HW5 SOLUTIONS

1)

a Ch 4, PROBLEM 14

LET'S FIRST DENOTE EVENTS

W_1 = {PLAYER 1 WINS FIRST COMPARISON}

W_2 = {PLAYER 1 WINS 2ND COMPARISON}

W_3 = {PLAYER 1 WINS 3RD COMPARISON}

W_4 = {PLAYER 1 WINS 4TH COMPARISON}

$P(X=0) = P(\bar{W}_1) = \frac{1}{2}$

because there are $5 \cdot 4 = 20$ ways players 1 and 2 can pick 2 out of 5 numbers, and exactly 10 of those outcomes will result in player 1 losing

$$\boxed{P(X=0) = \frac{1}{2}}$$

$$\boxed{P(W_1) = 1 - P(\bar{W}_1) = \frac{1}{2}}$$

NEXT $P(X=1) = P(\bar{W}_2 W_1)$

$$P(\bar{W}_2 W_1) = \frac{\left[\begin{array}{l}\text{\# OF OUTCOMES} \\ (n_1 > n_2) \;\&\; (n_1 < n_3)\end{array}\right]}{\left[\begin{array}{l}\text{\# OF TOTAL OUTCOMES} \\ \text{FOR } n_1, n_2, n_3\end{array}\right]}$$

$$= \frac{10}{5 \cdot 4 \cdot 3} = \frac{1}{6}$$

$$\Rightarrow \boxed{P(X=1) = \frac{1}{6}}$$

Let n_1 denote the 1st players number
Let n_2 denote the 2nd players number
Let n_3 denote the 3rd players number

$P(\bar{W}_2 W_1) = P((n_1 > n_2) \text{ and } (n_1 < n_3))$

If n_1 is 2, then n_2 must be 1 and n_3 has 3 possiblities

$\Rightarrow 1 \cdot 3$

if n_1 is 3, then n_2 has 2 possiblities and n_3 has 2 possiblities

$\Rightarrow 2 \cdot 2$

Generalize: n_2 has $(n_1 - 1)$ possiblities
n_3 has $(5 - n_1)$ possiblitis

$$\Rightarrow \boxed{\text{NUMBER OF OUTCOMES } \sum_{n_1=2}^{4} (n_1 - 1)(5 - n_1) = 10}$$

$$P(\overline{X}=2) = P(\overline{w}_3 w_2 w_1) \Rightarrow \left\{ \begin{array}{l} \left.\begin{array}{l} n_1 > n_2 \\ n_1 > n_3 \end{array}\right\} (n_1-1)\cdot(n_1-2) \text{ such outcomes} \\ n_1 < n_4 \} \ (5-n_1) \text{ such outcomes} \end{array} \right.$$

$$P(\overline{w}_3 w_2 w_1) = \frac{\left[\begin{array}{c}\text{\# OUTCOMES} \\ (n_1>n_2)\ \&\ (n_1>n_3)\ \&\ (n_1<n_4)\end{array}\right]}{\left[\begin{array}{c}\text{TOTAL \# OF} \\ \text{OUTCOMES FOR} \\ n_1, n_2, n_3, n_4\end{array}\right]}$$

total such outcomes

$$\sum_{n_1=3}^{4} (n_1-1)(n_1-2)\cdot(5-n_1)$$

$$= 2\cdot 1\cdot 2 + 3\cdot 2\cdot 1 = 10$$

$$P(\overline{w}_3 w_2 w_1) = \frac{10}{5\cdot 4\cdot 3\cdot 2} = \frac{1}{12}$$

$$\boxed{P(\overline{X}=2) = \frac{1}{12}}$$

$$P(\overline{X}=3) = P(\overline{w}_4 w_3 w_2 w_1) \Rightarrow$$

$$= \frac{\sum_{n_1=4}^{4} (n_1-1)\cdot(n_1-2)\cdot(n_1-3)\cdot(5-n_1)}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{3\cdot 2\cdot 1\cdot 1}{5!} = \frac{6}{120}$$

$$\boxed{P(\overline{X}=3) = \frac{1}{20}}$$

$$P(\overline{X}=4) = 1 - P(\overline{X}=0) - P(\overline{X}=1) - P(\overline{X}=2) - P(\overline{X}=3)$$

$$P(\overline{X}=4) = 1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12} - \frac{1}{20} = 1 - \frac{1}{2\cdot 1} - \frac{1}{3\cdot 2} - \frac{1}{4\cdot 3} - \frac{1}{5\cdot 4}$$

$$= 1 - \left[\left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \left(\tfrac{1}{4}-\tfrac{1}{5}\right)\right]$$

$$= 1 - \left[1 - \tfrac{1}{5}\right] = \frac{1}{5}$$

$$\boxed{P(\overline{X}=4) = \frac{1}{5}}$$

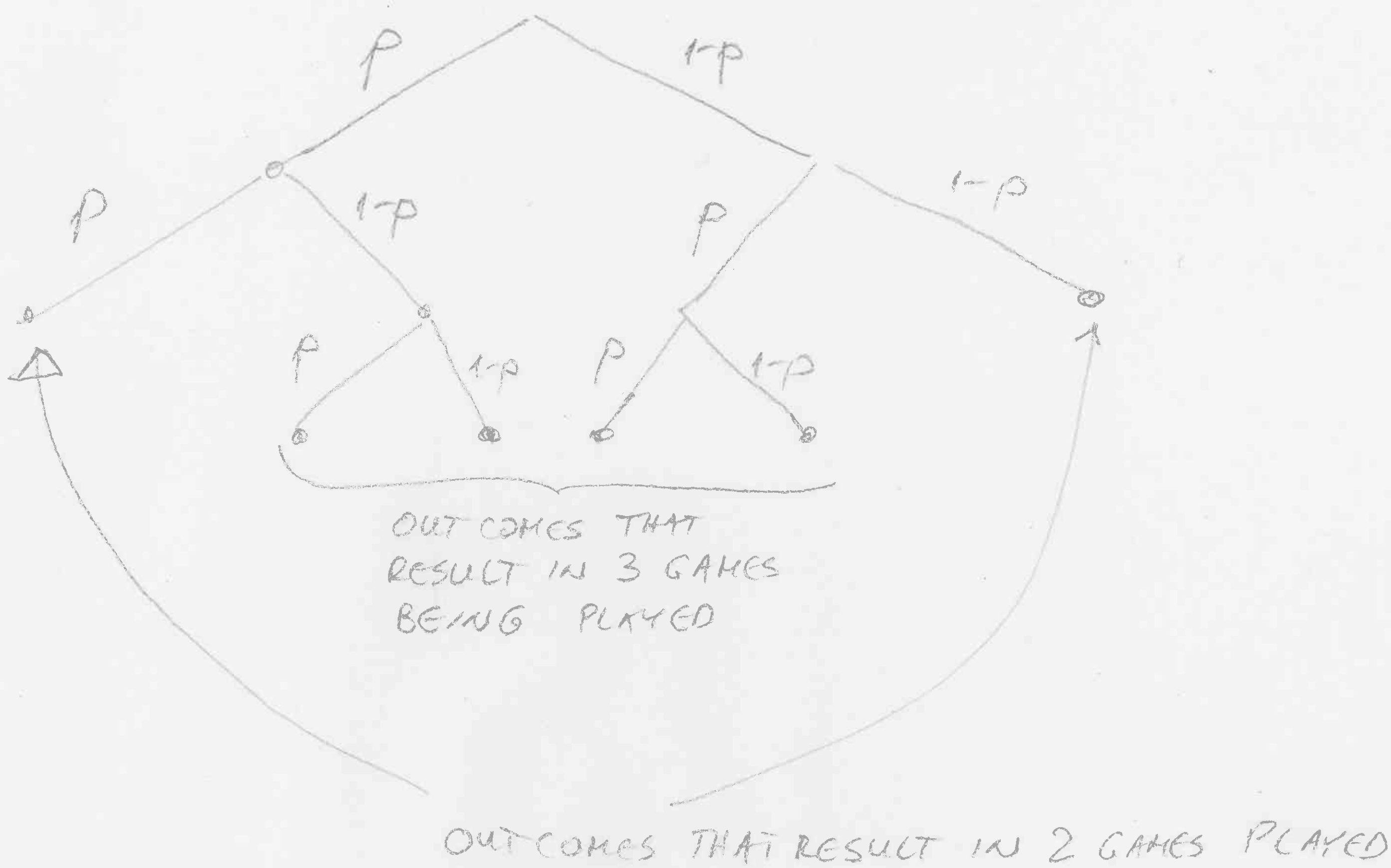
$\boxed{b}$ CH 4, PROBLEM 19

$$p_X(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{3}{5} - \frac{1}{2} = \frac{1}{10} & \text{if } x = 1 \\ \frac{4}{5} - \frac{3}{5} = \frac{1}{5} & \text{if } x = 2 \\ \frac{9}{10} - \frac{4}{5} = \frac{1}{10} & \text{if } x = 3 \\ 1 - \frac{9}{10} = \frac{1}{10} & \text{if } x = 3.5 \\ 0 & \text{for all other } x \end{cases}$$

<u>Proof</u>: verify that $P(X \leq b) = F(b)$

$\boxed{c}$ CH 4, PROBLEM 22

a) DRAW A PROBABILITY TREE FOR $i = 2$



$$P(X=2) = P\{2 \text{ games played}\}$$

$$\boxed{P(X=2) = p^2 + (1-p)^2}$$ ← obtained from probability tree

$$\boxed{P(X=3) = 1 - P(X=2) = 1 - p^2 - (1-p)^2}$$

$$E[X] = 2 \cdot P(X=2) + 3 \cdot P(X=3)$$

$$= 2p^2 + 2(1-p)^2 + 3\left[1 - p^2 - (1-p)^2\right]$$

$$E[X] = 3 - p^2 - (1-p)^2$$

$$\boxed{E[X] = 2 + 2p - 2p^2}$$

$$\boxed{E[X] = 2 + 2p(1-p)}$$ ← Maximum occurs when $p(1-p)$ is maximized, that is when

$$\frac{\partial\,[p(1-p)]}{\partial p} = 0$$

$$1 - 2p = 0$$

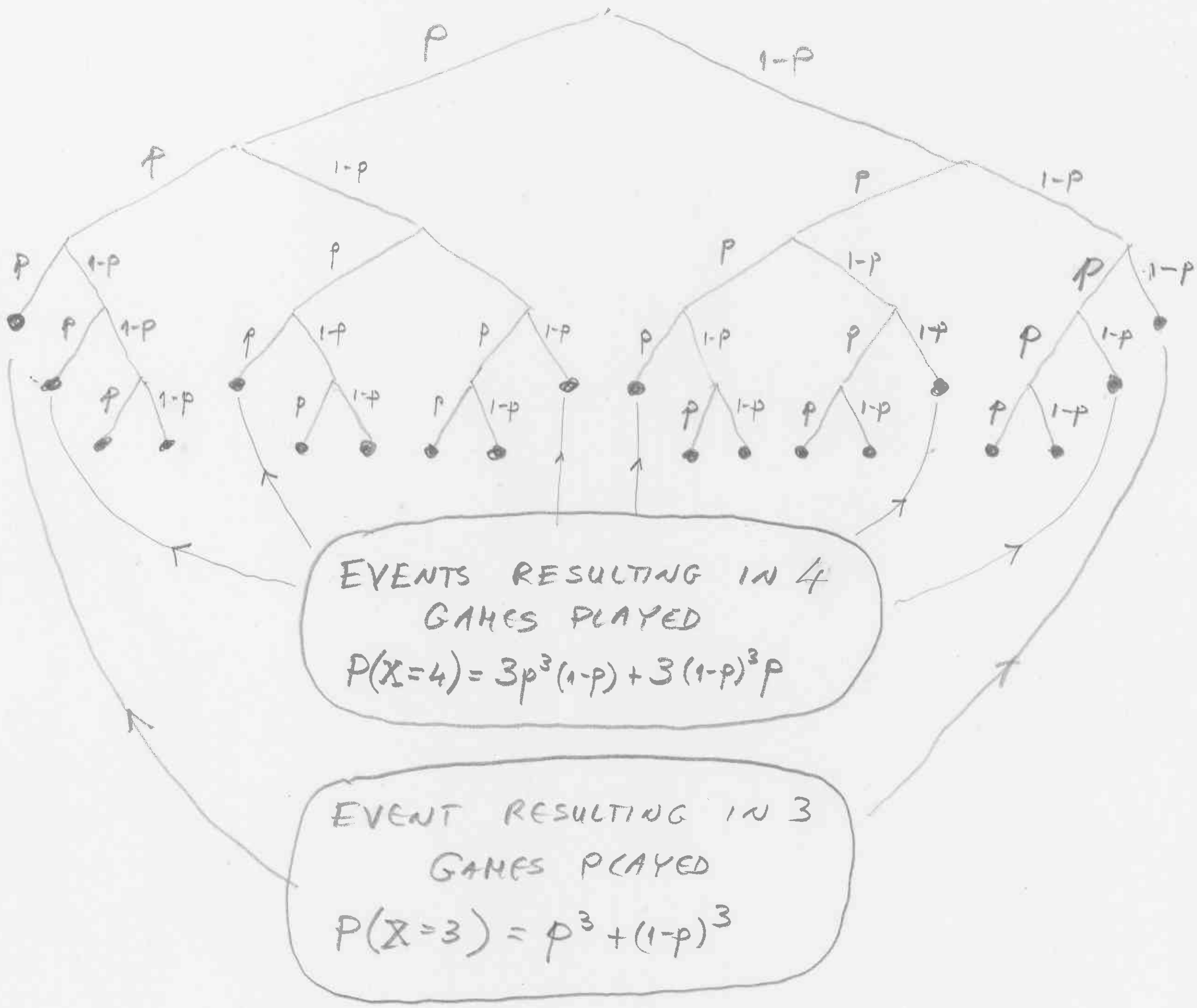
$$1 = 2p$$

$$\boxed{p = \tfrac{1}{2}}$$

$p(1-p)$ is maximized for $p = 1/2$

Therefore $E[X]$ is maximized for $p = 1/2$

b) DRAW A PROBABILITY TREE FOR $i = 3$



$$P_X(3) = P(X=3) = p^3 + (1-p)^3$$

$$P_X(4) = 3p^3(1-p) + 3(1-p)^3 p$$

$$P_X(5) = 1 - P_X(3) - P_X(4) = 1 - p^3 - (1-p)^3 - 3p^3(1-p) - 3(1-p)^3 p$$

$$E[X] = 3P_X(3) + 4P_X(4) + 5P_X(5)$$

$$= 3\left[p^3 + (1-p)^3\right] + (3+1)\left[3p^3(1-p) + 3(1-p)^3 p\right] + (3+2)\left[1 - p^3 - (1-p)^3 - 3p^3(1-p) - 3(1-p)^3 p\right]$$

$$= 3 + 1 \cdot \left[3p^3(1-p) + 3(1-p)^3 p\right] + 2\left[1 - p^3 - (1-p)^3 - 3p^3(1-p) - 3(1-p)^3 p\right]$$

$$= 3 + \left[3p^3(1-p) + 3(1-p)^3 p\right] + 12 \cdot p^2(1-p)^2$$

CONTINUED . . .

$$E[X] = 3 + 3\left[p^3(1-p) + (1-p)^3 p\right] + 12\,p^2(1-p)^2$$

$$= 3 + \underbrace{3\left[p^2(1-p) + (1-p)^3 p\right] + 6\,p^2(1-p)^2} + 6\,p^2(1-p)^2$$

$$= 3 + \qquad 3\,p(1-p) \qquad + 6\,p^2(1-p)^2$$

$$\boxed{E[X] = 3 + 3p(1-p) + 6\left[p(1-p)\right]^2}$$

- We know from part a) that $p(1-p)$ is maximized at $p = \frac{1}{2}$
- Likewise $\left[p(1-p)\right]^2$ is maximized at $p = \frac{1}{2}$
- Therefore $E[X] = 3 + 3p(1-p) + 6\left[p(1-p)\right]^2$ is maximized at $p = \frac{1}{2}$

$\boxed{d}$ CH 4, PROBLEM 25

$H_1 = \{\text{FIRST COIN "HEADS"}\}$
$H_2 = \{\text{2ND COIN "HEADS"}\}$

$P(H_1) = 0.6$ & $P(H_2) = 0.7$
$P(T_1) = 0.4$ $\quad$ $P(T_2) = 0.3$

$$P(X=0) = P(\bar{H}_1 \bar{H}_2) = P(\bar{H}_1) \cdot P(\bar{H}_2) = P(T_1) \cdot P(T_2) = 0.4 \times 0.3 = \underline{\underline{0.12}}$$

$$P(X=2) = P(H_1 H_2) = P(H_1) \cdot P(H_2) = 0.6 \times 0.7 = \underline{\underline{0.42}}$$

a) $P(X=1) = 1 - P(X=0) - P(X=2) = 1 - 0.12 - 0.42$

$$\boxed{P(X=1) = 0.46}$$

b) $E[X] = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$

$E[X] = 0 + 1 \times 0.46 + 2 \times 0.42$

$E[X] = 0.46 + 0.84$

$\boxed{E[X] = 1.3}$

$\boxed{e}$ CH 4, PROBLEM 38 GIVEN $E[X] = 1$ & $Var(X) = 5$

FIRST FIND $E[X^2]$

$Var(X) = E[X^2] - (E[X])^2$

$5 = E[X^2] - (1)^2 \Rightarrow \boxed{E[X^2] = 6}$

a) $E[(2+X)^2] = E[4 + 4X + X^2]$

$= E[4] + 4E[X] + E[X^2]$

$= 4 + 4 \cdot E[X] + E[X^2]$

$= 4 + 4 \times 1 + 6 = \underline{\underline{14}}$ //

$\boxed{E[(2+X)^2] = 14}$

b) $E[4+3X] = E[4] + 3E[X] = 4 + 3 \times E[X] = \underline{\underline{7}}$ //

$E[(4+3X)^2] = E[16 + 24X + 9X^2] = 16 + 24E[X] + 9E[X^2]$

$= 16 + 24 \times 1 + 9 \times 6 = \underline{\underline{94}}$ //

$Var(4+3X) = E[(4+3X)^2] - (E[4+3X])^2$

$= 94 - 7^2 = \underline{\underline{45}}$ // $\Rightarrow \boxed{Var(4+3X) = 45}$

f CH4, PROBLEM 58

a)

BINOMIAL	POISSON
$P(X=2) = \binom{8}{2} \cdot p^2 (1-p)^6$	$\lambda = n \cdot p = 0.8$
$P(X=2) = \frac{8!}{2!6!} \cdot (0.1)^2 \cdot (0.9)^6$	$P(X=2) = e^{-\lambda} \cdot \frac{\lambda^2}{2}$
$P(X=2) = 0.1488$	$P(X=2) = 0.1438$

b)

BINOMIAL	POISSON
$P(X=9) = \binom{10}{9} p^9 (1-p)$	$\lambda = 10 \times 0.95 = 9.5$
	$P(X=9) = e^{-\lambda} \cdot \frac{\lambda^9}{9!}$
$P(X=9) = 0.3151$	$P(X=9) = 0.1300$

c)

BINOMIAL	POISSON
$P(X=0) = 0.3487$	$P(X=0) = 0.3679$

d)

BINOMIAL	POISSON
$P(X=4) = 0.0661$	$P(X=4) = 0.0723$

$\boxed{9}$ CH 4, TH. EXCERCISE 4

$$E[N] = \sum_{k=0}^{\infty} k \cdot P(N=k)$$

$$= 0 \cdot P(N=0) + 1 \cdot P(N=1) + 2 \cdot P(N=2) + 3 \cdot P(N=3) + \cdots$$

$$= \left[\begin{array}{lllll} P(N=1) + & P(N=2) + & P(N=3) + & P(N=4) + \cdots & \leftarrow P(N\geq 1) \\ & + P(N=2) + & P(N=3) + & P(N=4) + \cdots & \leftarrow P(N\geq 2) \\ & & + P(N=3) + & P(N=4) + \cdots & \leftarrow P(N\geq 3) \\ & & & + P(N=4) + \cdots & \leftarrow P(N\geq 4) \\ & & & + \ddots & \vdots \end{array}\right]$$

notice: $P(N=1)+P(N=2)+P(N=3)+\cdots = P(N\geq 1)$

$P(N=2)+P(N=3)+\cdots = P(N\geq 2)$

$P(N=3)+\cdots = P(N\geq 3)$

$$= P(N\geq 1) + P(N\geq 2) + P(N\geq 3) + \cdots$$

$$= \sum_{i=1}^{\infty} P(N\geq 1)$$

$\boxed{h}$ CH 4, THEORETICAL EXCERCISE 8

$$E[X] = a \cdot P(X=a) + b \cdot P(X=b)$$

$$\boxed{E[X] = a \cdot p + b(1-p)} = m_X$$

$$\boxed{E[X^2] = a^2 p + b^2(1-p)}$$

$$Var(X) = E[X^2] - m_X^2$$

$$= a^2 \cdot p + b^2(1-p) - [ap + b(1-p)]^2$$

$$= a^2 p + b^2(1-p) - a^2p^2 - 2ab\,p(1-p) - b^2(1-p)^2$$

$$= a^2\,p(1-p) + b^2 p(1-p) - 2ab\,p(1-p)$$

$$= [a^2 + b^2 - 2ab]\,p(1-p)$$

$$\boxed{Var(X) = (a-b)^2\,p(1-p)}$$

MATLAB EXERCISES

2) a) n=20; p=0.5; k=0:20; pmf=gamma(n+1)./gamma(k+1)./gamma(n+1-k).*p.^k.*(1-p).^(n-k); stem(k,pmf); stairs((k,cumsum(pmf));
The commands generate the binomial pmf and CDF respectively for p=0.5 and n=20; The pmf has the shape of a symmetric Gaussian bell shaped curve centered at 10. The mean is 10 and the variance is 5. When we change the parameters to p=0.2 the pmf has a unimodal shape with maximum at n=4; The mean is 4 and the variance is 3.2 .

b) To simulate Binomial random variables generate Bernoulli random variables with parameter p and sum.
n=20; p=0.5; m=5000; ber= floor(rand(n,m) + p); bin=sum(ber); mx=max(bin); mn=min(bin); mean=mean(bin), var=std(bin)^2;
k=mn: mx;spmf=hist(bin,mx-mn+1); stem(k,spmf); stairs((k,cumsum(spmf));
For p=0.5 sample mean is 10.0446 and sample variance is 4.946. For p=0.2 sample mean is 4.0106 and sample variance is 3.1795. Sample pmfs, CDFs, means, and variances are close to true values.

3)a) p=0.5; k=1:100; pmf=p * (1-p).^(k-1); stem(k,pmf); stairs((k,cumsum(pmf));
The commands generate the geometric pmf and CDF respectively for p=0.5; The pmf has the shape of a decaying exponential function. The mean is 2 and the variance is 2. When we change the parameters to p=0.2 the pmf has the shape of a slower decaying exponential; The mean is 5 and the variance is 20 .

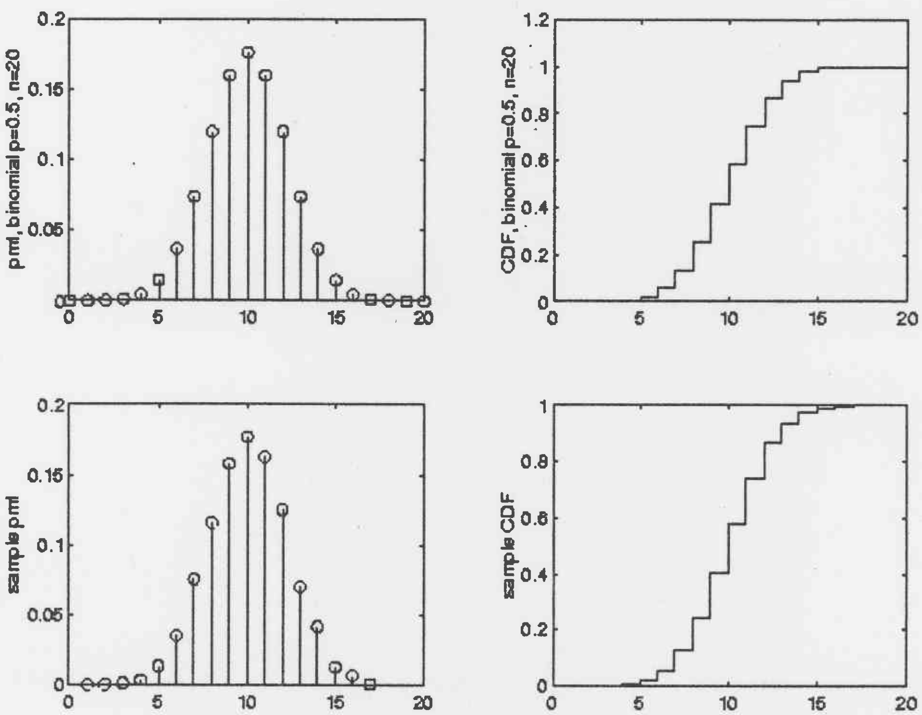
b) To simulate Geometric random variables generate Bernoulli random variables with parameter p. Then locate the time of all successes and find times between successes.
p=0.5; k=1:11000; m=11000; ber= floor(rand(1,m) + p); geom. = diff ([0 k(ber==1)]);
geom.= geom.(1:5000); max= max(geom.); mean=mean(geom), var=std(geom)^2;
k=0: max;spmf=hist(geom,max); spmf = [0 spmf]; stem(k,spmf);
stairs((k,cumsum(spmf));

For p=0.5 sample mean is 2.0188 and sample variance is 2.0921. For p=0.2 we need to use a longer sequence of Bernoulli random variables. Let m=26000; and k = 1:26000; Here we get sample mean is 5.0010 and sample variance is 20.758. Sample pmfs, CDFs, means, and variances are close to true values. An alternate way of generating Geometric random variables is using the inverse distribution method (map uniform to geometric pmf). geom. = ceil (log(1 –rand(1,5000))/log(1-p)); We will discuss this method later when we discuss continuous random variables.

plot for 2a)

$p = 0.5$
$n = 20$

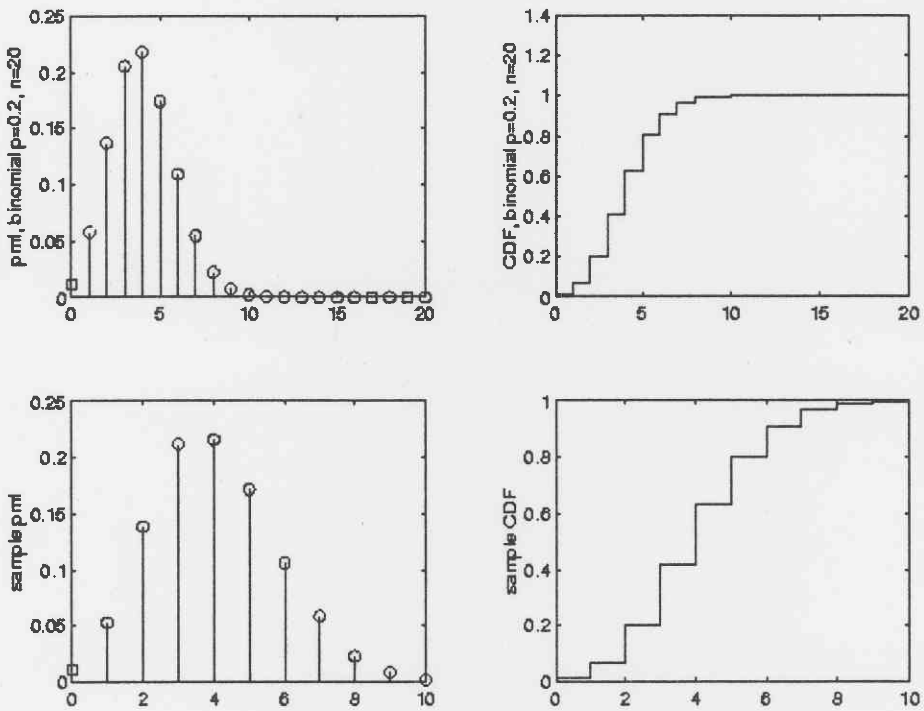
plot for 2b)



plot for 2a)

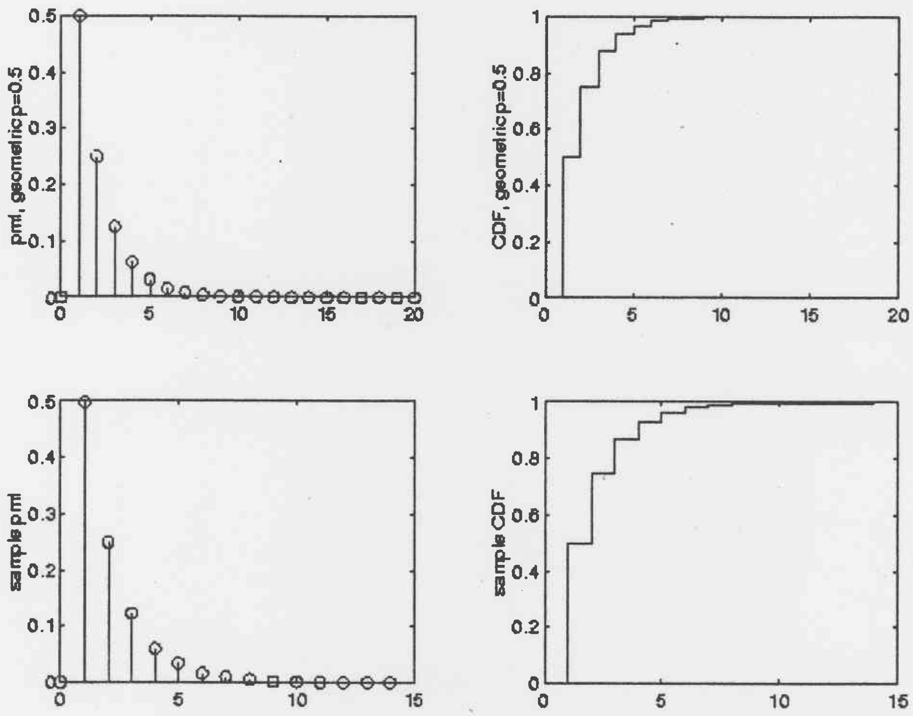
plot for 2b)

p=0.2
n=20



plot for 3a) plot for 3b)

$p = 0.5$



plot for 3a) plot for 3b)

p=0.2