Solutions to HW2

Problem 2.2

a) The Wienier-Hopf Equations are

$$R\underline{w}_o = \underline{p} \quad \text{where } R = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

By soloving the above equations using Gaussian Elimination

$$\left[\begin{array}{cc|c} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.25 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0.5 & 0.5 \\ 0 & 0.75 & 0 \end{array}\right],$$

the tap weights of the Wiener filter is obtained

$$\underline{w}_o = [0.5, \ 0]^H.$$

b) From Eqn. (2.49) in the textbook, the minimum mean-square error is

$$\begin{aligned} J_{min} &= \sigma_d^2 - \underline{w}_o^H R \underline{w}_o \\ &= \sigma_d^2 - \underline{w}_o^H \cdot \underline{p} \\ &= \sigma_d^2 - 0.25. \end{aligned}$$

c). Let $|\lambda I - R| = 0$. $\quad \begin{vmatrix} \lambda-1 & -0.5 \\ -0.5 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 0.25 = 0.$

Then the roots of the above equation are $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$.

So the eigenvalues of matrix R are $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$.

Case 1: $\lambda_1 = 1.5$.

$$0 = (\lambda I - R)\cdot \underline{u} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow u_1 = u_2 = 1 \xrightarrow{\text{nomalization}} \underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the unit eigenvector corresponding to eigenvalue $\lambda_1 = 1.5$ is $\underline{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Case 2: $\lambda_2 = 0.5$

$$0 = (\lambda I - R)\cdot \underline{v} = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow v_1 = 1, \ v_2 = -1 \xrightarrow{\text{nomalization}} \underline{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the unit eigenvector corresponding to $\lambda_2 = 0.5$ is $\underline{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then, if we let $\Lambda = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $Q = [\underline{u}\ \ \underline{v}] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,

We have $R = Q \Lambda Q^H$ and $QQ^H = I$.

Hence, from part a), we know that the tap weights of the Wiener filter is

$$\underline{w}_0 = R^{-1} \cdot \underline{p} = [Q\Lambda Q^H]^{-1} \cdot \underline{p} = Q\Lambda^{-1}Q^H \cdot \underline{p} \qquad \text{where } \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \left[Q \cdot \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & 0 \end{bmatrix} \cdot Q^H + Q \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} Q^H \right] \cdot \underline{p}$$

$$= \left(\frac{1}{\lambda_1} \underline{u} \cdot \underline{u}^H + \frac{1}{\lambda_2} \underline{v} \cdot \underline{v}^H \right) \cdot \underline{p}.$$

Problem 2.4.

The tap inputs of a transversal filter of length M and the desired response are $u(0), u(1), \dots, u(N)$ and $d(0), d(1), \dots, d(N)$, respectively.

Let $\underline{w}_0 = [w_{00}\ \ w_{01},\ w_{02}, \dots, w_{0(M-1)}]^H$ be the tap-weight vector of the Wiener filter.

Let $\underline{u}(n) = [u(n),\ u(n-1), \dots, u(n-M+1)]^H$, $(M \ll N)$

$$R = E[\underline{u}(n) \cdot \underline{u}^H(n)] \quad \text{and}$$

$$\underline{p} = E[\underline{u}(n) \cdot d^*(n)]$$

Since both of the processes $\{u(n)\}$ and $\{d(n)\}$ are jointly wide-sense stationary and ergodic, we have:

$$R \approx R(N) = \frac{1}{N+1} \sum_{n=0}^{N} \underline{u}(n) \cdot \underline{u}^H(n)$$

$$\underline{p} \approx \underline{p}(N) = \frac{1}{N+1} \sum_{n=0}^{N} \underline{u}(n) \cdot d^*(n).$$

Then the tap-weight vector is

$$\underline{w}_0 = \underline{w}_0(N) = (R(N))^{-1} \cdot \underline{p}(N) = \left(\sum_{n=0}^{N} \underline{u}(n) \cdot \underline{u}^H(n) \right)^{-1} \cdot \left(\sum_{n=0}^{N} \underline{u}(n) \cdot d^*(n) \right)$$

Problem 2.11

a): From Fig. P2.1 (a) & (b). we have

$$d(n) = v_1(n) - 0.8458\, d(n-1) \quad --- \quad (1)$$

$$u(n) = x(n) + v_2(n) \quad --- \quad (2)$$

$$x(n) = d(n) + 0.9458\, x(n-1) \quad --- \quad (3)$$

Eqn. (3) $\iff$ $d(n) = x(n) - 0.9458\, x(n-1) \quad ---- \quad (4)$

Substituting Eqn. (1) into Eqn. (3), we have

$$
\begin{aligned}
x(n) &= v_1(n) - 0.8458\, d(n-1) + 0.9458\, x(n-1) \\
&= v_1(n) - 0.8458\big(x(n-1) - 0.9458\, x(n-2)\big) + 0.9458\, x(n-1) \quad \text{using Eqn. (4)} \\
&= v_1(n) + 0.1\, x(n-1) + 0.8458 * 0.9458\, x(n-2) \\
&= v_1(n) + 0.1\, x(n-1) + 0.8\, x(n-2) \quad ----- \quad (5)
\end{aligned}
$$

Eqns. (2) and (5) tell us that the channel output is

$$u(n) = x(n) + v_2(n)$$

where

$$x(n) = 0.1\, x(n-1) + 0.8\, x(n-2) + v_1(n)$$

b). Since the two noise sources $v_1(n)$ and $v_2(n)$ are statistically independent,

from Eqn. (5), we can see that $x(n)$ and $v_2(n)$ are uncorrelated,

and from Eqn. (1), we can see that $d(n)$ and $v_2(n)$ are uncorrelated, too.

Therefore, for the Wiener filter of length two, we have

$$R_u = R_x + R_{v_2}$$

where $R_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}$ and $R_{v_2} = \begin{bmatrix} \sigma_{v_2}^2 & 0 \\ 0 & \sigma_{v_2}^2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$

From Eqn. (5) and that $v_1(n)$ is white with zero mean and variance $\sigma_1^2 = 0.27$,

we have $\mu_x = E[x(n)] = 0$ and $r_x(0) = \sigma_x^2$,

and $\sigma_{v_1}^2 = r_x(0) - 0.1\, r_x(1) - 0.8\, r_x(2)$ and $R_x \cdot \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix} = \begin{pmatrix} r_x(1) \\ r_x(2) \end{pmatrix}$. (Yule-Walker equations)

Solving the above equations, we have

$$r_x(0)=1 \quad r_x(1)=0.5\,r_x(0)=0.5 \quad r_x(2)=0.85\,r_x(0)=0.85.$$

Hence $R_u = R_x + R_{v_2} = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$

$$R_u = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}.$$

The cross-correlation vector is

$$\underline{p} = \begin{bmatrix} p(0) \\ p(-1) \end{bmatrix} \quad \text{where} \quad \begin{aligned} p(0) &= E[u(n)\cdot d^*(n)] \\ p(-1) &= E[u(n-1)\cdot d^*(n)]. \end{aligned}$$

$$p(0) = E\left[[x(n)+v_2(n)]\cdot d^*(n)\right] = E[x(n)\cdot d^*(n)] = E[x(n)\cdot(x^*(n)-0.9458\,x^*(n-1))]$$

$$= E[x(n)\,x^*(n)] - 0.9458\,E[x(n)\,x^*(n-1)] = r(0) - 0.9458\,r(1)$$

$$= 1 - 0.9458\times 0.5 = 0.5271$$

Similarly $p(-1) = r(-1) - 0.9458\,r(0) = r(1) - 0.9458\,r(0)$

$$= 0.5 - 0.9458$$

$$= -0.4458$$

$$\underline{p} = [0.5271 \quad -0.4458]^H.$$

c). The optimum weight vector is $\underline{w}_o$ satisfying $R_u\underline{w}_o = \underline{p}$.

Hence $\underline{w}_o = R_u^{-1}\underline{p} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5271 \\ -0.4458 \end{bmatrix} = \begin{bmatrix} 0.8362 \\ -0.7853 \end{bmatrix}.$

The minimum mean-square error is

$$J_{min} = \sigma_d^2 - \underline{w}_o^H\cdot\underline{p} = \sigma_d^2 - 0.7908$$

From Eqn. (1), we have $\mu_d = E[d(n)] = 0$. and $\sigma_d^2 = \dfrac{\sigma_{v_1}^2}{1-0.8458^2} = 0.9486$. So $\boxed{J_{min} = 0.1578}$.