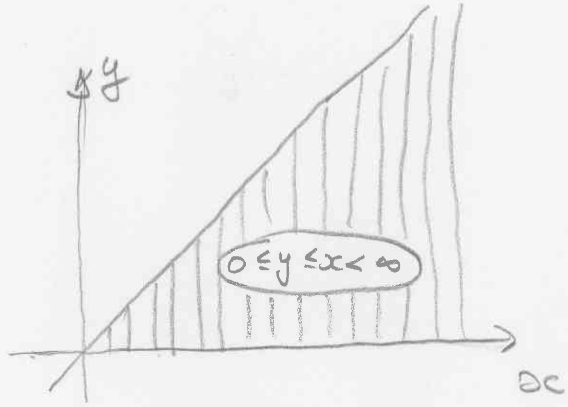
HW 10
(SOLUTIONS)

1 a CH 7, PROB 38



$$E[XY] = \int_{-\infty}^{\infty}\int_{0}^{\infty} xy\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{0}^{\infty}\int_{0}^{x} xy\, 2e^{-2x}\cdot\frac{1}{x}\,dy\,dx$$

$$= \int_{0}^{\infty} 2\, e^{-2x}\left[\int_{0}^{x} y\,dy\right]dx$$

$$= \int_{0}^{\infty} 2\, e^{-2x}\left[\frac{y^2}{2}\Big|_{0}^{x}\right]dx = \int_{0}^{\infty} 2\, e^{-2x}\cdot\frac{x^2}{2}\,dx$$

$$= \int_{0}^{\infty} x^2 e^{-2x}\,dx$$

$$= \int_{0}^{\infty} x\, e^{-2x}\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-2x}\,dx$$

$\Leftrightarrow$ THESE ARE TRUE BECAUSE

$$\int_{0}^{\infty} x^k e^{-2x}\,dx = \frac{k}{2}\int_{0}^{\infty} x^{k-1} e^{-2x}\,dx$$

WHICH CAN BE PROVED USING INTEGRATION BY PARTS

$$= \frac{1}{2}\,\frac{e^{-2x}}{(-2)}\Big|_{0}^{\infty}$$

$$\boxed{E[XY] = \frac{1}{4}}$$

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{x} x \cdot 2e^{-2x} \cdot \frac{1}{x}\,dy\,dx$$

$$= \int_0^{\infty} 2e^{-2x}\left[\int_0^{x} dy\right] \cdot dx$$

$$= \int_0^{\infty} 2e^{-2x}\left[y\Big|_0^{x}\right] dx$$

$$= 2\int_0^{\infty} x e^{-2x}\,dx \quad \text{because } \int_0^{\infty} x^k e^{-2x}\,dx = \frac{k}{2}\int_0^{\infty} x^{k-1} e^{-2x}\,dx$$

$$= 2 \cdot \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx$$

$$\boxed{E[X] = \frac{1}{2}}$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{x} y \cdot 2e^{-2x}\,\frac{1}{x}\,dy\,dx$$

$$= \int_0^{\infty} 2\,\frac{1}{x}\,e^{-2x}\left[\int_0^{x} y\,dy\right] dx$$

$$= \int_0^{\infty} 2 \cdot \frac{1}{x} \cdot e^{-2x}\left[\frac{x^2}{2}\right] dx$$

$$= \int_0^{\infty} x e^{-2x}\,dx \quad \text{because } \int_0^{\infty} x^k e^{-2x}\,dx = \frac{k}{2}\int_0^{\infty} x^{k-1} e^{-2x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx$$

$$\boxed{E[Y] = \frac{1}{4}}$$

FINALLY

$$Cov(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y]$$

$$= \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4}$$

$$\boxed{Cov(X,Y) = \frac{1}{8}}$$

b) CH 7, PROB 53

- LET X_n REPRESENT THE DOOR CHOSEN AT TRIAL n

CLEARLY $\boxed{\begin{array}{l} P(X_n = 1) = 0.5 \\ P(X_n = 2) = 0.3 \\ P(X_n = 3) = 0.2 \end{array}}$ $\Rightarrow P_{X_n}(i) = \begin{cases} 0.5 & \text{if } i = 1 \\ 0.3 & \text{if } i = 2 \\ 0.2 & \text{if } i = 3 \end{cases}$

- LET D_n REPRESENT THE DAYS OF TRAVEL AT TRIAL n

CLEARLY IF $X_n = 1$ THEN $D_n = 2$ $\Rightarrow$ $P(D_n = 2) = P(X_n = 1) = 0.5$

IF $X_n = 2$ THEN $D_n = 4$ $\Rightarrow$ $P(D_n = 4) = P(X_n = 2) = 0.3$

IF $X_n = 3$ THEN $D_n = 1$ $\Rightarrow$ $P(D_n = 1) = P(X_n = 3) = 0.2$

- NEXT NOTE $P(D_n = 1 \mid X_n = 3) = 1$ AND $P(D_n = i \mid X_n = 3) = 0$ for $i \neq 1$

THEREFORE $E[D_n \mid X_n = 3] = 1 \cdot P(D_n = 1 \mid X_n = 3) \Rightarrow \boxed{E[D_n \mid X_n = 3] = 1}$ (A)

- WE ALSO HAVE $P(D_n = 2 \mid X_n \neq 3) = P(X_n = 1 \mid X_n \neq 3) = \frac{P(X_n = 1)}{P(X_n \neq 3)} = \frac{5}{8}$

AND $P(D_n = 4 \mid X_n \neq 3) = P(X_n = 2 \mid X_n \neq 3) = \frac{P(X_n = 2)}{P(X_n \neq 3)} = \frac{3}{8}$

THEREFORE $E[D_n \mid X_n \neq 3] = 2 \cdot P(D_n = 2 \mid X_n \neq 3) + 4 \cdot P(D_n = 4 \mid X_n \neq 3)$

$= 2 \cdot \frac{5}{8} + 4 \cdot \frac{3}{8}$

$$\boxed{E[D_n \mid X_n \neq 3] = \frac{11}{4}} \quad (B)$$

- LET N REPRESENT THE NUMBER OF TRIALS UNTIL FREEDOM IS REACHED

LET'S FIND $E[N]$

$$P(N=n) = P(X_1 \neq 3, X_2 \neq 3, X_3 \neq 3, \ldots, X_{n-1} \neq 3 \ \& \ X_n = 3)$$

$$= P(X_i \neq 3)^{n-1} \cdot P(X_n = 3$$

$$\boxed{P(N=n) = (0.8)^{n-1} \cdot (0.2) \qquad \text{for} \quad 1 \leq n < \infty}$$

THEREFOR N IS A GEOMETRIC RANDOM VARIABLE

AND

$$E[N] = \sum_{n=1}^{\infty} n \cdot P(N=n)$$

$$= \sum_{n=1}^{\infty} n \cdot (0.2) \cdot (0.8)^{n-1}$$

$$= 0.2 \sum_{n=1}^{\infty} n \cdot (0.8)^{n-1}$$

we did this sum many times already, so you should know how to compute it

$$= 0.2 \times \frac{1}{(1-0.8)^2}$$

$$= \frac{1}{0.2}$$

$\boxed{E[N] = 5}$ (c)

- LET D REPRESENT THE TOTAL NUMBER OF DAYS THAT PASS UNTIL FREEDOM IS REACHED

CLEARLY $D = D_1 + D_2 + \cdots + D_N$

AND <u>WE NEED TO FIND $E[D]$</u>

- LET'S USE THE "CONDITION - UNCONDITION" RULE TO FIND $E[D]$

$$\boxed{E[D] = E\left[E[D|N]\right]} \quad (\text{?})$$

THEREFORE, LET'S FIRST FIND $E[D|N=n]$

$$E[D|N=n] = E[D_1 + D_2 + \dots + D_n \mid X_1 \neq 3, X_2 \neq 3, \dots, X_{n-1} \neq 3 \;\&\; X_n = 3]$$

$$= \underbrace{E[D_1|X_1 \neq 3]}_{\frac{11}{4}} + \underbrace{E[D_2|X_2 \neq 3]}_{\frac{11}{4}} + \dots + \underbrace{E[D_{n-1}|X_{n-1} \neq 3]}_{\frac{11}{4}} + \underbrace{E[D_n|X_n = 3]}_{1}$$

THESE FOLLOW FROM (A) & (B)

$$\boxed{E[D|N=n] = \frac{11}{4}(n-1) + 1}$$

$$\Downarrow$$

THEREFORE $\boxed{E[D|N] = \frac{11(N-1)}{4} + 1}$ (??)

NOW SUBSTITUTE (??) INTO (?)

$$E[D] = E\left[E[D|N]\right] = E\left[\frac{11(N-1)}{4} + 1\right] = E\left[\frac{11}{4}N - \frac{7}{4}\right]$$

$$= \frac{11}{4}E[N] - \frac{7}{4}$$

USE (C)

$$= \frac{11}{4} \cdot 5 - \frac{7}{4} = \frac{48}{4} = 12 \implies \boxed{E[D] = 12}$$

$\boxed{C}$ CH7, PROB 67

- LET X_i REPRESENT THE FORTUNE AFTER THE i-TH GAMBLE

CLEARLY $X_0 = x$

$$\text{AND FOR } i>0 \Rightarrow X_i = \begin{cases} X_{i-1} + (2p-1)X_{i-1} & \text{with probability } p \\ X_{i-1} - (2p-1)X_{i-1} & \text{with probability } (1-p) \end{cases}$$

- LET'S FIRST FIND $E[X_i | X_{i-1}]$

$$E[X_i|X_{i-1}] = p \cdot \left[X_{i-1} + (2p-1)X_{i-1}\right] + (1-p)\left[X_{i-1} - (2p-1)X_{i-1}\right]$$

$$\boxed{E[X_i|X_{i-1}] = \left[1 + (2p-1)^2\right] \cdot X_{i-1}} \quad (*)$$

- NEXT, UTILIZE THE "CONDITION-UNCONDITION" RULE

$$\begin{aligned} E[X_i] &= E\left[E[X_i|X_{i-1}]\right] \quad \text{use } (*) \\ &= E\left[\left[1+(2p-1)^2\right] \cdot X_{i-1}\right] \end{aligned}$$

$$\boxed{E[X_i] = \left(1+(2p-1)^2\right) E[X_{i-1}]} \quad (**)$$

$$\begin{aligned} \text{— Now } E[X_n] &= \left(1+(2p-1)^2\right) E[X_{n-1}] \quad \text{use } (**) \\ &= \left(1+(2p-1)^2\right)^2 E[X_{n-2}] \quad \text{use } (**) \\ &\vdots \\ &= \left(1+(2p-1)^2\right)^n E[X_0] \quad \text{use } (**) \end{aligned}$$

$$\boxed{E[X_n] = \left(1+(2p-1)^2\right)^n \cdot x}$$

$\boxed{d}$ CH7, PROB 73

KNOWN FROM EXAMPLE 6b WE HAVE THE FOLLOWING KNOWN FACTS

1° S IS NORMAL WITH $\boxed{E[S]=\mu}$, $\boxed{Var(S)=\sigma^2}$

2° $f_{R|S}(r|s)$ IS A NORMAL (GAUSSIAN) DISTRIBUTION WITH

$$\boxed{E[R|S=s]=s}$$

$$\boxed{Var[R|S=s]=1}$$

3° $f_{S|R}(s|r)$ IS NORMAL (GAUSSIAN) WITH

$$\boxed{E[S|R=r]=\frac{\mu}{1+\sigma^2}+\frac{\sigma^2}{1+\sigma^2}\cdot r}$$

$$\boxed{Var(S|R=r)=\frac{\sigma^2}{1+\sigma^2}}$$

a) SINCE $E[R|S=s]=s$

WE HAVE $E[R|S]=S$.

NOW APPLY THE "CONDITION-UNCONDITION" RULE

$$E[R]=E\big[E[R|S=s]\big]=E[S]=\mu$$ USE KNOWN FACT 1°

$$\Rightarrow \boxed{E[R]=\mu}$$

b) $Var(R) = E[R^2] - (E[R])^2$ from a)

(?) $\boxed{Var(R) = E[R^2] - \mu^2}$

NOW FIRST COMPUTE $E[R^2 | S = s]$

$E[R^2 | S = s] = Var(R | S = s) + (E[R | S = s])^2$ USE KNOWN FACT 2°

$\boxed{E[R^2 | S = s] = 1 + s^2}$

$\Rightarrow \boxed{E[R^2 | S] = 1 + S^2}$ (*)

NEXT USE THE "CONDITION - UNCONDITION" RULE

$$E[R^2] = E\left[E[R^2|S]\right]$$ USE (*)

$$= E[1 + S^2]$$

$$= 1 + E[S^2]$$

$$= 1 + Var(S) + (E[S])^2$$ USE KNOWN FACT 1°

$$= 1 + \sigma^2 + \mu^2$$

$\Rightarrow \boxed{E[R^2] = 1 + \sigma^2 + \mu^2}$ Substitute this into (?) AND GET

$$Var(R) = E[R^2] - \mu^2$$
$$= 1 + \sigma^2 + \mu^2 - \mu^2$$

$$\Rightarrow \boxed{Var(R) = 1 + \sigma^2}$$

c) - FROM FACT 1°

$f_S(s)$ IS GAUSSIAN $\Rightarrow$ HAS QUADRATIC FORM IN THE EXPONENT

- FROM FACT 2°

$f_{R|S}(r|s)$ IS GAUSSIAN $\Rightarrow$ HAS QUADRATIC FORM IN THE EXPONENT

$\Rightarrow$ WHEN WE MULTIPLY $f_S(s)$ AND $f_{R|S}(r|s)$ WE ADD THE TWO QUADRATIC FORMS IN THE EXPONENT TO GET A NEW QUADRATIC FORM IN THE EXPONENT OF THE PRODUCT.

SO, $$f_S(s) \cdot f_{R|S}(r|s) = f_{S,R}(s,r)$$

HAS A QUADRATIC FORM IN THE EXPONENT AND THEREFORE $\boxed{f_{S,R}(s,r) \text{ IS A } \underline{\text{JOINTLY}} \text{ GAUSSIAN PDF}}$

$\Rightarrow$ FROM CLASS WE KNOW THAT IF THE JOINT PDF IS GAUSSIAN, THEN THE MARGINALS $f_R(r)$ AND $f_S(s)$ MUST ALSO BE GAUSSIAN.

$\boxed{\text{THEREFORE } R \text{ IS GAUSSIAN}}$

d) FROM EXAMPLE 6b WE KNOW THAT THE BEST ESTIMATE OF S AFTER OBSERVING R IS

$$\boxed{E[S|R] = \frac{\sigma^2}{1+\sigma^2} R + \frac{\mu}{1+\sigma^2}}$$

USE KNOWN FACT 3°

BUT, WE SEE THAT THIS IS ALSO A LINEAR ESTIMATE OF THE FORM $\boxed{aR+b}$

THEREFORE THE BEST ESTIMATE IS ALSO THE BEST LINEAR ESTIMATE, i.E. $\Rightarrow$ $\boxed{a = \frac{\sigma^2}{1+\sigma^2}}$ & $\boxed{b = \frac{\mu}{1+\sigma^2}}$

BUT FROM CLASS, WE KNOW THAT FOR THE BEST LINEAR ESTIMATE, WE MUST HAVE

$$a = \frac{Cov(R,S)}{Var(R)} = \frac{Cov(R,S)}{1+\sigma^2} \Rightarrow \boxed{a = \frac{Cov(R,S)}{1+\sigma^2}}$$

$$b = E[S] - \frac{Cov(R,S)}{Var(R)} \cdot E[R]$$

$$\boxed{b = \mu - \frac{Cov(R,S)}{1+\sigma^2} \cdot \mu}$$

COMPARING $\boxed{a = \frac{Cov(R,S)}{1+\sigma^2}}$ TO $\boxed{a = \frac{\sigma^2}{1+\sigma^2}}$,

WE CONCLUDE $\boxed{Cov(R,S) = \sigma^2}$

$\boxed{c}$ CH 7, TH. EX. 38

minimize $E\left[(Y - a - bX_1 - cX_2)^2\right]$ WITH RESPECT TO a, b & c

$$F = E\left[(Y - a - bX_1 - cX_2)^2\right] =$$

$$= E[Y^2] + a^2 + b^2 E[X_1^2] + c^2 E[X_2^2] - 2aE[Y] - 2bE[X_1 Y] - 2cE[X_2 Y]$$
$$+ 2ab\,E[X_1] + 2ac\,E[X_2] + 2bc\,E[X_1 X_2]$$

$$\left.\begin{aligned}
\frac{\partial F}{\partial a} &= 2a - 2E[Y] + 2bE[X_1] + 2c\,E[X_2] = 0 \\
\frac{\partial F}{\partial b} &= 2b\,E[X_1^2] - 2E[X_1 Y] + 2a\,E[X_1] + 2cE[X_1 X_2] = 0 \\
\frac{\partial F}{\partial c} &= 2c\,E[X_2^2] - 2E[X_2 Y] + 2a\,E[X_2] + 2b\,E[X_1 X_2] = 0
\end{aligned}\right\}$$ Solve for a, b, c

$$a + E[X_1]\cdot b + E[X_2]\cdot c = E[Y] \quad \leftarrow (\nabla)$$
$$E[X_1]a + E[X_1^2]b + E[X_1 X_2]c = E[X_1 Y]$$
$$E[X_2]a + E[X_1 X_2]b + E[X_2^2]c = E[X_2 Y]$$

$$\begin{bmatrix} 1 & E[X_1] & E[X_2] \\ E[X_1] & E[X_1^2] & E[X_1 X_2] \\ E[X_2] & E[X_1 X_2] & E[X_2^2] \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E[Y] \\ E[X_1 Y] \\ E[X_2 Y] \end{bmatrix}$$

solve for a, b, c

$$\begin{bmatrix} 1 & E[X_1] & E[X_2] \\ 0 & Var(X_1) & Cov(X_1,X_2) \\ 0 & Cov(X_1,X_2) & Var(X_2) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E[Y] \\ Cov(X_1,Y) \\ Cov(X_2,Y) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) \\ Cov(X_1,X_2) & Var(X_2) \end{bmatrix}^{-1} \begin{bmatrix} Cov(X_1,Y) \\ Cov(X_2,Y) \end{bmatrix}$$

$$\begin{bmatrix} b \\ c \end{bmatrix} = \frac{\begin{bmatrix} Var(X_2) & -Cov(X_1,X_2) \\ -Cov(X_1,X_2) & Var(X_1) \end{bmatrix} \begin{bmatrix} Cov(X_1,Y) \\ Cov(X_2,Y) \end{bmatrix}}{Var(X_1)\cdot Var(X_2) - Cov(X_1,X_2)^2}$$

$$\Rightarrow \boxed{b = \frac{Var(X_2)\cdot Cov(X_1,Y) - Cov(X_1,X_2)\,Cov(X_2,Y)}{Var(X_1)Var(X_2) - Cov(X_1,X_2)^2}} \quad (*)$$

$$\Rightarrow \boxed{c = \frac{Var(X_1)\cdot Cov(X_2,Y) - Cov(X_1,X_2)\,Cov(X_1,Y)}{Var(X_1)Var(X_2) - Cov(X_1,X_2)^2}} \quad (**)$$

USE THE FOLLOWING NOTATION

$$(!!) \quad \begin{aligned} \mu_1 &= E[X_1] & \sigma_1^2 &= Var(X_1) & \rho_{12} &= \frac{Cov(X_1,X_2)}{\sigma_1 \cdot \sigma_2} \\ \mu_2 &= E[X_2] & \sigma_2^2 &= Var(X_2) & \rho_{1Y} &= \frac{Cov(X_1,Y)}{\sigma_1 \sigma_Y} \\ \mu_Y &= E[Y] & \sigma_Y^2 &= Var(Y) & \rho_{2Y} &= \frac{Cov(X_2,Y)}{\sigma_2 \sigma_Y} \end{aligned}$$

SUBSTITUTE (!!) INTO

(*) & (**) TO GET

$$(\alpha) \quad \boxed{b = \frac{\sigma_Y(\rho_{2Y} - \rho_{12}\rho_{1Y})}{\sigma_2(1-\rho_{12}^2)}}$$

$$(\beta) \quad \boxed{c = \frac{\sigma_Y(\rho_{1Y} - \rho_{12}\rho_{2Y})}{\sigma_1(1-\rho_{12}^2)}}$$

NOW SUBTITUTE THESE BACK INTO EQUATION (☺)

AND USE $\mu_1 = E[X_1]$
$\mu_2 = E[X_2]$
$\mu_Y = E[Y]$

TO GET

$$a = \mu_Y - b\cdot\mu_1 - c\cdot\mu_2$$

$$(\gamma) \quad \boxed{a = \mu_Y - \frac{\sigma_Y(\rho_{2Y} - \rho_{12}\rho_{1Y})}{\sigma_2(1-\rho_{12}^2)}\cdot\mu_1 - \frac{\sigma_Y(\rho_{1Y} - \rho_{12}\rho_{2Y})}{\sigma_1(1-\rho_{12}^2)}\,\mu_2}$$

$\boxed{f}$ CH 7, TH. EX. 39

* THIS PROBLEM IS EQUIVALENT TO THE PREVIOUS PROBLEM IF WE SUBSTITUTE $X_1 = X$ AND $X_2 = X^2$

* THEREFORE WE CAN USE SOLUTIONS (α), (β) & (γ) IF WE PROPERLY IDENTIFY THE CONSTANTS

$\{\mu_1, \mu_2, \mu_Y, \sigma_1, \sigma_2, \sigma_Y, \rho_{12}, \rho_{1Y}$ & $\rho_{2Y}\}$

$\boxed{\text{SO, LET'S FIND THESE CONSTANTS}}$

$$m_1 = E[X_1] = E[X] \Rightarrow \boxed{m_1 = E[X] = m_X}$$

$$m_2 = E[X_2] = E[X^2] = Var(X) + (E[X])^2 \Rightarrow \boxed{m_2 = Var(X) + m_X^2 = \sigma_X^2 + m_X^2}$$

$$\boxed{m_Y = E[Y]}$$

$$\sigma_1^2 = Var(X_1) = Var(X) = \sigma_X^2 \Rightarrow \boxed{\sigma_1 = \sqrt{Var(X)} = \sigma_X}$$

$$\sigma_2^2 = Var(X_2) = E[X_2^2] - (E[X_2])^2$$

$$= E[(X^2)^2] - m_2^2 = E[X^4] - (Var(X) + m_X^2)^2$$

$$\Rightarrow \boxed{\sigma_2 = \sqrt{E[X^4] - (\sigma_X^2 + m_X^2)^2}}$$

$$\boxed{\sigma_Y = \sqrt{Var(Y)}}$$

$$\rho_{12} = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2} = \frac{E[X_1 \cdot X_2] - E[X_1] \cdot E[X_2]}{\sigma_1 \sigma_2}$$

$$= \frac{E[X \cdot X^2] - m_1 \cdot m_2}{\sigma_1 \sigma_2} = \frac{E[X^3] - m_1 m_2}{\sigma_1 \sigma_2}$$

$$\boxed{\rho_{12} = \frac{E[X^3] - m_X(\sigma_X^2 + m_X^2)}{\sigma_X \cdot \sqrt{E[X^4] - (\sigma_X^2 + m_X^2)^2}}}$$

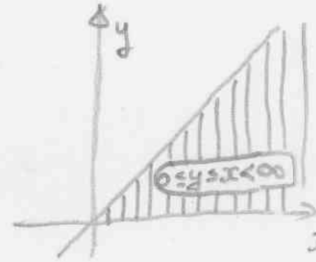
$$\rho_{1Y} = \frac{Cov(X_1, Y)}{\sigma_1 \; \sigma} = \frac{Cov(X, Y)}{\sigma_X \; \sigma_Y} \Rightarrow \boxed{\rho_{12} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}}$$

AND FINALLY

$$\boxed{\rho_{2Y} = \frac{Cov(X_2, Y)}{\sigma_2 \cdot \sigma_Y} = \frac{Cov(X^2, Y)}{\sigma_Y \cdot \sqrt{E[X^4] - (\sigma_x^2 + \mu_x^2)^2}}}$$

2

$$f_{X,Y}(x,y) = \begin{cases} e^{-x} & 0 \le y \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$



a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^x e^{-x} dy = e^{-x}\int_0^x dy = x e^{-x}$

for $0 \le x < \infty$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} x e^{-x} & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}}$$

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_y^{\infty} e^{-x} dx = -e^{-x}\Big|_y^{\infty} = e^{-y}$ for $0 \le y < \infty$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} e^{-y} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}}$$

c) X & Y ARE <u>NOT</u> INDEPENDENT BECAUSE

$$\boxed{f_X(x) f_Y(y) \ne f_{X,Y}(x,y)}$$

d) FROM CLASS, WE KNOW THAT THE MINIMUM MEAN SQUARE ESTIMATE OF X WITHOUT OBSERVING Y EQUALS

$$\boxed{g_{MMSE} = E[X]}$$

SO LET'S COMPUTE $E[X]$

$u = x^2$ $\quad du = 2x\,dx$

$dv = e^{-x}$ $\quad v = -e^{-x}$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x \cdot x e^{-x}\,dx$$

$$= (uv)\Big|_0^{\infty} - \int_0^{\infty} v\,du =$$

$$= \underbrace{x^2 e^{-x}\Big|_0^{\infty}}_{0} + \int_0^{\infty} 2x e^{-x}\,dx$$

$$= 2\int_0^{\infty} x e^{-x}\,dx = 2\int_0^{\infty} e^{-x}\,dx = 2 \qquad \Rightarrow \boxed{E[X] = 2}$$

$$\Rightarrow \boxed{g_{MMSE} = E[X] = 2}$$

e) FROM CLASS, WE KNOW THAT THE LINEAR MINIMUM MEAN SQUARE ESTIMATE OF X AFTER OBSERVING Y EQUALS

$$\boxed{g_{LMMSE}(Y) = aY + b}$$ WHERE

$$(*)\ \left\{ a = \frac{Cov(X,Y)}{Var(Y)} \right\} \quad \& \quad \left\{ b = E[X] - \frac{Cov(X,Y)}{Var(Y)} \cdot E[Y] \right\}\ (**)$$

SO, TO EVALUATE a & b WE MUST FIND

$E[X]$, $E[Y]$, $Cov(X,Y)$ & $Var(Y)$

$E[X]$ WAS ALREADY FOUND IN PART d) $\Rightarrow \boxed{E[X] = 2}$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^{\infty} y \cdot e^{-y}\, dy = \int_0^{y} e^{-y}\, dy = 1 \quad \Rightarrow \boxed{E[Y] = 1}$$

$$E[Y^2] = \int_0^{\infty} y^2 e^{-y}\, dy = 2\int_0^{\infty} y e^{-y}\, dy = 2\int_0^{y} e^{-y}\, dy = 2 \quad \Rightarrow \boxed{E[Y^2] = 2}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 2 - 1 = 1 \quad \Rightarrow \boxed{Var(Y) = 1}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{\infty} \int_0^{x} xy \cdot e^{-x}\, dy\, dx = \int_0^{\infty} x e^{-x} \left[\int_0^{x} y\, dy\right] \cdot dx = \int_0^{\infty} x e^{-x} \left[\frac{x^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3 e^{-x}\, dx = \frac{1}{2} \cdot 3 \int_0^{\infty} x^2 e^{-x}\, dx = \frac{1}{2} \cdot 3 \cdot 2 \int_0^{\infty} x e^{-x}\, dx = \frac{1}{2} \cdot 3 \cdot 2 \cdot 1 \cdot \int_0^{\infty} e^{-x}\, dx$$

$$= \frac{1}{2} \cdot 3 \cdot 2 \cdot 1 \left(-e^{-x}\right)\Big|_0^{\infty} = 3 \quad \Rightarrow \boxed{E[XY] = 3}$$

$$Cov(X,Y) = E[XY] - E[X] \cdot E[Y] = 3 - 2 \cdot 1 = 1 \quad \Rightarrow \boxed{Cov(X,Y) = 1}$$

NOW FIND

$$a = \frac{Cov(X,Y)}{Var(Y)} = \frac{1}{1} \quad \Rightarrow \boxed{a = 1}$$

$$b = E[X] - \frac{Cov(X,Y)}{Var(Y)} \cdot E[Y] = 2 - \frac{1}{1} \cdot 1 = 1 \quad \Rightarrow \boxed{b = 1}$$

$$\boxed{g_{LMMSE}(Y) = aY + b = Y + 1}$$

f) FROM CLASS, WE KNOW THAT THE MINIMUM MEAN SQUARE ESTIMATE OF X AFTER OBSERVING Y

EQUALS $\boxed{g_{MMSE}(Y) = E[X|Y]}$

So LET'S FIRST FIND $E[X|Y=y]$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dy$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dy \qquad y \geq 0$$

$$= \int_{y}^{\infty} x \cdot \frac{e^{-x}}{e^{-y}}\, dx \qquad 0 \leq x \leq y < \infty$$

$$= \int_{y}^{\infty} x e^{-(x-y)}\, dx = \qquad \text{sub } z = x-y,\ dz = dx,\ x = z+y$$

$$= \int_{0}^{\infty} (z+y) e^{-z}\, dz = \int_{0}^{\infty} z e^{-z} dz + y \int_{0}^{\infty} e^{-z}\, dz$$

$$= \int_{0}^{\infty} z e^{-z} dz + y \cdot 1$$

$$= \int_{0}^{\infty} e^{-z} dz + y$$

$$\boxed{E[X|Y=y] = 1+y} \Rightarrow \boxed{E[X|Y] = 1+Y}$$

$$\Rightarrow \boxed{g_{MMSE}(Y) = E[X|Y] = 1+Y}$$

NOTE THAT IN THIS SPECIAL CASE $\{ g_{LMMSE}(Y) = g_{MMSE}(Y) \}$

3

a) SINCE X & Y ARE JOINTLY GAUSSIAN,

THE MARGINAL $f_X(x)$ MUST ALSO BE GAUSSIAN

$$X \sim \mathcal{N}(E[X], Var(X)) = \mathcal{N}(1, 1)$$

$$\Rightarrow \boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}}$$

b) SIMILARLY, $f_Y(y)$ MUST ALSO BE GAUSSIAN

$$Y \sim \mathcal{N}(E[Y], Var(Y)) = \mathcal{N}(-1, 4)$$

$$\Rightarrow \boxed{f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{1}{2}\frac{(y+1)^2}{4}}}$$

c) X & Y ARE NOT INDEPENDENT BECAUSE

$$Cov(X, Y) \neq 0$$

d) THE MMSE ESTIMATE OF X WITHOUT OBSERVING Y IS

$$\boxed{g_{MMSE} = E[X] = 1}$$

e) THE LMMSE ESTIMATE OF X AFTER OBSERVING Y IS

$$g_{LMMSE}(Y) = \frac{Cov(X,Y)}{Var(Y)} \cdot Y + \left[E[X] - \frac{Cov(X,Y)}{Var(Y)} \cdot E[Y]\right]$$

$$= \frac{-1}{4} \cdot Y + \left[1 - \frac{(-1)}{4} \cdot (-1)\right]$$

$$\boxed{g_{LMMSE}(Y) = \frac{3}{4} - \frac{1}{4} Y}$$

f) WE ALREADY SAW IN EXAMPLE 6b (SEE PROBLEM 7.73 OF THIS HOMEWORK SET) THAT IF X & Y ARE JOINTLY GAUSSIAN THEN $\boxed{g_{MMSE}(Y) = g_{LMMSE}(Y)}$

THEREFORE $\boxed{g_{MMSE}(Y) = E[X|Y] = \frac{3}{4} - \frac{1}{4}Y}$

4

$$\boxed{\begin{aligned} E[X] &= 0 \\ E[Y] &= 0 \\ Var(X) &= 1 \\ Var(Y) &= 0 \end{aligned}} \quad \& \quad \boxed{Cov(X,Y) = 0}$$

THIS HOLDS BECAUSE X & Y ARE ZERO-MEAN UNIT-VARIANCE INDEPENDENT GAUSSIANS

a) $E[Z] = E[X+1] = E[X] + 1 = 1 \Rightarrow \boxed{E[Z] = 1}$

$E[W] = E[X+Y] = E[X] + E[Y] = 0 + 0 \Rightarrow \boxed{E[W] = 0}$

b) COVARIANCE MATRIX OF $\begin{bmatrix} Z \\ W \end{bmatrix}$ EQUALS

$$\boxed{\Lambda = \begin{bmatrix} Var(Z) & Cov(W,Z) \\ Cov(W,Z) & Var(W) \end{bmatrix}}$$

So, let's find $Var(Z)$, $Var(W)$ & $Cov(Z,W)$

$$Var(Z) = Var(X+1) = Var(X) = 1 \Rightarrow \boxed{Var(Z) = 1}$$

$$Var(W) = Var(X+Y) \overset{\uparrow}{=} Var(X) + Var(Y) = 1 + 1 = 2$$

because X & Y are independent

$$\Rightarrow \boxed{Var(W) = 2}$$

$$Cov(Z,W) = E[Z \cdot W] - E[Z] \cdot E[W]$$

$$= E[(X+Y)(X+1)] - E[Z] \cdot E[Y]$$

$$= \underbrace{E[X^2]}_{1} + \underbrace{E[XY]}_{0} + \underbrace{E[X]}_{0} + \underbrace{E[Y]}_{0} - \underbrace{E[Z]}_{1} \underbrace{E[W]}_{0}$$

$$E[X^2] = Var(X) + E[X] = 1 + 0 = 1$$

$$E[XY] \overset{\uparrow}{=} E[X] \cdot E[Y] = 0 \cdot 0 \Rightarrow \boxed{E[XY] = 0}$$

because X & Y are uncorrelated

$$\boxed{Cov(Z,W) = 1}$$

Finally,

$$\boxed{\Lambda = \begin{bmatrix} Var(Z) & Cov(Z,W) \\ Cov(Z,W) & Var(W) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$$

c) $$\boxed{f_{Z,W}(z,w) = \frac{e^{-\frac{1}{2}\left[(z-m_z)\ \ (w-m_w)\right]\Lambda^{-1}\begin{bmatrix}(z-m_z)\\(w-m_w)\end{bmatrix}}}{(2\pi)^{\frac{2}{2}}\cdot(\det\Lambda)^{\frac{1}{2}}}} \quad (!)$$

FIRST FIND $\boxed{\Lambda^{-1} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}}$ $\Rightarrow \boxed{\Lambda^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}}$ $(*)$

NEXT FIND $\boxed{\det\Lambda = \det\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = 2-1 = 1}$ $\Rightarrow \boxed{\det\Lambda = 1}$ $(**)$

FROM a) $\Rightarrow \boxed{m_z = E[Z] = 1}$ & $\boxed{m_W = E[W] = 0}$ $\Rightarrow \boxed{\begin{matrix}m_z = 1\\ m_w = 0\end{matrix}}$

NOW COMPUTE

$$\left[(z-m_z)\ \ (w-m_w)\right]\Lambda^{-1}\begin{bmatrix}(z-m_z)\\(w-m_w)\end{bmatrix}$$

$$= \left[(z-1)\ \ w\right]\cdot\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}(z-1)\\w\end{bmatrix}$$

$$= \left[(z-1)\ \ w\right]\cdot\begin{bmatrix}2(z-1)-w\\ w-(z-1)\end{bmatrix}$$

$$= 2(z-1)^2 - w(z-1) + w^2 - w(z-1)$$

$$= \boxed{2(z-1)^2 - 2w(z-1) + w^2 = \left[(z-m_z)\ \ w-m_w\right]\Lambda^{-1}\begin{bmatrix}(z-m_z)\\(w-m_w)\end{bmatrix}} \quad (***)$$

FINALLY SUBSTITUTE $(*)$, $(**)$ & $(***)$ INTO $(!)$ TO GET

$$f_{Z,W}(z,w) = \frac{e^{-\frac{1}{2}\left[2(z-1)^2 - 2w(z-1) + w^2\right]}}{(2\pi)\cdot\sqrt{1}}$$

$$\boxed{f_{Z,W}(z,w) = \frac{e^{\left[w(z-1) - (z-1)^2 - w^2/2\right]}}{2\pi}}$$

d) $Cov(Z,W) = 1 \neq 0$

$\Rightarrow$ Z & W ARE CORRELATED.

SINCE Z & W ARE CORRELATED, THEY CANNOT BE INDEPENDENT

$\Rightarrow$ $\boxed{\text{Z \& W ARE \underline{NOT} INDEPENDENT}}$

e) $Z = X + 1 \Rightarrow \boxed{X = Z - 1}$

$W = X + Y \Rightarrow W = (Z-1) + Y \Rightarrow \boxed{Y = W - Z + 1}$

ANSWER $\boxed{\begin{array}{l} X = Z - 1 \\ Y = -Z + W + 1 \end{array}}$ OR $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} Z \\ W \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$