Feedback Capacity of Finite-State Machine Channels

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Abstract

We consider a finite-state machine channel with a finite memory length (e.g., finite length intersymbol interference channels with finite input alphabets – also known as partial response channels). For such a finite-state machine channel, we show that feedback-dependent Markov sources achieve the feedback capacity, and that the required memory length of the Markov process matches the memory length of the channel. Further, we show that the whole history of feedback is summarized by the causal posterior channel state distribution, which is computed by the sum-product forward recursion of the BCJR (Baum-Welch, discrete-time Wonham filtering) algorithm. These results drastically reduce the space over which the optimal feedback-dependent source distribution needs to be sought. Further, the feedback capacity computation may then be formulated as an average-reward-per-stage stochastic control problem, which is solved by dynamic programming. With the knowledge of the capacity-achieving source distribution, the value of the capacity is easily estimated using Markov chain Monte Carlo methods. When the feedback is delayed, we show that the feedback capacity can be computed by similar procedures. We also show that the delayed feedback capacity is a tight upper bound on the feed-forward capacity by comparing it to tight existing lower bounds. We demonstrate the applicability of the method by computing the feedback capacity of partial response channels and the feedback capacity of run-length-limited (RLL) sequences over binary symmetric channels (BSCs).

Index Terms

channel capacity, delayed feedback capacity, directed information rate, dynamic programming, feedback capacity, finite-state machine channels, intersymbol interference, partial response channels, run-length-limited sequences

I. Introduction

The feedback capacity of a *memoryless* channel equals the feed-forward capacity of the same channel [1]. However, the computation (or characterization) of the feedback capacity of channels *with memory* has long remained an open problem [2]. In 1990, Massey [3] showed that the supremum of the directed information rate is an upper bound on the feedback capacity. In 2000, Tatikonda [4] proved that any directed information rate is achievable, and thus proved that the feedback capacity is the supremum of the directed information rate.

In this paper, we simplify the formulation and develop a dynamic programming procedure to compute the feedback capacity for channels that can be represented as finite-state machines with a finite input memory length. We prove two theorems that drastically simplify the computational problem. Namely, I) finite-memory feedback-dependent Markov sources achieve the feedback capacities, and II) the optimal feedback is computed by the forward sum-product recursion of the BCJR (or Baum-Petrie, Baum-Welch, also known as the discrete-time version of Wonham [5] filtering) algorithm [6], [7], [8], [9]. The feedback capacity is thus evaluated by combining three tools:

1) the forward sum-product recursion of the BCJR (Baum-Welch) algorithm [5], [6], [7], [8], [9],
2) dynamic programming for solving Bellman's equation [10] (a similar dynamic programming approach was introduced for channels where the channel state is known to the receiver [4]),
3) a Monte Carlo method (for computing the information rate), similar to the one proposed in [11], [12].

It is interesting to contrast the feedback capacity to the feed-forward capacity. For a *memoryless* channel, the feedback capacity equals the feed-forward capacity, and the capacity is achieved by a memoryless source [1]. It may be tempting to conjecture that the feed-forward capacity of a channel whose memory length is 1 symbol interval, is achieved by a 1-st order Markov process. This is clearly not correct as demonstrated in [11], [12], [13], where higher-order Markov processes are constructed to surpass the (feed-forward) information rates of lower-order Markov processes. If, however, we utilize feedback, we show in this paper that a feedback-dependent Markov process *does* achieve the feedback capacity. In other words, we show that the feedback capacity is achieved by a *feedback-dependent Markov source* whose memory length matches the channel memory length, i.e., given the state S_{t-1} at time $t-1$ and all prior feedback Y_1^{t-1}, the source state S_t at each time t is independent of all states S_0^{t-2} prior to time $t-1$. We get a generalized statement (which does not hold for the feed-forward capacity): *the feedback capacity of a channel whose memory length is L, is achieved by a feedback-dependent L-th order Markov source.* This generalizes a known fact that a memoryless source (i.e, a 0-th order Markov source) achieves the (feedback) capacity of a memoryless channel (i.e., a channel whose memory length is 0).

In this paper, we also show that when the feedback is delayed by ν symbol intervals, the corresponding delayed feedback capacity, C_ν^{fb}, is achieved by a feedback-dependent Markov source whose memory length is the sum of the channel memory length and the feedback delay ν. Note that as ν approaches infinity, there is no feedback information available to the transmitter, i.e., $\lim_{\nu\to\infty} C_\nu^{fb} = C$, where C is the capacity of the channel used without feedback. Thus, we can upper bound the feed-forward capacity C by the sequence of delayed feedback capacities C_ν^{fb} as follows

$$C \leq \cdots \leq C_1^{fb} \leq C_0^{fb} = C^{fb}. \tag{1}$$

Hence, we need only optimize feedback-dependent Markov sources with finite memory lengths to upperbound the feed-forward capacity.

Notation: Uppercase letters represent random variables (or vectors), while lowercase letters represent their realizations. An index t next to a random variable (e.g., X_t) denotes the random variable at discrete time $t \in \mathbb{Z}$. A vector of time-dependent variables is denoted as $X_t^n = [X_t, X_{t+1}, \ldots, X_n]$. Operators $\mathrm{E}[\cdot]$ and $\mathrm{E}_X[\cdot]$ represent the expectation and the expectation over the random variable X, respectively. The letters I and $\mathcal{I}$ are used for the information and the information *rate*, respectively. $H(X)$ represents the entropy of a discrete random variable X, while $h(Y)$ represents the differential entropy of a continuous random variable Y. The function $f_Y(y)$ is the probability density function (pdf) of the random variable Y, while $f_{Y|X}(y|x)$ stands for the conditional pdf of Y given X.

Structure: In Section II, we give the finite-memory finite-state channel model assumptions. In Section III, we study and simplify the feedback capacity computation problem for the finite-memory finite-state channels. In Section IV, we formulate the feedback capacity computation problem as a stochastic control problem and give a dynamic programming solution. In Section V, the computation of the delayed feedback capacity is discussed and a solution is given. The delayed feedback capacity can be used to upper bound the feed-forward capacity. In Section VI, we estimate the feedback capacity numerically and show results for the dicode channel and the binary symmetric channel with a run-length-limited channel input constraint. Section VII concludes the paper.

II. Channel Model

As depicted in Figure 1, we consider a discrete-time communication system consisting of a noisy forward channel and a noiseless causal feedback link. The forward channel is represented

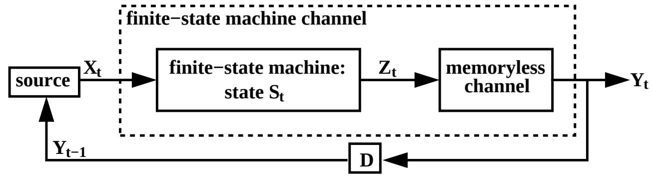

Fig. 1. A finite-state machine channel used with noiseless feedback.

by a finite-state machine observed through a memoryless noisy channel, see Figure 1. The state of the channel at time t is denoted by S_t, and the state realization is denoted by s_t. The state alphabet $\mathcal{S}$ is finite, $|\mathcal{S}| = M < \infty$, i.e., $s_t \in \mathcal{S} = \{0, 1, \ldots, M-1\}$. The channel input process is denoted by X_t, while the input process realization is denoted by x_t. The process X_t is drawn from a finite alphabet $\mathcal{X}$, i.e., $x_t \in \mathcal{X}$, and $|\mathcal{X}| < \infty$. The channel satisfies the following assumptions:

1) The state s_t is a function of s_{t-1} and x_t, i.e., $s_t = q\left(s_{t-1}, x_t\right)$. Some state pairs $(s_{t-1}, s_t) = (i, j)$ may not be valid, that is, the channel cannot be taken from state i to state j. We denote the set of all *valid* state pairs by $\mathcal{T}$.
2) The initial state of the channel $S_0 = s_0$ is known to both the transmitter and receiver, and there is a 1-to-1 correspondence between the state sequence S_1^t and the input sequence X_1^t for any given initial state s_0, that is
$$\left(s_0, X_1^t\right) \overset{1:1}{\longleftrightarrow} \left(s_0, S_1^t\right). \tag{2}$$
Hence, in the following, we interchangeably use the input symbol X for the state symbol S. Assumption (2) is a bit restrictive in the sense that given states s_{t-1} and s_t, the input x_t is uniquely determined, i.e., $q(s_{t-1}, \cdot)$ is invertible. In many instances (such as the partial response channel [14]) this assumption holds. Though assumption (2) is strictly not necessary[1] and can be relaxed (e.g., two different input sequences $X_1^t \neq \tilde{X}_1^t$ can map to the same state sequence S_1^t) it eases the derivations because we can substitute the input symbol X with the state symbol S.

[1]The result in this paper would also hold for, say finite-state machine channels, for which there are multiple branches for a given pair of states. Strictly speaking, such channels do not satisfy (2).

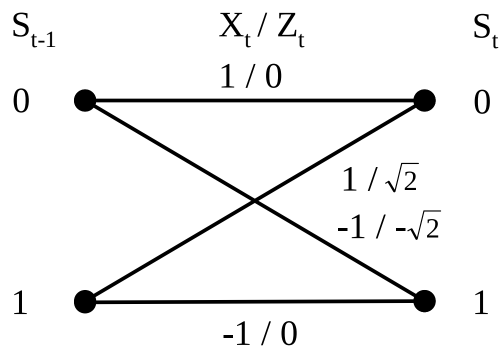

Fig. 2. Trellis representation of the dicode partial response channel.

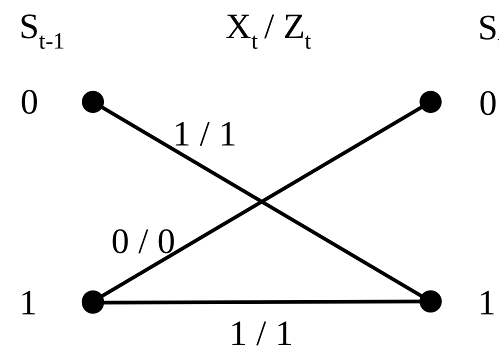

Fig. 3. The trellis of the RLL(0,1) constraint.

3) Conditioned on the channel state S_{t-1} and input X_t, the channel output Y_t is independent of the past channel states, inputs and outputs:

$$f_{Y_t|S_0,X_1^t,S_1^t,Y_1^{t-1}}\left(y_t \left| s_0, x_1^t, s_1^t, y_1^{t-1}\right.\right) = f_{Y_t|S_{t-1},X_t}(y_t \left| s_{t-1}, x_t\right.) \,. \tag{3}$$

4) The transmitter, before emitting symbol X_t, knows all previous channel output symbols Y_1^{t-1} without error.

We generally assume that the finite-state machine is *indecomposable* (or *irreducible*), i.e., every state is accessible from every other state within finite time [15], [16]. The channel model is illustrated with two examples.

Example 1 (The dicode partial response channel): Let $\mathcal{X} = \{-1, 1\}$ and let

$$Z_t = \frac{X_t - X_{t-1}}{\sqrt{2}} \tag{4}$$

$$Y_t = Z_t + W_t, \tag{5}$$

where W_t is white Gaussian noise with variance σ^2, shortly denoted by $W_t{\sim}\mathcal{N}\left(0, \sigma^2\right)$. Denote the state by $S_t = \frac{1-X_t}{2}$. So, $s_t \in \mathcal{S} = \{0, 1\}$. Clearly, $Z_t = \sqrt{2}\left(S_{t-1} - S_t\right)$, and

$$f_{Y_t|S_{t-1},S_t}\left(y_t \left| s_{t-1}, s_t\right.\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{\left[y_t - \sqrt{2}\left(S_{t-1} - S_t\right)\right]^2}{2\sigma^2}} . \tag{6}$$

The channel is represented by the trellis in Figure 2. As evident from the figure, the set of valid state pairs is $\mathcal{T} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. □

Example 2 (RLL(0,1) sequences over BSC): Let the channel input $X_t \in \mathcal{X} = \{0, 1\}$ be a run-length-limited (RLL) sequence with the constraint RLL(0,1), i.e., no consecutive zeros appear in the sequence $X_0, X_1, \cdots, X_t, \cdots$. The memory in the channel is exhibited by the constraint that if $X_{t-1} = 0$, then necessarily $X_t = 1$. The sequence X_t is transmitted over a binary symmetric channel (BSC) with cross-over probability p. We may define the state as $S_t = X_t$, so $s_t \in \mathcal{S} = \{0, 1\}$. The channel is represented by

$$Z_t = X_t \tag{7}$$

$$Y_t = Z_t \oplus W_t, \tag{8}$$

where $\oplus$ denotes binary addition, and W_t represents a sequence of binary independent and identically distributed (i.i.d.) random variables with $\Pr(W_t = 1) = p$. The channel law is

$$\Pr\left(Y_t = y_t \,|S_{t-1} = s_{t-1}, S_t = s_t\right) = \begin{cases} p & \text{if } y_t \neq s_t \\ 1-p & \text{if } y_t = s_t \end{cases}. \tag{9}$$

The channel is represented by the trellis in Figure 3. As evident from the figure, the set of valid state pairs is $\mathcal{T} = \{(0,1), (1,0), (1,1)\}$. Here, the pair $(0,0)$ is not a valid state pair, because this state pair would violate the RLL(0,1) constraint. $\square$

III. The Feedback Capacity

In [3], Massey proved that the supremum of the directed information rate is an upper bound on the feedback capacity. Tatikonda [4] proved that the directed information rate is achievable, and thus proved that the feedback capacity of a channel is the supremum of the directed information rate. Denoting the directed information by $I\left(X_1^n \to Y_1^n \,|s_0\right)$, the directed information *rate* is defined as[2]

$$\mathcal{I}\left(X \to Y \,|s_0\right) \triangleq \lim_{n\to\infty} \frac{1}{n} I\left(X_1^n \to Y_1^n \,|s_0\right) \tag{10}$$

$$\triangleq \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} I\left(X_1^t; Y_t \,\middle|\, s_0, Y_1^{t-1}\right). \tag{11}$$

[2]Here, we note that the limits in (10) and (11) may not exist for all source distributions, and "$\liminf$" should be used instead of "$\lim$". However, in this paper, we show that the problem of maximizing $\frac{1}{n} I\left(X_1^n \to Y_1^n \,|s_0\right)$, for an arbitrary value of $n > 0$, is a stochastic-control dynamic-programming problem, and thus the limits in (10) and (11) do exist for those source distributions that maximize the value $\frac{1}{n} I\left(X_1^n \to Y_1^n \,|s_0\right)$ for an infinite horizon ($n = \infty$), see [10], [17]. Thus, in the sequel, we use "lim" for defining the directed information rate, since we only need to consider source distributions such that the limits in (10) and (11) exist. See also [4].

The feedback capacity of the finite-memory finite-state machine channel may be expressed as

$$C^{fb} = \inf_{s_0} \sup_{\mathcal{P}} \mathcal{I}\left(X \to Y \left| s_0 \right.\right), \tag{12}$$

where the supremum is taken over the set of all (causal conditional) probability measure functions

$$\mathcal{P} = \left\{ \Pr\left(S_t = s_t \left| S_0^{t-1} = s_0^{t-1}, Y_1^{t-1} = y_1^{t-1} \right.\right), t = 1, 2, \cdots \right\}. \tag{13}$$

Note that due to channel assumption 2), we do not need to explicitly write the dependence on X_1^{t-1} in (13). Thus, to compute the feedback capacity is equivalent to finding the maximal directed information rate.

For an indecomposable finite-state machine channel, the initial state s_0 does not affect the directed information rate $\mathcal{I}(X \to Y \left| s_0 \right.)$ when induced by an optimal source distribution [16]. Hence, we will not need the dependence on s_0 in the directed information rate, that is

$$\mathcal{I}\left(X \to Y \left| s_0 \right.\right) = \mathcal{I}\left(X \to Y\right), \tag{14}$$

and

$$C^{fb} = \sup_{\mathcal{P}} \mathcal{I}\left(X \to Y\right). \tag{15}$$

If we were to apply (15) directly, the computation of the feedback capacity would be an enormously complex problem since we would need to specify $\Pr\left(S_t \left| S_0^{t-1}, Y_1^{t-1} \right.\right)$ for every time t and every realization $\left(S_0^t, Y_1^{t-1}\right) = \left(s_0^t, y_1^{t-1}\right)$. As the horizon t increases, the number of realizations $\left(S_0^t, Y_1^{t-1}\right) = \left(s_0^t, y_1^{t-1}\right)$ becomes unbounded, and the problem becomes intractable. The two theorems in this section will help us to reduce the problem to a manageable complexity.

Before stating and proving the theorems, we manipulate the expression of the directed information rate (14) for the finite-state channel in Section II as

$$\begin{aligned}
\mathcal{I}\left(X \to Y\right) &\stackrel{(a)}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} I\left(S_1^t; Y_t \left| s_0, Y_1^{t-1} \right.\right) && (16)\\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[h\left(Y_t \middle| s_0, Y_1^{t-1}\right) - h\left(Y_t \middle| s_0, S_1^t, Y_1^{t-1}\right) \right] && (17)\\
&\stackrel{(b)}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[h\left(Y_t \middle| s_0, Y_1^{t-1}\right) - h\left(Y_t \middle| s_0, S_{t-1}^t, Y_1^{t-1}\right) \right] && (18)\\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} I\left(S_{t-1}^t; Y_t \left| s_0, Y_1^{t-1} \right.\right) && (19)
\end{aligned}$$

where (a) and (b) are from the channel assumptions 2) and 3), respectively, given in Section II.

Definition 1 (A Condition-Dependent Markov Source): A source sequence S_t is said to be a *condition-dependent Markov source* (conditioned on the event $\mathcal{E}$) if

$$\Pr\left(S_t = s_t \left| S_0^{t-1} = s_0^{t-1}, \mathcal{E}\right.\right) = \Pr\left(S_t = s_t \left| S_{t-1} = s_{t-1}, \mathcal{E}\right.\right).$$

□

Definition 2 (A Feedback-Dependent Markov Source): A source sequence S_t is said to be a *feedback-dependent Markov source* if it is a condition-dependent Markov source, where the condition is the past channel output feedback $Y_1^{t-1} = y_1^{t-1}$, i.e., if the following holds

$$\Pr\left(S_t = s_t \left| S_0^{t-1} = s_0^{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right) = \Pr\left(S_t = s_t \left| S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right).$$

□

Theorem 1: For a finite-state machine channel, the feedback capacity is achieved by a feedback-dependent Markov source distribution determined by the following collection of state transition probabilities

$$\mathcal{P}^{\mathrm{Markov}} = \left\{\Pr\left(S_t = s_t \left| S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right), t = 1, 2, \cdots\right\}. \tag{20}$$

□

Proof: Here, we prove that given any feedback-dependent source distribution, there always exists at least one feedback-dependent *Markov* source distribution which induces the same directed information rate. By this argument, we show that for any feedback-capacity-achieving source distribution, there always exists a feedback-dependent Markov source distribution that also achieves the feedback capacity.

Let $\mathcal{P}_1$ be an arbitrary feedback-dependent source distribution as follows

$$\mathcal{P}_1 = \left\{\Pr\left(S_t = s_t \left| S_0^{t-1} = s_0^{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right) \triangleq P_t\left(s_t \left| s_0^{t-1}, y_1^{t-1}\right.\right), t = 1, 2, \cdots\right\}. \tag{21}$$

The source $\mathcal{P}_1$ induces the following distribution for the channel states S_{t-1}^t and outputs Y_1^t

$$f^{(\mathcal{P}_1)}_{S_{t-1}^t, Y_1^t | S_0}\left(s_{t-1}^t, y_1^t \left| s_0\right.\right) = \sum_{s_1^{t-2}} \prod_{\tau=1}^{t} P_\tau\left(s_\tau | s_0^{\tau-1}, y_1^{\tau-1}\right) f_{Y_\tau | S_{\tau-1}, S_\tau}\left(y_\tau \left| s_{\tau-1}, s_\tau\right.\right), \tag{22}$$

which takes non-zero values only if the state transition s_{t-1}^t is valid, i.e., $s_{t-1}^t \in \mathcal{T}$. Denote by $P_t^{(\mathcal{P}_1)}\left(s_t \left| s_0, s_{t-1}, y_1^{t-1}\right.\right)$ the conditional probability $\Pr\left(S_t = s_t \left| S_0 = s_0, S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right)$

when the state sequence S_t is generated by source $\mathcal{P}_1$. Since s_0 is chosen at the beginning of the transmission and it does not change, we will drop it from the notation to simplify the expression, that is

$$P_t^{(\mathcal{P}_1)}\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right) \triangleq P_t^{(\mathcal{P}_1)}\left(s_t \left| s_0, s_{t-1}, y_1^{t-1}\right.\right). \tag{23}$$

For $t = 1, 2, \cdots$, we define the conditional probability mass functions $Q_t\left(S_t \left| S_{t-1}, Y_1^{t-1}\right.\right)$ as

$$\begin{aligned} Q_t\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right) &\triangleq P_t^{(\mathcal{P}_1)}\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right) \\ &= \frac{\sum\limits_{s_1^{t-2}} \prod\limits_{\tau=1}^{t-1} P_\tau\left(s_\tau \left| s_0^{\tau-1}, y_1^{\tau-1}\right.\right) f_{Y_\tau|S_{\tau-1},S_\tau}\left(y_\tau \left| s_{\tau-1}, s_\tau\right.\right) P_t\left(s_t \left| s_0^{t-1}, y_1^{t-1}\right.\right)}{\sum\limits_{s_1^{t-2}} \prod\limits_{\tau=1}^{t-1} P_\tau\left(s_\tau \left| s_0^{\tau-1}, y_1^{\tau-1}\right.\right) f_{Y_\tau|S_{\tau-1},S_\tau}\left(y_\tau \left| s_{\tau-1}, s_\tau\right.\right)}. \end{aligned} \tag{24}$$

Here, the function $Q_t\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right)$ takes non-zero values only if the state transition s_{t-1}^t is valid, i.e., $s_{t-1}^t \in \mathcal{T}$. Now, using the above functions $Q_t\left(S_t \left| S_{t-1}, Y_1^{t-1}\right.\right)$, we construct a feedback-dependent Markov source distribution $\mathcal{P}_2$ as

$$\mathcal{P}_2 = \left\{ \Pr\left(S_t = s_t \left| S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right) \triangleq Q_t\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right), t = 1, 2, \cdots \right\}. \tag{25}$$

The distribution of the channel outputs Y_1^t and states S_{t-1}^t induced by the source $\mathcal{P}_2$ is

$$f_{S_{t-1}^t, Y_1^t|S_0}^{(\mathcal{P}_2)}\left(s_{t-1}^t, y_1^t \left| s_0\right.\right) = \sum_{s_1^{t-2}} \prod_{\tau=1}^{t} Q_\tau\left(s_\tau \left| s_{\tau-1}, y_1^{\tau-1}\right.\right) f_{Y_\tau|S_{\tau-1},S_\tau}\left(y_\tau \left| s_{\tau-1}, s_\tau\right.\right) \tag{26}$$

$$\begin{aligned} &\overset{(a)}{=} \sum_{s_1^{t-2}} \prod_{\tau=1}^{t} P_\tau\left(s_\tau | s_0^{\tau-1}, y_1^{\tau-1}\right) f_{Y_\tau|S_{\tau-1},S_\tau}\left(y_\tau \left| s_{\tau-1}, s_\tau\right.\right) \\ &= f_{S_{t-1}^t, Y_1^t|S_0}^{(\mathcal{P}_1)}\left(s_{t-1}^t, y_1^t \left| s_0\right.\right), \end{aligned} \tag{27}$$

where (a) is the result of substituting (24) into (26) and simplifying the expression.

Thus, the sources $\mathcal{P}_1$ and $\mathcal{P}_2$ induce the same distribution for the symbols S_{t-1}^t and Y_1^t since (27) equals (22). Noting that the expression of the directed information rate in (19) depends only on the pdf $f_{S_{t-1}^t, Y_1^t|S_0}(\cdot)$, we conclude that the feedback-dependent Markov source $\mathcal{P}_2$ achieves the same directed information rate $\mathcal{I}(X \to Y)$ as the non-Markov source $\mathcal{P}_1$. ■

By Theorem 1, without loss of optimality, we need only consider a feedback-dependent Markov source distribution

$$\mathcal{P}^{\text{Markov}} = \left\{ \Pr\left(S_t = s_t \left| S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right.\right) \triangleq P_t\left(s_t \left| s_{t-1}, y_1^{t-1}\right.\right), t = 1, 2, \cdots \right\}. \tag{28}$$

Corollary 1: A feedback-dependent Markov source of order L achieves the feedback capacity of a channel whose memory length is L. □

Proof: If the channel state is captured by the latest L channel inputs, i.e., $S_t = X_{t-L+1}^t$, then the corollary follows directly from Theorem 1. ■

Corollary 1 is a generalization of a well known result by Shannon [1] that a memoryless source (i.e., a source of memory length 0) achieves the (feedback) capacity of a memoryless channel. Note that a statement similar to Corollary 1 does *not* hold for the feed-forward capacity.

Definition 3: Let $\underline{A}_t$ be the random vector of the causal posterior probabilities of the channel state S_t, whose realization $\underline{\alpha}_t$ is

$$\underline{\alpha}_t = \left[\alpha_t(0), \alpha_t(1), \cdots, \alpha_t(M-1)\right], \tag{29}$$

where

$$\alpha_t(\ell) = \Pr\left(S_t = \ell \,\middle|\, S_0 = s_0, Y_1^t = y_1^t\right). \tag{30}$$

Note that if the output observations y_1^{t-1} and the feedback-dependent Markov transition probabilities $\left\{P_\tau\left(s_\tau | s_{\tau-1}, y_1^{\tau-1}\right), \tau = 1, 2, \ldots, t\right\}$ up to time t are given, the values of $\alpha_t(\ell)$ can be computed by the forward recursion of the BCJR (Baum-Welch, sum-product) algorithm [5], [6], [7], [8], [9] as follows

$$\alpha_t\left(\ell\right) = \frac{\sum\limits_i \alpha_{t-1}(i) P_t\left(\ell | i, y_1^{t-1}\right) f_{Y_t|S_{t-1},S_t}(y_t\,| i, \ell)}{\sum\limits_{i,j} \alpha_{t-1}(i) P_t\left(j | i, y_1^{t-1}\right) f_{Y_t|S_{t-1},S_t}(y_t\,| i, j)}. \tag{31}$$

□

Theorem 2: For a finite-state machine channel, the vector of causal posterior channel state probabilities $\underline{\alpha}_{t-1}$ can be used to replace the feedback y_1^{t-1} for the purpose of determining the optimal (i.e., feedback-capacity-achieving) feedback-dependent Markov source distribution at time t. That is, the feedback-capacity-achieving feedback-dependent Markov source (condition-dependent Markov source) satisfies

$$\Pr\left(S_t = s_t \,\middle|\, S_{t-1} = s_{t-1}, Y_1^{t-1} = y_1^{t-1}\right) = \Pr\left(S_t = s_t \,\middle|\, S_{t-1} = s_{t-1}, \underline{A}_{t-1} = \underline{\alpha}_{t-1}\right).$$

□

Proof: We consider the problem of maximizing the directed information $I\left(X_1^n \to Y_1^n \,| s_0\right)$ for an arbitrary finite horizon n. We show that we can replace y_1^{t-1} with the vector $\underline{\alpha}_{t-1}$ such that the transmitter can select the optimal feedback-dependent Markov source distribution beyond time $t-1$ as $\left\{P_\tau\left(s_\tau\left|s_{\tau-1},\underline{\alpha}_{t-1},y_t^{\tau-1}\right.\right),t\le\tau\le n\right\}$. Thus, without loss of optimality, we can let the feedback-dependent Markov source distribution at time $\tau=t$ be dependent on the vector $\underline{\alpha}_{t-1}$ instead of y_1^{t-1}, i.e., $\left\{P_t\left(s_t\left|s_{t-1},\underline{\alpha}_{t-1}\right.\right)\right\}$. Since the horizon n is arbitrary, the same result holds for maximizing the directed information rate as $n\to\infty$. In the following proof, the feedback-dependent Markov source transition probabilities are also referred to as the *policy* used by the transmitter to generate the channel inputs.

We first show that, given the optimal policy $\left\{P_\tau\left(s_\tau\left|s_{\tau-1},y_1^{\tau-1}\right.\right),1\le\tau\le t-1\right\}$ up to time $t-1$, the policy beyond time $t-1$, i.e., $\left\{P_\tau\left(s_\tau\left|s_{\tau-1},y_1^{\tau-1}\right.\right),t\le\tau\le n\right\}$, is optimal if and only if it maximizes $\sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau};Y_\tau\left|s_0,y_1^{t-1},Y_t^{\tau-1}\right.\right)$. This is observed by expressing the directed information $I\left(X_1^n\to Y_1^n\left|s_0\right.\right)$ in (19) as

$$\begin{aligned}I\left(X_1^n\to Y_1^n\left|s_0\right.\right)&=\sum_{\tau=1}^{n} I(S_{\tau-1}^{\tau};Y_\tau\left|s_0,Y_1^{\tau-1}\right.)\\&=\sum_{\tau=1}^{t-1} I\left(S_{\tau-1}^{\tau};Y_\tau\left|s_0,Y_1^{\tau-1}\right.\right)+\int f_{Y_1^{t-1}|S_0}(y_1^{t-1}|s_0)\sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau};Y_\tau\middle|s_0,y_1^{t-1},Y_t^{\tau-1}\right)\mathrm{d}y_1^{t-1}\end{aligned}\quad(32)$$

where we note that the distribution of the symbols S_1^{t-1} and Y_1^{t-1} is computed as

$$f_{S_1^{t-1},Y_1^{t-1}|S_0}\left(s_1^{t-1},y_1^{t-1}\left|s_0\right.\right)=\prod_{\tau=1}^{t-1} P_\tau\left(s_\tau\left|s_0^{\tau-1},y_1^{\tau-1}\right.\right)f_{Y_\tau|S_{\tau-1},S_\tau}\left(y_\tau\left|s_{\tau-1},s_\tau\right.\right)$$

and is not a function of the policy beyond time $t-1$. Therefore, given the optimal policy up to time $t-1$, the policy beyond $t-1$ is optimal if and only if the sum $\sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau};Y_\tau\left|s_0,y_1^{t-1},Y_t^{\tau-1}\right.\right)$ is maximized. We note that this is an instance of Bellman's principle of optimality [10].

Now, we show that, without loss of optimality, to maximize $\sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau};Y_\tau\left|s_0,y_1^{t-1},Y_t^{\tau-1}\right.\right)$, we can let the policy beyond time $t-1$ (for $t\le\tau\le n$) be

$$P_\tau\left(s_\tau\left|s_{\tau-1},y_1^{t-1},y_t^{\tau-1}\right.\right)=P_\tau\left(s_\tau\left|s_{\tau-1},\underline{\alpha}_{t-1},y_t^{\tau-1}\right.\right)\quad(33)$$

where the entire past y_1^{t-1} is substituted by the vector $\underline{\alpha}_{t-1}$. Suppose two vectors y_1^{t-1} and $\tilde{y}_1^{t-1}$ ($y_1^{t-1}\neq\tilde{y}_1^{t-1}$) induce *identical* causal posterior channel state distributions $\underline{\alpha}_{t-1}=\underline{\tilde{\alpha}}_{t-1}$, that is, for any possible state value $s_{t-1}=\ell$ we have

$$\Pr\left(S_{t-1}=\ell\left|S_0=s_0,Y_1^{t-1}=y_1^{t-1}\right.\right)=\Pr\left(S_{t-1}=\ell\left|S_0=s_0,Y_1^{t-1}=\tilde{y}_1^{t-1}\right.\right).$$

If we let the policies for y_1^{t-1} and $\tilde{y}_1^{t-1}$ beyond time $t-1$ be equal, i.e., (for $t\le\tau\le n$)

$$P_\tau\left(s_\tau\left|s_{\tau-1},y_1^{t-1},y_t^{\tau-1}\right.\right)=P_\tau\left(s_\tau\left|s_{\tau-1},\tilde{y}_1^{t-1},y_t^{\tau-1}\right.\right)\quad(34)$$

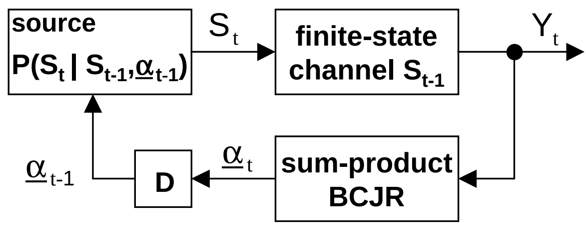

Fig. 4. Feedback loop that achieves the feedback capacity.

we then have

$$\sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau}; Y_\tau \left| s_0, y_1^{t-1}, Y_t^{\tau-1}\right.\right) = \sum_{\tau=t}^{n} I\left(S_{\tau-1}^{\tau}; Y_\tau \left| s_0, \tilde{y}_1^{t-1}, Y_t^{\tau-1}\right.\right), \tag{35}$$

which can be verified by the following equalities

$$\begin{aligned} & f_{S_{t-1}^n, Y_t^n \left| S_0, Y_1^{t-1}\right.}\left(s_{t-1}^n, y_t^n \left| s_0, \tilde{y}_1^{t-1}\right.\right) \\ &= \alpha_{t-1}(s_{t-1}) \prod_{\tau=t}^{n} P_\tau\left(s_\tau \left| s_{\tau-1}, y_1^{t-1}, y_t^{\tau-1}\right.\right) f_{Y_\tau \left| S_{\tau-1}^\tau\right.}\left(y_\tau \left| S_{\tau-1}^\tau\right.\right) \\ &= f_{S_{t-1}^n, Y_t^n \left| S_0, Y_1^{t-1}\right.}\left(s_{t-1}^n, y_t^n \left| s_0, y_1^{t-1}\right.\right). \end{aligned}$$

Therefore, the optimal policy for y_1^{t-1} must also be optimal for $\tilde{y}_1^{t-1}$ (and vice versa). We conclude that, without loss of optimality, we can let the feedback-dependent Markov source distribution beyond time $t-1$ be dependent on the vector $\underline{\alpha}_{t-1}$ instead of y_1^{t-1}. ■

Corollary 2 (Restatement of Theorem 1 and Theorem 2): For a channel with a finite input memory,

1) the feedback capacity is achieved by a feedback-dependent Markov source; the source memory length equals the memory length of the channel;
2) without loss of optimality, the channel output feedback can be replaced by the posterior channel state distribution $\underline{\alpha}_t$. □

By applying Theorem 2 to (28), without loss of optimality, we only consider the following feedback-dependent Markov source distributions

$$\mathcal{P}_\alpha^{\text{Markov}} = \left\{ \Pr\left(S_t = j \left| S_{t-1} = i, \underline{A}_{t-1} = \underline{\alpha}_{t-1}\right.\right) = P_t\left(j \left| i, \underline{\alpha}_{t-1}\right.\right), t = 1, 2, \ldots \right\}, \tag{36}$$

as candidate source distributions for achieving the feedback capacity.

IV. Stochastic Control Formulation and source optimization

By Theorem 2, at each time t, based on the value of the vector $\underline{\alpha}_{t-1}$, for each state pair $(i,j) \in \mathcal{T}$, the transmitter selects the optimal feedback-dependent *Markov* source transition probabilities $\left\{P_t\left(j|i,\underline{\alpha}_{t-1}\right)\right\}$ to maximize the directed information rate $\mathcal{I}(X \to Y)$. Here, we formulate this feedback-dependent Markov source optimization problem as a stochastic control problem, and then give a dynamic programming algorithm [10] which solves it and thus finds the optimal feedback-dependent Markov source distribution.

A. Stochastic control formulation

We consider the dynamic system whose *state* is the causal posterior channel state distribution $\underline{A}_{t-1}$. The state realization is the vector $\underline{\alpha}_{t-1}$, see Definition 3. The *control* or *policy* is the set of Markov transition probabilities $\left\{P_t\left(j|i,\underline{\alpha}_{t-1}\right)\right\}$, which is dependent on the state realization $\underline{\alpha}_{t-1}$. The system *disturbance* is Y_t which has the following conditional distribution

$$f_{Y_t|\underline{A}_{t-1}}\left(y_t\left|\underline{\alpha}_{t-1}\right.\right) = f_{Y_t|S_0,Y_1^{t-1}}\left(y_t\left|s_0,y_1^{t-1}\right.\right) = \sum_{i,j} \alpha_{t-1}(i) P_t\!\left(j|i,\underline{\alpha}_{t-1}\right) f_{Y_t|S_{t-1},S_t}(y_t\,|i,j\,)\,. \tag{37}$$

The system equation of this dynamic system is the forward recursion of the BCJR (Baum-Welch, sum-product) algorithm [5], [6], [7], [8], [9]

$$\underline{\alpha}_t = F_{\mathrm{BCJR}}\left(\underline{\alpha}_{t-1}, \left\{P_t\left(j|i,\underline{\alpha}_{t-1}\right)\right\}, y_t\right). \tag{38}$$

To be explicit, the entries $\alpha_t(\ell)$ of the state vector $\underline{\alpha}_t$ evolve as follows (substitute (36) into (31))

$$\alpha_t(\ell) = \frac{\sum\limits_{i} \alpha_{t-1}(i) P_t\left(\ell|i,\underline{\alpha}_{t-1}\right) f_{Y_t|S_{t-1},S_t}(y_t\,|i,\ell\,)}{\sum\limits_{i,j} \alpha_{t-1}(i) P_t\left(j|i,\underline{\alpha}_{t-1}\right) f_{Y_t|S_{t-1},S_t}(y_t\,|i,j\,)}. \tag{39}$$

Theorem 3: For a finite-state machine channel with a feedback-dependent Markov source (36),

1) the random process $\underline{A}_t$ is a Markov process, that is

$$f_{\underline{A}_t|\underline{A}_0^{t-1}}\left(\underline{\alpha}_t\left|\underline{\alpha}_0^{t-1}\right.\right) = f_{\underline{A}_t|\underline{A}_{t-1}}\left(\underline{\alpha}_t\left|\underline{\alpha}_{t-1}\right.\right); \tag{40}$$

2) the random process $(\underline{A}_t, S_t)$ is a Markov process, that is

$$f_{\underline{A}_t,S_t|\underline{A}_0^{t-1},S_0^{t-1}}\left(\underline{\alpha}_t, s_t\left|\underline{\alpha}_0^{t-1}, s_0^{t-1}\right.\right) = f_{\underline{A}_t,S_t|\underline{A}_{t-1},S_{t-1}}\left(\underline{\alpha}_t, s_t\left|\underline{\alpha}_{t-1}, s_{t-1}\right.\right); \tag{41}$$

$\square$

Proof: 1) From (37) we conclude that, given $\underline{A}_{t-1}$, the disturbance (channel output) Y_t is independent of $\underline{A}_0^{t-2}$. Thus from (39), given $\underline{A}_{t-1}$, the random vector $\underline{A}_t$ is independent of $\underline{A}_0^{t-2}$.

2) Combining (39) and the form of the feedback-dependent Markov source distribution (36), we can easily see that given $(\underline{A}_{t-1}, S_{t-1})$, the random vector $(\underline{A}_t, S_t)$ is independent of $(\underline{A}_0^{t-2}, S_1^{t-2})$. ∎

Theorem 3 can also be derived from a standard argument in stochastic control when the state is partially observed [17], Section 6. For an application to information rate computation, see also [4], Section 4.5. One can alternatively define a new state, in this case the pair $(S_t, \underline{A}_t)$, and show that it is Markov. This was also observed, though for a slightly different scenario, by Berger and Ying [18].

In this dynamic system, let the *reward* for each stage t be

$$\phi\left(\underline{\alpha}_{t-1}, \left\{P_t\left(j|i, \underline{\alpha}_{t-1}\right)\right\}\right) = I\left(S_{t-1}^t; Y_t \left|\underline{\alpha}_{t-1}\right.\right) = I\left(S_{t-1}^t; Y_t \left|s_0, y_1^{t-1}\right.\right), \tag{42}$$

where the distribution of S_{t-1}^t, Y_t given $\underline{A}_{t-1} = \underline{\alpha}_{t-1}$ (or $Y_1^{t-1} = y_1^{t-1}$) is determined as

$$f_{S_{t-1},S_t,Y_t|\underline{A}_{t-1}}\left(i, j, y_t \left|\underline{\alpha}_{t-1}\right.\right) = f_{S_{t-1},S_t,Y_t|S_0,Y_1^{t-1}}\left(i, j, y_t \left|s_0, y_1^{t-1}\right.\right) \tag{43}$$

$$= \alpha_{t-1}(i) P_t(j|i, \underline{\alpha}_{t-1}) f_{Y_t|S_{t-1},S_t}\left(y_t \left|i, j\right.\right). \tag{44}$$

According to (19), the expectation of the average reward per stage is equal to the directed information rate $\mathcal{I}(X \to Y)$. Thus, finding the optimal source distribution that maximizes the directed information rate is an *average-reward-per-stage* stochastic control problem [10].

Let λ be the *maximum average reward*, i.e., $\lambda = C^{fb}$, and let $\mathcal{J}(\underline{a})$ be the *optimal relative reward-to-go* function (or *return* function) for any possible value of the state vector $\underline{\alpha}_t = \underline{a}$. Then, Bellman's equation [10] for this stochastic control problem takes the following form

$$\lambda + \mathcal{J}(\underline{\alpha}_{t-1}) = \max_{\left\{P_t\left(j|i,\underline{\alpha}_{t-1}\right)\right\}} \Big\{\phi\left(\underline{\alpha}_{t-1}, \left\{P_t\left(j|i, \underline{\alpha}_{t-1}\right)\right\}\right) + \mathrm{E}_{Y_t}\left[\mathcal{J}\left(F_{\mathrm{BCJR}}\left(\underline{\alpha}_{t-1}, \left\{P_t\left(j|i, \underline{\alpha}_{t-1}\right)\right\}, Y_t\right)\right)\right]\Big\}. \tag{45}$$

For an indecomposable finite-state machine channel, there exists at least one optimal *stationary* policy $\{P_t\left(j|i, \underline{a}\right) = P\left(j|i, \underline{a}\right)\}$ that solves Bellman's equation (45) since the state process $\underline{A}_t$ forms a single steady-state recurrent class [10]. Further, there exist efficient dynamic programming algorithms, e.g., value iteration and policy iteration [10], which solve Bellman's equation (45) and thus find an optimal stationary policy $\{P(j\,|i, \underline{a})\}$ which achieves the feedback capacity.

Corollary 3: For an indecomposable finite-state machine channel, we have

$$C^{fb} = \sup_{\mathcal{P}_{\alpha}^{\text{Markov}}} \mathcal{I}(X \to Y), \tag{46}$$

where the supremum is over the *stationary feedback-dependent Markov* source distribution

$$\mathcal{P}_{\alpha}^{\text{Markov}} = \left\{ \Pr\left(S_t = j \left| S_{t-1} = i, \underline{A}_{t-1} = \underline{a}\right.\right) = P(j\,|i, \underline{a}) \right\}. \tag{47}$$

□

Proof: Directly apply the results for the average-cost-per-stage stochastic control problem in [10], section 7.4 to our stochastic control problem. Note that in (47), as opposed to (36), the probability $P_t\left(j \left|i, \underline{\alpha}_{t-1}\right.\right)$ no longer depends on t, i.e., we can set $P_t\left(j \left|i, \underline{\alpha}_{t-1} = \underline{a}\right.\right) = P(j\,|i, \underline{a})$. ■

We note that the directed information rate $\mathcal{I}(X \to Y)$ induced by a stationary source distribution $\{P(j\,|i, \underline{a})\}$ is equal to

$$\mathcal{I}(X \to Y) = \lim_{t\to\infty} \mathrm{E}_{\underline{A}_{t-1}}\left[\phi\left(\underline{A}_{t-1}, \left\{P\left(j|i, \underline{A}_{t-1}\right)\right\}\right)\right], \tag{48}$$

where the expectation is over the steady-state distribution of the random vector $\underline{A}_{t-1}$. When the steady state distribution of $\underline{A}_{t-1}$ is not available, by the law of large numbers, we can compute the directed information rate $\mathcal{I}(X \to Y)$ using the following Monte Carlo method (conceptually similar to [11], [12])

$$\mathcal{I}(X \to Y) \approx \frac{1}{N}\sum_{t=1}^{N}\left[\phi\left(\underline{\alpha}_{t-1}, \left\{P\left(j|i, \underline{\alpha}_{t-1}\right)\right\}\right)\right], \tag{49}$$

where N is a large integer.

B. Source optimization algorithm

We now describe a dynamic programming value iteration algorithm used to optimize the feedback-dependent Markov source distribution (47). Let $\underline{a}$ represent a possible value of $\underline{\alpha}_t$.

Define $J_0(\underline{a}) = 0$ to be the *terminal* reward function [10], and recursively generate the optimal k-stage reward-to-go functions $J_k(\underline{a})$ and sources $\mathcal{P}_{\alpha,k}^{\text{Markov}}$, for $k = 1, 2, \ldots$

$$J_k(\underline{a}) = \max_{\{P(j|i,\underline{a})\}} \left\{ \phi\left(\underline{a}, \{P\left(j|i,\underline{a}\right)\}\right) + \mathrm{E}_Y\left[J_{k-1}\left(F_{\mathrm{BCJR}}\left(\underline{a}, \{P\left(j|i,\underline{a}\right)\}, Y\right)\right)\right]\right\} \tag{50}$$

$$\mathcal{P}_{\alpha,k}^{\text{Markov}} = \arg\max_{\{P(j|i,\underline{a})\}} \left\{ \phi\left(\underline{a}, \{P\left(j|i,\underline{a}\right)\}\right) + \mathrm{E}_Y\left[J_{k-1}\left(F_{\mathrm{BCJR}}\left(\underline{a}, \{P\left(j|i,\underline{a}\right)\}, Y\right)\right)\right]\right\}, \tag{51}$$

where the reward-per-stage $\phi\left(\underline{a}, \{P\left(j|i,\underline{a}\right)\}\right)$ is defined as in (42), the maximization is over the stationary feedback-dependent Markov source $\{P\left(j|i,\underline{a}\right)\}$, and the expectation $\mathrm{E}_Y[\cdot]$ is over the random variable Y which has the following distribution

$$f_{Y|\underline{A}}\left(y\,|\underline{a}\right) = \sum_{i,j} a(i) P\left(j|i,\underline{a}\right) f_{Y_t|S_{t-1},S_t}\left(y\,|i,j\right). \tag{52}$$

The directed information rate $\mathcal{I}(X \to Y)$ induced by the source $\mathcal{P}_{\alpha,k}^{\text{Markov}}$ converges to the feedback capacity C^{fb} when $k \to \infty$ (see [10], p.390). Thus, the source distribution determined by (51) as $k \to \infty$ is an optimal source distribution.

In general, it is hard to find the optimal source distribution $\{P\left(j|i,\underline{a}\right)\}$ in closed form by applying the above value iteration algorithm. The following is the quantization-based *numerical approximation* of the value iteration method.

Algorithm 1 FOR OPTIMIZING FEEDBACK-DEPENDENT MARKOV SOURCE DISTRIBUTIONS

Initialization:

1) Choose a finite-level quantizer $\hat{\underline{a}} = \mathcal{Q}(\underline{a})$.
2) Initialize the terminal reward function as $J_0(\hat{\underline{a}}) = 0$.
3) Choose a large positive integer n.

Recursions:

For $1 \le k \le n$, numerically compute the k-stage reward-to-go function as

$$J_k(\hat{\underline{a}}) = \max_{\{P(j|i,\hat{\underline{a}})\}} \left\{ \phi\left(\hat{\underline{a}}, \{P\left(j|i,\hat{\underline{a}}\right)\}\right) + \mathrm{E}_Y\left[J_{k-1}\left(\mathcal{Q}\left(F_{\mathrm{BCJR}}\left(\hat{\underline{a}}, \{P\left(j|i,\hat{\underline{a}}\right)\}, Y\right)\right)\right)\right]\right\}. \tag{53}$$

Optimized source:

For $\hat{\underline{a}} = \mathcal{Q}(\underline{a})$, the optimized source $\mathcal{P}_{\alpha}^{\text{Markov}}$ is taken as

$$\{P\left(j|i,\underline{a}\right)\} = \arg\max_{\{P(j|i,\hat{\underline{a}})\}} \left\{ \phi\left(\hat{\underline{a}}, \{P\left(j|i,\hat{\underline{a}}\right)\}\right) + \mathrm{E}_Y\left[J_n\left(\mathcal{Q}\left(F_{\mathrm{BCJR}}\left(\hat{\underline{a}}, \{P\left(j|i,\hat{\underline{a}}\right)\}, Y\right)\right)\right)\right]\right\}. \tag{54}$$

End.

In (53) and (54), the maximization is taken by exhaustively searching the Markov source distribution space $\{P(j|i,\underline{\hat{a}})\}$ where finite-precision numerical approximations are necessary due to limited computing resources. The accuracy of the resulting optimal source distribution (54) is thus affected by the resolution of the quantizer $\mathcal{Q}(\cdot)$, by the finite value iteration number n and by the finite-precision approximation of $\{P(j|i,\underline{\hat{a}})\}$. So, strictly speaking, the directed information rate $\mathcal{I}(X \to Y)$ induced by the source $\mathcal{P}_{\alpha}^{\text{Markov}}$ optimized in (54) is only a lower bound on the feedback capacity. The information rate induced by $\mathcal{P}_{\alpha}^{\text{Markov}}$ is equal to the capacity only if Algorithm 1 is ideally executed with an infinitely fine quantizer $\mathcal{Q}(\cdot)$ for $n \to \infty$.

Example 3 (The dicode partial response channel): The channel model is given in Example 1. To illustrate the method, we first implement Algorithm 1 with crude numerical approximations. Then, we show the same results computed with a fine accuracy.

For the dicode channel, the vector $\underline{a} = [a(0), a(1)]^{\mathrm{T}} = [a(0), 1 - a(0)]^{\mathrm{T}}$ is completely determined by its first entry $a(0)$, so the reward-to-go functions $J_k(\underline{a})$ and the source transition probabilities $P(j|i,\underline{a})$ can be written as functions of $a(0)$ only, that is, $J_k(\underline{a}) = J_k(a(0))$ and $P(j|i,\underline{a}) = P(j|i,a(0))$.

1) [*A crude numerical solution*] We assume $\mathrm{SNR} = 10\log_{10}\left(\frac{1}{\sigma^2}\right) = -10$ dB, which corresponds to a noise variance of $\sigma^2 = 10$ (see Example 1 for a reference). We let the quantizer $\mathcal{Q}(\underline{a})$ be very crude, i.e.,

$$\begin{aligned}\underline{\hat{a}} = [\hat{a}(0), \hat{a}(1)]^{\mathrm{T}} &= \mathcal{Q}(\underline{a}) = \mathcal{Q}([a(0), a(1)]^{\mathrm{T}}) \\ &= \begin{cases} [0,1]^{\mathrm{T}} & \text{if } 0 \le a(0) < 0.25 \\ [0.5, 0.5]^{\mathrm{T}} & \text{if } 0.25 \le a(0) < 0.75 \\ [1,0]^{\mathrm{T}} & \text{if } 0.25 \le a(0) \le 1 \end{cases}\end{aligned}$$

and assume that the source transition probabilities only take 3 possible different values, that is, $P(j|i,\hat{a}(0)) \in \{0, 0.5, 1\}$. With these crude approximations, we implement the value iteration algorithm (53) for $n = 10$. The values of the reward-to-go functions $J_k(\hat{a}(0))$ are shown in Table I. We note that the k-stage relative reward-to-go functions, which are defined as $\mathcal{J}_k(\hat{a}(0)) \triangleq J_k(\hat{a}(0)) - J_k(0)$, converge when $k \to \infty$. For $n = 10$, the optimized source distribution is shown in Table II, which induces an information rate of $\mathcal{I}(X \to Y) \approx 0.107$ [bits/channel use]. The

TABLE I

IMPLEMENTING ALGORITHM 1 WITH CRUDE APPROXIMATIONS

k	$J_k(0)$	$J_k(0.5)$	$J_k(1)$	$\mathcal{J}_k(0.5)$	$\mathcal{J}_k(1)$
0	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0352	0.1314	0.0352	0.0962	0.0000
2	0.1650	0.2374	0.1650	0.0724	0.0000
3	0.2714	0.3497	0.2714	0.0783	0.0000
4	0.3836	0.4604	0.3836	0.0768	0.0000
5	0.4944	0.5715	0.4944	0.0772	0.0000
6	0.6055	0.6826	0.6055	0.0771	0.0000
7	0.7165	0.7936	0.7165	0.0771	0.0000
8	0.8275	0.9047	0.8275	0.0771	0.0000
9	0.9386	1.0157	0.9386	0.0771	0.0000
10	1.0496	1.1267	1.0496	0.0771	0.0000

TABLE II

OPTIMIZED SOURCE TRANSITION PROBABILITIES $P(j|i, \hat{a}(0))$

$\hat{a}(0)$	`Transition` $(i,j) \in \mathcal{T}$			
	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
0	0	1	0.5	0.5
0.5	0	1	1	0
1	0.5	0.5	1	0

information rate is estimated by the Monte Carlo method (49).

2) [*Feedback capacity estimation with a fine accuracy*] Choose a fine quantizer $\mathcal{Q}(\underline{a})$ as

$$\underline{\hat{a}} = [\hat{a}(0), \hat{a}(1)]^{\mathrm{T}} = \mathcal{Q}(\underline{a}) = \begin{cases} [0,1]^{\mathrm{T}} & \text{if } 0 \leq a(0) \leq 0.00125 \\ [0.0025, 0.9975]^{\mathrm{T}} & \text{if } 0.00125 < a(0) \leq 0.00375 \\ [0.0050, 0.9950]^{\mathrm{T}} & \text{if } 0.00375 < a(0) \leq 0.00625 \\ \vdots & \\ [1,0]^{\mathrm{T}} & \text{if } 0.99875 < a(0) \leq 1 \end{cases}$$

and assume that the values of the Markov source transition probabilities can only be integer multiples of 0.01. Then, by implementing Algorithm 1 with $n = 30$, we estimate the optimal relative reward-to-go functions $\mathcal{J}(a(0))$ and the optimal feedback-dependent Markov source transition probabilities $P(j|i, a(0))$. The feedback capacities for SNR=-10 dB, 0 dB and 10 dB are estimated as 0.111 [bits/channel use], 0.511 [bits/channel use] and 0.995 [bits/channel use], respectively. The actual plot of the feedback capacity is relegated to Section VI.

Figure 5 depicts the shapes of $P(0|0, a(0))$, $P(0|1, a(0))$, $\mathcal{J}(a(0))$, and the normalized histogram[3] of $a(0)$ over 10^7 simulation samples. According to the optimized source distribution

[3]The normalized histogram is computed as the number of simulation samples that fall into each quantization interval divided by the total number of simulation samples in all quantization intervals and then divided by the quantization interval size. Thus the normalized histogram is the estimated probability density function for the corresponding random variable.

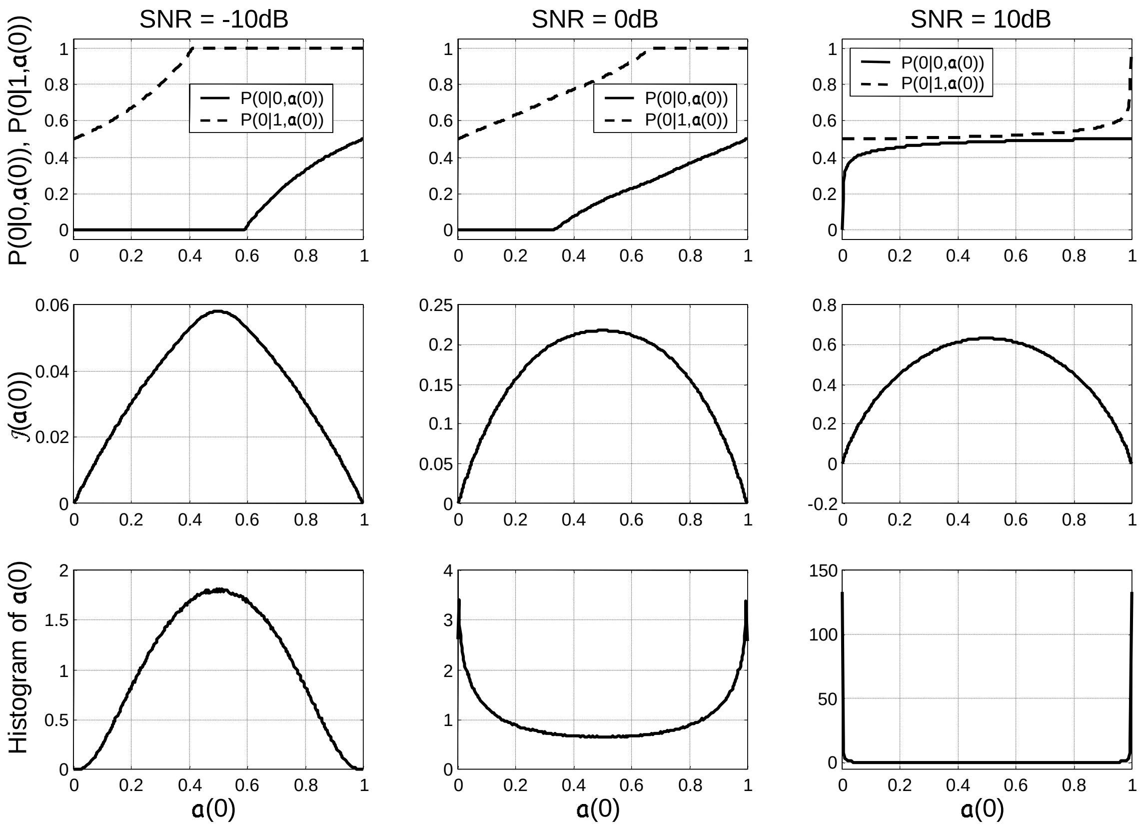

Fig. 5. Applying Algorithm 1 to the dicode channel with a fine accuracy.

in Figure 5, if the transmitter observes a *good* transmission, i.e., $\alpha_{t-1}(0) \approx 1$ for $x_{t-1} = 1$ (or $\alpha_{t-1}(0) \approx 0$ for $x_{t-1} = -1$), it tends to choose the input symbol X_t with more "new" information than redundancy, i.e., $P(0|0, \alpha_{t-1}(0)) \approx P(1|0, \alpha_{t-1}(0)) \approx 0.5$ (or $P(0|1, \alpha_{t-1}(0)) \approx P(1|1, \alpha_{t-1}(0)) \approx 0.5$). On the other hand, if the transmitter observes a *bad* transmission, i.e., $\alpha_{t-1}(0) \approx 0$ for $x_{t-1} = 1$ (or $\alpha_{t-1}(0) \approx 1$ for $x_{t-1} = -1$), it tends to choose the input symbol X_t with more redundancy in order to help the receiver, i.e., $P(0|0, \alpha_{t-1}(0)) \approx 0$ and $P(1|0, \alpha_{t-1}(0)) \approx 1$ (or $P(0|1, \alpha_{t-1}(0)) \approx 1$ and $P(1|1, \alpha_{t-1}(0)) \approx 0$). This way, the transmitter adjusts the amount of redundancy in the transmitted symbols according to the quality of previous transmissions.

At extremely low SNRs, e.g. SNR=-10dB, Figure 5 suggests a variant-rate repetition code (repeating the inverse of the previous transmitted symbol). The histogram of $a(0)$ at -10 dB shows that for most of the time, the transmitter observes an *ambiguous* channel state, i.e., $\alpha_{t-1}(0) \approx$

$\alpha_{t-1}(1) \approx 0.5$, and tends to let the next input X_t repeat the inverse of the previous symbol X_{t-1} until the channel state becomes good enough.

On the other hand, at extremely high SNRs, e.g. SNR=10dB, Figure 5 suggests a uniform and identically distributed (i.u.d.) channel input sequence. From the histogram of $a(0)$ at 10 dB, we observe that for most of the time, the channel is in a good state, i.e., $\alpha_{t-1}(0) \approx 1$ for $x_{t-1} = 0$ (or $\alpha_{t-1}(0) \approx 0$ for $x_{t-1} = 1$), and a uniform input distribution is preferred, i.e., $P(0|0, \alpha_{t-1}(0)) \approx P(1|0, \alpha_{t-1}(0)) \approx 0.5$ (or $P(0|1, \alpha_{t-1}(0)) \approx P(1|1, \alpha_{t-1}(0)) \approx 0.5$). □

V. Delayed Feedback Capacity

We show that the delayed feedback capacity is achieved by a finite-memory feedback-dependent Markov source distribution, and that the source distribution is optimized by dynamic programming methods. The delayed feedback capacity is an upper bound on the feed-forward capacity. By increasing the feedback delay, tight feed-forward capacity upper bounds can be obtained.

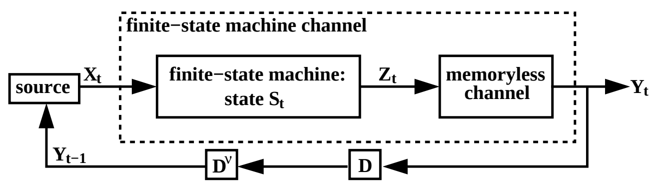

Fig. 6. A finite-state machine channel used with delayed feedback of ν symbol intervals.

A. Computing the delayed feedback capacity

We show that a finite-state machine channel used with *delayed* feedback can be regarded as a finite-state machine channel used with *instantaneous* feedback.

Assuming that the feedback delay is $\nu > 0$ as in Figure 6, we redefine the channel state as $\tilde{S}_{t-1} = S_{t-1-\nu}^{t-1}$, let the channel output be $\tilde{Y}_t = Y_{t-\nu}$, and keep the original channel input, i.e., $\tilde{X}_t = X_t$. The realizations of random variables $\tilde{S}_{t-1}$, $\tilde{X}_t$ and $\tilde{Y}_t$ are denoted by $\tilde{s}_{t-1}$, $\tilde{x}_t$ and $\tilde{y}_t$, respectively. The redefined channel states, inputs and outputs satisfy the assumptions 1)-4) in Section II for a finite-state machine channel used with instantaneous feedback

1) The state $\tilde{s}_t$ is a time-invariant function of $\tilde{s}_{t-1}$ and $\tilde{x}_t$, that is

$$\tilde{s}_t = \tilde{q}(\tilde{s}_{t-1}, \tilde{x}_t) = (s_{t-\nu}, \ldots, s_{t-1}, q(s_{t-1}, x_t)). \tag{55}$$

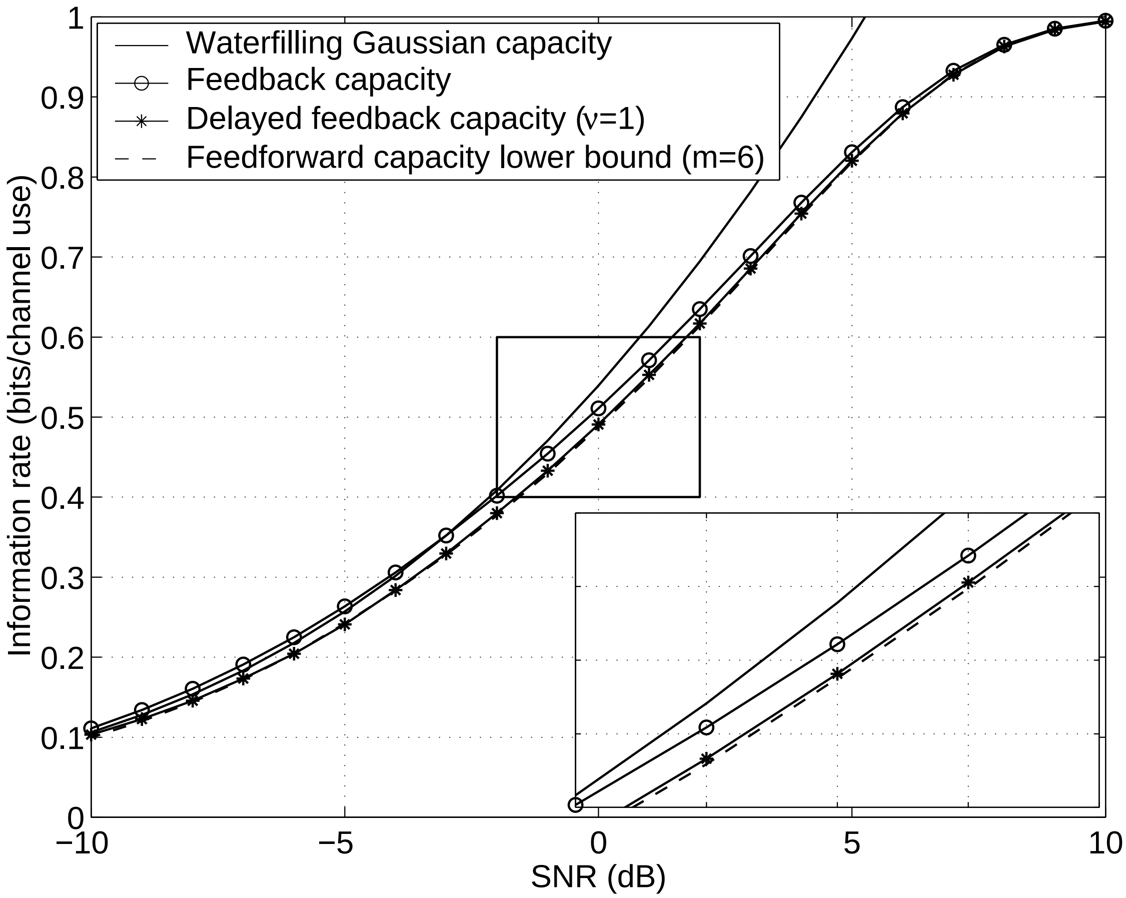

Fig. 7. Capacity bounds for the dicode $(1-D)$ channel in Example 1.

2) The initial channel state $\tilde{S}_0 = \tilde{s}_0$ is known to both the transmitter and receiver, and there is a 1-to-1 correspondence between $\tilde{S}_0^t$ and $(\tilde{S}_0, \tilde{X}_1^t)$.
3) Conditioned on the state pair $(\tilde{S}_{t-1}, \tilde{S}_t)$, the channel output $\tilde{Y}_t$ is independent of all other previous state and output variables, i.e.,

$$f_{\tilde{Y}_t \left| \tilde{S}_{-\infty}^t, \tilde{Y}_{-\infty}^{t-1} \right.}\left(\tilde{y}_t \left| \tilde{s}_{-\infty}^t, \tilde{y}_{-\infty}^{t-1} \right.\right) = f_{\tilde{Y}_t \left| \tilde{S}_{t-1}^t \right.}\left(\tilde{y}_t \left| \tilde{s}_{t-1}^t \right.\right) . \tag{56}$$

4) The transmitter, before emitting symbol $\tilde{X}_t$, knows all previous channel output symbols $\tilde{Y}_1^{t-1}$ without error.

If the original finite-state machine channel is indecomposable, then the reformulated channel model is also indecomposable. We apply Corollary 3 and conclude that the capacity C_ν^{fb} of the finite-state machine channel used with delayed feedback of ν symbol intervals is achieved by a

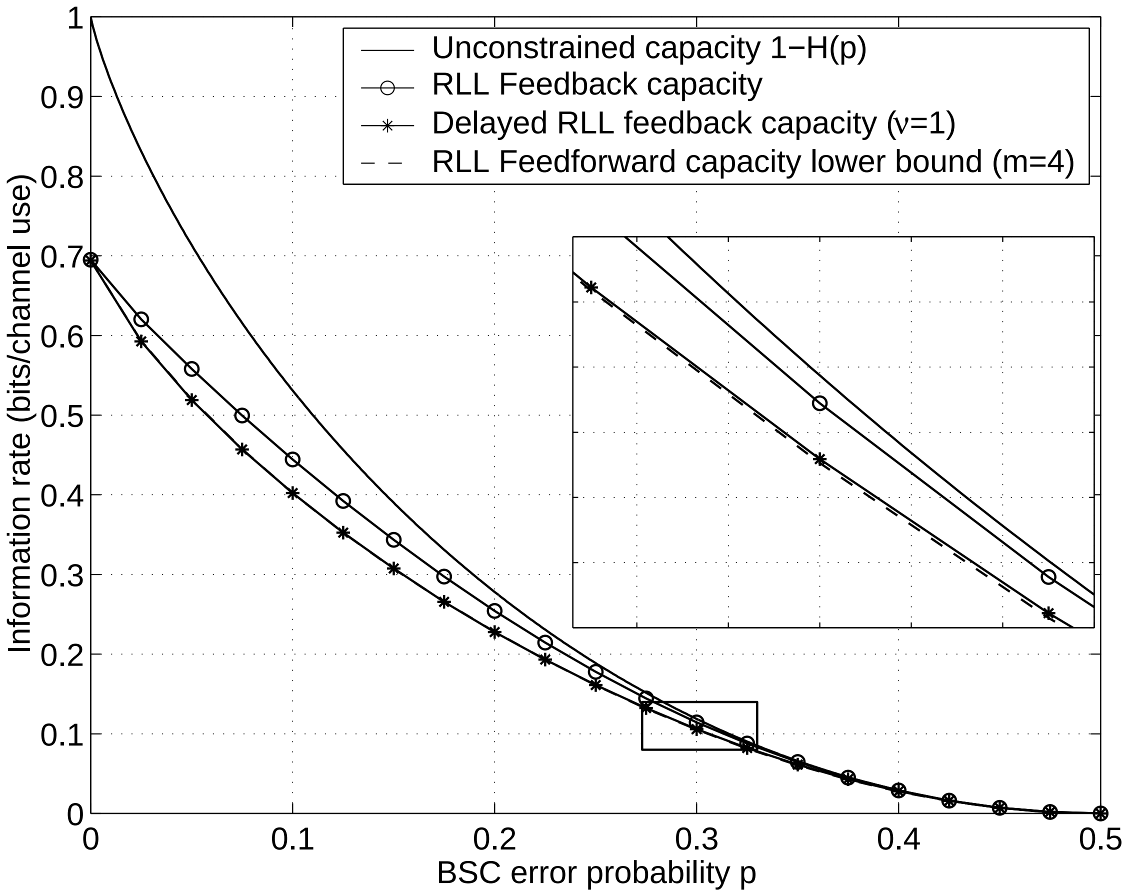

Fig. 8. Capacity bounds for the $RLL(0,1)$ sequence over the binary symmetric channel (BSC) in Example 2.

time-invariant feedback-dependent Markov source $\left\{P\left(\tilde{s}_t|\tilde{s}_{t-1},\underline{\tilde{\alpha}}_{t-1}\right)\right\}$, or $\left\{P\left(x_t|s_{t-1-\nu}^{t-1},\underline{\tilde{\alpha}}_{t-1}\right)\right\}$. That is, the required order of the feedback-dependent Markov source is linearly proportional to ν. Here, the optimal feedback is the vector of the posterior probabilities of the augmented channel states, i.e., $\underline{\tilde{\alpha}}_{t-1}$, whose entries are

$$\begin{aligned}\tilde{\alpha}_{t-1}(\tilde{s}_{t-1}) &= \Pr(\tilde{S}_{t-1}=\tilde{s}_{t-1}\,\Big|\,\tilde{S}_0=\tilde{s}_0,\tilde{Y}_1^{t-1}=\tilde{y}_1^{t-1})\\ &= \Pr(S_{t-1-\nu}^{t-1}=s_{t-1-\nu}^{t-1}\,\big|\,S_{-\nu}^0=s_{-\nu}^0,Y_1^{t-1-\nu}=y_1^{t-1-\nu}).\end{aligned}$$

Corollary 4: For an indecomposable finite-state machine channel, the ν-time delayed feedback capacity is

$$C_\nu^{fb}=\sup_{\mathcal{P}_{\tilde{\alpha}}^{\text{Markov}}}\mathcal{I}\left(\tilde{X}\to\tilde{Y}\right), \tag{57}$$

where ν is the feedback delay and the supremum is over the *stationary feedback-dependent Markov* source distribution

$$\mathcal{P}_{\tilde{\alpha}}^{\text{Markov}} = \left\{ \Pr\left(\tilde{S}_t = \tilde{s}_t \,\middle|\, \tilde{S}_{t-1} = \tilde{s}_{t-1}, \underline{\tilde{A}}_{t-1} = \underline{\tilde{\alpha}}_{t-1} \right) = P(\tilde{s}_t \,|\, \tilde{s}_{t-1}, \underline{\tilde{\alpha}}_{t-1}) \right\}. \tag{58}$$

□

Proof: Follows directly from Corollary 3. ■

Thus, we can compute the delayed feedback capacity by reformulating the channel model as a finite-state machine channel used with instantaneous feedback and then applying the methods presented in Section IV.

Corollary 5: If the channel has memory length L and is used with feedback delay ν, then a feedback-dependent Markov source, whose memory length is $L' = L + \nu$, achieves the delayed feedback capacity.

Proof: Follows directly from Corollary 1. ■

B. Bounding the feed-forward capacity C

For a finite-state machine channel, the feed-forward capacity C is usually achieved only by sources with an infinite memory length, and it is still an open problem to compute the feed-forward capacity C directly. However, by combining the delayed feedback capacity C_ν^{fb} and the feed-forward capacity lower bounds computed using the method in [13], we can tightly bound the feed-forward capacity C.

We note that as the feedback delay ν increases, the delayed feedback capacity C_ν^{fb} decreases because there is less feedback information to the transmitter. When the feedback delay ν becomes infinite, i.e., $\nu \to \infty$, the transmitter receives no feedback information. Thus, the delayed feedback capacity C_ν^{fb} decreasingly approaches the feed-forward capacity C as the delay ν increases, that is

$$C \leq \cdots \leq C_2^{fb} \leq C_1^{fb} \leq C_0^{fb} = C^{fb}. \tag{59}$$

The delayed feedback capacity C_ν^{fb} for any finite delay ν naturally upper bounds the feed-forward capacity C.

On the other hand, for a finite-state machine channel used without feedback, if we denote by C_m the maximal information rate achieved by a stationary m-th order Markov source, then we have

$$C_0 \leq C_1 \leq C_2 \cdots \leq C. \tag{60}$$

The achievable information rate C_m for any stationary Markov source with memory length m is thus a lower bound on the capacity C. We note that C_m can be estimated using the iterative method presented in [13].

Combining (59) and (60), we obtain tight bounds on the feed-forward capacity C by evaluating C_ν^{fb} and C_m for suitably chosen values of ν and m, that is

$$C_m \leq C \leq C_\nu^{fb}. \tag{61}$$

VI. Feedback Capacity Plots

Figure 7 depicts the capacity bounds of the dicode partial response channel in Example 1. The feedback capacity is compared to the waterfilling capacity [19], [16] for continuous-valued channel inputs, the delayed feedback capacity C_1^{fb}, and the feed-forward capacity lower bound C_6 estimated by the iterative algorithm in [13]. At low SNRs, the feedback capacity surpasses the waterfilling capacity, which numerically verifies that feedback increases the capacity of channels with memory. The delayed feedback capacity C_1^{fb} is very close to the feed-forward capacity lower bound C_6, resulting in a tightly bounded feed-forward capacity C, see Figure 7.

In Figure 8, the capacity bounds are plotted for RLL(0,1) sequences over BSCs in Example 2. The feedback capacity is compared to the capacity [19] of the BSCs without the channel input constraint which is $1 - H(p) = 1 + p\log(p) + (1+p)\log(1+p)$, the delayed feedback capacity C_1^{fb}, and the feed-forward capacity lower bound C_4 estimated by the iterative algorithm given in [13]. The delayed feedback capacity C_1^{fb} is very close to the feed-forward capacity lower bound C_4, which results in a tightly bounded feed-forward capacity C.

VII. Conclusion

The feedback capacity of a finite-state machine channel is achieved by a feedback-dependent Markov channel input distribution. The memory length of the feedback-capacity-achieving Markov channel input equals the sum of the channel memory length and the feedback delay. The optimized transition probabilities are only dependent on the posterior channel state distribution computed using the forward recursion of the BCJR (Baum-Welch, sum-product) algorithm [5],

[6], [7], [8], [9]. We formulated the feedback-dependent channel input distribution optimization problem as an average-reward-per-stage stochastic control problem and applied a dynamic programming algorithm to solve it. Using the proposed methods, we numerically evaluated the feedback capacities of partial response (PR) channels and BSC channels with run-length-limited (RLL) input constraints. The feed-forward capacity is tightly bounded by the delayed feedback capacity and the feed-forward capacity lower bound computed using the method in [13]. Closed form computations of the optimal feedback-dependent Markov transition probabilities and the feedback capacity are still open problems.

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