Computing Reliability Distributions of Windowed Max-log-map (MLM) Detectors : ISI Channels

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***Abstract*—In this paper, we consider sliding-window max-log-map (MLM) receivers, where for any integer m, the MLM detector is truncated to a length-m signaling neighborhood. For any number n of chosen time instants, we provide exact expressions for both i) the *joint* distribution of the MLM symbol reliabilities, and ii) the *joint* probability of the erroneous MLM symbol detections. The obtained expressions can be efficiently evaluated using Monte-Carlo techniques. Comparisons performed with empirical distributions reveal good match. Dynamic programming techniques are applied to simplify the procedures.**

***Index Terms*—detection, intersymbol inteference, max-log-map, probability distribution, reliability**

I. Introduction

The max-log-map (MLM) detector has well-known applications to the intersymbol interefence (ISI) channel [1]. It is the optimal sequence detector; it produces the same symbol estimates as the *Viterbi* detector [2]. On the other hand, it also additionally computes *symbol reliabilities* (also known as *soft-outputs*, *log-likelihood ratios*, etc) to be used for coding techniques, e.g. see [1]. The MLM is a well-accepted approximation of the *Bahl-Cocke-Jelinek-Raviv* (BCJR) algorithm [3].

In this paper, we consider a *sliding-window* implementation of an MLM receiver. We consider an m-truncated MLM receiver, i.e. only a signaling window of length m is considered around the time instant of interest. These truncations are observed to well-approximate the actual receiver. The analysis of truncated MLM receivers is tractable, and for any number n of chosen time instants, we present *exact, closed-form* expressions for *both* i) the *joint* distribution of the symbol reliabilities, and ii) the *joint* probability that the detected symbols are in error. While past work considered only marginal distributions (for convolutional codes, see [4], [5], and for approximations see [6], [7]), we provide analytic expressions for *joint* MLM receiver statistics.

Notation: Deterministic vectors and matrices are denoted using bold fonts (e.g., $\mathbf{a}$ and $\mathbf{A}$, respectively). Deterministic scalars are denoted using italic fonts (e.g., a). Random scalars are denoted using upper-case italics (e.g., A) and random vectors are denoted using upper-case bold italics (e.g., $\boldsymbol{A}$). We do not reserve specific notation for random matrices. We write $\max \mathbf{a}$ to denote the maximum (vector) component in $\mathbf{a}$.

A random sequence of symbols drawn from the set $\{-1,1\}$, denoted as $\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$, is transmitted across the ISI channel. Let the following random

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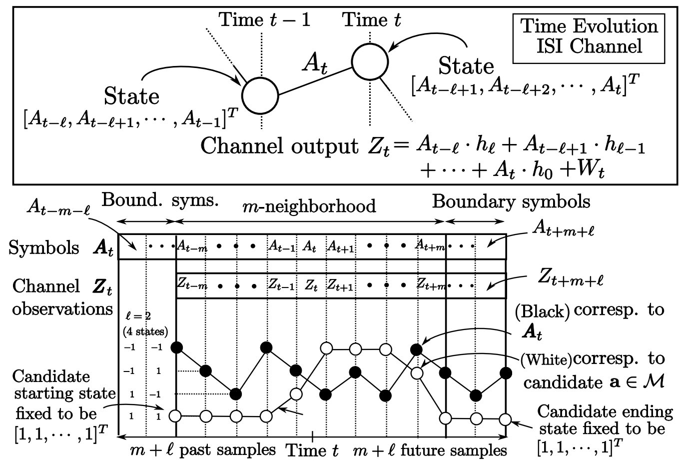

Fig. 1. The m-truncated Max-Log-Map (MLM) detector, illustrated for the case $\ell = 2$. All $2^\ell = 4$ possible states are shown. Channel states colored black and white, correspond respectively to the symbol neighborhood $\boldsymbol{A}_t$, and a candidate sequence $\mathbf{a}$ in the set $\mathcal{M}$ (see Definition 1). As shown, $\boldsymbol{A}_t$ and $\mathbf{a}$ may not have the same starting and/or end states.

sequence denoted as $\cdots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \cdots$ be the ISI *channel output* sequence. Let $h_0, h_1, \cdots, h_\ell$ denote the ISI *channel coefficients*; the constant ℓ is the ISI channel memory length. The input-output relationship of the ISI channel is

$$Z_t = \sum_{i=0}^{\ell} h_i A_{t-i} + W_t, \quad (1)$$

where the noise samples W_t are zero-mean jointly Gaussian. Figure 1 depicts the time evolution of the ISI channel states. The ISI channel state at time t equals the (length-ℓ) vector of input symbols $[A_{t-\ell+1}, A_{t-\ell+2}, \cdots, A_t]^T$.

A. The m-truncated max-log-map (MLM) detector

We proceed to describe the sliding-window MLM receiver. At time instant t, the m-truncated MLM detector considers the neighborhood of $2m+\ell+1$ channel outputs $\boldsymbol{Z}_t \triangleq [Z_{t-m}, Z_{t-m+1}, \cdots, Z_{t+m+\ell}]^T$. Define the symbol neighborhood $\boldsymbol{A}_t$ containing the following $2(m+\ell)+1$ input symbols $\boldsymbol{A}_t \triangleq [A_{t-m-\ell}, A_{t-m-\ell+1}, \cdots, A_{t+m+\ell}]^T$. Both $\boldsymbol{A}_t$ and $\boldsymbol{Z}_t$ are depicted in Figure 1. Let $\mathbf{h}_i$ denote the following length-$(2m+\ell+1)$ vector

$$\mathbf{h}_i \triangleq [\overbrace{0,0,\cdots,0}^{m+i}, h_0, h_1, \cdots, h_\ell, \overbrace{0,0,\cdots,0}^{m-i}]^T, \quad (2)$$

where i can take values $|i| \le m$. Let $\mathbb{0}$ denote an *all-zeros* vector $\mathbb{0} \triangleq [0,0,\cdots,0]^T$. Let both $\mathbf{H}$ and $\mathbf{T}$ denote the size

$2m+\ell+1$ by $2(m+\ell)+1$ matrices given as

$$\mathbf{H} \triangleq [\overbrace{\mathbb{0},\ \mathbb{0}, \cdots,\ \mathbb{0}}^{\ell}\ \overbrace{\mathbf{h}_{-m}, \mathbf{h}_{-m+1}, \cdots, \mathbf{h}_m}^{2m+1}\ \overbrace{\mathbb{0},\ \mathbb{0},\ \cdots,\ \mathbb{0}}^{\ell}],$$
$$\mathbf{T} \triangleq \begin{bmatrix} h_\ell, h_{\ell-1}, \cdots h_1 & & & & & \\ h_\ell \qquad \vdots & & & & & h_0, \\ \ddots\ \vdots & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & \vdots\ \ddots \\ h_1 & & & & & h_{\ell-2} \cdots h_0 \\ & & & & & h_{\ell-1} \cdots h_1 h_0 \end{bmatrix}. \tag{3}$$

Using (3), rewrite $\boldsymbol{Z}_t \triangleq [Z_{t-m}, Z_{t-m+1}, \cdots, Z_{t+m+\ell}]^T$ using (1) into the following form

$$\boldsymbol{Z}_t = (\mathbf{H}+\mathbf{T})\,\boldsymbol{A}_t + \boldsymbol{W}_t, \tag{4}$$

where here $\boldsymbol{W}_t$ denotes the neighborhood of (zero-mean) noise samples $\boldsymbol{W}_t \triangleq [W_{t-m}, W_{t-m+1}, \cdots, W_{t+m+\ell}]^T$. We assume stationary noise variance $\mathbb{E}\{W_t^2\} = \sigma^2$ for all times t; this is not a strict requirement, see [8].

Definition 1. *Denote the set $\mathcal{M}$ that contains the m-truncated MLM **candidate** sequences*

$$\mathcal{M} \triangleq \left\{\mathbf{a} \in \{-1,1\}^{2(m+\ell)+1} : a_i = 1 \text{ for all } |i| > m\right\}.$$

Each candidate $\mathbf{a} \in \mathcal{M}$ has boundary[1] symbols equal to 1; see Figure 1. Let the following sequence $\cdots, B_{-2}, B_{-1}, B_0, B_1, B_2, \cdots$ denote *symbol decisions* on the channel inputs $\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$. Let $\mathbb{1}$ denote the *all-ones* vector $\mathbb{1} \triangleq [1,1,\cdots,1]^T$. Let $|\mathbf{a}|$ denote the Euclidean norm of the vector $\mathbf{a}$.

Definition 2. *The symbol decision B_t on channel input A_t, is obtained by i) computing the sequence $\boldsymbol{B}^{[t]}$ that achieves the following minimum*

$$\begin{aligned} \boldsymbol{B}^{[t]} &\triangleq \arg\min_{\mathbf{a}\in\mathcal{M}} |\boldsymbol{Z}_t - (\mathbf{H}+\mathbf{T})\mathbf{a}|^2, \\ &= \arg\min_{\mathbf{a}\in\mathcal{M}} |\boldsymbol{Z}_t - \mathbf{T}\mathbb{1} - \mathbf{H}\mathbf{a}|^2, \end{aligned} \tag{5}$$

and ii) setting the symbol decision B_t to the 0-th component of $\boldsymbol{B}^{[t]}$ in (5), i.e. set $B_t \triangleq B_0^{[t]}$ where the sequence $\boldsymbol{B}^{[t]} = [\mathbb{1}, B_{-m}^{[t]}, B_{-m+1}^{[t]}, \cdots, B_{-1}^{[t]}, B_0^{[t]}, B_1^{[t]}, \cdots, B_m^{[t]}, \mathbb{1}]^T$.

The sequence $\boldsymbol{B}^{[t]}$ in (5), and therefore the symbol decision B_t, is obtained by considering the candidate sequences in the set $\mathcal{M}$, recall Definition 1 and refer to Figure 1. To obtain $\boldsymbol{B}^{[t]}$, we compare the squared Euclidean distances of each candidate $\mathbf{H}\mathbf{a}$ from the received neighborhood $\boldsymbol{Z}_t - \mathbf{T}\mathbb{1}$.

In addition to computing *hard*, i.e., $\{-1,1\}$, symbol decisions $\cdots, B_{-2}, B_{-1}, B_0, B_1, B_2, \cdots$, the m-truncated MLM also computes the symbol *reliabilities*, denoted as $\cdots, R_{-2}, R_{-1}, R_0, R_1, R_2, \cdots$. The reliability R_t is computed by considering *competing candidates* in the set $\mathcal{M}$; we compute R_t by considering those candidates $\mathbf{a} \in \mathcal{M}$ that *compete*, i.e. satisfy $a_0 \neq B_t$, with the symbol decision B_t. Denote the difference in the squared Euclidean distances

$$\begin{aligned} \Delta(\mathbf{a}, \bar{\mathbf{a}}) &= \Delta(\mathbf{a}, \bar{\mathbf{a}}; \boldsymbol{Z}_t) \\ &\triangleq |\boldsymbol{Z}_t - \mathbf{T}\mathbb{1} - \mathbf{H}\mathbf{a}|^2 - |\boldsymbol{Z}_t - \mathbf{T}\mathbb{1} - \mathbf{H}\bar{\mathbf{a}}|^2, \end{aligned} \tag{6}$$

where $\mathbf{a}$ and $\bar{\mathbf{a}}$ are sequences in $\{-1,1\}^{2(m+\ell)+1}$.

Definition 3. *Define the m-truncated MLM reliability*

$$R_t \triangleq \min_{\substack{\mathbf{a}\in\mathcal{M} \\ a_0 \neq B_t}} \frac{1}{2\sigma^2} \Delta(\mathbf{a}, \boldsymbol{B}^{[t]}), \tag{7}$$

where $\Delta(\mathbf{a}, \boldsymbol{B}^{[t]}) \geq 0$, see (6), is the difference in the obtained squared Euclidean distances corresponding to candidates $\mathbf{a}, \boldsymbol{B}^{[t]} \in \mathcal{M}$, and σ^2 is the noise variance.

Note that $\Delta(\mathbf{a}, \boldsymbol{B}^{[t]}) \geq 0$ for all $\mathbf{a} \in \mathcal{M}$, simply because $\boldsymbol{B}^{[t]}$ achieves the minimum squared Euclidean distance amongst all candidates in $\mathcal{M}$, see (5).

II. Distributions & Error Probabilities

In this section, we give closed-form expressions for the joint i) reliability distribution $F_{R_{t_1}, R_{t_2}, \cdots, R_{t_n}}(r_1, r_2, \cdots, r_n)$, and ii) symbol error probability $\Pr\{\cap_{i=1}^n \{B_{t_i} \neq A_{t_i}\}\}$. The result holds for any number n of arbitrarily chosen time instants $t_1, t_2, \cdots, t_n$.

For all times t, define the following two random variables X_t and Y_t as

$$X_t \triangleq \max_{\substack{\mathbf{a}\in\mathcal{M} \\ a_0 \neq A_t}} \frac{1}{4} \Delta(\boldsymbol{A}_t, \mathbf{a}), \text{ and } Y_t \triangleq \max_{\substack{\mathbf{a}\in\mathcal{M} \\ a_0 = A_t}} \frac{1}{4} \Delta(\boldsymbol{A}_t, \mathbf{a}) \geq 0, \tag{8}$$

where $\Delta(\boldsymbol{A}_t, \mathbf{a})$ is the difference in obtained squared Euclidean distances, corresponding to the transmitted sequence $\boldsymbol{A}_t$ and a candidate $\mathbf{a} \in \mathcal{M}$, see (6). Note that $Y_t \geq 0$, because there must exist a candidate $\mathbf{a} \in \mathcal{M}$ that satisfies $\Delta(\boldsymbol{A}_t, \mathbf{a}) = 0$, see (6); this particular candidate $\mathbf{a} \in \mathcal{M}$ satisfies $a_i = A_{t+i}$ for all values of i satisfying $|i| \leq m$.

Lemma 1. *The m-truncated MLM reliability R_t in (7) satisfies*

$$R_t = \frac{2}{\sigma^2} |X_t - Y_t|, \tag{9}$$

where both random variables X_t and Y_t are given in (8). □

The proof of Lemma 1 will appear in the journal version [8]. Lemma 1 is important as it relates R_t to random variables X_t and Y_t in (8); equation (9) allows us to obtain both reliability distribution and error probability, from the distribution of $X_t - Y_t$ (see [8] and upcoming Corollaries 1 and 2).

A. Closed-form expressions and evaluation procedure

For any n number of arbitrarily chosen time instants $t_1, t_2, \cdots, t_n$, we wish to obtain the distribution of the vector of reliabilities $\boldsymbol{R}_{\mathbf{t}_1^n} \triangleq [R_{t_1}, R_{t_2}, \cdots, R_{t_n}]^T$.

Definition 4. *Define the* binary *vector $\mathbf{e}_i$ of size $2(m+\ell)+1$*

$$\mathbf{e}_i \triangleq [\overbrace{0,0,\cdots,0}^{m+\ell+i}, 1, \overbrace{0,0,\cdots,0}^{m+\ell-i}]^T, \tag{10}$$

[1] Alternatively, the boundary symbols can be specified to be any sequence of choice in the set $\{-1,1\}^\ell$; here we choose the boundary sequence $[1,1,\cdots,1] = \mathbb{1}$ simply for clearer exposition.

where i can take values $|i| \leq m+\ell$. Further define the matrix $\mathbf{E}$ *of size* $2(m+\ell)+1$ *by* $2m$ *as*

$$\mathbf{E} \triangleq [\mathbf{e}_{-m}, \mathbf{e}_{-m+1}, \cdots, \mathbf{e}_{-1}, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m]. \tag{11}$$

Definition 5. *Define the matrix* $\mathbf{S}$ *of size* $2m$ *by* 2^{2m} *as*

$$\mathbf{S} \triangleq [\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{2^{2m}-1}], \tag{12}$$

where the columns s_i make up all 2^{2m} possible, length-$(2m)$ binary vectors, i.e., $\{\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{2^{2m}-1}\} = \{0,1\}^{2m}$.

Let $\operatorname{diag}(\boldsymbol{A}_t)$ denote the diagonal matrix, whose diagonal equals $\boldsymbol{A}_t$. Recall the size $2m+\ell+1$ by $2(m+\ell)+1$ channel matrix $\mathbf{H}$ given in (3). Define the matrix $\mathbf{G}(\boldsymbol{A}_t)$ of size $2m+\ell+1$ by 2^{2m} as

$$\mathbf{G}(\boldsymbol{A}_t) \triangleq \mathbf{H}\operatorname{diag}(\boldsymbol{A}_t)\mathbf{E}. \tag{13}$$

Let $\boldsymbol{W}_{\mathbf{t}_1^n}$ denote the concatenation

$$\boldsymbol{W}_{\mathbf{t}_1^n} \triangleq [\boldsymbol{W}_{t_1}^T, \boldsymbol{W}_{t_2}^T, \cdots, \boldsymbol{W}_{t_n}^T]^T, \tag{14}$$

where $\boldsymbol{W}_t$ appears in (4). Note that (14) is a length $n\cdot(2m+\ell+1)$ vector. Define the noise covariance matrix

$$\mathbf{K}_{\boldsymbol{W}} \triangleq \mathbb{E}\{\boldsymbol{W}_{\mathbf{t}_1^n}\boldsymbol{W}_{\mathbf{t}_1^n}^T\}. \tag{15}$$

Note, $\mathbf{K}_{\boldsymbol{W}}$ is generally not Toeplitz even if W_t is stationary.

Let $\boldsymbol{A}_{\mathbf{t}_1^n}$ denote the concatenation of $\boldsymbol{A}_{t_1}, \boldsymbol{A}_{t_2}, \cdots, \boldsymbol{A}_{t_n}$, i.e. similar to (14), we have $\boldsymbol{A}_{t_i}$ in place of $\boldsymbol{W}_{t_i}$. Let $\mathbf{I}$ denote the identity matrix; in particular $\mathbf{I}_{2m}$ has size $2m$ by $2m$. The matrix $\mathbf{S}\mathbf{S}^T$ can be verified to equal

$$\mathbf{S}\mathbf{S}^T = \sum_{k=0}^{2^{2m}-1} \mathbf{s}_k\mathbf{s}_k^T = 2^{2(m-1)} \cdot [\mathbf{I}_{2m} + \mathbb{1}\mathbb{1}^T], \tag{16}$$

where $\mathbf{S}$ is given in Definition 5 and $\mathbb{1} \triangleq [1, 1, \cdots, 1]^T$. We denote the matrix *Kronecker product* using the operation $\otimes$. Let $\mathbf{diag}(\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n}))$ denote a *block diagonal matrix*, whose block-diagonal entries are $\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n})$, see (13).

Definition 6. *Let the square matrix $\mathbf{Q} = \mathbf{Q}(\boldsymbol{A}_{\mathbf{t}_1^n})$ of size $2mn$ by $2mn$ satisfy the following two conditions:*

i) *the matrix* $\mathbf{Q}$ ***decomposes*** *the following size $2mn$ matrix*

$$\mathbf{Q}\boldsymbol{\Lambda}^2\mathbf{Q}^T = \mathbf{diag}(\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n}))^T \mathbf{K}_{\boldsymbol{W}} \cdot \mathbf{diag}(\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n})), \tag{17}$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{A}_{\mathbf{t}_1^n})$ on the l.h.s. of (17) is a diagonal matrix. The number of non-zero diagonal elements in $\boldsymbol{\Lambda}$, equals the rank of the matrix on the r.h.s. of (17).

ii) *the matrix* $\mathbf{Q}$ ***diagonalizes*** *the matrix $\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T$, i.e.,*

$$\mathbf{Q}^T(\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T)\mathbf{Q} = \mathbf{I}. \tag{18}$$

See Appendix A on how to compute both matrices $\mathbf{Q} = \mathbf{Q}(\boldsymbol{A}_{\mathbf{t}_1^n})$ and $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{A}_{\mathbf{t}_1^n})$ in (17). Denote matrices $\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_n$ that vertically concatenate (similar to (14)) to form $\mathbf{Q}$; each $\mathbf{Q}_i$ is an equal-sized ($2m$ by $2mn$) partition of $\mathbf{Q}$. Let $\operatorname{diag}(A_{t_1}, A_{t_2}, \cdots, A_{t_n})$ denote a diagonal matrix similarly as $\operatorname{diag}(\boldsymbol{A}_t)$. Define the size n by $2mn$ matrix

$$\mathbf{F}(\boldsymbol{A}_{\mathbf{t}_1^n}) \triangleq \left(\operatorname{diag}(A_{t_1}, A_{t_2}, \cdots, A_{t_n}) \otimes \mathbf{h}_0^T\right) \cdot \mathbf{K}_{\boldsymbol{W}} \cdot \begin{bmatrix} \mathbf{G}(\boldsymbol{A}_{t_1}) & & & \\ & \mathbf{G}(\boldsymbol{A}_{t_2}) & & \\ & & \ddots & \\ & & & \mathbf{G}(\boldsymbol{A}_{t_n}) \end{bmatrix} \begin{bmatrix} \mathbf{S}\mathbf{S}^T\mathbf{Q}_1 \\ \mathbf{S}\mathbf{S}^T\mathbf{Q}_2 \\ \vdots \\ \mathbf{S}\mathbf{S}^T\mathbf{Q}_n \end{bmatrix} \boldsymbol{\Lambda}^\dagger, \tag{19}$$

where $\mathbf{h}_0$ is given in (2), and $\boldsymbol{\Lambda}^\dagger$ is formed by reciprocating only the *non-zero* diagonal elements of $\boldsymbol{\Lambda}$. Define the following length-2^{2m} vectors $\boldsymbol{\mu}(\boldsymbol{A}_t)$ and $\boldsymbol{\nu}(\boldsymbol{A}_t)$ as

$$\begin{aligned} \boldsymbol{\mu}(\boldsymbol{A}_t) &= [\mu_1, \mu_2, \cdots, \mu_{2^{2m}-1}]^T \\ &\triangleq [\mathbf{G}(\boldsymbol{A}_t)\mathbf{S}]^T \cdot \mathbf{T}(\mathbb{1} - \boldsymbol{A}_t) \\ &\quad - \left[|\mathbf{G}(\boldsymbol{A}_t)\mathbf{s}_0|^2, |\mathbf{G}(\boldsymbol{A}_t)\mathbf{s}_1|^2, \cdots, |\mathbf{G}(\boldsymbol{A}_t)\mathbf{s}_{2^{2m}-1}|^2\right]^T, \end{aligned} \tag{20}$$

$$\begin{aligned} \boldsymbol{\nu}(\boldsymbol{A}_t) &= [\nu_1, \nu_2, \cdots, \nu_{2^{2m}-1}]^T \\ &\triangleq \boldsymbol{\mu}(\boldsymbol{A}_t) - 2A_t \cdot \mathbf{h}_0^T\mathbf{G}(\boldsymbol{A}_t)\mathbf{S}, \end{aligned} \tag{21}$$

where $\mu_k = \mu_k(\boldsymbol{A}_t)$ and $\nu_k = \nu_k(\boldsymbol{A}_t)$ denote the k-th components of $\boldsymbol{\mu}_k(\boldsymbol{A}_t)$ and $\boldsymbol{\nu}_k(\boldsymbol{A}_t)$ respectively, and $\mathbf{T}$ is given in (3). Let $\Phi_{\mathbf{K}}(\mathbf{r})$ denote the distribution function of a zero-mean Gaussian random vector with covariance matrix $\mathbf{K}$. Finally define the following length-n random vectors $\boldsymbol{X}_{\mathbf{t}_1^n} \triangleq [X_{t_1}, X_{t_2}, \cdots, X_{t_n}]^T$ and $\boldsymbol{Y}_{\mathbf{t}_1^n} \triangleq [Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}]^T$, see (8). Let $\mathbb{R}$ denote the set of real numbers.

Theorem 1. *The distribution of $\boldsymbol{X}_{\mathbf{t}_1^n} - \boldsymbol{Y}_{\mathbf{t}_1^n}$ equals*

$$F_{\boldsymbol{X}_{\mathbf{t}_1^n} - \boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r}) = \mathbb{E}\left\{\Phi_{\mathbf{K}_{\boldsymbol{V}}(\boldsymbol{A}_{\mathbf{t}_1^n})}\left(\mathbf{r} + \boldsymbol{\delta}(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n}) - \boldsymbol{\eta}(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n})\right)\right\} \tag{22}$$

for all $\mathbf{r} \in \mathbb{R}^n$, where the following random vectors and matrices appear in (22)

- $\boldsymbol{U}$ *is a standard zero-mean identity-covariance Gaussian random vector of length-$(2mn)$.*
- $\boldsymbol{\delta}(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n}) = [\delta_1, \delta_2, \cdots, \delta_n]^T$ *is a length-n vector in $\mathbb{R}^n$, where*

$$\begin{aligned} \delta_i = \delta_i(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n}) &\triangleq \max(\mathbf{S}^T\mathbf{Q}_i\boldsymbol{\Lambda}\boldsymbol{U} + \boldsymbol{\mu}(\boldsymbol{A}_{t_i})) \\ &\quad - \max(\mathbf{S}^T\mathbf{Q}_i\boldsymbol{\Lambda}\boldsymbol{U} + \boldsymbol{\nu}(\boldsymbol{A}_{t_i})) \end{aligned} \tag{23}$$

- $\boldsymbol{\eta}(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n}) = [\eta_1, \eta_2, \cdots, \eta_n]^T$ *is a length-n vector in $\mathbb{R}^n$, where*

$$\begin{aligned} \boldsymbol{\eta}(\boldsymbol{U}, \boldsymbol{A}_{\mathbf{t}_1^n}) \triangleq\ & \operatorname{diag}(A_{t_1}, A_{t_2}, \cdots, A_{t_n}) \cdot \mathbf{T} \\ & \cdot \left(\mathbb{1}\cdot\mathbb{1}^T - [\boldsymbol{A}_{t_1}, \boldsymbol{A}_{t_2}, \cdots, \boldsymbol{A}_{t_n}]\right)^T \mathbf{h}_0 \\ & - |\mathbf{h}_0|^2 \cdot \mathbb{1} + \mathbf{F}(\boldsymbol{A}_{\mathbf{t}_1^n})\boldsymbol{U}. \end{aligned} \tag{24}$$

- $\mathbf{K}_{\boldsymbol{V}}(\boldsymbol{A}_{\mathbf{t}_1^n})$ *is the n by n matrix*

$$\begin{aligned} \mathbf{K}_{\boldsymbol{V}}(\boldsymbol{A}_{\mathbf{t}_1^n}) \triangleq\ & \left(\operatorname{diag}(A_{t_1}, A_{t_2}, \cdots, A_{t_n}) \otimes \mathbf{h}_0^T\right) \cdot \mathbf{K}_{\boldsymbol{W}} \\ & \cdot \left(\operatorname{diag}(A_{t_1}, A_{t_2}, \cdots, A_{t_n}) \otimes \mathbf{h}_0\right) \\ & - \mathbf{F}(\boldsymbol{A}_{\mathbf{t}_1^n})\mathbf{F}(\boldsymbol{A}_{\mathbf{t}_1^n})^T. \end{aligned} \tag{25}$$

Procedure 1: Evaluate Joint Distribution $F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r})$

Initialize: Set $F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r}) := 0$ for all $\mathbf{r} \in \mathbb{R}^n$;

1 **while** $F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r})$ *not converged* **do**

2 Sample $\boldsymbol{A}_{\mathbf{t}_1^n} = \mathbf{a}_1^n$ using $\Pr\left\{\boldsymbol{A}_{\mathbf{t}_1^n} = \mathbf{a}_1^n\right\}$; Sample the length-$n$, standard zero-mean identity-covariance Gaussian vector $\boldsymbol{U} = \mathbf{u}$;

3 Using the sampled realization $\boldsymbol{A}_{\mathbf{t}_1^n} = \mathbf{a}_1^n$, obtain the matrices $\mathbf{Q} = \mathbf{Q}(\mathbf{a}_1^n)$ and $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\mathbf{a}_1^n)$ satisfying Definition 6, see Appendix A;

4 Compute $\delta_i = \delta_i(\mathbf{u}, \mathbf{a}_1^n)$ for all $i \in \{1, 2, \cdots, n\}$. For δ_i compute

$$\max_{k\in\{0,1,\cdots,2^{2m}-1\}} \mathbf{s}_k^T \mathbf{Q}_i \boldsymbol{\Lambda} \mathbf{u} + \mu_k(\mathbf{a}),$$

$$\max_{k\in\{0,1,\cdots,2^{2m}-1\}} \mathbf{s}_k^T \mathbf{Q}_i \boldsymbol{\Lambda} \mathbf{u} + \nu_k(\mathbf{a}),$$

see (23). Here $\mathbf{a}$ is the sampled realization $\boldsymbol{A}_{t_i} = \mathbf{a}$, and both $\mu_k(\mathbf{a})$ and $\nu_k(\mathbf{a})$ are the k-th components of $\boldsymbol{\mu}(\mathbf{a})$ and $\boldsymbol{\nu}(\mathbf{a})$, see (20) and (21);

5 Compute $\mathbf{F}(\boldsymbol{A}_{\mathbf{t}_1^n})$ in (19); Also compute $\boldsymbol{\eta}(\mathbf{u}, \mathbf{a}_1^n)$ in (24) and $\mathbf{K}_{\boldsymbol{V}}(\mathbf{a}_1^n)$ in (25);

6 For all $\mathbf{r} \in \mathbb{R}^n$, update the result

$F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r})$

7 $:= F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r}) + \Phi_{\mathbf{K}_{\boldsymbol{V}}(\mathbf{a}_1^n)}\left(\mathbf{r} + \boldsymbol{\delta}(\mathbf{u}, \mathbf{a}_1^n) - \boldsymbol{\eta}(\mathbf{u}, \mathbf{a}_1^n)\right)$

8 **end**

The proof of Theorem 1 is lengthly and appears in [8]. Both i) the joint distribution of the reliabilities $\boldsymbol{R}_{\mathbf{t}_1^n} \triangleq [R_{t_1}, R_{t_1}, \cdots, R_{t_n}]^T$, and ii) the joint error probability $\Pr\{\bigcap_{i=1}^n \{B_{t_i} \neq A_{t_i}\}\}$, follow as corollaries from Theorem 1; also see [8] for proofs. In the following we denote an index subset $\{\tau_1, \tau_2, \cdots, \tau_j\} \subseteq \{t_1, t_2, \cdots, t_n\}$ of size j, written compactly in vector form as $\boldsymbol{\tau}_1^j = [\tau_1, \tau_2, \cdots, \tau_j]^T$.

Corollary 1. *The distribution of* $\boldsymbol{R}_{\mathbf{t}_1^n} \triangleq 2/\sigma^2 \cdot |\boldsymbol{X}_{\mathbf{t}_1^n} - \boldsymbol{Y}_{\mathbf{t}_1^n}|$*, see Proposition 1, is given as*

$$\begin{aligned} F_{\boldsymbol{R}_{\mathbf{t}_1^n}}(\mathbf{r}) &= F_{|\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}|}(\sigma^2/2 \cdot \mathbf{r}) \\ &= \sum_{j=0}^{n} \sum_{\substack{\{\tau_1,\tau_2,\cdots,\tau_j\}\subseteq \\ \{t_1,t_2,\cdots,t_n\}}} (-1)^j \cdot F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}\left(\frac{\sigma^2}{2} \cdot \boldsymbol{\alpha}(\boldsymbol{\tau}_1^j, \mathbf{r})\right) \end{aligned}$$

where the length-n vector $\boldsymbol{\alpha}(\boldsymbol{\tau}_1^j, \mathbf{r}) = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ *satisfies*

$$\alpha_i = \alpha_i(\boldsymbol{\tau}_1^j, r_i) = \begin{cases} -r_i & \textit{if } t_i \in \{\tau_1, \tau_2, \cdots, \tau_j\}, \\ r_i & \textit{otherwise}, \end{cases}$$

and $F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r})$ *has closed form as in Theorem 1.* □

Corollary 2. *The probability* $\Pr\{\bigcap_{i=1}^n \{B_{t_i} \neq A_{t_i}\}\}$ *that **all** symbol decisions* $B_{t_1}, B_{t_2}, \cdots, B_{t_n}$ *are in error, equals*

$$\begin{aligned} \Pr\left\{\bigcap_{i=1}^{n} \{B_{t_i} \neq A_{t_i}\}\right\} &= \Pr\left\{\boldsymbol{X}_{\mathbf{t}_1^n} \geq \boldsymbol{Y}_{\mathbf{t}_1^n}\right\} \\ &= 1 + \sum_{j=1}^{n} \sum_{\substack{\{\tau_1,\tau_2,\cdots,\tau_j\}\subseteq \\ \{t_1,t_2,\cdots,t_n\}}} (-1)^j \cdot F_{\boldsymbol{X}_{\boldsymbol{\tau}_1^j}-\boldsymbol{Y}_{\boldsymbol{\tau}_1^j}}(\mathbf{0}), \end{aligned}$$

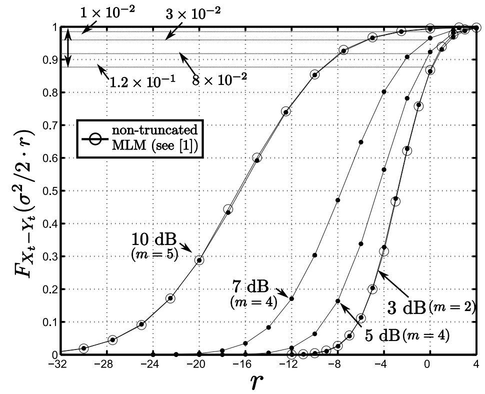

Fig. 2. Comparing the distributions $F_{X_t-Y_t}(\sigma^2/2 \cdot r)$ across different SNRs, for the PR1 channel and various truncation lengths m. Also shown are comparisons with empirical MLM distributions.

where the probability

$$F_{\boldsymbol{X}_{\boldsymbol{\tau}_1^j}-\boldsymbol{Y}_{\boldsymbol{\tau}_1^j}}(\mathbf{0}) = \Pr\left\{\bigcap_{\tau\in\{\tau_1,\tau_2,\cdots,\tau_j\}} \{X_\tau - Y_\tau \leq 0\}\right\}$$

has the similar closed form as in Theorem 1. □

Denote the realizations of $\boldsymbol{A}_{\mathbf{t}_1^n}$, $\boldsymbol{A}_t$ and $\boldsymbol{U}$, as $\boldsymbol{A}_{\mathbf{t}_1^n} = \mathbf{a}_1^n$, and $\boldsymbol{A}_t = \mathbf{a}$, and $\boldsymbol{U} = \mathbf{u}$. The Monte-Carlo Procedure 1 evaluates the closed-form of $F_{\boldsymbol{X}_{\mathbf{t}_1^n}-\boldsymbol{Y}_{\mathbf{t}_1^n}}(\mathbf{r})$ in Theorem 1. Direct evaluations of the maximizations in Line 4 may be difficult for large truncation length m; see Appendix B for an efficient dynamic programming technique for this task.

III. Numerical Computations

We now present numerical computations performed for various ISI channels. Due to space constraints, the results presented here are limited, see [8] for a more general discussion. We consider only i.i.d. noise W_t and uniform input symbol distribution $\Pr\{\boldsymbol{A}_t = \mathbf{a}\}$, i.e. $\Pr\{\boldsymbol{A}_t = \mathbf{a}\} = 2^{-2(m+\ell)-1}$. The signal-to-noise (SNR) ratio is $10\log_{10}(\sum_{i=0}^{\ell} h_i^2/\sigma^2)$.

Figure 2 shows the marginal distribution $F_{X_t-Y_t}(\sigma^2/2 \cdot r)$ computed for the PR1 channel with memory $\ell = 1$, i.e. $h_0 = h_1 = 1$. We fix a truncation length $m = 4$. As SNR increases, the distributions $F_{X_t-Y_t}(\sigma^2/2 \cdot r)$ appear to concentrate more probability mass over negative values of $X_t - Y_t$. This is expected, because the symbol error probability $\Pr\{B_t \neq A_t\} = 1 - F_{X_t-Y_t}(0)$ decreases as SNR increases. From Figure 2, the (error) probabilities $\Pr\{X_t \geq Y_t\}$ are found to be approximately $1.2 \times 10^{-1}, 8 \times 10^{-2}, 3 \times 10^{-2}$, and 1×10^{-2}, respectively for SNRs 3 to 10 dB. Finally, comparisons with empirical distributions, obtained from a non-truncated MLM (see [8]) reveal a good match.

Figure 3 shows joint distributions $F_{\boldsymbol{X}_{\mathbf{t}_1^2}-\boldsymbol{Y}_{\mathbf{t}_1^2}}(\sigma^2/2 \cdot \mathbf{r})$ computed for two time lags $|t_1 - t_2| = 1$ (i.e. neighboring symbols) and $|t_1 - t_2| = 7$. The SNR is moderate at 5 dB, and truncation length $m = 2$. The difference between both cases is subtle (but nevertheless inherent); observe the differently labeled points in Figure 3. The joint symbol error probability $\Pr\{B_{t_1} \neq A_{t_1}, B_{t_2} \neq A_{t_2}\}$ is $\approx 6 \times 10^{-2}$ and 2×10^{-2} for $|t_1 - t_2| = 1$ and 7, respectively. Note that when $|t_1 - t_2| = 7$,

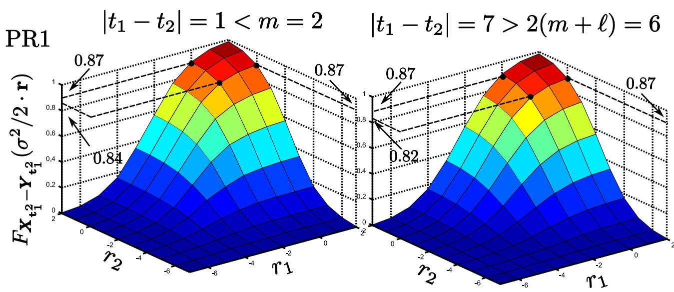

Fig. 3. Joint reliability distribution $F_{\mathbf{X}_{t_1^2} - \mathbf{Y}_{t_1^2}}(\sigma^2/2 \cdot \mathbf{r})$ computed for the PR1 channel at SNR 5 dB, for a fixed truncation length $m = 2$.

both reliabilities R_{t_1} and R_{t_2} are *independent*; this is because then $|t_1 - t_2| = 7 > 2(m + \ell) = 6$, refer to Figure 1.

IV. Conclusion

In this paper, we presented closed-form expressions for both i) the reliability distributions $F_{\mathbf{X}_{t_1^n} - \mathbf{Y}_{t_1^n}}(\sigma^2/2 \cdot \mathbf{r})$, and ii) the symbol error probabilities Pr $\{\bigcap_{i=1}^{n} \{B_{t_i} \neq A_{t_i}\}\}$, for the m-truncated MLM detector. Our results hold jointly for any number n of arbitrarily chosen time instants $t_1, t_2, \cdots, t_n$. Efficient Monte-Carlo procedures have been given; these procedures can be used to numerically evaluate the closed-form expressions.

Appendix

A. Computing the matrix $\mathbf{Q} = \mathbf{Q}(\boldsymbol{A}_{\mathbf{t}_1^n})$ in Definition 6

In this appendix, we show that the size $2mn$ square matrix $\mathbf{Q}$ with both properties i) and ii) as stated in Definition 6, can be easily found. We begin by noting from (16) that $\mathrm{rank}(\mathbf{S}\mathbf{S}^T) = 2m$, therefore the matrix $\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T$ has rank $2mn$ and is *positive definite*.

Lemma 2. *Let* $\mathbf{S}$ *be given as in Definition 5. Let the size* $2mn$ *by* $2mn$ *square matrix* $\boldsymbol{\alpha}$ *diagonalize*

$$\boldsymbol{\alpha}^T (\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T) \boldsymbol{\alpha} = \mathbf{I}. \tag{26}$$

Let $\boldsymbol{\beta}$ *be the size* $2mn$ *by* $2mn$ *eigenvector matrix* $\boldsymbol{\beta}$ *in the following decomposition*

$$\begin{aligned}&\boldsymbol{\alpha}^T (\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T) \cdot \mathbf{diag}\left(\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n})\right)^T \\ &\cdot \mathbf{K}_{\boldsymbol{W}} \cdot \mathbf{diag}\left(\mathbf{G}(\boldsymbol{A}_{t_1}), \mathbf{G}(\boldsymbol{A}_{t_2}), \cdots, \mathbf{G}(\boldsymbol{A}_{t_n})\right) \cdot (\mathbf{I}_n \otimes \mathbf{S}\mathbf{S}^T)\boldsymbol{\alpha} \\ &= \boldsymbol{\beta}\boldsymbol{\Lambda}^2\boldsymbol{\beta}^T, \end{aligned} \tag{27}$$

see (17), and $\boldsymbol{\Lambda}^2$ *is the eigenvalue matrix of (27), therefore* $\boldsymbol{\Lambda}^2$ *in (27) is diagonal of size* $2mn$*. Then* $\mathbf{Q} = \boldsymbol{\alpha}\boldsymbol{\beta}$ *satisfies both properties i) and ii) stated in Definition 6.*

See [8] for the proof. To summarize Lemma 2, the matrix $\mathbf{Q} = \mathbf{Q}(\boldsymbol{A}_{\mathbf{t}_1^n})$ in Definition 6, is obtained by first computing two size $2mn$ matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively satisfying (26) and (27), and then setting $\mathbf{Q} = \boldsymbol{\alpha}\boldsymbol{\beta}$. The matrix $\boldsymbol{\beta}$ is obtained from an eigenvalue decomposition of the size-$2mn$ matrix (27), and clearly $\boldsymbol{\beta}$ depends on the symbols $\boldsymbol{A}_{\mathbf{t}_1^n}$.

Remark 1. *The matrix* $\boldsymbol{\alpha}$ *in (26) is obtained from the eigenvectors of the matrix* $\mathbf{S}\mathbf{S}^T$ *in (16). The first* $2m - 1$ *eigenvectors of* $\mathbf{S}\mathbf{S}^T$ *are*

$$(i + i^2)^{-\frac{1}{2}} \cdot [\overbrace{1, 1, \cdots, 1}^{i}, -i, \overbrace{0, 0, \cdots, 0}^{2m-(i+1)}]^T$$

where i *can take values* $1 \leq i < 2m$*, and the last eigenvector is simply* $\mathbb{1}/|\mathbb{1}| = \mathbb{1}/(2m)$*.*

Procedure 2: Solve $\max_{\mathbf{s} \in \{0,1\}^{2m}} \mathbf{s}^T\boldsymbol{\mathcal{C}} - |\mathbf{G}(\mathbf{a})\mathbf{s}|^2$ by DP

Convention: Set $\mathcal{C}_0 := -\infty$, and $\mathcal{C}_j := 0$ for all $|j| > m$;
: Binary vector $\bar{\mathbf{s}} \triangleq [\bar{s}_{\ell-1}, \bar{s}_{\ell-2}, \cdots, \bar{s}_0]^T$;
Input: Matrix $\mathbf{G}(\mathbf{a})$; Vector of constants $\boldsymbol{\mathcal{C}} = [\mathcal{C}_{-m}, \mathcal{C}_{-m+1}, \cdots, \mathcal{C}_{-1}, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m]^T$;
Output: Value stored in $\beta_{m+\ell}(\bar{\mathbf{s}}) = \beta_{m+\ell}(\mathbf{0})$;
Initialize: For all $\bar{\mathbf{s}} \in \{0,1\}^\ell$, set the values
$$f\beta_{-m-1}(\bar{\mathbf{s}}) := \begin{cases} 0 & \text{if } \bar{\mathbf{s}} = \mathbf{0}, \\ -\infty & \text{otherwise} \end{cases}.$$

```
forall the τ ∈ {−m, −m+1, ··· , m+ℓ } do
    forall the s̄ ∈ {0,1}^ℓ do
        Set the value α = α(s̄) := Σ_{j=0}^{ℓ−1} h_j a_{τ−j} s̄_j. Set
        the states s̄_0 and s̄_1 as
            s̄_0 := [0, s̄_{ℓ−1}, ··· , s̄_2, s̄_1]^T,
            s̄_1 := [1, s̄_{ℓ−1}, ··· , s̄_2, s̄_1]^T. ;
        Compute β_τ(s̄) := max{−α^2 + β_{τ−1}(s̄_0), C_{τ−ℓ} −
        [h_ℓ a_{τ−ℓ} + α]^2 + β_{τ−1}(s̄_1)};
    end
end
```

B. On efficient execution of Line 4 of Procedure 1

Define the length-$(2m)$ vector $\boldsymbol{\mathcal{C}} \triangleq [\mathcal{C}_{-m}, \mathcal{C}_{-m+1}, \cdots, \mathcal{C}_{-1}, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m]^T$. Set $\mathcal{C}_0 := -\infty$ and $\mathcal{C}_\tau := 0$ for all $|\tau| > m$. It can be verified from (20) and (21), that by setting

$$\begin{aligned}\boldsymbol{\mathcal{C}} &:= \mathbf{Q}_i\boldsymbol{\Lambda}\mathbf{u} + [\mathbf{G}(\mathbf{a})]^T \cdot \mathbf{T}(\mathbb{1} - \mathbf{a}) \quad \text{and} \\ \boldsymbol{\mathcal{C}} &:= \mathbf{Q}_i\boldsymbol{\Lambda}\mathbf{u} + [\mathbf{G}(\mathbf{a})]^T \cdot [\mathbf{T}(\mathbb{1} - \mathbf{a}) - 2a_0 \cdot \mathbf{h}_0],\end{aligned}$$

respectively, we can solve both maximizations in Line 4, Procedure 1 as

$$\max_{\mathbf{s} \in \{0,1\}^{2m}} \mathbf{s}^T\boldsymbol{\mathcal{C}} - |\mathbf{G}(\mathbf{a})\mathbf{s}|^2. \tag{28}$$

Matrix $\mathbf{G}(\mathbf{a})$ is $(\ell + 1)$-banded, see [8], and therefore (28) is solved using the (dynamic programming) Procedure 2.

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