Delayed Feedback Capacity of Finite-State Machine Channels: Upper Bounds on the Feedforward Capacity

Shaohua Yang[1]
Harvard University
e-mail: yangsh@deas.harvard.edu

Aleksandar Kavčić
Harvard University
e-mail: kavcic@hrl.harvard.edu

Sekhar Tatikonda
Yale University
e-mail: sekhar.tatikonda@yale.edu

Abstract — **The delayed feedback capacity of a finite-state machine channel is achieved by a Markov source whose memory length is equal to the sum of the channel memory length and the feedback delay. Further, the optimal feedback is achieved by computing the causal posterior distribution of the Markov source state. We formulate the optimization of the Markov source as a stochastic optimal control problem, and solve it by dynamic programming. The delayed feedback capacity is a tight upper bound on the feedforward capacity.**

I. Main Results

We consider a finite-state machine channel (X_t, S_t, Y_t). The channel input X_t takes its value from a finite alphabet $\mathcal{X}$. The channel output Y_t is induced by the input sequence $X_{t-M_c}^t = [X_{t-M_c}, X_{t-M_c+1}, \ldots, X_t]$ of length $M_c + 1$ and is corrupted by additive white noise N_t. Thus the conditional probability density function of the channel output Y_t satisfies

$$f_{Y_t|X_{-\infty}^t, Y_{-\infty}^{t-1}}(y_t|x_{-\infty}^t, y_{-\infty}^{t-1}) = f_{Y_t|X_{t-M_c}^t}(y_t|x_{t-M_c}^t), \quad (1)$$

and $S_t \stackrel{\triangle}{=} X_{t-M_c+1}^t = [X_{t-M_c+1}, \ldots, X_t] \in \mathcal{X}^{M_c}$ captures the channel state at time t. Further, we assume that the channel is used with ν-time-delayed noiseless feedback ($\nu \geq 0$), i.e., the encoder, before sending out X_t, knows without error the realizations of $Y_1^{t-1-\nu}$.

A finite-state machine channel used with delayed feedback can always be alternatively reformulated as a finite-state machine channel used with instantaneous feedback. This is achieved by redefining the channel input, state and output at time t as $\tilde{X}_t = X_t$, $\tilde{S}_t \stackrel{\triangle}{=} X_{t-M_c-\nu+1}^t$ and $\tilde{Y}_t = Y_{t-\nu}$, respectively. We apply the results in [1] and get the following theorem.

Theorem 1: The delayed feedback capacity of a finite-state machine channel is

$$C_\nu^{fb} = \sup_{\mathcal{P}} \mathcal{I}(X \to Y), \quad (2)$$

where the supremum is over the Markov source distribution

$$\mathcal{P} = \left\{ \Pr\left(X_t \left| X_{t-M_c-\nu}^{t-1}, \underline{\alpha}_{t-1}\right.\right) \right\}, \quad (3)$$

and $\underline{\alpha}_{t-1}$ is the posterior source state distribution, i.e., the set of probabilities $\left\{ \Pr\left(X_{t-M_c-\nu}^{t-1} = \tilde{s} \left| Y_1^{t-1-\nu} = y_1^{t-1-\nu}\right.\right) \right\}$ for all possible Markov source states $\tilde{s} \in \mathcal{X}^{M_c+\nu}$. $\square$

Further, the delayed feedback capacity C_ν^{fb} equals to the non-delayed feedback capacity $\tilde{C}^{fb}$ of the channel $(\tilde{X}_t, \tilde{S}_t, \tilde{Y}_t)$, which can be computed by solving the related stochastic optimal control problem using dynamic programming algorithms [1].

Let C be the channel capacity without feedback, then $C = \lim_{\nu\to\infty} C_\nu^{fb}$ and

$$C \leq \cdots \leq C_\nu^{fb} \leq \cdots C_1^{fb} \leq C^{fb}. \quad (4)$$

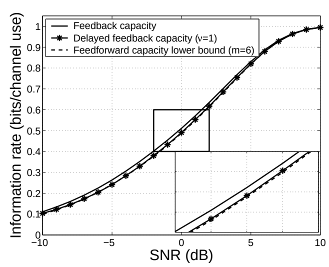
On the other hand, if we let C_m be the maximal information rate achievable without feedback by a Markov source whose memory length is m, then $C = \lim_{m\to\infty} C_m$ and

$$C \geq \cdots \geq C_m \geq \cdots C_1 \geq C_0. \quad (5)$$

The value of C_m can be computed by the algorithm in [2]. Thus, we can tightly bound the channel capacity C of a finite-state machine

$$C_m \leq C \leq C_\nu^{fb} \quad (6)$$

by properly selecting the values of m and ν.



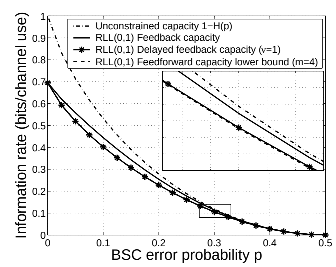


Figure 1 shows capacity bounds for the dicode $(1 - D)$ channel. By taking $\nu = 1$ and $m = 6$, the delayed feedback capacity C_1^{fb} and the capacity lower bound C_6 are almost indistinguishable.

Figure 2 depicts capacity bounds for the binary symmetric channel (BSC) with a run-length-limited $RLL(0,1)$ input constraint. The channel capacity C is tightly bounded between C_1^{fb} and C_4.

References

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[2] A. Kavčić, "On the capacity of Markov sources over noisy channels," in *Proceedings IEEE Global Communications Conference 2001*, (San Antonio, Texas), pp. 2997 –3001, November 2001.

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