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Optimal Quantization for Soft-Decision Decoding Revisited

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Abstract

A new criterion for quantization is proposed. The criterion is optimal in the sense that the resulting discrete channel has the highest information rate.

1. Mutual Information

Let $\mathcal{X} = \{a_1, a_2, \cdots, a_J\}$ and $\mathcal{Y} = \{b_1, b_2, \cdots, b_K\}$ be two finite sets. Let X be the "source" (a random variable) characterized by a probability mass function $P_X(a_j)$, $1 \leq j \leq J$. Let $P_{Y|X}(b_k|a_j)$, $1 \leq k \leq K$, $1 \leq j \leq J$ be the "channel" (a probability transition matrix). The mutual information (MI) is defined as (for general definition, see [1], pp.33-37)

$$I(X;Y) = \sum_{j=1}^{J} \sum_{k=1}^{K} P_X(a_j) P_{Y|X}(b_k|a_j) \log \frac{P_{Y|X}(b_k|a_j)}{P_Y(b_k)} \tag{1}$$

where $P_Y(b_k) = \sum_{1 \leq j \leq J} P_X(a_j) P_{Y|X}(b_k|a_j)$. By definition, we may denote $I(X;Y)$ by $I(P_X, P_{Y|X})$ to emphasize that the MI is a function of the "source" distribution P_X and the "channel" transition probability matrix $P_{Y|X}$. The following two problems are well-known,

P1. For given $P_{Y|X}$,

$$C = \max_{P_X \in \mathcal{P}} I(P_X, P_{Y|X}) \tag{2}$$

where $\mathcal{P}$ is a given subset of probability distributions.

P2. For given P_X,

$$R = \min_{P_{Y|X} \in \mathcal{W}} I(P_X, P_{Y|X}) \tag{3}$$

where $\mathcal{W}$ is a given subset of probability transition matrices.

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The quantities C and R defined above represent the transmission limits and the compression limits (under certain constraints), respectively. In this paper, we attempt to interpret the following two problems,

P3. For given $P_{Y|X}$,

$$R_3 = \min_{P_X \in \mathcal{P}} I(P_X, P_{Y|X}) \tag{4}$$

where $\mathcal{P}$ is a given subset of probability distributions.

P4. For given P_X,

$$R_4 = \max_{P_{Y|X} \in \mathcal{W}} I(P_X, P_{Y|X}) \tag{5}$$

where $\mathcal{W}$ is a given subset of probability transition matrices.

For most problems of interest, problems P1 and P2 can be solved by the Blahut-Arimoto algorithm [2] [3]. On the other hand, it is not clear how to solve problems P3 and P4. However, if the subset $\mathcal{W}$ ($\mathcal{P}$, respectively) can be parameterized by one or two parameters, one can always find satisfactory numerical solutions.

2. Optimal Quantization

2.1. Problem Statement

As an example, we choose the additive white Gaussian noise (AWGN) channel. Assume that the input is constrained to $\mathcal{X} = \{-1, +1\}$, and the output has to be quantized to $\mathcal{Y} = \{-1, 0, +1\}$. The original channel (binary input/continuous output) is described as

$$Z_t = X_t + N_t, \; t = 0, 1, ... \tag{6}$$

where N_t is an independent identically distributed (i.i.d.) Gaussian noise with mean zero and variance σ^2. The signal-to-noise-ratio (SNR) is defined as $\mathrm{SNR} = 10 \log_{10}(1/\sigma^2)$.

The quantizer is described as

$$Y_t = \begin{cases} -1, & Z_t < -T \\ 0, & -T \le Z_t \le T \\ +1, & Z_t > T \end{cases} \tag{7}$$

where the threshold $T \ge 0$ is a parameter. An immediate question is how to choose T.

2.2. Forney's Criterion

The following argument is based on an Exercise in [4]. Let $\mathcal{C}$ be a binary linear block code $[n, k, d]$ with $d > 1$. Assume that a codeword is transmitted over an AWGN channel with BPSK signaling and the received signals are quantized to ternary outputs according to Equation (7), where a zero "0" is interpreted as an erasure. Assume that there occur s erasures and t errors. It is well known that such a code is able to recover the transmitted codeword if $2t + s < d$. On the other hand, for any integers t and s such that $2t + s \ge d$ and for any decoding rule, there exists some pattern of t errors and s erasures that will cause a decoding error. Therefore, the minimum squared distance from any codeword to its decoding boundary is equal to

$$L(T) = \min_{2t+s \ge d} \{s(1-T)^2 + t(1+T)^2\}. \tag{8}$$

Forney's criterion [4] is to choose $T = T_f$ such that

$$T_f = \arg \max_{0 \le T \le 1} L(T) \tag{9}$$

It can be verified that the solution is

$$T_f = \frac{\sqrt{2}-1}{\sqrt{2}+1} \tag{10}$$

which is independent of SNR.

2.3. The Cutoff Rate Criterion

Evidently, the original AWGN channel is transformed into a discrete memoryless channel (DMC) parameterized by the quantization threshold T. The resulting DMC is shown in Figure 1. The erasure probability is

$$\alpha = Q\left(\frac{1-T}{\sigma}\right) - Q\left(\frac{1+T}{\sigma}\right) \tag{11}$$

and the error probability is

$$\beta = Q\left(\frac{1+T}{\sigma}\right), \tag{12}$$

where Q function is defined as

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt. \tag{13}$$

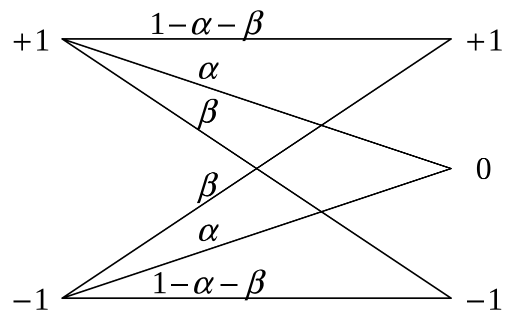

Figure 1: The binary-input/ternary-output DMC model.

The *symmetric cutoff rate* $\tilde{R}_0$ for this DMC is evaluated as [5]

$$\tilde{R}_0(T) = 1 - \log_2\left(1 + \alpha + 2\sqrt{(1-\alpha-\beta)\beta}\right). \tag{14}$$

The *cutoff rate* criterion is to choose threshold T that maximizes the cutoff rate R_0, for details, see [5][6][7][8]. A simpler criterion [5] is to choose $T = T_c$ such that

$$T_c = \arg \max_{T \ge 0} \tilde{R}_0(T). \tag{15}$$

2.4. The MI criterion

Let X be an independent uniformly distributed (i.u.d.) random variable over $\mathcal{X} = \{-1, +1\}$. Applying Equation (1) to the DMC shown in Figure 1, we obtain the mutual information (rate) between X and Y as follows

$$R(T) = (1-\alpha-\beta)\log(1-\alpha-\beta) + \beta\log\beta - (1-\alpha)\log\frac{1-\alpha}{2}. \tag{16}$$

We propose to choose $T = T_m$ according to the i.u.d. mutual information (MI) criterion as

$$T_m = \arg \max_{T \ge 0} R(T). \tag{17}$$

The MI criterion is optimal in the sense that the resulting DMC has the highest (i.u.d.) information rate.

3. Numerical Results

Unlike Forney's criterion, closed-form solutions to both the cutoff rate criterion and the MI criterion (see Equations (15) and (17)) are not available. The numerical solutions to these two criterion are plotted in Figure 2. From Figure 2, a simple approximation for the MI criterion can be written as

$$T_m \approx \exp(a \cdot \mathrm{SNR} + b), \tag{18}$$

where a and b are two yet-to-be determined parameters which are independent of SNR.

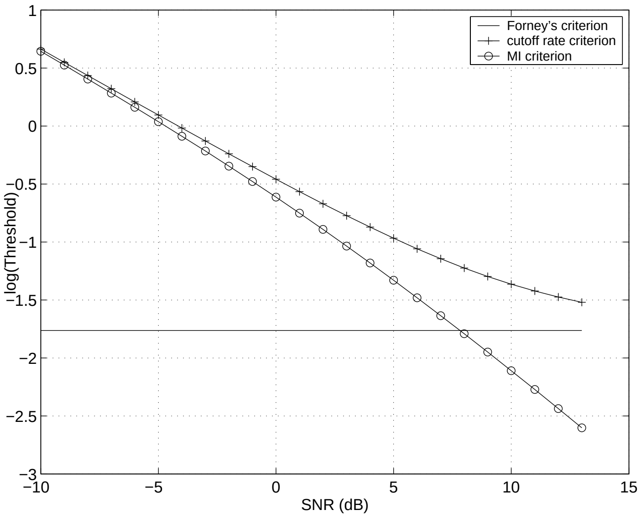

Figure 2: The quantization threshold versus SNR.

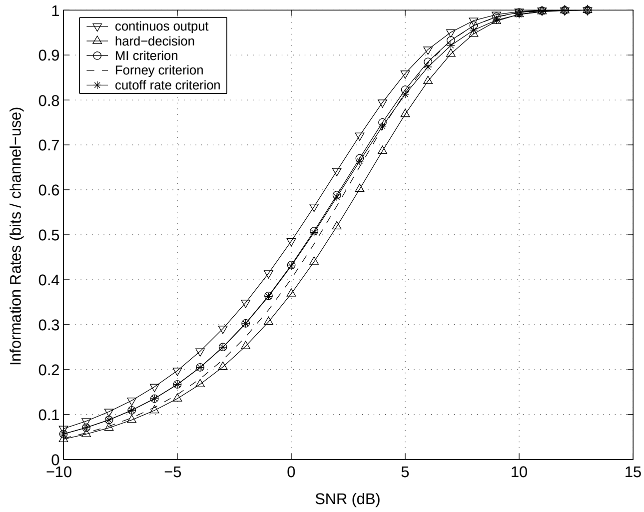

Figure 3: Information rates for different quantization criteria.

For a given SNR, we may first determine the thresholds T using different criteria, and then evaluate the mutual information rates using Equation (16). Numerical results (see Fig. 3) show that, 1) the cut-off rate criterion is almost equivalent to the optimal MI criterion at low SNRs, and 2) Forney's criterion is almost equivalent to the optimal MI criterion at high SNRs.

4. Conclusions

A quantizer can be viewed as a post-channel processor. The MI criterion is to choose the post-channel processor (under given complexity constraints) such that the resulting "channel" has the highest information rate. Similar ideas appear in [9] [10] [11], where our goal was to choose a "pre-channel" processor such that the resulting super-channel has a high (i.u.d.) mutual information rate. The (i.u.d.) MI criterion is optimal in the sense that it gives the highest information rate. Furthermore, since the inputs are (i.u.d.), this information rate can be achieved by an (outer) linear code.

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