# Optimized LDPC Codes for Partial Response Channels†

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*Abstract*— **We construct codes that can closely approach (and possibly ultimately achieve) the i.i.d. capacity of an intersymbol interference (ISI) channel with inputs confined to binary values. We use the low-density-parity-check (LDPC) degree sequences optimization method proposed by Richardson, Shokrollahi and Urbanke and appropriately modify it for ISI channels.**

## I. Introduction

The probabilistic law of a binary-input/continuous-output AWGN channel with ISI memory is captured by $Y_i = \sum_{k=0}^{\nu} h_k X_{i-k} + N_i$. Here $\nu$ is the length of the channel memory, $\{N\}$ represents a zero-mean white Gaussian random process with variance $\sigma^2$, $\{X\}$ denotes the antipodal binary input process and $\{Y\}$ is the discrete-time continuous-valued output process. The channel memory is captured by the vector of coefficients $\underline{h} = [h_0, ..., h_\nu]^T$. The maximal i.i.d. mutual information rate (the i.i.d. capacity) of a channel is defined as

$$C_{i.i.d} = \lim_{n\to\infty} \frac{1}{n} \sup_{Pr(X_1^n = x_1^n) = \prod_{t=1}^{n} Pr(X = x_t)} I(X_1^n, Y_1^n) \qquad (1)$$

where $x_1^n$ denotes the vector $[x_1, x_2, ..., x_n]^T$. This capacity represents an upper bound for the achievable rates of random binary linear (coset) codes over channels with memory, see Proposition 1 in [1].

An ensemble of LDPC codes is often defined by the variable and check degree sequence vectors $\underline{\lambda} = [\lambda_2, ..., \lambda_L]^T$ and $\underline{\rho} = [\rho_2, ..., \rho_R]^T$. Here, $\lambda_i$ denotes the fraction of edges connected to variable nodes with the degree equal to $i$, and $\rho_j$ denotes the fraction of edges connected to check nodes with the degree equal to $j$. The code rate equals $r = 1 - \frac{\sum_{i=2}^{R} \rho_i / i}{\sum_{i=2}^{L} \lambda_i / i}$, where $L$ and $R$ denote the maximal variable node degree and the maximal check node degree, respectively.

## II. Optimization

Our optimization method is based on the approach by Richardson *et al.* [2], modified appropriately to fit the density evolution for ISI channels [1]. We slightly change the scenario of density evolution described in [1]. Instead of running the BCJR algorithm after each variable-to-check density update [1], we apply the BCJR step once, then the variable-to-check density update $K \geq 1$ times, then the BCJR step once, and so on. The integer $K$ plays an important role in the optimization and varies from channel to channel. Experiments show that larger channel memories imply lower $K$'s.

We perform the optimization in a similar fashion to the optimization in [2]. We denote the average probability density function (pdf) of the log-likelihood-ratio message emitted from a variable to a check node after $l$ iterations by $f_v^{(l)}$. The average pdf of the message emitted from a check to a variable node is denoted by $f_c^{(l)}$. The average pdf obtained by evolving the density $f_v^{(l)}$ through the trellis section of the code/trellis graph is denoted by $f_o^{(l)}$. The average probability that a variable-to-check edge carries an erroneous message after $l$ iterations is [1][2]

$$p_l = \int_{-\infty}^{0} f_v^{(l)}(\xi) d\xi$$

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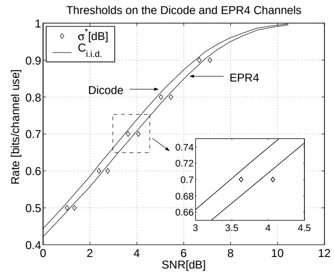


Fig. 1. The i.i.d. capacity curves and LDPC thresholds for the Dicode and the EPR4 channels.

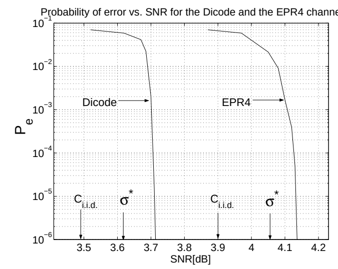


Fig. 2. Probability of error vs. SNR for the Dicode and the EPR4 channels; $r = 0.7$, code length $10^6$.

$$= \int_{-\infty}^{0} \left( \sum_{i=2}^{L} \lambda_i (\otimes_1^{i-1} (f_c^{(l-1)}(\xi)) \right) \otimes f_o^{(l-1)}(\xi) d\xi = \sum_{i=2}^{L} A_{l,i} \lambda_i,$$

where $\otimes$ is the standard convolution operator. When $l-1$ iterations are performed with parameters $\{\underline{\lambda}, \underline{\rho}\}$ and in the $l^{th}$ iteration $\underline{\lambda}$ is changed to $\underline{\lambda}'$, the probability of error equals $p_l' = \sum_{i=2}^{L} A_{l,i} \lambda_i'$. The last equation holds if $f_o^{(l)}$ is not updated through the trellis after the change of $\underline{\lambda}$. We optimize the variable degree sequence by minimizing the function $L(\underline{\lambda}') = \sum_{l=1}^{m} \frac{p_{l-1} - p_l'}{p_{l-1} - p_l}$ under the following constraints

$$\lambda_i' \geq 0, \quad \sum_{i=2}^{L} \lambda_i' = 1, \quad \sum_{i=2}^{L} \frac{\lambda_i'}{i} = \sum_{i=2}^{L} \frac{\lambda_i}{i}, \quad p_l' \leq p_{l-1}, \quad (1 \leq l \leq m)$$

$$|p_l - p_l'| \leq \delta \cdot \max(0, p_{l-1} - p_l), \quad (1 \leq l \leq m;\ 0 < \delta \ll 1).$$

Initially, we start with a very low noise variance $\sigma^2$ and then increment it as we update the degree sequences. We accomplish the optimization by means of linear programming. The optimization of the check degree sequences is analogous to the method described above for $\underline{\lambda}$. Since this is a local optimization method, we need to restart with different initial $\underline{\lambda}$ and $\underline{\rho}$ until a good sequence is found.

## III. Results

We performed the code optimization on the Dicode and the EPR4 channels. We used $K = 5$ for the Dicode channel, and $K = 3$ for the EPR4 channel. The noise tolerance *thresholds* ($\sigma^*$) are within 0.2dB of $C_{i.i.d.}$ over the entire range of rates from 0.5 to 0.9 (see Figure 1). We set the maximum-degree lengths to $L = 50$ and $R = 50$. Naturally, increasing $L$ and $R$ results in a slight improvement. However, when the maximum-degree lengths are larger the construction of the code with a finite block length is more difficult.

We used the optimized degree sequences to construct irregular codes with the block length $10^6$ and the code rate $r = 0.7$. These codes achieve the probability of error $10^{-6}$ at the SNR's less than 0.25dB away from the capacity (see Figure 2).

## Acknowledgment

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## References

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