EE 342, HW 6

Solutions

1) a CH5, PROB. 1

$$f_X(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-1}^{1} c(1-x^2)\,dx = 1$

$$c\int_{-1}^{1}(1-x^2)\,dx = 1$$

$$c\cdot\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 1$$

$$c\cdot\left[1-\frac{1^3}{3}-\left[(-1)-\frac{(-1)^3}{3}\right]\right] = 1$$

$$c\cdot\left(2-\frac{2}{3}\right) = 1$$

$$c\cdot\frac{4}{3} = 1 \Rightarrow \boxed{c=\frac{3}{4}}$$

b) $F_X(z) = \int_{-\infty}^{z} f_X(x)\,dx = \begin{cases} 0 & z \le -1 \\ \int_{-1}^{z}\frac{3(1-x^2)}{4}\,dx & -1<z<1 \\ 1 & z \ge 1 \end{cases}$

Now solve $\int_{-1}^{z}\frac{3(1-x^2)}{4}\,dx = \frac{3}{4}\left(x-\frac{x^3}{3}\right)_{-1}^{z} = \frac{3}{4}\left[z-\frac{z^3}{3}-\left((-1)-\frac{(-1)^3}{3}\right)\right]$

$$= \frac{3}{4}\left(z-\frac{z^3}{3}\right)+\frac{3}{4}\left[1-\frac{1}{3}\right]$$

$$= \frac{3z-z^3}{4}+\frac{1}{2}$$

$$\boxed{F_X(z) = \begin{cases} 0 & z \le -1 \\ \frac{2+3z-z^3}{4} & -1<z<1 \\ 1 & z \ge 1 \end{cases}}$$

$\boxed{b}$ CH5, PROB. 4

a) $$P(X > 20) = \int_{20}^{\infty} f_X(x)\,dx = \int_{20}^{\infty} \frac{10}{x^2}\,dx$$

$$= 10 \cdot \left.\frac{x^{-1}}{(-1)}\right|_{20}^{\infty} = \frac{10}{(-1)}\left[0 - 20^{-1}\right] = \frac{10}{20} = \boxed{\frac{1}{2}}$$

b) $$F_X(z) = P(X \le z) = \int_{-\infty}^{z} f_X(x)\,dx)$$

$$= \begin{cases} 0 & z \le 10 \\ \int_{10}^{z} \frac{10}{x^2}\,dx & z > 10 \end{cases}$$

$$= \begin{cases} 0 & z \le 10 \\ \frac{10}{(-1)} \left.\frac{1}{x}\right|_{10}^{z} & z > 10 \end{cases}$$

$$= \begin{cases} 0 & z \le 10 \\ 1 - \frac{10}{z} & z > 10 \end{cases}$$

c) FIRST FIND THE PROBABILITY THAT ONE DEVICE WILL FUNCTION FOR AT LEAST 15 HOURS

$$p = P(X \ge 15) \underset{\text{from a)}}{=} 10 \cdot \left.\frac{x^{-1}}{(-1)}\right|_{15}^{\infty} = \frac{10}{15} = \boxed{\frac{2}{3} = p}$$

NOW THE PROBABILITY THAT

1° EXACTLY 3 OF 6 WILL FUNCTION IS $\binom{6}{3} p^3 \cdot (1-p)^3$

2° EXACTLY 4 OF 6 WILL FUNCTION IS $\binom{6}{4} p^4 (1-p)^2$

3° EXACTLY 5 OF 6 WILL FUNCTON IS $\binom{6}{5} p^5 (1-p)$

4° EXACTLY 6 OF 6 WILL FUNCTION IS $\binom{6}{6} p^6$

FINALLY, THE PROBABILITY THAT AT LEAST 3 OF 6 WILL FUNCTION IS

$$q = \binom{6}{3} p^3 (1-p)^3 + \binom{6}{4} p^4 (1-p)^2 + \binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6$$

$$q = 20 \cdot \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3 + 15 \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^2 + 6 \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^6$$

$$q = \frac{20 \times 8 + 15 \times 16 + 6 \times 32 + 64}{3^6}$$

$$\boxed{q \cong 0.8999}$$

$\boxed{c}$ CH 5, PROB. 6

a) $$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_0^{\infty} \frac{x^2}{4} e^{-\left(\frac{x}{2}\right)} dx$$

$$= \left[-\frac{1}{2} x^2 e^{-x/2} - 2x e^{-x/2} - 4 e^{-x/2} \right] \Big|_0^{\infty}$$

$$\boxed{E[X] = 4}$$

b) $$E[X] = \int_{-\infty}^{\infty} x f_X(x) = \int_{-1}^{1} c \cdot x (1 - x^2) dx = 0$$

because this is an integral of an odd function with integration limits symmetric with respect to 0

$$\boxed{E[X] = 0}$$

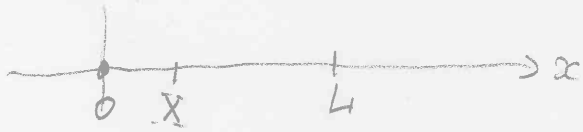
c) $$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_5^{\infty} x \cdot \frac{5}{x^2} dx = \int_5^{\infty} \frac{5}{x} dx$$ ← this is a diverging integral

so $\boxed{E[X] = \infty}$

$\boxed{d}$ CH 5, PROB 11 $\Rightarrow$ THIS IS EQUIVALENT TO CHOOSING A POINT UNIFORMLY AT RANDOM BETWEEN 0 AND L

$$f_X(x) = \begin{cases} \dfrac{1}{L} & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ X IS UNIFORMLY DISTRIBUTED ON THE INTERVAL $[0, L]$



LENGTH OF LEFT-HAND SEGMENT $\Rightarrow (X - 0)$

LENGTH OF RIGHT-HAND SEGMENT $\Rightarrow (L - X)$

LENGTH OF SHORTER SEGMENT $\quad MIN(X, L-X)$

LENGTH OF LONGER SEGMENT $\quad MAX(X, L-X)$

WHEN IS $\dfrac{MIN(X, L-X)}{MAX(X, L-X)} < \dfrac{1}{4}$?

THE ANSWER IS EITHER WHEN $X < 0.2 \times L$

OR WHEN $X > 0.8 \times L$

HENCE $P\left(\dfrac{MIN(X, L-X)}{MAX(X, L-X)} < \dfrac{1}{4}\right) = P(0 \le X < 0.2L \text{ OR } 0.8L < X \le L)$

$$= \int_0^{0.2L} f_X(x)\,dx + \int_{0.8L}^{L} f_X(x)\,dx$$

$$= \frac{0.2L}{L} + \frac{L - 0.8L}{L} = \boxed{0.4}$$

$\boxed{e}$ CH 5, PROB. 16

ASSUME THAT THE RAINFALL IN EACH YEAR IS DISTRIBUTED AS $\mathcal{N}(\mu, \sigma^2)$ AND THAT RAINFALL IN ANY YEAR DOES NOT INFLUENCE RAINFALL IN ANY OTHER YEAR

FIRST, COMPUTE THE PROBABILITY THAT IN A GIVEN YEAR THE RAINFALL IS OVER 50 INCHES

$$p = P(X > 50) = \int_{50}^{\infty} f_X(x)\,dx = \int_{50}^{\infty} \frac{e^{-(x-40)^2/32}}{\sqrt{2\pi \times 16}}\,dx$$

$$= \int_{\frac{50-40}{4}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = 1 - \int_{-\infty}^{\frac{50-40}{4}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

$$= 1 - \phi\left(\frac{50-40}{4}\right)$$

$$= 1 - \phi(2.5)$$

$$= 1 - 0.9938 = 0.0062$$

$$\boxed{p = 0.0062}$$

NOW, THE PROBABILITY THAT IT WILL TAKE OVER 10 YEARS BEFORE A YEAR OCCURS WHOSE RAINFALL IS HIGHER THAN 50 INCHES IS:

$$Q = (1-p)^{10} p + (1-p)^{11} p + (1-p)^{12} p + \cdots$$

$$Q = p(1-p)^{10}\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots\right]$$

$$Q = p \cdot (1-p)^{10} \cdot \frac{1}{p} = (1-p)$$

$$Q = \left[\Phi(2.5)\right]^{10}$$

$$\boxed{Q = 0.9938^{10} = 0.9629}$$

$\boxed{f}$ CH. 5, PROB 21

$$\mu = 71, \quad \sigma^2 = 6.25 \qquad X \sim N(\mu, \sigma^2)$$

19 $$P(X > 74) = \int_{74}^{\infty} \frac{e^{-\frac{(x-71)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} = \int_{\frac{74-71}{\sqrt{\sigma^2}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$= \int_{\frac{3}{2.5}}^{\infty} e^{-z^2/2} / \sqrt{2\pi}\, dz = 1 - \Phi(1.2)$$

$$= 1 - 0.8849$$

$$= 0.1151 \Rightarrow \boxed{11.5\ \%}$$

2° $P(X > 77 \mid X > 72) =$

$$= \frac{P(X>77 \,\&\, X>72)}{P(X>72)}$$

$$= \frac{P(X>77)}{P(X>72)}$$

$$= \frac{1-\Phi\left(\frac{77-71}{2.5}\right)}{1-\Phi\left(\frac{72-71}{2.5}\right)}$$

$$= \frac{1-\Phi(2.4)}{1-\Phi(0.4)} = \frac{1-0.9918}{1-0.6554} = 0.0238 \Rightarrow \boxed{2.4\%}$$

$\boxed{9}$ CH5, THEOR. EX. 8

note: $x \le c$

$$E[X^2] = \int_0^c x^2 f_X(x)\,dx = \int_0^c x \cdot x \cdot f_X(x)\,dx \le \int_0^c x \cdot c \cdot f_X(x)\,dx = c \cdot \int_0^c x f_X(x)\,dx = c \cdot E[X]$$

hence $\boxed{E[X^2] \le c\,E[X]}$

Now.

$$Var(X) = E[X^2] - (E[X])^2 \le c\,E[X] - (E[X])^2$$

$$= c^2 \cdot \frac{E[X]}{c} - c^2 \frac{(E[X])^2}{c^2}$$

$$= c^2\left[\frac{E[X]}{c} - \left(\frac{E[X]}{c}\right)^2\right]$$

$$= c^2\left[\alpha - \alpha^2\right] = \boxed{c^2 \alpha \cdot (1-\alpha)}$$

$\Rightarrow \boxed{Var(X) \le c^2 \alpha (1-\alpha)}$

$\boxed{h}$ CH 5, TH. EX. 9

$$Z \sim \mathcal{N}(0,1)$$

a) $$P(Z > x) = \int_x^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz \underset{\substack{\uparrow \\ \text{sub } u=-z}}{=} \int_{-x}^{-\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} (-du)$$

$$= \int_{-\infty}^{-x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du = P(Z < -x)$$

b) $$P(|Z| > x) = P(Z < -x \text{ or } Z > x)$$

$$= P(Z < -x) + P(Z > x)$$

by a)

$$= P(Z > x) + P(Z > x)$$

$$= 2P(Z > x)$$

c) $$P(|Z| < x) = 1 - P(|Z| > x)$$

by b)

$$= 1 - 2P(Z > x)$$

$$= 2 - 2P(Z > x) - 1$$

$$= 2\left[1 - P(Z > x)\right] - 1$$

$$= 2P(Z < x) - 1$$

2) a) mean is 1 and variance is 4/3.

b) sample pdf and sample CDF approximate pdf and CDF with plots shown below. Sample mean is 1.0052 and sample variance is 1.3343.

Matlab commands:

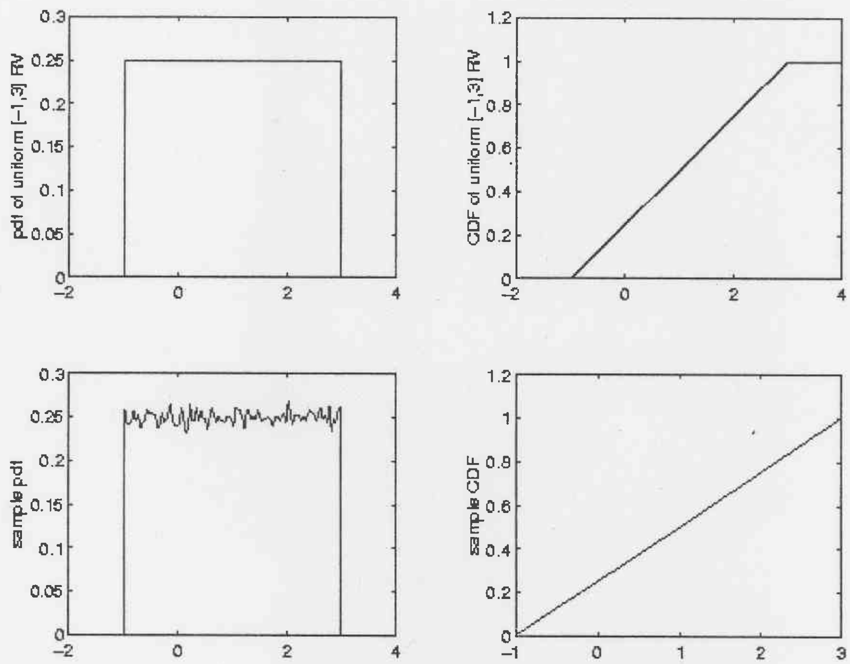
```
dt=.002; t=-2:dt:4;
pdfu=.25*(t>= -1 & t<3);
subplot(2,2,1);
plot(t,pdfu)
axis([-2 4 0 .3])
ylabel('pdf of uniform [-1,3] RV')
cdfu=cumsum(pdfu)*(dt);
subplot(2,2,2)
plot(t,cdfu)
axis([-2 4 0 1.2])
ylabel('CDF of uniform [-1,3] RV')
u=4*rand(1,100000)-1;
[histu tu]= hist(u,100);
tu = [-1 tu 3];
histu= [0 histu/4000 0];
subplot(2,2,3)
plot(tu,histu)
axis([-2 4 0 0.3])
ylabel('sample pdf')
cdfus= cumsum(histu);
subplot(2,2,4)
plot(tu,cdfus/25)
axis([-1 3 0 1.2])
ylabel('sample CDF')
```

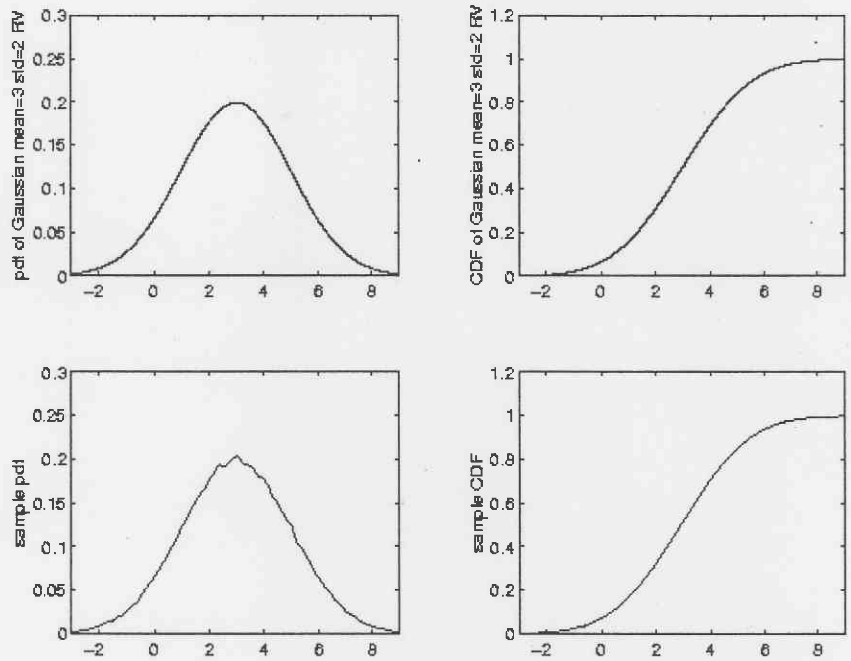
3) a) mean is 3 and variance is 4.

b) sample pdf and sample CDF closely approximate pdf and CDF with plots shown below. Sample mean is 3.0037 and sample variance is 3.9822.

Matlab commands:

```
dt=.002; t=-3:dt:9; mg=3; vg=4;
pdfg= 1/sqrt(2*pi*vg)*exp(-.5*(t-3).^2/vg);
subplot(2,2,1);
plot(t,pdfg)
axis([-3 9 0 .3])
ylabel('pdf of Gaussian mean=3 std=2 RV')
cdfg=cumsum(pdfg)*(dt);
subplot(2,2,2)
plot(t,cdfg)
axis([-3 9 0 1.2])
ylabel('CDF of Gaussian mean=3 std=2 RV')
g=2*randn(1,100000)+3;
[histg tg]= hist(g,100);
histg= histg/100000/(tg(2)-tg(1));
subplot(2,2,3)
plot(tg,histg)
axis([-3 9 0 0.3])
ylabel('sample pdf')
cdfgs= cumsum(histg);
subplot(2,2,4)
plot(tg,cdfgs*(tg(2)-tg(1)))
axis([-3 9 0 1.2])
ylabel('sample CDF')
```


pdf of uniform [−1,3] RV
CDF of uniform [−1,3] RV
sample pdf
sample CDF
0.3
0.25
0.2
0.15
0.1
0.05
0
1.2
1
0.8
0.6
0.4
0.2
−2
2
4
−1
3


pdf of Gaussian mean=3 std=2 RV
CDF of Gaussian mean=3 std=2 RV
sample pdf
sample CDF