,

1. $|\mathcal{U}^n_{X_0^k}| \le (n+1)^{|\mathcal{X}|^{k+1}}$
2. $n^{-|\mathcal{X}|^k}(n+1)^{-|\mathcal{X}|^{k+1}} 2^{nH(X_k|X_0^{k-1})} \le |T(U_{X_0^k})| \le |\mathcal{X}|^k 2^{nH(X_k|X_0^{k-1})}$
3. $\alpha 2^{-n[D(U_{X_0^k}||\Gamma_{X_k|X_0^{k-1}})+H(X_k|X_0^{k-1})]} \le P[x_1^n] \le \beta^{-k} 2^{-n[D(U_{X_0^k}||\Gamma_{X_k|X_0^{k-1}})+H(X_k|X_0^{k-1})]}$,

where for notational convenience, we write $H(X_k|X_0^{k-1})$ in place of $H(U_{X_0^k}) - H(U_{X_0^{k-1}})$ and $U_{X_0^{k-1}}$ is a marginalization of $U_{X_0^k}$. Furthermore, if $\Gamma_{X_k|X_0^{k-1}}[x_k|x_0^{k-1}] = 0$ for some x_0^k, then the lower bound in 3 still holds. Finally, if $\Gamma_{X_k|X_0^{k-1}}$ is irreducible and $U_{X_0^k} \ll \Gamma_{X_k|X_0^{k-1}}$, then the upperbound in 3 still holds.

Unfortunately, Theorem 1 does not provide an upperbound on the probability of a sequence x_1^n when $\Gamma_{X_k|X_0^{k-1}}$ is irreducible and $U_{X_0^k}$, the type of x_1^n, does not satisfy $U_{X_0^k} \ll \Gamma_{X_k|X_0^{k-1}}$. The difficulty here is that the sequence x_1^n (or its cyclic extension) contains transitions which are forbidden by $\Gamma_{X_k|X_0^{k-1}}$.

There are two scenarios in which these may occur. First, a forbidden transition may occur in the sequence x_1^n and is not due to the cyclic extension by which $U_{X_0^k}$ is derived. In this case, the upperbound in part 3 of Theorem 1 is still valid since $P[x_1^n] = 0$. In the case that the only forbidden transitions in $U_{X_0^k}$ are due to the cyclic extension, clearly we have $P[x_1^n] > 0$, yet the right-hand side in part 3 of Theorem 1 is 0.

Despite this, it is noted in [7] that for such a type $U_{X_0^k}$ we may find another type $U'_{X_0^k}$ with no forbidden transitions and for which $U'_{X_0^k}$ is sufficiently "close" to $U_{X_0^k}$ that the probability of a sequence of type $U'_{X_0^k}$ is "close" to the probability of x_1^n. In particular, for any irreducible Markov chain with $K = |\mathcal{X}|^k$ states, there is a sequence of at most K allowable transitions between any two states. Hence, by replacing the (at most) last K transitions of x_1^n appropriately and possibly shortening the sequence by up to K terms, we may eliminate the forbidden transitions entirely. Furthermore, since we only replaced (and removed) up to a fixed number K of terms in the sequence, we can upperbound the ratio between the probability of the sequence x_1^n to that of a sequence of type $U'_{X_0^k}$. We may thus complement the upperbound in Theorem 1 with

Theorem 2. *Let x_1^n be a sequence of type $U_{X_0^k}$ and $P[x_1^n]$ a probability mass function on $\mathcal{X}^n$ defined by*

$$P[x_1^n] = \mu_{X_1^k}[x_1^k] \prod_{m=k+1}^{n} \Gamma_{X_k|X_0^{k-1}}[x_m|x_{m-k}^{m-1}]. \tag{3}$$

with $\mu_{X_1^k} > 0$ and $\Gamma_{X_k|X_0^{k-1}}$ irreducible. Let $K = |\mathcal{X}|^k$. Then there exists a type $U'_{X_0^k} \in \mathcal{U}^n_{X_0^k} \bigcup \cdots \bigcup \mathcal{U}^{n-K}_{X_0^k}$ which does not depend on $\Gamma_{X_k|X_0^{k-1}}$ such that

types. A $k+1$ higher order type is a kth order Markov type. We prefer the notation 'kth order Markov type' as it is the type of interest when the sequence is generated by a kth order Markov chain.

1. $|T(U_{X_0^k})| \leq 2^{n\rho_n} 2^{nH(X_k|X_0^{k-1})|_{U'}}$

2. $P[x_1^n] \leq 2^{n\sigma_n} 2^{-n[D(U'_{X_0^k}||\Gamma_{X_k|X_0^{k-1}})+H(X_k|X_0^{k-1})|_{U'}]}$,

with $\rho_n \to 0$ and $\sigma_n \to 0$ as $n \to \infty$ and these do not depend on $U_{X_0^k}$.

Proof outline. 1) Follows from the fact that we have only changed (at most) K terms in the sequence x_1^n. Hence, by the triangle inequality $||U_{X_0^k} - U'_{X_0^k}|| \leq 2(K+k)/n+K/n$, and the result follows by the uniform continuity of $H(X_k|X_0^{k-1})$ on $\mathcal{F}_{X_0^k}$.

2) If the sequence x_1^n has a forbidden transition that is not due to the cyclic extension of x_1^n, then the statement is clearly true for any $U'_{X_0^k}$ since $P[x_1^n] = 0$.

If $U_{X_0^k} \ll \Gamma_{X_k|X_0^{k-1}}$ then we may take $U'_{X_0^k} = U_{X_0^k}$ and by Theorem 1, the bound holds. If the only forbidden transitions are due to the cyclic extension of x_1^n, then by replacing the last K transitions of the cyclic sequence x_1^n, to form a new cyclic sequence $\hat{x}_1^{\hat{n}}$ with no forbidden transitions, we have $P[x_1^n] \leq \alpha^{-1}\beta^{-(K+k)}P[\hat{x}_1^{\hat{n}}]$ (α is the smallest term in $\mu_{X_1^k}$ and β is the smallest non-zero transition probability of $\Gamma_{X_k|X_0^{k-1}}$). The type $U'_{X_0^k}$ of the sequence $\hat{x}_1^{\hat{n}}$ depends on the sequence x_1^n, not just its type $U_{X_0^k}$. However, due to the choice of α and β, the bound derived for a particular x_1^n holds for all $x_1^n \in T(U_{X_0^k})$. Finally, since $U'_{X_0^k} \ll \Gamma_{X_k|X_0^{k-1}}$, the fact that $\hat{n}$ may be up to K terms smaller than n can be universally absorbed by the constants σ_n since the exponent in the upperbound of 2) is continuous over such $U'_{X_0^k}$ and the set of such $U'_{X_0^k}$ is compact. $\square$

Finally, we have the following result, which is a direct extension of Natarajan's result for 1st order Markov types [8].

Lemma 1. *(Natarajan) Given any $F_{X_0^k} \in \mathcal{F}_{X_0^k}$, there exists a sequence of kth order Markov types $U^n_{X_0^k} \in \mathcal{U}^n_{X_0^k}$ such that $U^n_{X_0^k} \to F_{X_0^k}$ as $n \to \infty$.*

Proof idea: It suffices to prove the result for $F_{X_0^k} \in \mathbf{i}(\mathcal{F}_{X_0^k})$. Since the latter is associated with an ergodic Markov chain $F_{X_k|X_0^{k-1}}$, by the strong law of large numbers, a sequence of types $U^n_{X_0^k} \in \mathcal{U}^n_{X_0^k}$ convergent to $F_{X_0^k}$ must exist. $\square$

2.2. Definitions

In general, a Markov policy of order k is given by the time-invariant conditional probabilities $Q_{X_k|X_0^{k-1},Y_0^{k-1}}[x_k|x_0^{k-1}, y_0^{k-1}]$. Employing such a policy, if a decision is made at time n after observing sequences x_1^n and y_1^n, a MAP decision maker will choose,

$$\hat{h}(x_1^n, y_1^n) \triangleq \max_{h\in\mathcal{H}} P[h|x_1^n, y_1^n]. \quad (4)$$

The probability of correct detection is then $P[H = \hat{h}|x_1^n, y_1^n]$. If each successive input is determined according to the policy $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$, then the average probability of correct detection is,

$$P_c^n \triangleq E_{X_1^n,Y_1^n} P[H = \hat{h}(X_1^n, Y_1^n)|X_1^n, Y_1^n], \quad (5)$$

where the distribution on the sequence (X_1^n, Y_1^n) under hypothesis h is given by the policy $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$,the channel $P^h_{Y_\ell|X_0^\ell,Y_0^{\ell-1}}$ and the method by which the initial state X_0^{k-1}, Y_0^{k-1} is chosen.

Definition 1. *The* **feedback error exponent** *of a policy $\{Q_t\}$ is defined to be*

$$E_f(\{Q_t\}) = \lim_{n\to\infty} -\frac{1}{n}\log(1 - P_c^n). \quad (6)$$

Finally, in Section 3, we will find it convenient to generalize the usual max and min operators.

Definition 2. (**minN, maxN**) *Consider an M-tuple of real numbers $(a_1, \ldots, a_M)$ which may be (not necessarily uniquely) ordered as $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_M}$. Let N be an integer between 1 and M. Then, we define*

$$\mathrm{minN}\{a_1, \ldots, a_M\} \triangleq a_{i_N} \quad (7)$$

$$\mathrm{maxN}\{a_1, \ldots, a_M\} \triangleq a_{i_{M-N+1}}. \quad (8)$$

We note that if the numbers $a_1, \ldots, a_M$ are all distinct, then the minN and maxN operators may be interpreted as evaluating the Nth smallest and Nth largest number, respectively. We also note the following fact which is easy to verify:

$$\mathrm{min2}\{a_1, \ldots, a_M\} = \min_{i\neq j} \max\{a_i, a_j\}. \quad (9)$$

3. MAIN RESULTS

We denote the Markov order of the channel by ℓ and that of the policy by k. Since any policy of order less than k is a policy of order k, without loss of generality, we shall assume that $\ell \leq k$. We first derive the exact exponent for a particular class of policies. In particular, we first consider policies for which

$$Q_{X_t|X_1^{t-1},Y_1^{t-1}}[x_t|x_1^{t-1}, y_1^{t-1}] > 0 \quad (10)$$

when $t \leq k$ and

$$Q_{X_t|X_1^{t-1},Y_1^{t-1}}[x_t|x_1^{t-1},x_1^{t-1}] = Q_{X_k|X_0^{k-1},Y_0^{k-1}}[x_t|x_{t-k}^{t-k+1},y_{t-k}^{t-k+1}] \quad (11)$$

otherwise. In other words, we consider the class of time-invariant policies with finite memory k We further restrict ourselves to policies $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$ for which $W^h_{(X,Y)_k|(X,Y)_0^{k-1}} \triangleq P^h_{Y_k|X_{k-\ell}^k,Y_{k-\ell}^{k-1}} Q_{X_k|X_0^{k-1},Y_0^{k-1}}$ is irreducible. We note that since for all h we have assumed that $P^h_{Y_k|X_{k-\ell}^k,Y_{k-\ell}^{k-1}} > 0$ it follows that if $W^h_{(X,Y)_k|(X,Y)_0^{k-1}}$ is irreducible for a particular h, then it is for every $h \in \mathcal{H}$. Furthermore, all W^h have the same state transitions; they differ only in the probabilities of these transitions. Because of this, we denote such a policy $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$ as irreducible even though strictly speaking it is the $W^h_{(X,Y)_k|(X,Y)_0^{k-1}}$ which are irreducible.

Although each successive input is based on the previous k inputs and outputs only, the MAP decision is based on the entire realizations of x_1^n and y_1^n. We now state the following theorem.

Theorem 3. *For a given finite k memory time-invariant irreducible policy $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$, and channels $P^h_{Y_k|X_{k-\ell}^k,Y_{k-\ell}^{k-1}} > 0$ with $\ell \leq k$, define*

$$E_f^*(Q) \triangleq \min_{\substack{h,h' \in \mathcal{H} \\ h \neq h'}} \min_{U_{(X,Y)_0^k} \in \mathcal{F}_{(X,Y)_0^k}} \max\{D(U_{(X,Y)_0^k}||W^h_{(X,Y)_k|(X,Y)_0^{k-1}}), D(U_{(X,Y)_0^k}||W^{h'}_{(X,Y)_k|(X,Y)_0^{k-1}})\}. \quad (12)$$

Then, the error exponent is $E_f(Q) = E_f^*(Q)$.

Proof. If we let $Z_i = (X_i, Y_i)$, then Z_i is a Markov chain of order k which under hypothesis h has time-invariant transition probabilities $W^h_{(X,Y)_k|(X,Y)_0^{k-1}}$.

Case 1: We will first consider the case $Q_{X_k|X_0^{k-1},Y_0^{k-1}}$ is irreducible and show that $E_f(Q) \geq E_f^*(Q)$. Carefully expanding the expression for the probability of correct detection yields,

$$P_c^n = \sum_{(x_1^n,y_1^n)} \sum_h P^h[x_1^n,y_1^n]P[h] \times \max_{h'} \left\{ \frac{P^{h'}[x_1^n,y_1^n]P[h']}{\sum_{h''} P^{h''}[x_1^n,y_1^n]P[h'']} \right\} \quad (13)$$

$$= \sum_{z_1^n \in \mathcal{Z}^n} \max_h \{W^h[z_1^n]P[h]\}, \quad (14)$$

where the summation in (13) is over all sequences $z_1^n = (x_1^n, y_1^n)$ with strictly positive probability under W^h. In (14), we may sum over all sequences z_1^n since those with probability 0 contribute nothing. Employing Theorem 2, the probability of error may be upperbounded as,

$$1 - P_c^n \leq |\mathcal{H}| \sum_{z_1^n} \text{max2}\{W^h[z_1^n]P[h]\} \quad (15)$$

$$\leq |\mathcal{H}| 2^{n\sigma_n} \sum_{U_{Z_0^k} \in \mathcal{U}^n_{Z_0^k}} |T(U_{Z_0^k})| \times \quad (16)$$

$$\max_h 2 \left\{ 2^{-n[D(U'_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}}) + H(Z_k|Z_0^{k-1})|_{U'}]} \right\},$$

where we have employed part 2 of Theorem 2 and $U'_{Z_0^k}$ is an implicit function of $U_{Z_0^k}$ as discussed in Theorem 2. Continuing and using part 1 of Theorem 2,

$$1 - P_c^n \leq |\mathcal{H}| 2^{n(\sigma_n + \rho_n)} \times \sum_{U_{Z_0^k} \in \mathcal{U}^n_{Z_0^k}} \max_h 2\{2^{-nD(U'_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})}\} \quad (17)$$

$$\leq |\mathcal{H}| 2^{n(\sigma_n + \rho_n)} |\mathcal{U}^n_{Z_0^k}| \max_{U''_{Z_0^k} \in \mathcal{U}^n_{Z_0^k} \cup \cdots \cup \mathcal{U}^{n-K}_{Z_0^k}} \Big[\max_h 2\{2^{-nD(U''_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})}\} \Big] \quad (18)$$

$$\leq |\mathcal{H}| 2^{n(\sigma_n + \rho_n)} |\mathcal{U}^n_{Z_0^k}| \times \max_{U_{Z_0^k} \in \mathcal{F}_{Z_0^k}} \max_h 2\{2^{-nD(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})}\}, \quad (19)$$

where the last inequality follows for the fact that $\mathcal{U}^n_{Z_0^k} \subset \mathcal{F}_{Z_0^k}$. Furthermore, we are justified in writting max as opposed to sup since on the subset A of $\mathcal{F}_{Z_0^k}$ defined by $U_{Z_0^k} \ll W^h_{Z_k|Z_0^{k-1}}$, $D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})$ is continuous and the subset A is compact while on $\mathcal{F}_{Z_0^k} \backslash A$, $D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}}) = \infty$. Hence, since $\sigma_n \to 0$ and $\rho_n \to 0$,

$$E_f(Q) \geq \min_{U_{Z_0^k} \in \mathcal{F}_{Z_0^k}} \min_h 2\{D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})\}. \quad (20)$$

Finally, from (9) and (20), this shows that (12) is a lowerbound to $E_f(Q)$.

Case 2: We consider $Q_{X_k|X_0^{k-1},Y_0^{k-1}} > 0$ and show that $E_f(Q) \leq E_f^*(Q)$. From (14), we may lowerbound the probability of error as

$$P_e^n \triangleq 1 - P_c^n \geq \sum_{z_1^n} \max_h 2\{W^h[z_1^n]P[h]\} \quad (21)$$

$$\geq \alpha \sum_{U_{Z_0^k} \in \mathcal{U}^n_{Z_0^k}} |T(U_{Z_0^k})| \times \quad (22)$$

$$\max 2\{2^{-n[D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})+H(Z_k|Z_0^{k-1})]}\}$$
$$\geq \alpha n^{-|\mathcal{Z}|^k}(n+1)^{-|\mathcal{X}|^{k+1}}\times$$
$$\max_{U_{Z_0^k}\in\mathcal{U}^n_{Z_0^k}} \max 2\{2^{-nD(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})}\}, \quad (23)$$

where (22) follows from Theorem 1 part 3 and (23) follows from Theorem 1 part 2. One then obtains

$$E_f(Q) \leq \lim_{n\to\infty} \min_{U_{Z_0^k}\in\mathcal{U}^n_{Z_0^k}} \min_h 2\{D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})\} \quad (24)$$
$$= \min_{U_{Z_0^k}\in\mathcal{F}_{Z_0^k}} \min_h 2\{D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})\} \quad (25)$$

where the last equality follows from Lemma 1 and the continuity of $D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})$ in $U_{Z_0^k}$. Again, (9) and (25) show that (12) is an upperbound to $E_f(Q)$.
Comment: The proofs in case 1 and case 2 combine to show that $E_f(Q) = E^*_f(Q)$ if $Q_{X_k|X_0^{k-1},Y_0^{k-1}} > 0$. Now we look at the case $Q_{X_k|X_0^{k-1},Y_0^{k-1}} \not> 0$.

Case 3: In this case $W^h_{Z_k|Z_0^{k-1}}[z_k|z_0^{k-1}] = 0$ for some z_0^k, but $W^h_{Z_k|Z_0^{k-1}}$ is assumed irreducible (and hence ergodic). Then the expression in (24) is still valid but insufficient to derive (25). We note that since we are performing MAP detection, $P^n_e = \max\{P^n_e, \ldots, P^{n+K}_e\}$. It then follows that (24) is still valid if we minimize over $U_{Z_0^k} \in \mathcal{U}^n_{Z_0^k} \cup \cdots \cup \mathcal{U}^{n+K}_{Z_0^k}$ instead. Then carefull evaluation yields (25). This is not entirely trivial as $D(U_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})$ is not continuous on $\mathcal{F}_{Z_0^k}$. In particular, since all $W^h_{Z_k|Z_0^{k-1}}[z_k|z_0^{k-1}]$ have the same allowable state transitions (i.e., transitions with strictly positive probability), evaluation of (25) for any $U_{Z_0^k}$ associated with a $U_{Z_k|Z_0^{k-1}}$ with exactly the same allowable state transitions as $W^h_{Z_k|Z_0^{k-1}}$ will result in a finite bound on $E_f(Q)$. Consider $U'_{Z_0^k}$ to be any such distribution.

Now, let $U^*_{Z_0^k}$ be a minimizer in (25). $U^*_{Z_0^k}$ is not necessarily associated with an irreducible $U^*_{Z_k|Z_0^{k-1}}$. However by convexity of divergence, for $0 \leq \lambda \leq 1$, the distribution $U^\lambda_{Z_0^k} = (1-\lambda)U^*_{Z_0^k} + \lambda U'_{Z_0^k}$ results in a finite valuation of the expression $D(U^\lambda_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}})$ for any λ. Hence, for $0 < \lambda \leq 1$, there are no non-zero terms in $U^\lambda_{Z_0^k}$ which are not present in $U'_{Z_0^k}$ and vice-versa. Hence, both $U^\lambda_{Z_0^k}$ (for $\lambda > 0$) and $U'_{Z_0^k}$ have the same allowable state transitions and since $U'_{Z_0^k}$ is ergodic, so is $U^\lambda_{Z_0^k}$ for each $\lambda > 0$.

Now, by employing methods similar to Lemma 1 and appropriately changing (and removing) up to the last K terms of any generated sequence with forbidden transitions in its cyclic extension, for each $\lambda > 0$, we can find a sequence of circular types $U^n_{Z_0^k} \in \mathcal{U}^n_{Z_0^k} \cup \cdots \cup \mathcal{U}^{n+K}_{Z_0^k}$ which converges to $U^\lambda_{Z_0^k}$ and for which (for all n greater than some N) both $U^n_{Z_0^k}$ and $U^\lambda_{Z_0^k}$ share exactly the same allowable transitions. Hence, for each $\lambda > 0$, $E_f(Q) \leq \min 2_h\{D(U^\lambda_{Z_0^k}||W^h_{Z_k|Z_0^{k-1}}\}$ and (25) follows by continuity. □

Since every finite state time-invariant Markov chain can be decomposed into ergodic classes, by suitably redefining our notion of state, we may evaluate the error exponent associated with each class. It is then clear that a good policy Q has only one class and all other states transit to the class in at most a finite number of transitions, all of which are deterministic. In this case, direct evaluation of $E^*_f(Q)$ still provides the exponent of the class and hence the policy. If a policy Q^* has several classes, then direct evaluation of $E^*_f(Q^*)$ provides a lower bound to the exponent of any of its classes.

Theorem 4. *The optimal time-invariant Markov policy for discriminating between two Markov channels h and h' of order $\ell = 0$ is a Markov policy of order 0.*

Proof. We will show that from any policy Q of order $k \geq 0$ we may construct another policy Q^* of order $k^* = 0$ such that $E^*_f(Q) \leq E^*_f(Q^*)$. Without loss of generality, it suffices to consider policies Q with only one class and for which all other states transit to the class in a finite number of transitions, all of which are deterministic.

Since the constructed Q^* will have order $k^* = 0$, it is then trivially irreducible. Hence, we will have produced a policy such that $E_f(Q) = E^*_f(Q) \leq E^*_f(Q^*) = E_f(Q^*)$.

We first observe that any $U_{(X,Y)_0^k}$ may be factored into the product $U_{(X,Y)_0^k} = U_{(X,Y)_0^{k-\ell-1}} U_{(X,Y)_{k-\ell}^k|(X,Y)_0^{k-\ell-1}}$. If $U_{(X,Y)_0^{k-\ell-1}}[(x,y)_0^{k-\ell-1}] = 0$ for some $(x,y)_0^{k-\ell-1}$, then the choice of $U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}$ is not unique.

Now, by the definition of Kullback-Leibler divergence [4], we have that for any $U_{(X,Y)_0^k}$,

$$D(U_{(X,Y)_0^k}||W^h_{(X,Y)_k|(X,Y)_0^{k-1}})$$
$$= \sum_{(x,y)_0^{k-\ell-1}} U_{(X,Y)_0^{k-\ell-1}}[(x,y)_0^{k-\ell-1})]\times \quad (26)$$
$$D(U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}||W^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}})$$

$$\leq \max_{(x,y)_0^{k-\ell-1}} \tag{27}$$
$$D(U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}||W^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}})$$

Furthermore, if $U_{(X,Y)_0^{k-\ell-1}}[(x,y)_0^{k-\ell-1}] = 0$ for some $(x,y)_0^{k-\ell-1}$, the bound in (27) holds for any choice of $U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}$.

Then, we note that from (12) and (27), E_f^* may be bounded above as

$$\begin{aligned}
&E_f^*(Q) \\
&\leq \min_{U_{(X,Y)_0^k}\in\mathcal{F}_{(X,Y)_0^k}} \max_{(x,y)_0^{k-\ell-1}} \max\{ \\
&D(U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}||W^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}}), \\
&D(U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}||W^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}})\}.
\end{aligned} \tag{28}$$

Again, if $U_{(X,Y)_0^{k-\ell-1}}[(x,y)_0^{k-\ell-1}] = 0$ for some $(x,y)_0^{k-\ell-1}$, the bound in (28) holds for any choice of $U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}}$.

If instead of minimizing over all $U_{(X,Y)_0^k} \in \mathcal{F}_{(X,Y)_0^k}$, we minimize over $U_{(X,Y)_0^k}$ such that each conditional $U_{(X,Y)_{k-\ell}^k|(x,y)_0^{k-\ell-1}} \in \mathcal{F}_{(X,Y)_{k-\ell}^k}$, then we may interchange the min over such $U_{(X,Y)_0^k}$ with the max over $(x,y)_0^{k-\ell-1}$. Since $\ell = 0$, this set of $U_{(X,Y)_0^k}$ is a subset of $\mathcal{F}_{(X,Y)_0^k}$. Hence, we can only have increased the upperbound.

$$\begin{aligned}
&E_f^*(Q) \leq \\
&\max_{(x,y)_0^{k-\ell-1}} \Big[\min_{U_{(X,Y)_{k-\ell}^k}\in\mathcal{F}_{(X,Y)_{k-\ell}^k}} \max\{ \\
&D(U_{(X,Y)_{k-\ell}^k}||W^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}}), \\
&D(U_{(X,Y)_{k-\ell}^k}||W^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(x,y)_0^{k-\ell-1}})\}\Big].
\end{aligned} \tag{29}$$

Let $(\hat{x},\hat{y})_0^{k-\ell-1}$ denote the maximizing $(x,y)_0^{k-\ell-1}$ in (29) and define

$$V^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}} \triangleq W^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(\hat{x},\hat{y})_0^{k-\ell-1}} \tag{30}$$
$$V^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}} \triangleq W^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1},(\hat{x},\hat{y})_0^{k-\ell-1}}. \tag{31}$$

Then,

$$\begin{aligned}
E_f^*(Q) \leq &\min_{U_{(X,Y)_{k-\ell}^k}\in\mathcal{F}_{(X,Y)_{k-\ell}^k}} \\
&\max\{D(U_{(X,Y)_{k-\ell}^k}||V^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}}), \\
&D(U_{(X,Y)_{k-\ell}^k}||V^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}})\}.
\end{aligned} \tag{32}$$

However, if we further define

$$\begin{aligned}
&Q^*_{X_k|(X,Y)_{k-\ell}^{k-1}}[x_k|(x,y)_{k-\ell}^{k-1}] \\
&\triangleq Q[x_k|(x,y)_{k-\ell}^{k-1},(\hat{x},\hat{y})_0^{k-\ell-1}]
\end{aligned} \tag{33}$$

Then we note that since the channel is only order ℓ,

$$V^h_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}} = P^h_{Y_k|X_{k-\ell}^k,Y_{k-\ell}^{k-1}} Q^*_{X_k|(X,Y)_{k-\ell}^{k-1}} \tag{34}$$
$$V^{h'}_{(X,Y)_k|(X,Y)_{k-\ell}^{k-1}} = P^{h'}_{Y_k|X_{k-\ell}^k,Y_{k-\ell}^{k-1}} Q^*_{X_k|(X,Y)_{k-\ell}^{k-1}}. \tag{35}$$

Hence, we may rewrite the last upperbound in (32) as

$$E_f^*(Q) \leq E_f^*(Q^*) \tag{36}$$

where we have employed the fact the right side of (32) is simply a time shift of (12). In other words, every order $k \geq \ell = 0$ policy has its error exponent upperbounded by that of an order $k^* = 0$ policy. □

4. CONCLUSION

We have considered the problem of designing probing policies for identifying Markov channels. We have derived the exact Bayesian error exponent for irreducible Markov probing policies. Finally, we have shown that for two memoryless channels (i.e., Markov of order $\ell = 0$), there is an optimal policy of order 0 in the sense that no Markov policy of any finite order achieves a better exponent.

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