Design of Unitary Precoders for ISI Channels

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Abstract— **Redundancy introduced using filterbank precoders at the transmitter builds a unified framework for modulation schemes. Taking advantage of this diversity can offer a powerful tool for removing interblock interference and devising simple but effective precoders for suppressing the intersymbol interference (ISI) and being robust to frequency selective channels. In this paper, we assume that the transmitter knows the autocorrelation sequences of the channel impulse response. Under this assumption, we derive a lower and an upper bound on the free distance for a precoded channel and with this, design the precoder that maximizes the lower bound subject to a power constraint. It turns out that the optimal precoder is a unitary matrix which makes QR decomposition of its super-channel exhibit equal diagonal entries in R-factor and the lower and the upper bounds of its free distance equal. We show that for the optimal precoded channel, the detection performance using the decision feedback equalizer (DFE) based on QR decomposition is asymptotically equivalent to that of the maximum likelihood detector (MLD) when the signal to noise ratio (SNR) is large.**

I. Introduction

Block transmission is commonly used for communicating over dispersive channels affected by intersymbol interference. The transmitted data stream is parsed into consecutive equal-size blocks and redundancy is added to each block in order to remove interblock interference. Filterbank precoding framework [1], [2], [3] unifies existing modulations including orthogonal frequency division (OFDM), discrete multione (DMT), time division multiplex acess (TDMA) and code division multiplex acess (CDMA) schems encountered with single and multiuser communications. Taking advantage of this diversity can offer a powerful tool for removing interblock interference and devising simple but effective precoders for suppressing ISI and being robust to frequency selective channels. Existing precoder design criteria are based on maximum output SNR and minimum mean-square error criteria under zero-forcing and fixed transmitted power constrains [3], [4]. Idealy, an optimal precoder should minimize the detection error probability of the MLD. But we know that the average probability of error over all blocks is dominated by the free distance for high SNR [5]. However, directly maximizing the free distance is too expensive for the complexities of both its design and detection to be affordable. In this paper we are interested in designing the precoder which maximizes the lowerbound of its free distance.

The main contribution in this paper is to derive the lowerbound and upperbound of the free distance in terms of the diagonal entries in the R-factor of the QR decomposition of a channel and with this, reduce the precoder design to the design of a unitary matrix whose QR decomposition of the super-channel matrix exhibits equal diagonal entries in the R-factor. We not only give a necessary and sufficient condition to check if a matrix possesses this property, but also give a recursive algorithm to construct the matrix. For this optimal precoder, the free distance of the super-channel has a simple formula. In addition, we derive an analytic expression for the block error probability of the DFE detection based on QR decomposition. By comparison, we find that the detection performance of the optimal precoded channel is asymptotically equivalent to that of the MLD when using the same precoder.

Notation: The columns of an $M \times N$ matrix $\mathbf{V}$ are denoted by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N$. $\mathbf{V}_k$ denotes a matrix consisting of the first k colums of $\mathbf{V}$, i.e., $\mathbf{V}_k = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$. By convention, $\mathbf{V}_0 = 1$. $\mathbf{R_V}$ denotes the autocorrelation matrix of $\mathbf{V}$, i.e., $\mathbf{R_V} = \mathbf{V}^H\mathbf{V}$.

II. Channel Model

Fig.1 shows the discrete-time equivalent model of the baseband communication system using filterbank precoders [2], [3]. The blocked signal $\mathbf{s}(n) = [s(nN), s(nN+1), \cdots, s(nN+N-1)]$ is precoded by an $N \times N$ matrix $\mathbf{F}$ and then transmitted through an equivalent $M \times N$ matrix channel, where the channel matrix is

$$\mathbf{H} = \begin{pmatrix} h(0) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ h(L) & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & h(0) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & h(L) \end{pmatrix}$$

In this case, a received signal $\mathbf{r}(n)$ can be expressed as

$$\mathbf{r}(n) = \mathbf{HFs}(n) + \boldsymbol{\xi}(n) \tag{1}$$

Our task is to find the precoder $\mathbf{F}$ that maximizes the lower bound of its free distance.

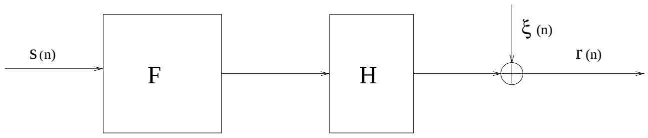

Fig. 1. Equivalent matrix receiver model

III. Review of the decision feedback equalizer

We briefly review a decision feedback equalization scheme based on QR decomposition. Consider a channel model similar to (1),

$$\mathbf{r}(n) = \mathbf{C}\mathbf{s}(n) + \boldsymbol{\xi}(n) \tag{2}$$

with the only difference from (1) being that $\mathbf{C}$ is an $M \times N$ full rank channel matrix with $M \geq N$. The QR decision feedback is captured by these 3 steps:

- Perform QR decomposition, $\mathbf{C} = \mathbf{QR}$, where $\mathbf{Q}$ is a tall orthonormal matrix and $\mathbf{R}$ is an upper triangular square matrix,

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1N} \\ 0 & R_{22} & \dots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{NN} \end{pmatrix} \qquad R_{kk} \geq 0$$

For $k = N, \cdots, 1$, this reduces (2) to

$$\tilde{r}_k(n) = R_{k,k}s_k(n) + \sum_{m=k+1}^{N} R_{k,m}s_m(n) + \tilde{\xi}_k(n)$$

where $\tilde{\mathbf{r}}(n) = \mathbf{Q}^H\mathbf{r}(n)$ and $\tilde{\boldsymbol{\xi}}(n) = \mathbf{Q}^H\boldsymbol{\xi}(n)$.

- Quantization, $\hat{s}_N(n) = \text{Quant}\left[\tilde{r}_N(n)/R_{N,N}\right]$ (for binary signaling, Quant[$\cdot$]=Sign[$\cdot$]);
- Remove the interference term in $r_k(n)$ to obtain the estimate of $s_k(n)$ using the decision on $s_{k+1}(n), \dots, s_M(n)$, i.e., for $k = N, \cdots, 1$,

$$\hat{s}_k(n) = \text{Quant}\left[\frac{\tilde{r}_k(n) - \sum_{m=k+1}^{N} R_{k,m}\hat{s}_m(n)}{R_{k,k}}\right].$$

With the help of QR decomposition, we have:

Theorem 1: Let the free distance of a MLD detector for a channel $\mathbf{C}$ be defined as

$$d^2_{\text{free}}(\mathbf{C}) = \min_{\mathbf{s}_1 \neq \mathbf{s}_2} (\mathbf{s}_1 - \mathbf{s}_2)^H \mathbf{R}_{\mathbf{C}} (\mathbf{s}_1 - \mathbf{s}_2) \tag{3}$$

for any $\mathbf{s_1}, \mathbf{s_2} \in \mathcal{S} \times \mathcal{S} \cdots \times \mathcal{S}$. Then,

$$d_{\min} \min_{1 \leq k \leq N} R_{kk} \leq d_{free}(\mathbf{C}) \leq R_{11} d_{\min}, \tag{4}$$

where $d_{\min}$ is the minimum distance between input symbols, i.e.,

$$d_{\min} = \min_{s_1 \neq s_2} |s_1 - s_2| \qquad \text{for any } s_1, s_2 \in \mathcal{S}$$

IV. Design of maximum free distance lowerbound precoder

The following assumptions characterize channel (1).
1. For $k < 0, k > L$ the channel coefficients are $h_k = 0$;
2. $M = N + L$ and the auto-correlation sequences of the channel impuse response is known to the transmitter;

Our problem is now stated as

Problem 1: : Find the precoder matrix $\mathbf{F}$ that maximizes the lower bound of the free distance, $\min R_{kk}$, subject to the power constraint, tr $\left(\mathbf{F}^H\mathbf{F}\right) \leq p$. More precisly, it is formulated as the following optimization problem,

$$\mathbf{F}^\star = \arg \max_{\text{tr}(\mathbf{F}^H\mathbf{F}) \leq p} \min_{1 \leq k \leq N} R_{kk},$$

First we note that

$$\min_{1 \leq k \leq N} R^2_{kk} \leq \left(\prod_{k=1}^{N} R^2_{kk}\right)^{1/N} \tag{5}$$

The equality here holds if and only if

$$R_{11} = R_{22} = \cdots = R_{NN}. \tag{6}$$

On the other hand, note that the diagonal elements of the upper triangular matrix $\mathbf{R}$ in the QR decomposition of matrix $\mathbf{HF}$ can be expressed by the determinants of the submatrices of the autocorrelation matrix of $\mathbf{F}^H\mathbf{R_H}\mathbf{F}$, i.e.,

$$R_{kk} = \sqrt{\frac{det(\mathbf{F}^H_k\mathbf{R_H}\mathbf{F}_k)}{det(\mathbf{F}^H_{k-1}\mathbf{R_H}\mathbf{F}_{k-1})}} \tag{7}$$

for $k = 1, 2, \cdots, N$. Since $det(\mathbf{F}^H_0\mathbf{R_H}\mathbf{F}_0) = 1$ and $det(\mathbf{F}^H_N\mathbf{R_H}\mathbf{F}_N) = det(\mathbf{F}^H\mathbf{R_H}\mathbf{F})$, combining (7) with (5) yields

$$\min_{1 \leq k \leq N} R^2_{kk} \leq det\left(\mathbf{F}^H\mathbf{R_H}\mathbf{F}\right) = det\left(\mathbf{F}^H\mathbf{F}\right) det\left(\mathbf{R_H}\right) \tag{8}$$

The equality holds if and only if (6) holds. Applying Hadamard's inequality leads to

$$det\left(\mathbf{F}^H\mathbf{F}\right) \leq \prod_{k=1}^{N} \mathbf{f}^H_k\mathbf{f}_k. \tag{9}$$

The equality here holds if and only if $\mathbf{F}^H\mathbf{F}$ is a diagonal matrix. Furthermore, under the power constraint $\text{tr}(\mathbf{F^H F}) \leq \text{p}$, we have

$$\prod_{k=1}^{N} \mathbf{f}^H_k\mathbf{f}_k \leq \left(\frac{\sum_{k=1}^{N} \mathbf{f}^H_k\mathbf{f}_k}{N}\right)^N \leq \left(\frac{p}{N}\right)^N \tag{10}$$

The equalities hold if and only if the diagonal entries of $\mathbf{F}^H\mathbf{F}$ are equal, i.e.,

$$\mathbf{f}^H_1\mathbf{f}_1 = \mathbf{f}^H_2\mathbf{f}_2 = \cdots = \mathbf{f}^H_N\mathbf{f}_N = \frac{p}{N} \tag{11}$$

Combining (8) and (9) with (10) results in

$$\min_{1\le k\le N} R_{kk}^2 \le \frac{p}{N}\left(det(\mathbf{R_H})\right)^{1/N}. \quad (12)$$

The equality holds if and only if (6) holds and the precoder $\mathbf{F}$ has the following structure,

$$\mathbf{F} = \sqrt{\frac{p}{N}}\mathbf{V}, \quad (13)$$

where $\mathbf{V}$ is a unitary matrix. Actually, (6) is equivalent to

$$\frac{det(\mathbf{F}_k^H \mathbf{R_H} \mathbf{F}_k)}{det(\mathbf{F}_{k-1}^H \mathbf{R_H} \mathbf{F}_{k-1})} = \frac{det(\mathbf{F}_{k+1}^H \mathbf{R_H} \mathbf{F}_{k+1})}{det(\mathbf{F}_k^H \mathbf{R_H} \mathbf{F}_k)}$$

for $k = 1, 2, \cdots, N-1$. This shows that $det(\mathbf{F}_k^H \mathbf{R_H} \mathbf{F}_k)$ is a geometric sequence, i.e.,

$$det(\mathbf{F}_k^H \mathbf{R_H} \mathbf{F}_k) = det(\mathbf{F}^H \mathbf{R_H} \mathbf{F})^{k/N}. \quad (15)$$

Therefore, our optimization problem is reduced to find a unitary matrix $\mathbf{V}$ satisfying

$$det(\mathbf{V}_k^H \mathbf{R_H} \mathbf{V}_k) = det(\mathbf{R_H})^{k/N}. \quad (16)$$

That is equivalent to finding a unitary matrix $\mathbf{V}$ that makes the QR decomposition of the super-channel $\mathbf{C} = \mathbf{HV}^\star$ exhibit equal diagonal entries in the R-factor. Now a natural question is whether such unitary matrix exist and how to construct it if exists. The following theorem, whose proof is omitted, gives an answer.

Theorem 2: The solution $\mathbf{F}^\star$ to Problem 1 exists. For this precoder $\mathbf{F}^\star$, let the QR decomposition of $\mathbf{HF}^\star$ be $\mathbf{HF}^\star = \mathbf{Q}^\star\mathbf{R}^\star$. Then, we have
1. The diagonal elements in the upper triangular factor $\mathbf{R}^\star$ are all equal.
2. For the optimal precoder $\mathbf{F}^\star$,

$$d_{\text{free}}(\mathbf{HF}^\star) = \sqrt{\frac{p}{N}} d_{\min}\left(det(\mathbf{R_H})\right)^{1/2N}. \quad (17)$$

3. The optimal solution of Problem 1, $\mathbf{F}^\star$, has the following structure,

$$\mathbf{F}^\star = \sqrt{\frac{p}{N}}\mathbf{V_H}\mathbf{W} \quad (18)$$

where $\mathbf{H}^H\mathbf{H} = \mathbf{V_H}\mathbf{\Lambda}\mathbf{V_H}^H$ and $\mathbf{W}$ is determined by the following recursive algorithm,

(a) $\mathbf{w}_1 = (w_{11}, \cdots, w_{N1})^T$ is determined by

$$\sum_{k=1}^N \lambda_k w_{k1}^2 = det\left(\mathbf{R_H}\right)^{1/N} \quad (19)$$

$$\mathbf{w}_1^H \mathbf{w}_1 = 1 \quad (20)$$

(b) $\mathbf{w}_{k+1} = \mathbf{W}_k^\perp \mathbf{z}_{k+1}$, where $\mathbf{z}_{k+1}$ is determined by

$$\mathbf{z}_{k+1}^H \mathbf{A}^{(k)} \mathbf{z}_{k+1} = det\left(\mathbf{R_H}\right)^{1/N} \quad (21)$$

$$\mathbf{z}_{k+1}^H \mathbf{z}_{k+1} = 1 \quad (22)$$

for $k = 1, 2, \cdots, N-1$, where $\mathbf{B}^{(k)} = \mathbf{W}_k^H \mathbf{R_H} \mathbf{W}_k$ and

$$\mathbf{A}^{(k)} = (\mathbf{W}_k^\perp)^H \mathbf{R_H}\left(\mathbf{I} - \mathbf{W}_k(\mathbf{B}^{(k)})^{-1}\mathbf{W}_k^H\mathbf{R_H}\right)\mathbf{W}_k^\perp.$$

Remarks: To appreciate the physical meaning of the solution $\mathbf{F}^\star$, we make the following comments.

• Equation (17) is equivalent to

$$\log\left[d_{\text{free}}(\mathbf{HF}^\star)\right] = \frac{1}{2}\log\frac{p}{N} + logd_{\min} + \frac{1}{2N}\sum_{k=1}^N \log\lambda_k. \quad (23)$$

For N large, $\frac{1}{N}\sum_{k=1}^N \log\lambda_k$ tends to $\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|H(\omega)|^2 d\omega$. Hence,

$$d_{\text{free}}(\mathbf{HF}^\star) = \sqrt{\frac{p}{N}} 2^{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log|H(\omega)|^2 d\omega} d_{\min},$$

where we have used Kolmogorov's result ([6], p. 274).

• For the optimal precoder $\mathbf{F}^\star$ and for BPSK input symbols, we derive an analytic expression for P_c, the probability of correctly detecting a block using the DFE based on QR decomposition. By the chain rule, we have

$$\begin{aligned} P_c &= Pr(s_1^c(n), s_2^c(n), \cdots, s_N^c(n)) \\ &= \prod_{k=1}^N Pr(s_k^c(n)|s_{k+1}^c(n), \cdots, s_N^c(n)) \end{aligned} \quad (25)$$

where $s_k^c(n)$ indicates that the detected symbol is correct, i.e., $s_k(n) = \hat{s}_k(n)$. If s_k^e denotes that the detected symbol is not correct, then (25) can be rewritten as

$$P_c = \prod_{k=1}^N \left\{1 - Pr\left[s_k^e(n)|s_{k+1}^c(n), \cdots, s_N^c(n)\right]\right\} \quad (26)$$

For BPSK input symbols, we know

$$Pr(s_k^e(n)|s_{k+1}^c(n), \cdots, s_N^c(n)) = \frac{1}{2}erfc\left(\frac{R_{kk}^\star}{\sigma\sqrt{2}}\right) \quad (27)$$

In this case, since $R_{kk}^\star$ are all equal, i.e., $R_{kk}^\star = \sqrt{\frac{p}{N}}\left(det(\mathbf{R_H})\right)^{1/2N}, k = 1, 2, \cdots, N$, substituting (27) into (26) yields

$$P_c = \left(1 - \frac{1}{2}erfc\left(\sqrt{\frac{snr}{2}}det(\mathbf{R_H})^{1/2N}\right)\right)^N \quad (28)$$

or equivalently, the block error probability is

$$P_e = 1 - \left(1 - \frac{1}{2}erfc\left(\sqrt{\frac{snr}{2}}det(\mathbf{R_H})^{1/2N}\right)\right)^N, \quad (29)$$

where $snr = \frac{p}{N\sigma^2}$. Combining this with (17), we see that the detection perforamnce of the DFE is asymptotically equivalent to that of the MLD.

• From the structure of the optimal precoder $\mathbf{F}^{\star}$ we see that $\mathbf{V_H}$ diagonizes the autocorrelation matrix of the channel matrix $\mathbf{H}$. The role of $\mathbf{W}$ is to adjust the diagonal entries of $\mathbf{\Lambda}$ so as to equalize all the diagonal elements of the upper triangular matrix $\mathbf{R}$ in the QR decomposition of the super-channel $\mathbf{\Lambda W}$.

• When $N = 2$, equations (19)-(22) have four solutions, one of which is

$$\mathbf{W} = \mathbf{V} \begin{pmatrix} \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1+\lambda_2}} & -\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1+\lambda_2}} \\ \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1+\lambda_2}} & \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1+\lambda_2}} \end{pmatrix}$$

For more general cases when $N > 2$, there are an infinite number of solutions, among which, we can obtain a special solution as follows. Let the singular value decomposition of $\mathbf{A}^{(k)}$ be $\mathbf{A}^{(k)} = \mathbf{U}^{(k)}\mathbf{\Lambda}^{(k)}(\mathbf{U}^{(k)})^H$ with $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_{N-k}^{(k)}$, and let $\mathbf{z}_{k+1} = \mathbf{U}^{(k)}\mathbf{t}_k$. Then, $\mathbf{w}_{k+1} = \mathbf{W}_k^{\perp}\mathbf{U}^{(k)}\mathbf{t}_k$ for $k = 1, 2, \cdots, N-1$, where

$$\begin{aligned} t_{11} &= \sqrt{\frac{det(\mathbf{R_H})^{1/N} - \lambda_{N-k}^{(k)}}{\lambda_1^{(k)} - \lambda_{N-k}^{(k)}}} \\ t_{(N-k)1} &= \sqrt{\frac{\lambda_1^{(k)} - det(\mathbf{R_H})^{1/N}}{\lambda_1^{(k)} - \lambda_{N-k}^{(k)}}} \\ t_{\ell 1} &= 0 \qquad for\ \ell = 2, \cdots, N-k-1 \end{aligned}$$

$$\begin{aligned} w_{11} &= \sqrt{\frac{det(\mathbf{R_H})^{1/N} - \lambda_N}{\lambda_1 - \lambda_N}} \\ w_{N1} &= \sqrt{\frac{\lambda_1 - det(\mathbf{R_H})^{1/N}}{\lambda_1 - \lambda_N}} \\ \widetilde{w}_{k1} &= 0 \qquad for\ k = 2, \cdots, N-1 \end{aligned}$$

$$\mathbf{W}_1^{\perp} = \begin{pmatrix} -w_{N1} & \mathbf{0_{1\times(N-2)}} \\ \mathbf{0}_{(N-2)\times 1} & \mathbf{I}_{(N-2)\times(N-2)} \\ w_{11} & \mathbf{0}_{1\times(N-2)} \end{pmatrix}$$

V. Simulation

We give an example to show the detection performances of the DFE and the MLD for the optimal precoded channel. For simplicity of comparison to the maximum likelihood detector, we consider the case where $\mathbf{h} = [0.407, 0.815, 0.407]$ and $N = 4$. Simulation results for this channel are shown in Fig.2, where the solid line denotes the theoretic result determined by (29), the stars and circles denote simulation results of DFE detector and MLD detector for the optimal precoded channel, respectively.

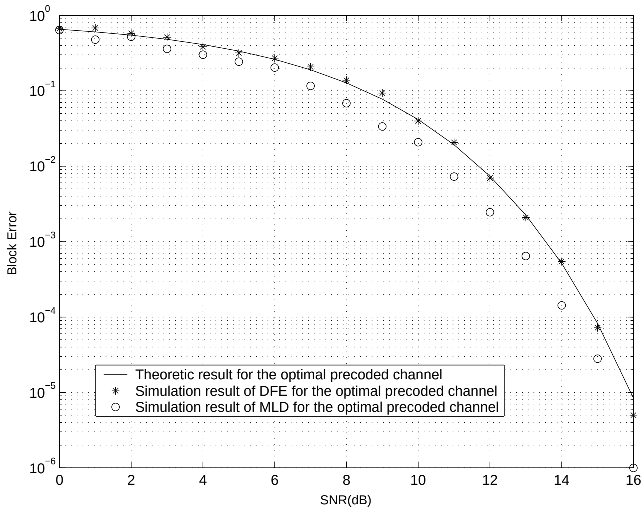

Fig. 2. Performance comparison of DFE with MLD for the optimal precoded channel

VI. Conclusion

For high SNRs and for the autocorrelation sequences of the channel's impulse response known to the transmitter, we have designed the precoder that maximizes the lowerbound of the free distance for ISI channel in block transmission. The optimal precoder turns out to be a unitary matrix whose super-channel has a QR decomposition that exhibits equal diagonal entries in the R-factor. At the same time, the lowerbound and the upperbound of the free distance become equal. For the optimal precoded channel, the DFE detection performance based on QR decomposition is comparable to that of the MLD. Our simulation verified the claim.

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