Optimal Soft-output Detector for Channels with Intersymbol Interference and Timing Errors

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***Abstract*— We address the issue of detecting transmitted (or recorded) symbols when there is a timing error in the sampled waveform, along with intersymbol interference (ISI) and additive channel noise. We built a special trellis to jointly characterize the timing uncertainty and the channel ISI. The method derived here provides the joint maximum a posteriori probability detection of each input symbol and the quantized timing offset of each sample. It also generates "soft" decisions, which are essential for soft-decision iterative decoding algorithms.**

***Index Terms*— Markov chain, intersymbol interference, trellis, maximum a posteriori detection, timing error.**

I. Introduction

Reliable transmission of information over channels with synchronization errors and intersymbol interference is important in many data storage systems. In magnetic recording systems, for example, random fluctuations in the motion of the recording medium, during both the read and write process, cause timing uncertainty in the read-back signal. The resulting received sequence, therefore, has a *random* length. Our goal in this paper, is to formulate the maximum *a posteriori* probability (MAP) detector of each input symbol after observing the received sequence (similar to BCJR [1]).

Previous work [**?**] has been done on a similar problem, but only considered the case of a *fixed* random timing offset. In [**?**], a special case of timing error channel, namely the channel with insertion and deletion was studied. Literature ([**?**],[**?**]) has also analyzed the problem of phase estimation, which is a related, but different problem.

II. Source/Channel Model

We assume that the source is a stationary, discrete-time, first order Markov source defined on a binary alphabet, i.e., we have for any integer $m \geq 0$

$$P(X_{t+1}|X_{t-m}^{t}) = P(X_{t+1}|X_t). \tag{1}$$

The transition probabilities of the Markov source are

$$\pi_{ij} = P(X_{t+1} = j|X_t = i), \tag{2}$$

where $(i, j) \in \{(\pm 1, \pm 1)\}$.

The basic assumption in this work is that the timing error can be modelled by a first-order Markov chain. Let T be the symbol interval. An ideal receiver would sample the waveform at times $0, T, 2T, 3T, \ldots$. However, because of the timing errors, the receiver samples the waveform at times $(0+\varepsilon_0)T$, $(1+\varepsilon_1)T$, $(2+\varepsilon_2)T$, $\ldots$, $(k+\varepsilon_k)T$, and so on. We assume that the random variable ε_k forms a first order Markov chain, and its value is determined by uniformly quantizing the symbol interval T. For example, if we quantize ε_k to 5 levels per symbol interval, the variable ε_k can take values in the countable set $\{\ldots, -0.4, -0.2, 0, 0.2, 0.4, \ldots\}$. The number of quantization levels can be chosen arbitrarily.

We also assume that ε_k is a slowly varying independent increment process and we limit the value of the increment. For example, for 5 levels of quantization, ε_k can take only three values: ε_{k-1} or $\varepsilon_{k-1} \pm 0.2$. Define $\phi_k = \varepsilon_k T$. We assume

$$P(\phi_k|\phi_{k-1}) = \begin{cases} \delta & \text{if} \quad \phi_k = \phi_{k-1} \pm 0.2T \\ 1 - 2\cdot\delta & \text{if} \quad \phi_k = \phi_{k-1} \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

Under such assumptions, there are three possibilities for the number of samples in each symbol interval: no sample, one sample, or two samples.

Define the $k-th$ symbol interval as the semi-open segment $((k-1)T, kT]$ on the time axis. Our states for timing uncertainty at each symbol interval are defined so as to represent all sampling possibilities during $((k-1)T, kT]$. In the case of 5 quantization levels per symbol interval, we have 7 states:

- no sample in $((k-1)T, kT]$. We will refer to this as state $(\bullet, \bullet)$.
- 1 sample in $((k-1)T, kT]$ at time $(k-1)T + \frac{i}{5}T$, $i \in \{1, 2, \ldots, 5\}$. We will refer to this state as $(\bullet, i)$.
- 2 samples in $((k-1)T, kT]$ at times $(k-1)T + \frac{1}{5}T$ and $((k-1)T, kT]$. We will refer to this as state $(1, 5)$. By our assumption, this is the only two-sample state.

Generally, if we quantize each symbol interval to Q levels, we form a timing uncertainty trellis with $Q+2$ states. Fig.1 illustrates the trellis for 5 quantization levels. The left column of states denotes where the interval $((k-2)T, (k-1)T]$ is sampled, while the right state denotes where the interval $((k-1)T, kT]$ is sampled. Most values of the state transition probabilities are δ or $1-2\delta$, and are easily determined. Some state transition probabilities are not so "obvious", and are provided in Fig. 1.

We assume that the partial response polynomial, without timing error, has the form $h(D) = 1 + D$, see Fig.2 (b). However, if there is a timing offset, the ISI length is 2. Thus, we generally assume that the ISI length is 2, and impose a standard 4-state symbol trellis as shown in Fig.2 (a).

The final trellis is obtained as a cross product of the two trellises in Fig.1 and 2 (a), and therefore has 28 states.

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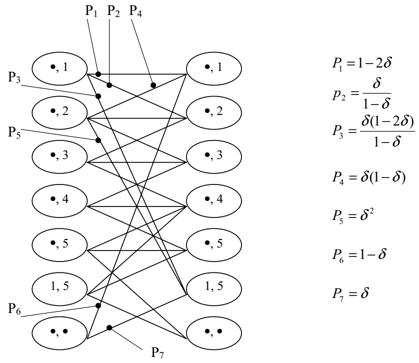

Fig. 1. Trellis for timing uncertainty

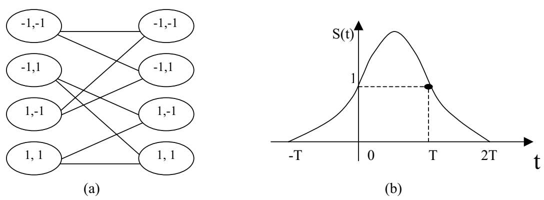

Fig. 2. ISI trellis and pulse shape

III. A Soft-Output Algorithm

In this section, we consider the problem of MAP detection of each input symbol based on the received sequence Y_1^ℓ, where ℓ denotes the number of waveform samples that correspond to a block of n input symbols.

Suppose the timing error process is quantized to Q values per symbol interval. Define $\mathcal{T} = \{(\bullet,\bullet),(\bullet,1),(\bullet,2),\ldots,(\bullet,Q),(1,Q)\}$ to be the set of timing states, and $\mathcal{B} = \{(i,j)\}$ to be the set of ISI states, where $i,j=\pm 1$. The final state set in our trellis is $\mathcal{F} = \mathcal{B}\times\mathcal{T}$.

Denote $S_t \in \mathcal{F}$ as the state at time t (or, a state in the $t-th$ column of the trellis). We calculate the following probabilities:

$$\lambda_t(m) = P(S_t = m; y_1^\ell), \tag{4}$$

$$\sigma_i(\psi) = P(\phi_i = \psi; y_1^\ell), \tag{5}$$

where ϕ_i is the overall sampling offset of the $i-th$ sample relative to $i{\cdot}T$, with $m \in \mathcal{F}$ and $\psi \in \{0, \pm\frac{T}{Q}, \pm\frac{2T}{Q}, \ldots\}$. From these soft-outputs, we can readily calculate the *a posteriori* probabilities of each input symbol and the timing uncertainty at each sampling point. For example:

$$\begin{aligned} P(X_t = 1|y_1^\ell) &= \sum_{m\in\{(\pm 1,1)\}\times\mathcal{T}} \lambda_t(m)/P(y_1^\ell) \\ P(\phi_i = \psi|y_1^\ell) &= \sigma_i(\psi)/P(y_1^\ell), \end{aligned} \tag{6}$$

where $\psi \in \{0, \pm\frac{T}{Q}, \pm\frac{2T}{Q}, \ldots\}$. $P(y_1^\ell)$ is a constant for a given received sequence, so it need not be calculated.

To calculate the soft-outputs (4), (5), we give some definitions. We use Z_t to represent the vector of output samples corresponding to input symbol X_t. (Note: Z_t may include one symbol, two symbols, or no symbol, depending on how many samples we have in the interval $((t-1)T, tT]$.) We use Z_t^n to represent the sequence of output samples corresponding to the input sequence X_t^n. We define the following functions:

$$\alpha(t,m,i) = P(S_t = m; Z_1^t = y_1^i) \tag{7}$$
$$\beta(t,m,i) = P(Z_{t+1}^n = y_{i+1}^\ell | S_t = m) \tag{8}$$

In words, $\alpha(t,m,i)$ is the probability that the state at time t (or, the $t-th$ column in the trellis) is m, and that the first t inputs x_1^t gave rise to the first i outputs y_1^i. We also define the following branch metric function:

$$\gamma(t,m,m',i) = \begin{cases} P(S_t{=}m';Z_t{=}y_i|S_{t-1}{=}m) & \text{if} \quad m'\in\mathcal{B}\times\{(\bullet,q)\} \\ P(S_t{=}m';Z_t{=}y_{i-1}^i|S_{t-1}{=}m) & \text{if} \quad m'\in\mathcal{B}\times\{(1,Q)\} \\ P(S_t{=}m';Z_t{=}\emptyset|S_{t-1}{=}m) & \text{if} \quad m\in\mathcal{B}\times\{(\bullet,\bullet)\}, \end{cases} \tag{9}$$

where $q \in \{1,2,\ldots,Q\}$. If we know the *a priori* statistics of the Markov source and timing uncertainty, as well as the statistics of the channel noise, it is straightforward to calculate the above branch metrics $\gamma(t,m,m',i)$.

We can now rewrite the soft-output functions as follows:

$$\lambda_t(m) = \sum_{i=1}^{\ell} \alpha(t,m,i)\beta(t,m,i), \tag{10}$$

$$\begin{aligned} \sigma_i(\psi) = & \sum_{\substack{t,m\in\mathcal{B}\times\{(\bullet,q)\}\\(t-i)T+qT/Q=\psi}} \alpha(t,m,i)\beta(t,m,i) \\ & + \sum_{\substack{t,m\in\mathcal{B}\times\{(1,Q)\}\\(t-i)T+T=\psi}} \alpha(t,m,i)\beta(t,m,i) \\ & + \sum_{\substack{t,m\in\mathcal{B}\times\{(1,Q)\}\\(t-i)T+T/Q=\psi}} \alpha(t,m,i+1)\beta(t,m,i+1). \end{aligned} \tag{11}$$

Equations (10), (11) indicate that the soft-outputs can be calculated in terms of $\alpha(t,m,i)$, and $\beta(t,m,i)$. Next, from the Markovian property of the trellis, we have

$$\alpha(t,m,i) = \begin{cases} \sum_{m'\in\mathcal{F}} \alpha(t{-}1,m',i{-}1)\gamma(t,m',m,i) & \text{if} \quad m\in\mathcal{B}\times\{(\bullet,q)\} \\ \sum_{m'\in\mathcal{F}} \alpha(t{-}1,m',i)\gamma(t,m',m,i) & \text{if} \quad m\in\mathcal{B}\times\{(\bullet,\bullet)\} \\ \sum_{m'\in\mathcal{F}} \alpha(t{-}1,m',i{-}2)\gamma(t,m',m,i{-}1) & \text{if} \quad m\in\mathcal{B}\times\{(1,Q)\}. \end{cases} \tag{12}$$

From (12), we can recursively calculate $\alpha(t,m,i)$, provided that $\alpha(t-1,m,i)$ is known for all m and i.

The initial statistics depend on the real system. For example, we can assume that the intersymbol interference state stays at $(-1,-1)$ before the transmission of data symbols, and we know the a priori statistics of the initial sampling phase offset. If we assume the sampling clock starts sampling from the "0" phase offset, we will have the following initial condition:

$$\alpha(1,m,i) = \begin{cases} \gamma(1,m_0,m,1), \\ \qquad \text{if} \quad {\scriptstyle m\in\{(-1,\pm1)\}\times\{(\bullet,Q-1),(\bullet,Q)\};} \\ \gamma(1,m_0,m,0), \\ \qquad \text{if} \quad {\scriptstyle m\in\{(-1,\pm1)\}\times\{(\bullet,\bullet)\};} \\ 0, \qquad \text{otherwise}. \end{cases} \tag{13}$$

where $m_0 = (-1,-1)\times(\bullet,Q) = [-1,-1,(\bullet,Q)]$.

Similarly, using the Markovian property of the trellis, we have for $t = 1,2,\cdots,n-1$

$$\beta(t,m,i) = \begin{cases} \sum_{m'\in\mathcal{B}\times\{(\bullet,d)\}} \gamma(t+1,m,m',i+1)\beta(t+1,m',i+1) \\ + \sum_{m'\in\mathcal{B}\times\{(\bullet,\bullet)\}} \gamma(t+1,m,m',i)\beta(t+1,m',i) \\ + \sum_{m'\in\mathcal{B}\times\{(1,D)\}} \gamma(t+1,m,m',i+2)\beta(t+1,m',i+2). \end{cases} \tag{14}$$

To calculate $\beta(t,m,i)$ using the backward recursion in (14), we set the initial conditions at $t = n$

$$\beta(n,m,i) = \begin{cases} 1, & i = \ell; \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

Similar to the BCJR algorithm [1], this algorithm has a forward-backward recursion process, given by (12) and (14).

IV. Simulation Results

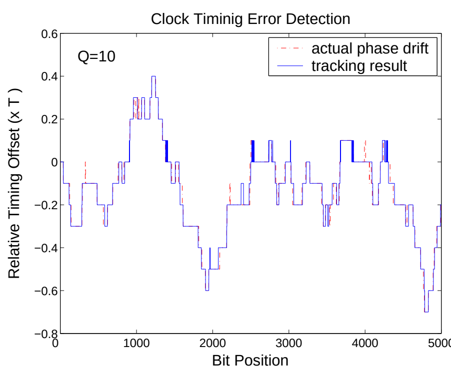

Fig. 3. Using the soft-output detection to track the discrete random timing offset in the presence to intersymbol interference and additive Gaussian noise

Fig. 3 and Fig. 4 demonstrate the timing offset tracking results of the soft output algorithm when the timing error is quantized to 10 and 5 levels per sampling period, respectively. In Fig. 3 the random phase change is generated by a discrete Markov chain in (3) with $\delta = 0.01$ and $Q = 10$ quantization levels. Fig. 4 shows the results of a more realistic case that models the timing offset increment at each sample to be an independent and identically distributed Gaussian random variable with mean 0 and variance $\frac{\sigma}{T} = 3\%$. We still track the timing uncertainty by our discrete quantized timing offset model with $Q = 5$ levels, and we choose $\delta = 0.01$ so that it has the same variance for the independent timing offset increment. For each case, we transmit 5000 bits from a uniform binary source, the signal-to-noise ratio (SNR) is 8dB and all the information bits are detected without errors.

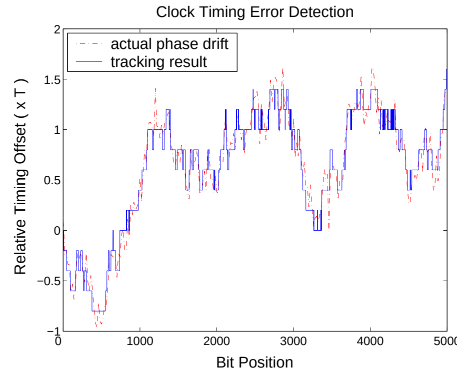

Fig. 4. Using the discrete Markov phase model to track more realistic continuous random timing offset in the presence of ISI and AWGN.

Fig. 5 compares the average bit error probability of the soft-output algorithm with different timing error probabilities under additive white Gaussian noise and intersymbol interference.

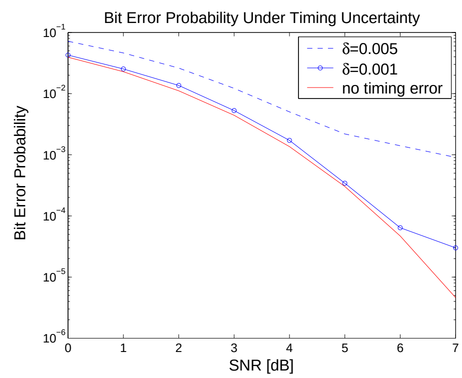

Fig. 5. Comparison of bit error probability under different timing uncertainties.

V. Conclusion

In this paper, we presented a soft-output symbol detection algorithm when a binary Markov sequence is transmitted over a noisy channel with ISI, timing uncertainty and additive white Gaussian noise. Under the assumption of ”slowly varying” timing error, this algorithm provides *maximum a posteriori probability* detection for both the input symbol and the timing offset if the latter is discrete. When the timing offset is not discrete (say, Gaussian) it gives satisfactory result in tracking the timing offset as shown by simulations.

Acknowledgments

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