Solutions of HW6

Problem 3.22:

(a): The forward prediction error is $f_m(n) = f_{m-1}(n) + \kappa_m^* b_{m-1}(n-1)$.

The backward prediction error is $b_m(n) = b_{m-1}(n-1) + \kappa_m f_{m-1}(n)$.

Then the cost function $J_m(\kappa_m)$ can be expressed as:

$$J_m(\kappa_m) = a\,E[|f_m(n)|^2] + (1-a)\,E[|b_m(n)|^2]$$

$$= a\,E\left[\left(f_{m-1}(n) + \kappa_m^* b_{m-1}(n-1)\right)\left(f_{m-1}^*(n) + \kappa_m b_{m-1}^*(n-1)\right)\right]$$
$$+ (1-a)\,E\left[\left(b_{m-1}(n-1) + \kappa_m f_{m-1}(n)\right)\left(b_{m-1}^*(n-1) + \kappa_m^* f_{m-1}^*(n)\right)\right]$$

$$= a\cdot\left\{E[|f_{m-1}(n)|^2] + |\kappa_m|^2\cdot E[|b_{m-1}(n-1)|^2] + \kappa_m^* E[b_{m-1}(n-1) f_{m-1}^*(n)] + \kappa_m E[f_{m-1}(n) b_{m-1}^*(n-1)]\right\}$$
$$+ (1-a)\left\{E[|b_{m-1}(n-1)|^2] + |\kappa_m|^2 E[|f_{m-1}(n)|^2] + \kappa_m E[f_{m-1}(n) b_{m-1}^*(n-1)] + \kappa_m^* E[b_{m-1}(n-1) f_{m-1}^*(n)]\right\}$$

$$= a\,E[|f_{m-1}(n)|^2] + (1-a)\,E[|b_{m-1}(n-1)|^2]$$
$$+ |\kappa_m|^2\left\{a\,E[|b_{m-1}(n-1)|^2] + (1-a)\,E[|f_{m-1}(n)|^2]\right\}$$
$$+ \kappa_m^* E[b_{m-1}(n-1) f_{m-1}^*(n)] + \kappa_m E[f_{m-1}(n) b_{m-1}^*(n-1)]$$

Differentiating J_m w.r.t. κ_m and letting $\frac{\partial J_m}{\partial \kappa_m} = 0$, we have

$$0 = \frac{\partial J_m}{\partial \kappa_m} = 2\kappa_m\left\{a\,E[|b_{m-1}(n-1)|^2] + (1-a)\,E[|f_{m-1}(n)|^2]\right\}$$
$$+ 2E[b_{m-1}(n-1) f_{m-1}^*(n)]. \quad \left(\text{using } \frac{\partial J_m}{\partial \kappa_m} \triangleq \frac{\partial J_m}{\partial[\mathrm{Re}(\kappa_m)]} + j\frac{\partial J_m}{\partial[\mathrm{Im}(\kappa_m)]}\right)$$

Hence, the optimal $\kappa_{m,o}$ is

$$\kappa_{m,o}(a) = -\frac{E[b_{m-1}(n-1) f_{m-1}^*(n)]}{a\,E[|b_{m-1}(n-1)|^2] + (1-a)\,E[|f_{m-1}(n)|^2]}$$

(b) (1) Case 1: $a=1$. (forward method)

$$K_{m,0}(1) = -\frac{E[b_{m-1}(n-1) f_{m-1}^*(n)]}{E[|b_{m-1}(n-1)|^2]}$$ --- the forward method shown in Notes.

(2) Case 2: $a=2 \notin [0,1]$ beyond the problem. X

(3) Case3 : $a=1/2$

$$K_{m,0}(\tfrac{1}{2}) = -\frac{2E[b_{m-1}(n-1) f_{m-1}^*(n)]}{E[|b_{m-1}(n-1)|^2] + E[|f_{m-1}(n)|^2]}$$ --- the Burg formula shown in Notes

$\left(E[|b_{m-1}(n-1)|^2] + E[|f_{m-1}(n)|^2] = E[|b_{m-1}(n-1)|^2 + |f_{m-1}(n)|^2]\right)$

(4) Case 4: $a=0$ (From Notes)

$$K_{m,0}(0) = -\frac{E[b_{m-1}(n-1) f_{m-1}^*(n)]}{E[|f_{m-1}(n)|^2]}$$ ... the backward method shown in Notes.

Problem 3.23:

(a) From part (b) of Problem 3.22, we have:

$$\frac{2}{K_{m,0}} = -\frac{E[|b_{m-1}(n-1)|^2] + E[|f_{m-1}(n)|^2]}{E[b_{m-1}(n-1) f_{m-1}^*(n)]}$$ where $K_{m,0}$ is the Burg formula,

and

$$\frac{1}{K_{m,0}^{(1)}} + \frac{1}{K_{m,0}^{(0)}} = -\frac{E[|b_{m-1}(n-1)|^2]}{E[b_{m-1}(n-1) f_{m-1}^*(n)]} - \frac{E[|f_{m-1}(n)|^2]}{E[b_{m-1}(n-1) f_{m-1}^*(n)]}$$

where $K_{m,0}^{(1)}$ is referred the forward method and $K_{m,0}^{(0)}$ is referred the backward method.

Hence,
$$\frac{2}{K_{m,0}} = \frac{1}{K_{m,0}^{(1)}} + \frac{1}{K_{m,0}^{(0)}}$$

(b) Use the partial correlation coefficient between the forward prediction error $f_{m-1}(n)$ and the delayed backward prediction error $b_{m-1}(n-1)$.

$$\rho_m \triangleq \frac{E[b_{m-1}(n-1)\, f^*_{m-1}(n)]}{\sqrt{E[|b_{m-1}(n-1)|^2]\cdot \cancel{E[|f_m(n)|^2]}}} .$$

E[|f_{m-1}(n)|^2]

We have $|\rho_m| \le 1$ for all m from the Cauchy-Schwarz inequality.

Let $$\alpha \triangleq \sqrt{\frac{\cancel{E[|f_m(n)|^2]}}{E[|b_{m-1}(n-1)|^2]}} \;\longleftarrow \text{(*)}$$

E[|f_{m-1}(n)|^2]

Then $\kappa^{(1)}_{m,0} = -\rho_m \cdot \alpha$ $\qquad \kappa^{(0)}_{m,0} = -\frac{\rho_m}{\alpha}$

$$\Rightarrow \frac{1}{\kappa_{m,0}} = -\frac{1}{2\rho_m}\left(\frac{1}{\alpha}+\alpha\right) \quad \text{(from the result of part (a))}$$

Let $A = \frac{1}{\alpha} + \alpha$. ~~We have $A_{min} = 2$, which can be obtained by setting $\frac{\partial A}{\partial \alpha} = 0$.~~

~~And A_{min} is achieved at $\alpha = 1$ since $\alpha \geq 0$.~~

Hence, from the definition of \alpha in Eqn. (*), we have \alpha = 1. Then A = 2.

Hence $$\left|\frac{1}{\kappa_{m,0}}\right| = \left|\frac{1}{2\rho_m}\left(\frac{1}{\alpha}+\alpha\right)\right| \cancel{\geq} = \left|\frac{1}{\rho_m}\right| \geq 1 \quad \text{for all } m$$

$$\Rightarrow |\kappa_{m,0}| \le 1, \quad \text{for all } m.$$

(c) The forward prediction error $f_m(n)$ has expression: $f_m(n) = f_{m-1}(n) + \kappa_m^* b_{m-1}(n-1)$

The mean-square value of the forward prediction error is

$$E[|f_m(n)|^2] = E[f_m(n) f_m^*(n)] = E[(f_{m-1}(n) + \kappa_m^* b_{m-1}(n-1))(f^*_{m-1}(n) + \kappa_m b^*_{m-1}(n-1))]$$

$$= E[|f_{m-1}(n)|^2] + |\kappa_m|^2 E[|b_{m-1}(n-1)|^2] + \kappa_m^* E[b_{m-1}(n-1) f^*_{m-1}(n)] + \kappa_m E[f_{m-1}(n) b^*_{m-1}(n-1)]$$

Using the Burg formula:

$$\kappa_m = \kappa_{m,o}\left(\tfrac{1}{2}\right) = -\frac{2E\left[b_{m-1}(n-1) f_{m-1}^*(n)\right]}{E\left[|b_{m-1}(n-1)|^2\right] + E\left[|f_{m-1}(n)|^2\right]}$$

$$\Leftrightarrow \left\{E\left[f_{m-1}(n)\, b_{m-1}^*(n-1)\right]\right\}^* = E\left[b_{m-1}(n-1) f_{m-1}^*(n)\right] = -\frac{1}{2}\kappa_{m,o}\cdot\left\{E\left[|b_{m-1}(n-1)|^2\right] + E\left[|f_{m-1}(n)|^2\right]\right\},$$

we have

$$\begin{aligned} E\left[|f_m(n)|^2\right] &= E\left[|f_{m-1}(n)|^2\right] + |\kappa_{m,o}|^2 E\left[|b_{m-1}(n-1)|^2\right] \\ &\quad - |\kappa_{m,o}|^2\left\{E\left[|b_{m-1}(n-1)|^2\right] + E\left[|f_{m-1}(n)|^2\right]\right\} \\ &= \left(1-|\kappa_{m,o}|^2\right) E\left[|f_{m-1}(n)|^2\right]. \end{aligned}$$

Similarly, from $b_m(n) = b_{m-1}(n-1) + \kappa_m f_{m-1}(n)$ and $\kappa_m = \kappa_{m,o} = \kappa_{m,o}(\frac{1}{2})$

we have

$$\begin{aligned} E\left[|b_m(n)|^2\right] &= E\left[|b_{m-1}(n-1)|^2\right] + |\kappa_{m,o}|^2 E\left[|f_{m-1}(n)|^2\right] \\ &\quad + \kappa_{m,o} E\left[f_{m-1}(n)\, b_{m-1}^*(n-1)\right] + \kappa_{m,o}^* E\left[b_{m-1}(n-1) f_{m-1}^*(n)\right] \\ &= E\left[|b_{m-1}(n-1)|^2\right] + |\kappa_{m,o}|^2 E\left[|f_{m-1}(n)|^2\right] \\ &\quad - |\kappa_{m,o}|^2\left\{E\left[|b_{m-1}(n-1)|^2 + E\left[|f_{m-1}(n)|^2\right]\right.\right\} \\ &= \left(1-|\kappa_{m,o}|^2\right) E\left[|b_{m-1}(n-1)|^2\right]. \end{aligned}$$

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Problem 10.5:

∵ $\lim_{n\to\infty} K(n+1,n) = K$. ∴ We can set $K = K(n+1,n) = K(n,n-1)$ when n approaches infinity.

Since $F(n+1,n) = I$ and $\lim_{n\to\infty} C(n) = C$, $\lim_{n\to\infty} Q_1(n) = Q_1$, $\lim_{n\to\infty} Q_2(n) = Q_2$, and letting $\lim_{n\to\infty} G(n) = G$

the predicted state-error correlation matrix in Eqn. (10.54) in textbook can be expressed as

$$K = [I - GC]K[I - GC]^H + Q_1 + GQ_2G^H$$

$$= K - GCK - KC^HG^H + GCKC^HG^H + Q_1 + GQ_2G^H$$

$$\Rightarrow GCK + KC^HG^H - G[CKC^H + Q_2]G^H - Q_1 = 0 \quad ---- (1)$$

From Eqns. (10.35) and (10.49) in textbook, we have

$$G = KC^HR^{-1} = KC^H[CKC^H + Q_2]^{-1}. \quad ---- (2)$$

Substituting Eqn. (2) into the first and third terms in Eqn. (1), we have

$$KC^H[CKC^H + Q_2]^{-1}CK + \cancel{KC^HG^H} - \cancel{KC^H[CKC^H + Q_2]^{-1}[CKC^H + Q_2]G^H} - Q_1 = 0$$

$$\Rightarrow KC^H[CKC^H + Q_2]^{-1} - Q_1 = 0.$$

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Problem 10.7:

(a) ① ∵
$$\begin{cases} \underline{x}(n) = [a_1(n), \dots, a_M(n), \dots, a_{M+N}(n)]^T \\ \underline{x}(n+1) = [a_1(n+1), \dots, a_M(n+1), \dots, a_{M+N}(n)]^T \\ a_k(n+1) = a_k(n) + w_k(n), \quad k = 1, \dots, M+N \end{cases}$$

∴ $\underline{x}(n+1) = \underline{x}(n) + \underline{w}(n)$ where $\underline{w}(n) = [w_1(n), w_2(n) \dots w_M(n), w_{M+N}(n)]^T$ and

$w_k(n)$ is a zero mean, white Gaussian noise process that is independent of $w_j(n)$ for $j \neq k$.

② ∵
$$\begin{cases} y(n) = -\sum_{k=1}^{M} a_k(n) y(n-k) + \sum_{k=1}^{N} a_{M+k}(n) v(n-k) + v(n) \\ C(n) = [-y(n-1), \dots, -y(n-M), v(n-1), \dots, v(n-N)] \end{cases}$$

∴ $y(n) = C(n) \cdot \underline{x}(n) + v(n)$

①&② ⇒ the state-space equations for the ARMA process are

$$\begin{cases} \underline{x}(n+1) = \underline{x}(n) + \underline{w}(n) \triangleq F(n+1,n)\underline{x}(n) + \underline{v}_1(n) \\ y(n) = C(n) \cdot x(n) + v(n) \triangleq C(n)\underline{x}(n) + \underline{v}_2(n) \end{cases} \Leftrightarrow \begin{cases} F(n+1,n) = I \\ \underline{v}_1(n) = \underline{w}(n), \ \underline{v}_2(n) = v(n), \ \underline{y}(n) = y(n) \end{cases}$$

(b) From part (a) we have $F(n+1, n) = I$.

So the recursive equation to compute the predicted value of $\underline{x}(n+1)$ is

$\hat{\underline{x}}(n+1|\mathcal{Y}_n) = \hat{\underline{x}}(n|\mathcal{Y}_{n-1}) + G(n)\,\alpha(n)$ by applying Eqn. (10.45) in the textbook,

where (i) $G(n) = K(n,n-1)\,C^H(n)\left[C(n)K(n,n-1)C^H(n) + Q_2(n)\right]^{-1}$ (by applying Eqns. (10.35) & (10.49) in the textbook)

(ii) $\alpha(n) = y(n) - C(n)\,\hat{\underline{x}}(n|\mathcal{Y}_{n-1})$ (by applying Eqn. (10.31) in the textbook)

(iii) $K(n,n-1) = K(n-1) + Q_1(n)$ & $K(n-1) = K(n-1,n-2) - G(n-1)C(n-1)K(n-1,n-2)$ (Using Eqns. (10.55) & (10.56))

(iv) $Q_1(n) = E[\underline{v}_1(n)\underline{v}_1^H(n)] = E[\underline{w}(n)\,\underline{w}^H(n)] = \sigma_W^2 \cdot I$ & $\sigma_W^2 = E[w_k(n)\,w_k^*(n)]$

$Q_2(n) = E[v_2(n)v_2^H(n)] = E[v(n)\,v(n)] = \sigma^2$

(c). To initialize the recursive algorithm in part (b), we can set

(i) $$\hat{\underline{x}}(1|\mathcal{Y}_0) = E[\underline{x}(1)] = E\begin{bmatrix} a_1(1) \\ \vdots \\ a_M(1) \\ \vdots \\ a_{M+N}(1) \end{bmatrix} = E\begin{bmatrix} w_1(0) \\ \vdots \\ w_M(0) \\ \vdots \\ w_{M+N}(0) \end{bmatrix} = \underline{0}$$ (Assuming $a_k(0)=0$ for all k) & ($E[w_k(0)]=0$ for all k)

(ii) $K(1,0) = E\left[[\underline{x}(1) - \hat{\underline{x}}(1|\mathcal{Y}_0)][\underline{x}(1) - \hat{\underline{x}}(1|\mathcal{Y}_0)]^H\right]$

$= E[\underline{x}(1)\,\underline{x}^H(1)] = E[\underline{w}(0)\,\underline{w}^H(0)]$ (from $a_k(1) = a_k(0) + w_k(0)$ and assumption of $a_k(0)=0$ for all k)

$= \sigma_W^2 \cdot I = Q_1(n)$

(d). The Kalman filter used in this problem is not optimal.

Problem 10.X1:

Prove that (10.54) and (10.55) in the textbook are equivalent.

proof: Eqn. (10.54) in the textbook is

$$K(n+1,n) = [F(n+1,n) - G(n)C(n)]\, K(n,n-1)\, [F(n+1,n) - G(n)C(n)]^H + Q_1(n) + G(n)Q_2(n)G^H(n)$$

$$= F(n+1,n)K(n,n-1)F^H(n+1,n) - G(n)C(n)K(n,n-1)F^H(n+1,n) - F(n+1,n)K(n,n-1)C^H(n)G^H(n)$$

$$+ G(n)C(n)K(n,n-1)C^H(n)G^H(n) + G(n)Q_2(n)G^H(n) + Q_1(n)$$

$$= F(n+1,n)K(n,n-1)F^H(n+1,n) - G(n)C(n)K(n,n-1)F^H(n+1,n) - F(n+1,n)K(n,n-1)C^H(n)G^H(n)$$

$$+ G(n)\left[C(n)K(n,n-1)C^H(n) + Q_2(n)\right]G^H(n) + Q_1(n) \quad ---- \quad (*)$$

From Eqns. (10.35) and (10.49) in the textbook, we have

$$G(n) = F(n+1,n)K(n,n-1)C^H(n)\left[C(n)K(n,n-1)C^H(n) + Q_2(n)\right]^{-1}.$$

Then, $G(n)\left[C(n)K(n,n-1)C^H(n) + Q_2(n)\right]G^H(n) = F(n+1,n)K(n,n-1)C^H(n)G^H(n)$.

Hence Eqn. $(*) \Leftrightarrow$

$$K(n+1,n) = F(n+1,n)K(n,n-1)F^H(n+1,n) - G(n)C(n)K(n,n-1)F^H(n+1,n) + Q_1(n)$$

$$= F(n+1,n)\left[K(n,n-1) - F(n,n+1)G(n)C(n)K(n,n-1)\right]F^H(n+1,n) + Q_1(n)$$

$$(\because F(n+1,n) \cdot F(n,n+1) = I)$$

$$= F(n+1,n)K(n)F^H(n+1,n) + Q_1(n) \quad \text{as shown in Eqn. (10.55)}$$

where $K(n) = K(n,n-1) - F(n,n+1)G(n)C(n)K(n,n-1)$. as shown in Eqn. (10.56). #