Grid Monitoring: Bounds on Performances of Sensor Placement Algorithms

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***Abstract*—The objective of this paper is to find numerical bounds on the performances of algorithms for the placements of phasor measurement units (PMUs) in the power grid. Given noisy measurements and knowledge of the state correlation matrix, we use a linear minimum mean squared error estimator as the state estimator to formulate the PMU placement problem as an integer programming problem. Finding the optimal placements of a fixed number of PMUs in a large network is prohibitively complex, forcing us to look for suboptimal algorithms and bounds on the optimal performance. In this paper, we present a sequence of nested bounds using matrix pencils and generalized eigenvalues, that upper bound the optimal performance. Finally, we numerically compare the performances of the suboptimal solutions with the upper bounds using the IEEE 30- and 57-bus test systems, showing that the proposed bounds provide a valid basis for determining the quality of different suboptimal solutions.**

Keywords–phasor measurement units; matrix pencils; generalized eigenvalues.

I. Introduction

State estimation (SE) is a key function in modern energy management systems, where various crucial control tasks depend on the accurate snapshots of the system state [1]. Conventional state estimators rely on the redundant measurements captured by supervisory control and data acquisition (SCADA) systems [1], which can only take non-synchronized measurements. These measurements are too infrequent to capture the dynamics of the power grid [1]. With the advent of phasor technology, time synchronized measurements can be obtained using phasor measurement units (PMUs) [2]. These devices take advantage of the global positioning system (GPS) technology to provide time-stamped measurements of the bus voltage magnitudes and phase angles [2].

Traditional SE using SCADA measurements is nonlinear, and is solved using iterative algorithms [3]. The PMUs, on the other hand, can directly measure the states at the PMU-installed buses, and the states of all the connected buses (if enough channels are available). In fact, given the high measurement precision and reliability of the PMUs, we can consider the PMU measurements to be low-noise refinements of certain states (exactly those states that are measured by the PMUs) [2]. Since the PMUs can refine only a small subset of all state estimates, a common task is to refine the remaining state estimates (corresponding to the buses not carrying PMUs) using the sparse PMU measurements.

To measure all the state variables, the PMUs need to be installed at around one third of all the buses [4]. Since this goal is unlikely to be achieved in the near future, researchers look for the best solutions to deploy PMUs at a smaller subset of the buses, such that the state estimation error is minimized.

In this paper, we consider the optimization problem where we have n bus locations (where we can deploy PMUs) and m PMUs to place ($m << n$). We formulate the optimization problem to minimize the mean squared estimation error. Finding the optimal solution for the PMU placement problem is very difficult. In fact, it has been shown that the problem is NP-complete [4]. This means that there is no known efficient method to solve this problem with computational complexity that is polynomial in n. For this reason, heuristic approaches (e.g., greedy algorithm [5], gradient projection algorithm [6] etc.) are typically applied to search for good suboptimal solutions. But the question is, how can we guarantee that a heuristic solution is close to the optimal one, when we have no computationally feasible method of computing the optimal solution? The only way to guarantee the quality of a heuristic solution is to compare it to a provable and computationally feasible performance bound. However, for the PMU placement problem, no tight bounds are available either. Hence, in this paper, we propose upper bounds on the optimal solution that allow us to bound the difference between optimal and suboptimal solutions. Thereby, we significantly extend our prior work in [7] by presenting nested upper bounds with complete analysis of the bounds using matrix pencils and their generalized eigenvalues.

Related Work on PMU Placement: There has been much work done on finding reasonable suboptimal solutions to the PMU placement problem. In [5], the PMU placement

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problem is formulated as an optimal experimental design problem. A greedy approach is proposed to solve this problem suboptimally and a lower bound is obtained on the performance of the greedy solution when the objective function is submodular. An estimation-theoretic approach is proposed in [6]; after posing the optimization problem as a linear regression problem, a convex relaxation is developed to find a suboptimal solution. Other research has considered PMU placement for system observability and measurement redundancy [8][9]. Dua et al. [9] considered the phasing of PMU deployment in an integer linear programming framework, while [8] proposed a binary particle swarm optimization based algorithm. In [10], the authors present a unified description of different algorithms proposed to solve the PMU placement problem. However, for a large network, in the absence of the optimal solution there is no benchmark to compare the performances of these suboptimal algorithms.

Paper Organization: The paper is organized as follows: Section II gives the measurement model used for our analysis. The optimization problem is formulated in Section III. We review two suboptimal algorithms in Section IV. In Section V, a family of upper bound on the optimal solution is found and the results are verified numerically using IEEE 30- and 57-bus test systems in Section VI. Section VII summarizes the results of this paper and suggests directions of future work.

Notation: Upper and lower case letters denote random variables and their realizations, respectively; underlined letters stand for vectors; boldface upper case letters denote matrices, and $\mathbf{I}_n$ denotes the $n \times n$ identity matrix; $\langle \mathbf{A}, \mathbf{B} \rangle$ denotes a matrix pencil formed by matrices $\mathbf{A}$ and $\mathbf{B}$; $(\cdot)^T$ and $\mathbf{E}(\cdot)$ stand for transposition and expectation, respectively.

II. Model

We consider voltage magnitudes and phase angles as state variables that are initially estimated using nonlinear SE from SCADA data, and then further refined using sparse PMU measurements. For introduction and justification of this approach, see [5]. We further explain this scenario below.

Assume there are n_b buses. Let V_k and Δ_k denote the voltage magnitude and angle of the kth bus, $k = 1, \ldots, n_b$. Let $\underline{V} = [V_1, V_2, \ldots, V_{n_b}]^T$ be the state vector representing the bus voltage magnitudes and $\underline{\Delta} = [\Delta_1, \Delta_2, \ldots, \Delta_{n_b}]^T$ be the state vector representing the corresponding phase angles. To make the state estimation more efficient in terms of storage and computational costs, we assume the voltage magnitudes and phases to be statistically independent random vectors [1][11]. We further assume that all the PMUs are identical and take statistically independent voltage magnitude and phase measurements with variances σ_v^2 and σ_δ^2, respectively.

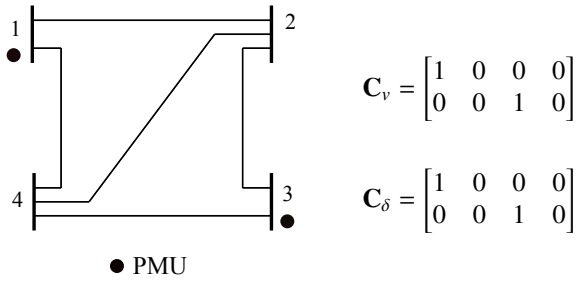
Let m_v and m_δ be the number of PMU voltage magnitude measurements and the number of PMU phase angle measurements, respectively, where $m_v \leq n_b$ and $m_\delta \leq n_b$. Let $\underline{Z}_v \in \mathbb{R}^{m_v}$ and $\underline{Z}_\delta \in \mathbb{R}^{m_\delta}$ be the PMU voltage magnitude measurement vector and PMU phase angle measurement vector, respectively. Then the PMU measurement model is:

$$\underline{Z}_v = \mathbf{C}_v(\underline{V} + \sigma_v \underline{N}_v), \tag{1}$$

$$\underline{Z}_\delta = \mathbf{C}_\delta(\underline{\Delta} + \sigma_\delta \underline{N}_\delta), \tag{2}$$

where $\underline{N}_v$ and $\underline{N}_\delta$ are random noise vectors with mean zero and covariance matrix $\mathbf{I}_{n_b}$. We assume that the random noise vectors $\underline{N}_v$ and $\underline{N}_\delta$ are statistically independent of the state vectors. $\mathbf{C}_v$ and $\mathbf{C}_\delta$ are matrices that represent the positions of the PMU placements (see Example 1). These are binary matrices with orthonormal rows, where each row has one '1'. The positions of ones in the matrix $\mathbf{C}_v$ and $\mathbf{C}_\delta$ denote the position of the sensors. In order to provide a reference point for the phase angle measurements, we assume that a PMU is always placed at the swing bus [2].

Example 1. *Figure 1 shows a* 4*-bus system [3] and matrices* $\mathbf{C}_v$ *and* $\mathbf{C}_\delta$ *when two PMUs are placed on buses* 1 *and* 3*.*



$$\mathbf{C}_v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_\delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1. PMU placement in a 4 bus system

Equations (1) and (2) represent separate models for PMU voltage magnitude and PMU phase angle measurements. We can combine (1) and (2) into a single model equation as

$$\begin{bmatrix} \underline{Z}_v \\ \underline{Z}_\delta \end{bmatrix} = \begin{bmatrix} \mathbf{C}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\delta \end{bmatrix} \left(\begin{bmatrix} \underline{V} \\ \underline{\Delta} \end{bmatrix} + \begin{bmatrix} \sigma_v \underline{N}_v \\ \sigma_\delta \underline{N}_\delta \end{bmatrix} \right). \tag{3}$$

Next, we argue that we can capture the natures of models (1)-(3) using a single state vector $\underline{X} \in \mathbb{R}^n$ and a single measurement vector $\underline{Z} \in \mathbb{R}^m$ ($m \leq n$) as

$$\underline{Z} = \mathbf{C}(\underline{X} + \sigma \underline{N}), \tag{4}$$

where $\underline{N}$ and $\underline{X}$ are statistically independent zero-mean random vectors with covariance matrices $\mathbf{I}_n$ and $\mathbf{\Sigma}_{\underline{X}}$, respectively. The following two examples illustrate this concept.

Example 2. *Under the following transformations,* (1) *and* (4) *are equivalent.*

$$\begin{array}{ll} \underline{X} = \underline{V} - \mathbf{E}(\underline{V}), & \sigma = \sigma_v, \\ \underline{Z} = \underline{Z}_v - \mathbf{E}(\underline{Z}_v), & n = n_b, \\ \underline{N} = \underline{N}_v, & m = m_v, \\ \mathbf{C} = \mathbf{C}_v. & \end{array}$$

Example 3. *Under the following transformations,* (3) *and*

(4) *are equivalent.*

$$\underline{X} = \begin{bmatrix} \underline{V} - \mathbf{E}(\underline{V}) \\ \frac{\sigma_v}{\sigma_\delta}\left(\underline{\Delta} - \mathbf{E}(\underline{\Delta})\right) \end{bmatrix}, \qquad \sigma = \sigma_v,$$

$$\underline{Z} = \begin{bmatrix} \underline{Z}_v - \mathbf{E}(\underline{Z}_v) \\ \frac{\sigma_v}{\sigma_\delta}\left(\underline{Z}_\delta - \mathbf{E}(\underline{Z}_\delta)\right) \end{bmatrix}, \qquad n = 2n_b,$$

$$\underline{N} = \begin{bmatrix} \underline{N}_v \\ \underline{N}_\delta \end{bmatrix}, \qquad m = m_v + m_\delta,$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\delta \end{bmatrix}.$$

We assume that the network is observable using conventional SCADA measurements and $\mathbf{\Sigma}_{\underline{X}}$ is the state covariance matrix (estimated using the traditional nonlinear SE approaches), $\underline{X}$ and $\underline{N}$ are statistically independent and $\mathbf{C}$ is composed of m rows of the $n \times n$ identity matrix $\mathbf{I}_n$.

III. Sensor placement problem

We desire to utilize the newly obtained low-noise PMU measurement $\underline{Z}$ to make a refined estimate $\hat{\underline{X}}(\underline{Z})$ of the entire state vector $\underline{X}$. To this end, we use the linear minimum mean squared error estimator $\hat{\underline{X}}(\underline{Z})$ given by [12]

$$\hat{\underline{X}}(\underline{Z}) = \mathbf{E}\left(\underline{X} \,|\, \underline{Z}\right) = \mathbf{E}\left(\underline{X}\,\underline{Z}^T\right)\mathbf{E}\left(\underline{Z}\,\underline{Z}^T\right)^{-1}\underline{Z}. \tag{5}$$

The estimation error vector is defined as $\underline{\mathcal{E}} = \underline{X} - \hat{\underline{X}}(\underline{Z})$ and the error covariance matrix is given by [12]

$$\mathbf{E}\left(\underline{\mathcal{E}}\,\underline{\mathcal{E}}^T\right) = \mathbf{\Sigma}_{\underline{X}} - \mathbf{E}\left(\underline{X}\,\underline{Z}^T\right)\mathbf{E}\left(\underline{Z}\,\underline{Z}^T\right)^{-1}\mathbf{E}\left(\underline{Z}\,\underline{X}^T\right), \tag{6}$$

where $\mathbf{E}\left(\underline{X}\,\underline{Z}^T\right) = \mathbf{\Sigma}_{\underline{X}}\mathbf{C}^T$ and $\mathbf{E}\left(\underline{Z}\,\underline{Z}^T\right) = \mathbf{C}\mathbf{\Sigma}_{\underline{X}}\mathbf{C}^T + \sigma^2\mathbf{I}_m$.

Our task is to find the matrix $\mathbf{C}^*$ that minimizes the total expected estimation error $\mathrm{tr}\,\mathbf{E}\left(\underline{\mathcal{E}}\,\underline{\mathcal{E}}^T\right)$. We now define the optimization problem as an integer programming problem of choosing m rows of $\mathbf{I}_n$ that minimize $\mathrm{tr}\,\mathbf{E}\left(\underline{\mathcal{E}}\,\underline{\mathcal{E}}^T\right)$.

Definition 1. *Let $C^{[m\times n]}$ denote the set of all $m \times n$ matrices composed of m rows of $\mathbf{I}_n$.* ■

The optimization problem is then given by

$$\mathbf{C}^* = \underset{\mathbf{C}\in C^{[m\times n]}}{\arg\min}\,\mathrm{tr}\,\mathbf{E}\left(\underline{\mathcal{E}}\,\underline{\mathcal{E}}^T\right) = \underset{\mathbf{C}\in C^{[m\times n]}}{\arg\min}\,\mathbf{E}\left(\underline{\mathcal{E}}^T\underline{\mathcal{E}}\right). \tag{7}$$

Since the first term in (6) (i.e., $\mathbf{\Sigma}_{\underline{X}}$) does not depend on the choice of matrix $\mathbf{C}$, we can express the minimization problem in (7) as an equivalent maximization problem using the following definition.

Definition 2. *Let the* **efficacy** *of a matrix* $\mathbf{C}$ *be defined as*

$$J(\mathbf{C}) \stackrel{\Delta}{=} \mathrm{tr}\left\{\mathbf{E}\left(\underline{X}\,\underline{Z}^T\right)\mathbf{E}\left(\underline{Z}\,\underline{Z}^T\right)^{-1}\mathbf{E}\left(\underline{Z}\,\underline{X}^T\right)\right\} \tag{8}$$

$$= \mathrm{tr}\left\{\left[\mathbf{C}\left(\mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I}\right)\mathbf{C}^T\right]^{-1}\mathbf{C}\mathbf{\Sigma}_{\underline{X}}^2\mathbf{C}^T\right\}. \tag{9}$$

[Note: the form in (9) *is a generalized Rayleigh quotient.]* ■

The optimization problem in (7) is then equivalent to

$$\mathbf{C}^* = \underset{\mathbf{C}\in C^{[m\times n]}}{\arg\max}\,J(\mathbf{C}), \tag{10}$$

which is an integer programming problem of choosing m rows of the identity matrix $\mathbf{I}_n$ that maximize the efficacy. The optimum solution to (10) requires an exhaustive search by testing all $\binom{n}{m}$ possible choices of m rows. Even for a moderately sized n and m, this becomes computationally infeasible. In fact, the sensor placement problem in the power grid is NP-complete [4].

IV. Suboptimal solutions

Since the optimization in (10) is difficult to perform, several algorithms that seek suboptimal but computationally feasible solutions have been reported [5][6][7][10]. If $\mathbf{C}$ is a suboptimal solution to (10), then it provides a lower bound on the optimal efficacy $J(\mathbf{C}^*)$, i.e., $J(\mathbf{C}) \le J(\mathbf{C}^*)$. Therefore, the search for suboptimal solutions to closely approach (10) is equivalent to constructing tight lower bounds on $J(\mathbf{C}^*)$. A number of algorithms can be applied to find suboptimal solutions. Here, for completeness, we review two such algorithms requiring a much lower search complexity than $O\left(\binom{n}{m}\right)$.

A. Expedient solution

This is a trivial approximate solution to consider [7]. Let $J(\underline{e}_k)$ be the efficacy of the k-th unit row vector, i.e., the efficacy of the sensor placed at the k-th bus when $m = 1$. Then using (9),

$$J(\underline{e}_k) = \sum_{i=1}^{n} \frac{\left(\underline{e}_k\,\mathbf{\Sigma}_{\underline{X}}\,\underline{e}_i^T\right)^2}{\underline{e}_k\,\mathbf{\Sigma}_{\underline{X}}\,\underline{e}_k^T + \sigma^2}. \tag{11}$$

We rank the vectors $\underline{e}_k$ in descending order of their efficacies $J(\underline{e}_k)$. For any arbitrary m, we pick the m highest ranked vectors $\underline{e}_k$ and stack them to be the rows of the approximate solution $\mathbf{C}_E$. Since this algorithm requires sorting and picking m highest ranked vectors $\underline{e}_k$, it has search complexity at most $O(n \log n)$.

This algorithm provides a useful insight into problem (10). From (11), it is clear that we want to place sensors at buses where the measured state is maximally correlated to the remaining states.

B. Greedy solution

A greedy algorithm obtains an approximate solution to (10) by making a sequence of choices [13]. At each step t, it assumes that t sensor locations are fixed, and makes a greedy choice where to place the $(t+1)$-st sensor. Let $\mathbf{C}_G$ denote the solution provided by the greedy algorithm. The algorithm can be described by the following.

Greedy Algorithm [13]

1) Set $t = 1$ and $\mathbf{C}_t = \underline{e}^*$ such that $\underline{e}^* = \arg\max_{\underline{e}\in C^{[1\times n]}} J(\underline{e})$.
2) Find $\underline{e}^* = \underset{\underline{e}\in C^{[1\times n]}:\,\mathbf{C}_t\underline{e}^T = \mathbf{0}}{\arg\max}\; J\left(\begin{bmatrix}\mathbf{C}_t \\ \underline{e}\end{bmatrix}\right)$.
3) Set $\mathbf{C}_{t+1} = \begin{bmatrix}\mathbf{C}_t \\ \underline{e}^*\end{bmatrix}$.
4) Increment: $t \leftarrow t+1$.
5) if $t = m$ set $\mathbf{C}_G = \mathbf{C}_t$ and stop, else go to 2. ■

Note that the greedy solution may not be optimal even for $m = 2$, but it has search complexity $O(mn)$, which is much smaller than $O\left(\binom{n}{m}\right)$ required to find the optimal solution $\mathbf{C}^*$.

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V. Upper bounds on the optimal efficacy

In Section IV, we considered algorithms to obtain lower bounds on the optimal efficacy $J(\mathbf{C}^*)$. However, it is hard to evaluate how well these algorithms perform compared to $J(\mathbf{C}^*)$ since the optimal solution is not available for comparison. Therefore, we want to obtain numerically computable upper bound, $\bar{J}$, on the optimal efficacy $J(\mathbf{C}^*)$ such that

$$J(\mathbf{C}^*) - J(\mathbf{C}) \leq \bar{J} - J(\mathbf{C}).$$

We devote this section to finding a family of upper bounds $\bar{J}_k$ on $J(\mathbf{C}^*)$ by relaxing conditions on $\mathbf{C}$.

A. Definitions

To develop a family of upper bounds on the optimal efficacy, we generalize the reward function (efficacy), and generalize the optimization problem and its constraints. Instead of considering two matrices $\mathbf{\Sigma}_{\underline{X}}^2$ and $\mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I}$, in this section we consider a general matrix pencil $\langle \mathbf{A}, \mathbf{B} \rangle$, where $\mathbf{A}$ and $\mathbf{B}$ do not necessarily equal $\mathbf{\Sigma}_{\underline{X}}^2$ and $\mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I}$, respectively. Next, instead of considering matrix $\mathbf{C}$ whose entries take values in the set $\{0, 1\}$, in this section we consider an unconstrained matrix $\mathbf{F}$ whose entries take values in $\mathbb{R}$. Finally, we introduce a modified optimization problem (different from the one in Section III) that leads to the upper bounds. The following definitions set the stage.

Definition. *For two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, define the* **efficacy** *of a matrix $\mathbf{F}$, with respect to the pencil $\langle \mathbf{A}, \mathbf{B} \rangle$, as*

$$J_{\langle \mathbf{A}, \mathbf{B} \rangle}(\mathbf{F}) \triangleq \operatorname{tr}\left\{\left(\mathbf{F}\mathbf{B}\mathbf{F}^T\right)^{-1}\mathbf{F}\mathbf{A}\mathbf{F}^T\right\}, \tag{12}$$

[should the inverse $(\mathbf{F}\mathbf{B}\mathbf{F}^T)^{-1}$ exist]. ■

Definition. *For $m \leq n$, let $\mathcal{F}^{[m\times n]}$ be the* **set** *of all $m \times n$ matrices with rank m.* ■

Definition. *We define $\mathbf{F}^*_{\langle \mathbf{A}, \mathbf{B} \rangle}$ to be the argument that solves the following optimization problem*

$$\begin{aligned}\mathbf{F}^*_{\langle \mathbf{A}, \mathbf{B} \rangle} &\triangleq \arg\max_{\mathbf{F}\in\mathcal{F}^{[m\times n]}} J_{\langle \mathbf{A}, \mathbf{B} \rangle}(\mathbf{F}) \\ &= \arg\max_{\mathbf{F}\in\mathcal{F}^{[m\times n]}} \operatorname{tr}\left\{\left(\mathbf{F}\mathbf{B}\mathbf{F}^T\right)^{-1}\mathbf{F}\mathbf{A}\mathbf{F}^T\right\}.\end{aligned} \tag{13}$$

■

Definition. *We define $J^*_{\langle \mathbf{A}, \mathbf{B} \rangle}$ as the solution to the optimization problem in (13).*

$$J^*_{\langle \mathbf{A}, \mathbf{B} \rangle} \triangleq \max_{\mathbf{F}\in\mathcal{F}^{[m\times n]}} J_{\langle \mathbf{A}, \mathbf{B} \rangle}(\mathbf{F}) = J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\mathbf{F}^*_{\langle \mathbf{A}, \mathbf{B} \rangle}\right).$$

■

B. Canonic theorem

Definition. *For a matrix pencil $\langle \mathbf{A}, \mathbf{B} \rangle$, if a number γ_j and a vector $\underline{u}_j$ satisfy the equation*

$$\mathbf{A}\underline{u}_j = \gamma_j \mathbf{B}\underline{u}_j, \tag{14}$$

then γ_j is called the generalized eigenvalue and $\underline{u}_j$ is called the generalized eigenvector of the pencil $\langle \mathbf{A}, \mathbf{B} \rangle$. ■

If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ symmetric matrices, there exist n generalized eigenvectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$, corresponding to generalized eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_n$. [Note: $\gamma_1, \gamma_2, \ldots, \gamma_n$ need not be distinct.] We arrange the generalized eigenvalues as the diagonal elements of a diagonal matrix $\mathbf{D}$,

$$\mathbf{D} \triangleq \begin{bmatrix} \gamma_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \gamma_n \end{bmatrix}, \tag{15}$$

and we arrange the generalized eigenvectors as the columns of a matrix $\mathbf{U}$,

$$\mathbf{U} \triangleq \left[\underline{u}_1, \ldots, \underline{u}_n\right]. \tag{16}$$

Theorem 1 (see [14])**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric and $\mathbf{B}$ be positive definite, and let $\mathbf{D}$ and $\mathbf{U}$ denote the generalized eigenvalue matrix and generalized eigenvector matrix as in (15) and (16), respectively. If the eigenvalues are ordered as $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$, then*

$$J^*_{\langle \mathbf{A}, \mathbf{B} \rangle} = \operatorname{tr}\left\{\begin{bmatrix}\mathbf{I}_m & \mathbf{0}\end{bmatrix}\mathbf{D}\begin{bmatrix}\mathbf{I}_m & \mathbf{0}\end{bmatrix}^T\right\} = \sum_{j=1}^{m}\gamma_j, \tag{17}$$

and

$$\mathbf{F}^*_{\langle \mathbf{A}, \mathbf{B} \rangle} = \begin{bmatrix}\mathbf{I}_m & \mathbf{0}\end{bmatrix}\mathbf{U}^T = \left[\underline{u}_1, \ldots, \underline{u}_m\right]^T. \tag{18}$$

Remark 1 (see [7])**.** *If $\mathbf{A} = \mathbf{\Sigma}_{\underline{X}}^2$ and $\mathbf{B} = \mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I}$, and $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues of $\mathbf{\Sigma}_{\underline{X}}$, then the generalized eigenvalues of the pencil $\left\langle \mathbf{\Sigma}_{\underline{X}}^2, \mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I} \right\rangle$ are $\gamma_j = \lambda_j^2/(\lambda_j + \sigma^2)$. Thus, using Theorem 1 we can write*

$$J^*_{\left\langle \mathbf{\Sigma}_{\underline{X}}^2, \mathbf{\Sigma}_{\underline{X}} + \sigma^2\mathbf{I} \right\rangle} = \sum_{j=1}^{m}\frac{\lambda_j^2}{\lambda_j + \sigma^2}. \tag{19}$$

To develop a family of upper bounds, we find it useful to solve a series of modified efficacy maximization problems for all $k \leq m$. The next definition addresses the modified efficacy maximization problem.

Definition. *For any $k \leq m$, we define $\mathbf{F}^{(k)*}_{\langle \mathbf{A}, \mathbf{B} \rangle}$ and $J^{(k)*}_{\langle \mathbf{A}, \mathbf{B} \rangle}$ as the solution pair of the following modified efficacy maximization*

$$\mathbf{F}^{(k)*}_{\langle \mathbf{A}, \mathbf{B} \rangle} \triangleq \arg\max_{\mathbf{F}\in\mathcal{F}^{[(m-k)\times(n-k)]}} J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right), \tag{20}$$

and

$$\begin{aligned}J^{(k)*}_{\langle \mathbf{A}, \mathbf{B} \rangle} &\triangleq \max_{\mathbf{F}\in\mathcal{F}^{[(m-k)\times(n-k)]}} J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right) \\ &= J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{(k)*}_{\langle \mathbf{A}, \mathbf{B} \rangle}\end{bmatrix}\right).\end{aligned} \tag{21}$$

■

In order to solve the modified efficacy maximization problem in (20) - (21), it is convenient to split the efficacy

$$J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right)$$

into two terms such that

1) the first term does not depend on $\mathbf{F}$, and
2) the second term equals the efficacy of $\mathbf{F}$ with respect to a modified pencil of lower dimensions.

We formulate the split in the following lemma.

Lemma A. *In the optimization problem (21), the efficacy can be expressed as*

$$J_{\langle \mathbf{A}, \mathbf{B} \rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right) = t_k + J_{\langle \mathbf{A}_k, \mathbf{B}_k \rangle}(\mathbf{F}), \tag{22}$$

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where the term t_k and the modified pencil $\langle \mathbf{A}_k, \mathbf{B}_k \rangle$ satisfy

$$t_k = \mathrm{tr}\left\{\mathbf{A}\begin{bmatrix}\mathbf{P}_k^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\} \tag{23}$$

$$\mathbf{A}_k = \begin{bmatrix}\mathbf{P}_k^{-1}\mathbf{Q}_k\\ -\mathbf{I}_{n-k}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{P}_k^{-1}\mathbf{Q}_k\\ -\mathbf{I}_{n-k}\end{bmatrix} \quad \textit{and} \quad \mathbf{B}_k = \mathbf{R}_k - \mathbf{Q}_k^T\mathbf{P}_k^{-1}\mathbf{Q}_k \tag{24}$$

$$\mathbf{P}_k = \begin{bmatrix}\mathbf{I}_k\\ \mathbf{0}\end{bmatrix}^T \mathbf{B}\begin{bmatrix}\mathbf{I}_k\\ \mathbf{0}\end{bmatrix} \textit{ and } \mathbf{Q}_k = \begin{bmatrix}\mathbf{I}_k\\ \mathbf{0}\end{bmatrix}^T \mathbf{B}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{n-k}\end{bmatrix} \textit{ and } \mathbf{R}_k = \begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{n-k}\end{bmatrix}^T \mathbf{B}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{n-k}\end{bmatrix} \tag{25}$$

Proof: For any $\mathbf{F} \in \mathcal{F}^{[(m-k)\times(n-k)]}$, using (12) we have

$$J_{\langle \mathbf{A},\mathbf{B}\rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right) = \mathrm{tr}\left\{\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\mathbf{B}\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}^T\right)^{-1}\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\mathbf{A}\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}^T\right\}. \tag{26}$$

Using (25), we express $\mathbf{B}$ in terms of $\mathbf{P}_k$, $\mathbf{Q}_k$ and $\mathbf{R}_k$ as

$$\mathbf{B} = \begin{bmatrix}\mathbf{P}_k & \mathbf{Q}_k\\ \mathbf{Q}_k^T & \mathbf{R}_k\end{bmatrix}. \tag{27}$$

Substituting $\mathbf{B}$ into (26) and using the partitioned matrix inversion lemma [15], we write (26) as

$$\begin{aligned} J_{\langle \mathbf{A},\mathbf{B}\rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right) &= \mathrm{tr}\left\{\mathbf{A}\begin{bmatrix}\mathbf{P}_k^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\} \\ &+ \mathrm{tr}\left\{\begin{bmatrix}\mathbf{P}_k^{-1}\mathbf{Q}_k\mathbf{F}^T\\ -\mathbf{I}_{m-k}\end{bmatrix}\left\{\mathbf{F}\left(\mathbf{R}_k - \mathbf{Q}^T\mathbf{P}_k^{-1}\mathbf{Q}_k\right)\mathbf{F}^T\right\}^{-1}\begin{bmatrix}\mathbf{P}_k^{-1}\mathbf{Q}_k\mathbf{F}^T\\ -\mathbf{I}_{m-k}\end{bmatrix}^T\right. \\ &\qquad \left.\times\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\mathbf{A}\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}^T\right\}. \end{aligned}$$

Using (23)-(24), we simplify the above equation as

$$\begin{aligned} J_{\langle \mathbf{A},\mathbf{B}\rangle}\left(\begin{bmatrix}\mathbf{I}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\right) &= t_k + \mathrm{tr}\left\{\left(\mathbf{F}\mathbf{B}_k\mathbf{F}^T\right)^{-1}\mathbf{F}\mathbf{A}_k\mathbf{F}^T\right\} \\ &= t_k + J_{\langle \mathbf{A}_k,\mathbf{B}_k\rangle}(\mathbf{F}). \end{aligned}$$

■

Lemma A now lets us express the solution of the the modified optimization problem (20)-(21) equivalently as the solution of a regular efficacy maximization (i.e., using Theorem 1), but for a modified matrix pencil. Hence, Lemma A yields the following corollary of Theorem 1.

Corollary 1.1. $\mathbf{F}^{(k)*}_{\langle \mathbf{A},\mathbf{B}\rangle} = \mathbf{F}^{*}_{\langle \mathbf{A}_k,\mathbf{B}_k\rangle}$ *and* $J^{(k)*}_{\langle \mathbf{A},\mathbf{B}\rangle} = t_k + J^{*}_{\langle \mathbf{A}_k,\mathbf{B}_k\rangle}$.

Proof: In (22), t_k does not depend on $\mathbf{F}$. Therefore, the equalities hold. ■

Remark 2. $J^{(0)*}_{\langle \mathbf{A},\mathbf{B}\rangle} = J^{*}_{\langle \mathbf{A},\mathbf{B}\rangle}$ *and* $J^{(m)*}_{\langle \mathbf{A},\mathbf{B}\rangle} = t_m$.

C. Nested bounds

Now, using the results of Section V-B, we are ready to develop upper bounds to the original optimization problem (10) defined in Section III. These nested upper bounds are obtained under the assumption that the optimal solution to (10) is calculable for some $k \le m$. The nested upper bounds $\bar{J}_k$ are defined as follows.

Definition.

$$\bar{J}_k \triangleq \max_{\mathbf{C}\in C^{[k\times n]}}\left\{\max_{\substack{\mathbf{F}\in\mathcal{F}^{[(m-k)\times n]}\\ \mathbf{F}\mathbf{C}^T=\mathbf{0}}} J_{\langle \mathbf{\Sigma}_{\underline{X}}^2,\,\mathbf{\Sigma}_{\underline{X}}+\sigma^2\mathbf{I}\rangle}\left(\begin{bmatrix}\mathbf{C}\\ \mathbf{F}\end{bmatrix}\right)\right\} \tag{28}$$

■

From Remark 2, we clearly see that $\bar{J}_0 \ge J(\mathbf{C}^*)$. We next show that $\bar{J}_k \ge J(\mathbf{C}^*)$ for any $k \le m$.

Theorem 2. *For any $k \le m$, we have $\bar{J}_k \ge J(\mathbf{C}^*)$.*

Proof:

$$\begin{aligned} J(\mathbf{C}^*) &= \max_{\mathbf{C}\in C^{[m\times n]}} J(\mathbf{C}) \\ &= \max_{\mathbf{C}_1\in C^{[k\times n]}}\ \max_{\substack{\mathbf{C}_2\in C^{[(m-k)\times n]}\\ \mathbf{C}_2\mathbf{C}_1^T=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{bmatrix}\right) \\ &\le \max_{\mathbf{C}_1\in C^{[k\times n]}}\ \max_{\substack{\mathbf{F}\in\mathcal{F}^{[(m-k)\times n]}\\ \mathbf{F}\mathbf{C}_1^T=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}_1\\ \mathbf{F}\end{bmatrix}\right) = \bar{J}_k, \end{aligned} \tag{29}$$

where the inequality follows from $C^{[(m-k)\times n]} \subset \mathcal{F}^{[(m-k)\times n]}$. ■

Remark 3. *For $k=0$, we have* $\bar{J}_0 = J^*_{\langle \mathbf{\Sigma}_{\underline{X}}^2,\,\mathbf{\Sigma}_{\underline{X}}+\sigma^2\mathbf{I}\rangle} = \sum_{j=1}^{m}\frac{\lambda_j^2}{\lambda_j+\sigma^2}$.

Remark 4. *When $k = m$, we have $\bar{J}_m = J(\mathbf{C}^*)$.*

We now show that the upper bounds are nested.

Theorem 3. *For any $k \le m-1$, we have $\bar{J}_k \ge \bar{J}_{k+1}$.*

Proof:

$$\begin{aligned} \bar{J}_{k+1} &= \max_{\mathbf{C}\in C^{[(k+1)\times n]}}\ \max_{\substack{\mathbf{F}\in\mathcal{F}^{[(m-k-1)\times n]}\\ \mathbf{F}\mathbf{C}^T=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}\\ \mathbf{F}\end{bmatrix}\right) \\ &= \max_{\mathbf{C}_1\in C^{[k\times n]}}\ \max_{\substack{\underline{e}\in C^{[1\times n]}\\ \mathbf{C}_1\underline{e}^T=\mathbf{0}}}\ \max_{\substack{\mathbf{F}\in\mathcal{F}^{[(m-k-1)\times n]}\\ \mathbf{F}\left[\mathbf{C}_1^T\ \ \underline{e}^T\right]=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}_1\\ \underline{e}\\ \mathbf{F}\end{bmatrix}\right) \\ &\le \max_{\mathbf{C}_1\in C^{[k\times n]}}\ \max_{\substack{\underline{f}\in\mathcal{F}^{[1\times n]}\\ \mathbf{C}_1\underline{f}^T=\mathbf{0}}}\ \max_{\substack{\mathbf{F}\in\mathcal{F}^{[(m-k-1)\times n]}\\ \mathbf{F}\left[\mathbf{C}_1^T\ \ \underline{f}^T\right]=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}_1\\ \underline{f}\\ \mathbf{F}\end{bmatrix}\right) \\ &= \max_{\mathbf{C}_1\in C^{[k\times n]}}\ \max_{\substack{\mathbf{F}_1\in\mathcal{F}^{[(m-k)\times n]}\\ \mathbf{F}_1\mathbf{C}_1^T=\mathbf{0}}} J\left(\begin{bmatrix}\mathbf{C}_1\\ \mathbf{F}_1\end{bmatrix}\right) = \bar{J}_k, \end{aligned}$$

where the inequality follows from $C^{[1\times n]} \subset \mathcal{F}^{[1\times n]}$. ■

Corollary 3.1. $\sum_{j=1}^{m}\frac{\lambda_j^2}{\lambda_j+\sigma^2} = \bar{J}_0 \ge \bar{J}_1 \ge \cdots \ge \bar{J}_m = J(\mathbf{C}^*)$.

Proof: Combine Remark 3-4 and Theorem 3. ■

We next want to utilize Theorem 1 (more specifically, Corollary 1.1) to efficiently compute the upper bounds $\bar{J}_k$. To that end, we define the matrix pencil $\langle \mathbf{A}_{(\mathbf{C})}, \mathbf{B}_{(\mathbf{C})}\rangle$ as a permutation of the matrix pencil $\left\langle \mathbf{\Sigma}_{\underline{X}}^2,\ \mathbf{\Sigma}_{\underline{X}}+\sigma^2\mathbf{I}\right\rangle$.

Definition. *Let $\bar{\mathbf{C}}$ denote the* **complement** *of $\mathbf{C}$, with constraints $\bar{\mathbf{C}} \in C^{[(n-m)\times n]}$ and $\bar{\mathbf{C}}\mathbf{C}^T = \mathbf{0}$.* ■

Definition. *We define*

$$\mathbf{A}_{(\mathbf{C})} = \begin{bmatrix}\mathbf{C}\\ \bar{\mathbf{C}}\end{bmatrix}\mathbf{\Sigma}_{\underline{X}}^2\begin{bmatrix}\mathbf{C}\\ \bar{\mathbf{C}}\end{bmatrix}^T \textit{ and } \mathbf{B}_{(\mathbf{C})} = \begin{bmatrix}\mathbf{C}\\ \bar{\mathbf{C}}\end{bmatrix}\left(\mathbf{\Sigma}_{\underline{X}}+\sigma^2\mathbf{I}\right)\begin{bmatrix}\mathbf{C}\\ \bar{\mathbf{C}}\end{bmatrix}^T. \tag{30}$$

■

We now reformulate the upper bounds $\bar{J}_k$ so that it easily relates back to Corollary 1.1.

Corollary 3.2. $\bar{J}_k = \max_{\mathbf{C}\in C^{[k\times n]}} J^{(k)*}_{\langle \mathbf{A}_{(\mathbf{C})},\,\mathbf{B}_{(\mathbf{C})}\rangle}$.

Proof: Let $\mathbf{F} \in \mathcal{F}^{[(m-k)\times(n-k)]}$ and let $\mathbf{F}_1 = \mathbf{F}\bar{\mathbf{C}}$, for any

$\mathbf{C} \in C^{[k\times n]}$. Then, from (12) and (30), it follows that

$$J_{\langle \mathbf{A}_{(\mathbf{C})}, \mathbf{B}_{(\mathbf{C})} \rangle} \left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \right) = J_{\langle \underline{\Sigma}_{\underline{X}}^2, \underline{\Sigma}_{\underline{X}} + \sigma^2 \mathbf{I} \rangle} \left(\begin{bmatrix} \mathbf{C} \\ \mathbf{F}_1 \end{bmatrix} \right). \quad (31)$$

Since $\text{rank}(\mathbf{F}_1) = \text{rank}(\mathbf{F}) = m - k$, and $\mathbf{F}_1 \mathbf{C}^T = \mathbf{0}$, using (28), (21) and (31) we can write

$$\begin{aligned} \bar{J}_k &= \max_{\mathbf{C} \in C^{[k\times n]}} \max_{\substack{\mathbf{F}_1 \in \mathcal{F}^{[(m-k)\times n]} \\ \mathbf{F}_1 \mathbf{C}^T = \mathbf{0}}} J_{\langle \underline{\Sigma}_{\underline{X}}^2, \underline{\Sigma}_{\underline{X}} + \sigma^2 \mathbf{I} \rangle} \left(\begin{bmatrix} \mathbf{C} \\ \mathbf{F}_1 \end{bmatrix} \right) \\ &= \max_{\mathbf{C} \in C^{[k\times n]}} \max_{\mathbf{F} \in \mathcal{F}^{[(m-k)\times(n-k)]}} J_{\langle \mathbf{A}_{(\mathbf{C})}, \mathbf{B}_{(\mathbf{C})} \rangle} \left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \right) \\ &= \max_{\mathbf{C} \in C^{[k\times n]}} J^{(k)*}_{\langle \mathbf{A}_{(\mathbf{C})}, \mathbf{B}_{(\mathbf{C})} \rangle}. \end{aligned}$$

■

For any $k \leq m$, the computation of the upper bound $\bar{J}_k$ requires searching over all $\binom{n}{k}$ matrices $\mathbf{C} \in C^{[k\times n]}$. Therefore, computation of $\bar{J}_k$ has search complexity $O\left(\binom{n}{k}\right)$.

VI. Numerical performance bounds

We performed evaluations on standard IEEE test cases [16][17] to evaluate the performances of the approximate solutions and the family of upper bounds $\bar{J}_k$ (for $k = 0$ to 5). We construct the sample correlation matrix $\mathbf{\Sigma}_{\underline{X}}$ by running traditional SE algorithms 1000 times, where each state estimate is extracted from a single scan of SCADA measurements. We used the MATPOWER package [18] as the traditional state estimator, which assumes that the physical network model, based on bus-section/switching device representation, is exact [19]. In the state estimation process, the standard deviations of voltage magnitudes, bus power injections, and line power flow measurements used are 0.01, 0.015, and 0.02, respectively, according to the default setups in [18]. For simplicity, we consider only the voltage magnitudes of the buses as the state variables. Furthermore, we assume that a PMU is always placed at the reference bus [2], and therefore, the reference bus is not considered in our sensor placement algorithms.

A. IEEE 30-bus system

First, we consider the standard IEEE 30-bus test system [16]. The simulation results are shown in Figure 2, where the upper bounds (developed in Section V) are compared to efficacy of the expedient solution $J(\mathbf{C}_E)$ and the efficacy of the greedy solution $J(\mathbf{C}_G)$. Due to the large number of buses, the exhaustive search for the optimal solution $J(\mathbf{C}^*)$ is prohibitively complex. We observe that the expedient algorithm and the greedy algorithm perform very close to each other. The upper bound $J(\mathbf{F}^*) = \bar{J}_0$ is not tight, but the nested bounds $\bar{J}_k$ become tighter as k increases.

B. IEEE 57-bus system

Next, we consider the IEEE 57-bus system [17] and the simulations results are shown in Figure 3. Again, the size of the system prevents us from calculating the optimal efficacy $J(\mathbf{C}^*)$. Similar to the IEEE 30-bus system, the expedient solution and the greedy solutions are indistinguishable. The

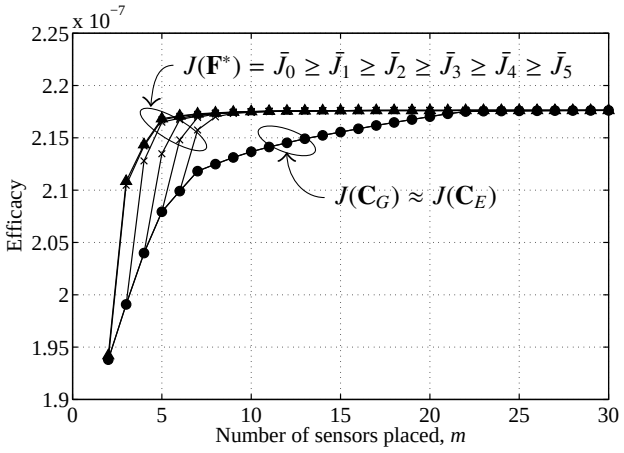

Figure 2. IEEE 30-bus test system: efficacies of approximate solutions and the family of upper bounds $\bar{J}_k$

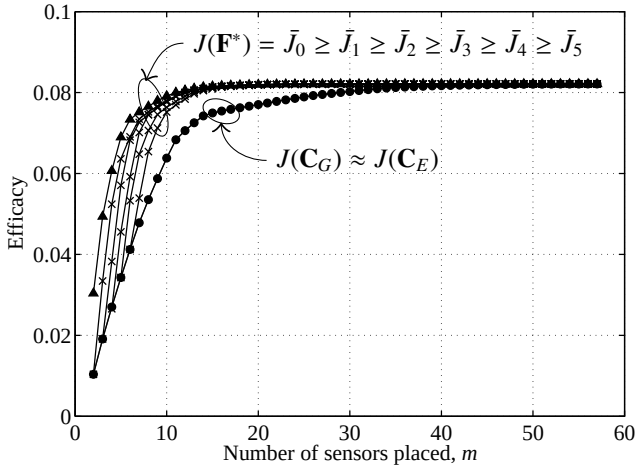

Figure 3. IEEE 57-bus test system: efficacies of approximate solutions and the family of upper bounds $\bar{J}_k$

nested bounds $\bar{J}_k$ also get tighter as k is increased, confirming that the expedient and the greedy solutions are very close to (if not) the optimal solution in this scenario.

VII. Conclusion and Future Work

In this paper, we considered the optimal placement of m PMUs among n bus locations and formulated the optimization problem as an integer programming problem. To understand the performance of the (sub)optimal solutions, we presented a series of nested upper bounds (using generalized eigenvectors and matrix manipulations) that give tighter upper bounds at increasing complexity. These bounds were then numerically evaluated for the IEEE 30- and 57- bus test systems. Some further directions for this research may be finding tighter theoretical bounds when the covariance matrix, $\mathbf{\Sigma}_{\underline{X}}$ has some structure (e.g., diagonally block dom-

inant), or when there are additional constraints (e.g., PMUs are not allowed at certain bus locations).

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References

[1] Y.-F. Huang, S. Werner, J. Huang, N. Kashyap, and V. Gupta, "State estimation in electric power grids: Meeting new challenges presented by the requirements of the future grid," IEEE Signal Process. Mag., vol. 29, no. 5, pp. 33–43, Sep. 2012.

[2] A. G. Phadke and J. S. Thorp, Synchronized Phasor Measurements and Their Applications, Springer, 2008.

[3] A. Abur and A. Gómez-Expósito, Power System State Estimation: Theory and Implementation, New York, NY: Marcel Dekker, Inc., 2004.

[4] D. Brueni and L. Heath, "The PMU placement problem," SIAM Journal on Discrete Mathematics, vol. 19, no. 3, pp. 744–761, Dec. 2005.

[5] Q. Li, R. Negi, and M. D. Ilić, "Phasor measurement units placement for power system state estimation: A greedy approach," in 2011 IEEE PES General Meeting, Detroit, MI, July 2011, pp. 1–8.

[6] V. Kekatos, G. B. Giannakis, and B. Wollenberg, "Optimal placement of phasor measurement units via convex relaxation," IEEE Trans. Power Syst., vol. 27, no. 3, pp. 1521–1530, Aug. 2012.

[7] M. Uddin, A. Kuh, A. Kavcic, and T. Tanaka, "Approximate solutions and performance bounds for the sensor placement problem," in Proc. 2012 Third IEEE Int. Conf. Smart Grid Commun. (SmartGridComm), Tainan, Taiwan, Nov. 2012.

[8] S. Chakrabarti, G. K. Venayagamoorthy, and E. Kyriakides, "PMU Placement for power system observability using binary particle swarm optimization," in Proc. Australasian Universities Power Engineering Conf., Sydney, Australia, Dec. 2008, pp. 1–5.

[9] D. Dua, S. Dambhare, R. K. Gajbhiye, and S. A. Soman, "Optimal multistage scheduling of PMU placement: an ILP approach," IEEE Trans. Power Del., vol. 23, no. 4, pp. 1812–1820, Oct. 2008.

[10] N. M. Manousakis, G. N. Korres, and P. S. Georgilakis, "Taxonomy of PMU placement methodologies," IEEE Trans. Power Syst., vol. 27, no. 2, pp. 1070–1077, May 2012.

[11] A. Monticelli, "Electric power system state estimation," Proc. IEEE, vol. 88, no. 2, pp. 262–282, 2000.

[12] S. M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory, Englewood Cliffs, NJ: PTR Prentice Hall, 1993.

[13] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 3rd ed., Cambridge, MA: MIT Press, 2009.

[14] P.-A. Absil, R. Mahony, and R. Sepulchre, Optimization Algorithms on Matrix Manifolds, New York, NY: Cambridge University Press, 2005.

[15] K. M. Abadir and J. R. Magnus, Matrix Algebra, Princeton, NJ: Princeton University Press, 2008.

[16] R. W. Ferrero, S. M. Shahidehpour, and V. C. Ramesh, "Transaction analysis in deregulated power systems using game theory," IEEE Trans. Power Syst., vol. 12, no. 3, pp. 1340–1347, Aug. 1997.

[17] Power system test case archive. University of Washington. [Online]. Available: http://www.ee.washington.edu/research/pstca/ [retrieved: Feb. 2013].

[18] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education," IEEE Trans. Power Syst., vol. 26, no. 1, pp. 12–19, Feb. 2011.

[19] F. C. Schweppe and J. Wildes, "Power System Static-State Estimation, Parts I–III," IEEE Trans. Power Apparatus and Systems, vol. PAS-89, no. 1, pp. 120–135, Jan. 1970.

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