Solution to HW1

Problem 1.6

For the first-order AR process $u(n)$

$$u(n) + a_1 u(n-1) = v(n),$$

we get the following equations recursively

$$\begin{aligned}
u(n) &= v(n) - a_1 u(n-1) \\
&= v(n) - a_1 v(n-1) + a_1^2 u(n-2) \\
&= v(n) - a_1 v(n-1) + a_1^2 v(n-1) - a_1^3 u(n-3) \\
&= \cdots \\
&= v(n) - a_1 v(n-1) + a_1^2 v(n-1) + \cdots + (-a_1)^{n-2} v(2) + (-a_1)^{n-1} u(1).
\end{aligned}$$

Assume that $u(0)=0$. Then $u(1)=v(1)$ and

$$u(n) = v(n) - a_1 v(n-1) + a_1^2 v(n-1) + \cdots + (-a_1)^{n-2} v(2) + (-a_1)^{n-1} v(1). \quad \text{----} (1)$$

(a) Let $E[v(n)] = \mu$ for all n. Then

$$\begin{aligned}
E[u(n)] &= \mu - a_1\mu + a_1^2 \mu + \cdots + (-a_1)^{n-1}\mu \\
&= \begin{cases} \mu \cdot \dfrac{1-(-a_1)^n}{1+a_1}, & \text{if } a_1 \neq -1 \\ n\mu, & \text{if } a_1 = -1 \end{cases}
\end{aligned}$$

If $\mu \neq 0$, then $E[u(n)]$ is a function of n. Hence, the AR process $u(n)$ is not stationary.

(b) Let $E[v(n)] = \mu = 0$ and $Var[v(n)] = \sigma_v^2$. From (a), we have $E[u(n)] = 0$.

Since $v(n)$ is a white-noise process, we have

$$E[v(n)v(m)] = \begin{cases} \sigma_v^2, & \text{if } n=m \\ 0, & \text{if } n \neq m \end{cases}$$

Then using Eqn.(1), the variance of $u(n)$ is obtained

$$\mathrm{Var}(u(n)) = E[u^2(n)]$$

$$= \sigma_v^2 (1 + a_1^2 + a_1^4 + \cdots + a_1^{2(n-2)} + a_1^{2(n-1)})$$

$$= \begin{cases} \sigma_v^2 \cdot \dfrac{1-a_1^{2n}}{1-a_1^2}, & \text{if } a_1 \neq \pm 1 \\ n\sigma_v^2 & \text{if } a_1 = \pm 1 \end{cases}$$

If $|a_1| < 1$, then,

$$\mathrm{Var}(u(n)) \longrightarrow \sigma_v^2 / 1-a_1^2 \qquad (n \to \infty).$$

(c) Let $E[u(n)u(n+k)]$ be the autocorrelation fuction of the AR process $u(n)$.

From Eqn. (1),

$$\begin{cases} u(n) = v(n) - a_1 v(n-1) + a_1^2 v(n-2) + \cdots + (-a_1)^{n-1} v(1) \\ u(n+k) = v(n+k) - a_1 v(n+k-1) + a_1^2 v(n+k-2) + \cdots + (-a_1)^k v(n) + (-a_1)^{k+1} v(n-1) + \cdots + (-a_1)^{n+k-1} v(1). \end{cases}$$

Using the property of the white-noise process $v(n)$, we have

$$r(k) \triangleq E[u(n)u(n+k)] = \sigma_v^2 \left[(-a_1)^k + (-a_1)^{k+2} + (-a_1)^{k+4} + \cdots + (-a_1)^{k+2(n-1)}\right]$$

$$= \begin{cases} \sigma_v^2 \cdot (-a_1)^k \cdot \dfrac{1-a_1^{2n}}{1-a_1^2}, & \text{if } a_1 \neq \pm 1 \\ (-1)^k \cdot n\sigma_v^2 & \text{if } a_1 = +1 \\ n \cdot \sigma_v^2 & \text{if } a_1 = -1 \end{cases}$$

If $|a_1| < 1$, then, for large n, we have,

$$r(k) \approx (-a_1)^k \cdot \sigma_v^2 / 1-a_1^2, \qquad k \geq 0$$

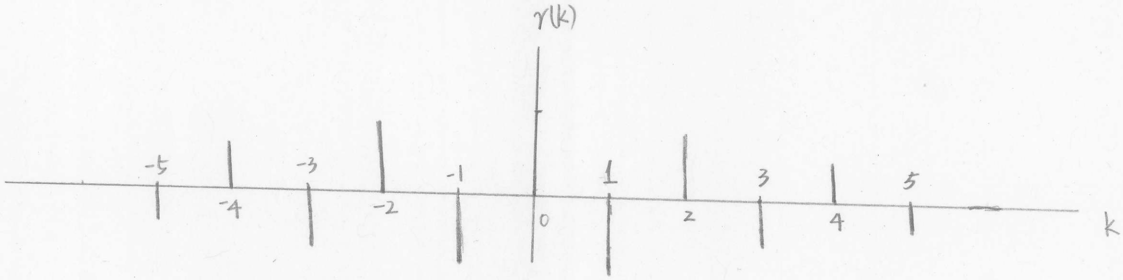
and

$$r(-k) = r(k)$$

i). Case 1: $0 < a_1 < 1$. The autocorrelation function is

$$r(k) = (-1)^k \cdot \frac{\sigma_v^2 \cdot a_1^k}{1 - a_1^2} \quad (k \geq 0)$$

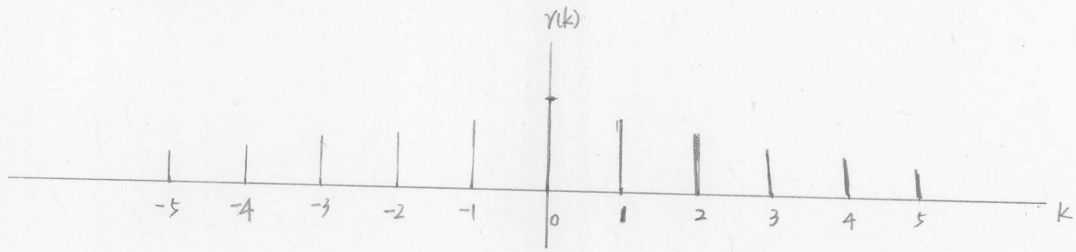
and $r(-k) = r(k)$



ii). Case 2: $-1 < a_1 < 0$. The autocorrelation function is

$$r(k) = \frac{\sigma_v^2 \cdot b_1^k}{1 - b_1^2} \quad (k \geq 0) \text{ where } b_1 = -a_1 \text{ and } 0 < b_1 < 1$$

and $r(-k) = r(k)$



Problem 1.7

(a) For the AR process of oder two

$$u(n) = u(n-1) - 0.5u(n-2) + v(n),$$

the AR coefficients are $a_1 = -1$, $a_2 = 0.5$. To get Yule-Walker equations, we let $w_1 = -a_1 = 1$, $W_2 = -a_2 = -0.5$. Then the Yule-Walker equations can be written as

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix} \quad \text{------} \quad (2)$$

where $r(l)$ (for $l = 0, 1, 2$) is the autocorrelation fuction of $u(n)$.

(b) From the Yule-Walker equations in Eqn. (2), we have

$$\begin{cases} r(1) = \frac{2}{3} r(0) \\ r(2) = \frac{1}{6} r(0) \end{cases}.$$

where $r(0) = E[u^2(n)]$

(c) Assume that $u(1) = u(2) = 0$. From the AR process

$$u(n) = u(n-1) - 0.5\,u(n-2) + v(n).$$

we have the following equations

$$u(3) = v(3)$$
$$u(4) = v(4) + u(3)$$
$$u(5) = v(5) + u(4) - 0.5\,u(3)$$
$$u(6) = v(6) + u(5) - 0.5\,u(4)$$
$$\vdots$$

Then $E[u(n)] = 0$ for $n = 1, 2, 3, \dots$, since the noise process $v(n)$ has zero mean.

Hence the variance of $u(n)$ is

$$Var[u(n)] = E[u^2(n)] = r(0).$$

According to Eqn. (1.71) in Page 57 in the textbook, we have

$$Var[v(n)] = \sigma_v^2 = r(0) + a_1 r(1) + a_2 r(2).$$

So $$r(0) = \frac{\sigma_v^2}{1 + \frac{2}{3}a_1 + \frac{1}{6}a_2} = \frac{0.5}{1 + \frac{2}{3}\cdot(-1) + \frac{1}{6}\cdot(0.5)} = 1.2$$

Problem 1.11

(a) Let $\underline{u}(n)$ be the input vector. Then the filter output is

$$y(n) = \underline{w}^H \underline{u}(n).$$

Hence the average power of the filter output is

$$E[|y(n)|^2] = E[\underline{w}^H \underline{u}(n) \cdot \underline{u}^H(n)\, \underline{w}] = \underline{w}^H E[\underline{u}(n)\, \underline{u}^H(n)]\, \underline{w} = \underline{w}^H R\, \underline{w}$$

since R is the correlation matrix of the wide-sense stationary process.

(b) If the filter input is a white-noise process with zero mean and variance σ^2, then we have $R = \sigma^2 I$ wher I is the identity matrix.

Hence, in this case, the average power of the filter output is

$$E[|y(n)|^2] = \sigma^2 \underline{w}^H \underline{w}.$$

Problem 1.13

(a) From the Gaussian moment-factoring theorem (GMFT), we have

$$E[(u_1^* u_2)^k] = E[\underbrace{u_1^* u_1^* \cdots u_1^*}_{\#\text{ of } u_1^* = k}\ \underbrace{u_2 u_2 \cdots u_2}_{\#\text{ of } u_2 = k}]$$

$$= E[v_1^* v_2^* \cdots v_k^* v_{l_1} v_{l_2} \cdots v_{l_k}] \quad \text{where } v_i^* = u_1^* \text{ and } v_{l_i} = u_2 \text{ for } i = 1, 2, \ldots, k.$$

$$\overset{\text{GMFT}}{=} \sum_{\pi} \prod_{i=1}^{k} E[v_{\pi(i)}^* v_{l_i}]$$

$$= \sum_{\pi} \prod_{i=1}^{k} E[u_1^* u_2]$$

$$= k! \{E[u_1^* u_2]\}^k$$

(b) Using the result of part (a), we have

$$E[|u|^{2k}] = E[(u^* u)^k] = E[(u_1^* u_2)^k] \quad \text{where } u_1 = u_2 = u$$

$$= k! \{E[u_1^* u_2]\}^k$$

$$= k! \{E[|u|^2]\}^k.$$

Problem 15.

Let $u(n)$ be the filter input. The filter output is

$$y(n) = \sum_{i=-\infty}^{+\infty} h(i)\, u(n-i),$$

or

$$y(m) = \sum_{k=-\infty}^{+\infty} h(k)\, u(m-k).$$

Then the autocorrelation function of the output process is

$$\begin{aligned}
r_y(n,m) &= E\left[y(n)\, y^*(m) \right] \\
&= E\left[\left(\sum_i h(i)\, u(n-i) \right) \cdot \left(\sum_k h^*(k)\, u^*(m-k) \right) \right] \\
&= E\left[\sum_i \sum_k h(i)\, h^*(k)\, u(n-i)\, u^*(m-k) \right] \\
&= \sum_i \sum_k h(i)\, h^*(k)\, E\left[u(n-i)\, u^*(m-k) \right] \\
&= \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} h(i)\, h^*(k)\, r_u(n-i,\, m-k). \qquad \cdots\cdots (*)
\end{aligned}$$

Let $l=n-m$. Then Eqn. $(*)$ is Eqn. (1.126) in Page 74 in the textbook.