Problem 1 [25 points]

A type C battery is in working condition with probability 0.7, whereas a type D battery is in working condition with probability 0.4. A bin consists of 8 type C batteries and 6 type D batteries. One battery is randomly chosen from the bin.

a) What is the probability that the chosen battery works?

b) Given that the chosen battery does not work, what is the conditional probability that the chosen battery was of type C?

[For full credit, properly justify your answer and give final numeric value!]

$P(W_c) = 0.7 \qquad P(\overline{W_c}) = 0.3$

$P(W_D) = 0.4 \qquad P(\overline{W_D}) = 0.6$

$$P(C) = \frac{8}{6+8} = \frac{8}{14}$$

$$P(D) = \frac{6}{6+8} = \frac{6}{14}$$

a) $P(W) = P(W|C) \cdot P(C) + P(W|D) \cdot P(D)$

$= P(W_c) \cdot P(C) + P(W_D) \cdot P(D)$

$= 0.7 \times \frac{8}{14} + 0.4 \times \frac{6}{14}$

$$\boxed{P(W) = 0.5714}$$

b) $$P(C|\bar{W}) = \frac{P(C \cdot \bar{W})}{P(\bar{W})}$$

$$= \frac{P(\bar{W}|C) \cdot P(C)}{P(\bar{W})}$$

$$= \frac{P(\bar{W}_C) \cdot P(C)}{P(\bar{W})}$$

$$= \frac{P(\bar{W}_C) \cdot P(C)}{P(\bar{W}|C) \cdot P(C) + P(\bar{W}|D) \cdot P(D)}$$

$$= \frac{P(\bar{W}_C) \cdot P(C)}{P(\bar{W}_C) \cdot P(C) + P(\bar{W}_D) \cdot P(D)}$$

$$= \frac{0.3 \times \frac{8}{14}}{0.3 \times \frac{8}{14} + 0.6 \times \frac{6}{14}}$$

$$= \frac{3 \times 8}{3 \times 8 + 6 \times 6} = \frac{24}{24 + 36} = \frac{24}{60} = \frac{4}{10}$$

$$\boxed{P(C|\bar{W}) = 0.4}$$

Problem 2 [30 points]

You are entered to play a game that proceeds as follows. You must play at least 5 rounds of the game. If you lose round 5, you are removed from the game. However, if you win round 5, you must play additional rounds until you eventually lose a round, at which point you are removed from the game.

In each round you play, the probability of winning the round is $p = 0.7$

a) Find the expected number of rounds that you will **play**.

b) Find the expected number of rounds that you will **lose**.

[For full credit, give a final numerical value for both a) and b).]

a) THE PLAYER MUST PLAY AT LEAST 5 ROUNDS

* FIND THE PROBABILITY THAT THE PLAYER PLAYS EXACTLY 5 ROUNDS.

(THIS MEANS THAT THE PLAYER LOSES ROUND 5)

$$\boxed{P(X=5) = (1-p)}$$

* FIND THE PROBABILITY THAT THE PLAYER PLAYS EXACTLY 6 ROUNDS

(THIS MEANS THAT THE PLAYER WINS ROUND 5, BUT LOSES ROUND 6)

$$\boxed{P(X=6) = p\cdot(1-p)}$$

* SIMILARY $\boxed{P(X=7) = p^2(1-p)}$

* $\boxed{P(X=8) = p^3(1-p)}$

$\vdots$

* $\boxed{P(X=n) = p^{n-5}(1-p)}$ $\boxed{\text{for } n \geq 5}$

$$E[X] = \sum_{k=5}^{\infty} k \cdot P(X=k)$$

$$= 5 \cdot (1-p) + 6 \cdot p(1-p) + 7p^2(1-p) + 8p^3(1-p) + \cdots$$

$$= (1-p) \cdot [5 + 6p + 7p^2 + 8p^3 + \cdots]$$

$$= 5 + (6-5)p + (7-6)p^2 + (8-7)p^3 + \cdots$$

$$= 5 + p + p^2 + p^3 + \cdots$$

$$= 5 + p[1 + p + p^2 + p^3 + \cdots]$$

$$= 5 + p \cdot \frac{(1+p+p^2+p^3+\cdots)(1-p)}{1-p}$$

$$= 5 + \frac{p}{1-p}\left[1 + \cancel{(1-1)}^{0}p + \cancel{(1-1)}^{0}p^2 + \cancel{(1-1)}^{0}p^3 + \cdots\right]$$

$$= 5 + \frac{p}{1-p} = 5 + \frac{0.7}{0.3} =$$

$$\boxed{E[X] = 7\tfrac{1}{3} \approx 7.333}$$

b) IN THE FIRST 4 ROUNDS, THE PLAYER WILL LOSE:

0 ROUNDS WITH PROBABILITY p^4

1 ROUND WITH PROBABILITY $\binom{4}{1}(1-p)\,p^3$

2 ROUNDS WITH PROBABILITY $\binom{4}{2}(1-p)^2p^2$

3 ROUNDS WITH PROBABILITY $\binom{4}{3}(1-p)^3\,p$

4 ROUNDS WITH PROBABILITY $\binom{4}{4}(1-p)^4$

SO IN THE FIRST 4 ROUNDS, THE PLAYER WILL HAVE THE FOLLOWING EXPECTED NUMBER OF LOSSES

$$E[L_4] = 0\times p^4 + 1\cdot\binom{4}{1}(1-p)p^3 + 2\cdot\binom{4}{2}(1-p)^2p^2 + 3\binom{4}{3}(1-p)^3p + 4\binom{4}{4}(1-p)^4$$

$$= 1\cdot 4\cdot(1-p)p^3 + 2\cdot 6\,(1-p)^2p^2 + 3\cdot 4\cdot(1-p)^3p + 4\,(1-p)^4$$

$$= 4(1-p)\left[p^3 + 3(1-p)p^2 + 3(1-p)^2p + (1-p)^3\right]$$

$$= 4(1-p)\,\left[p + (1-p)\right]^3$$

$$= 4(1-p) = 4\cdot 0.3 = 1.2$$

NOW, AFTER ROUND 4, THE PLAYER WILL LOSE EXACTLY <u>1</u> MORE ROUND (REGARDLESS OF HOW MANY EXTRA ROUNDS ARE PLAYED)

So. $\boxed{E[L] = 1 + E[L_4] = 2.2}$

For the following 3 (three) multiple choice problems, <u>no justification</u> of the answers is necessary. Simply circle the correct answer.

Problem 3 [15 points]

Let A and B be events. If P(A) = 0.3 and P(B) = 0.5 and the probability that neither event occurs is 0.4. Circle the correct statement:

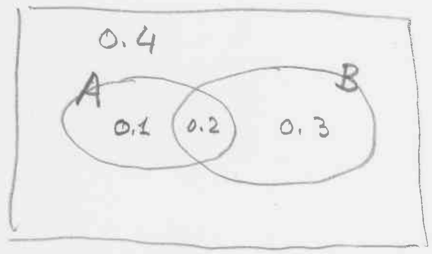
a) $P(A \cup B) = 0.7$ and $P(A \mid B) = 2/3$

b) $P(A \cup B) = 0.6$ and $P(A \mid B) = 2/3$

c) $P(A \cup B) = 0.7$ and $P(A \mid B) = 0.4$

d) $P(A \cup B) = 0.6$ and $P(A \mid B) = 0.4$

e) $P(A \cup B) = 0.65$ and $P(A \mid B) = 0.3$



$P(A \cup B) = 0.6$

$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{0.2}{0.5} = 0.4$

Problem 4 [15 points]

At UH 20% of the computers have outdated anti-viral software, 70% have current anti-viral software and 10% have a new improved beta version of the anti-viral software. A novel software virus enters the UH system. The computers with the outdated anti-viral software have 0.9 probability of being infected. The computers with the current anti-viral software have 0.3 probability of being infected. The computers with the beta version anti-viral software have 0.1 probability of being infected. Circle the correct statement.

a) Probability of a UH computer <u>not</u> being infected is 0.32

b) Probability of a UH computer <u>not</u> being infected is 0.40

c) Probability of a UH computer <u>not</u> being infected is 0.48

d) Probability of a UH computer <u>not</u> being infected is 0.60

e) Probability of a UH computer <u>not</u> being infected is 0.68

$P(\text{not infected}) = P(\bar{I} \mid \text{outdated}) \cdot P(\text{outdated}) + P(\bar{I} \mid \text{current})\, P(\text{current}) + P(\bar{I} \mid \text{beta}) \cdot P(\text{beta})$

$= 0.1 \times 0.2 + 0.7 \times 0.7 + 0.9 \times 0.1$

$= 0.02 + 0.49 + 0.09 = 0.6$

Problem 5 [15 points]

Let X be a Bernoulli random variable that can have two possible outcomes: 0 or 1.

Suppose $P(X=1) = p$.

If $E[X] = 4\ Var(X)$, circle the correct answer.

a) $p = 0$

b) $p = 0.25$

c) $p = 0.5$

d) $p = 0.75$

e) $p = 1$

$P(X=1) = p \qquad P(X=0) = (1-p)$

$E[X] = 0 \times (1-p) + 1 \times p = p$

$E[X^2] = 0^2 \times (1-p) + 1^2 \times p = p$

$Var(X) = E[X^2] - (E[X])^2 = p - p^2$

$E[X] = 4 \cdot Var(X)$

$p = 4 \cdot [p - p^2] \Rightarrow 4p^2 = 3p$

$\Downarrow$ or $\Downarrow$

$4p = 3 \qquad p = 0$

$\boxed{p = \frac{3}{4}}$ or $\boxed{p = 0}$