Solutions of HW4

Problem 3.4:

a). We have following relations:

$$\begin{cases} a_{m,k} = a_{m-1,k} + K_m a^*_{m-1,m-k}, & k=0, 1, \cdots, m \quad \cdots\cdots (1) \\ P_m = P_{m-1}(1-|K_m|^2), & m=1,2,\cdots M. \quad \cdots\cdots (2) \\ K_m = -\frac{1}{P_{m-1}} \sum_{k=0}^{m-1} a_{m-1,k}\, r(k-m), & m=1,2,\cdots M. \quad \cdots\cdots (3) \end{cases}$$

where
$$\begin{cases} a_{m-1,0}=1, \quad a_{m-1,m}=0 \quad \text{and} \quad K_m = a_{mm} \quad \cdots\cdots (4) \\ P_0 = r(0) \end{cases}$$

We also know that
$$\begin{cases} r(0)=1 \\ r(1)=0.8 \\ r(2)=0.6 \\ r(3)=0.4 \end{cases}$$

i) Case 1: $m=1$.

From Eqn. (3), we have: $K_1 = -\frac{1}{P_0} a_{00}\, r(-1) = -\frac{r^*(1)}{r(0)} = -0.8,$

From Eqn. (4), we have $a_{11} = K_1 = -0.8$.

From Eqn. (2), we have $P_1 = P_0(1-|K_1|^2) = r(0)\cdot(1-0.8^2) = 0.36.$

ii) Case 2: $m=2$.

From Eqns. (3) and (4), $K_2 = -\frac{1}{P_1}\left[a_{10}\, r(-2) + a_{11}\, r(-1)\right] = -\frac{1}{P_1}\left[r^*(2) - 0.8\, r^*(1)\right]$

$$= -\frac{1}{0.36}\left[0.6 - 0.8\cdot 0.8\right] = \frac{1}{9} = 0.1111.$$

From Eqn. (2), $P_2 = P_1(1-|K_2|^2) = 0.36\cdot(1-\frac{1}{81}) = \frac{16}{45} = 0.3556$

From Eqn. (1), $a_{21} = a_{11} + K_2 \cdot a^*_{11} = -0.8 - \frac{1}{9}\cdot 0.8 = -\frac{8}{9} = -0.8889.$

and $a_{20} = 1$, $a_{22} = K_2 = \frac{1}{9} = 0.1111.$

iii). Case 3: $m=3$

From Eqn. (3), $$K_3 = -\frac{1}{P_2}\left[a_{20}\,r(-3) + a_{21}\,r(-2) + a_{22}\,r(-1)\right]$$

$$= -\frac{1}{P_2}\left[a_{20}\,r^*(3) + a_{21}\,r^*(2) + a_{22}\,r^*(1)\right]$$

$$= -\frac{45}{16} * \left[1\cdot 0.4 + (-\tfrac{8}{9})\cdot 0.6 + \tfrac{1}{9}\cdot 0.8\right]$$

$$= \frac{1}{8} = 0.125.$$

From Eqn. (2), $$P_3 = P_2\,(1-|K_3|^2) = \frac{16}{45}\cdot(1-\frac{1}{64}) = \frac{7}{20} = 0.35$$

From Eqn. (1), $$a_{31} = a_{21} + K_3\,a_{22}^* = -\frac{8}{9} + \frac{1}{8}\cdot\frac{1}{9} = -\frac{7}{8} = -0.875$$

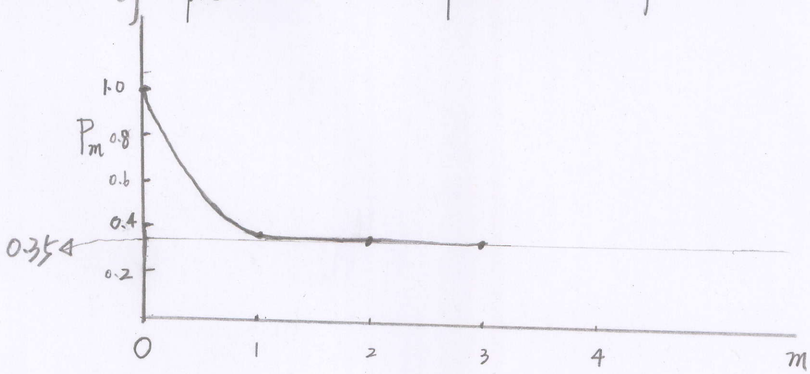
$$a_{32} = a_{22} + K_3\,a_{21}^* = \frac{1}{9} + \frac{1}{8}\cdot(-\frac{8}{9}) = 0.$$

and $a_{30} = 1$, $a_{33} = K_3 = 0.125$.

Therefore: $K_1 = -0.8$, $K_2 = 0.1111$, $K_3 = 0.125$.

C). From Part a), we have the average power of the prediction error for each stage as $P_1 = 0.36$, $P_2 = 0.3556$, $P_3 = 0.35$. And $P_0 = r(0) = 1$.

Then the picture of prediction-error power v.s. prediction order is



From the above plot, we find that the average power P_m decreases with the prediction order m.

Problem 3.5.

From Fig. P3.1, we know that the estimation error is

$$e(n) = u(n) - y(n) \quad \text{where } y(n) = \underline{w}^H \cdot \underline{u}(n-\Delta) \text{ and } \underline{u}(n-\Delta) = [u(n-\Delta), u(n-\Delta-1), \cdots, u(n-\Delta-M)]^T.$$

The mean-square value of the estimation error is

$$J \triangleq E[|e(n)|^2] = E[(u(n) - \underline{w}^H \underline{u}(n-\Delta)) \cdot (u(n) - \underline{w}^H \underline{u}(n-\Delta))^*]$$

$$= E[(u(n) - \underline{w}^H \underline{u}(n-\Delta)) \cdot (u^*(n) - \underline{u}^H(n-\Delta) \cdot \underline{w})]$$

$$= E[u(n) u^*(n)] - \underline{w}^H E[\underline{u}(n-\Delta) \cdot u^*(n)] - E[u(n) \underline{u}^H(n-\Delta)] \cdot \underline{w} + \underline{w}^H E[\underline{u}(n-\Delta) \underline{u}^H(n-\Delta)] \cdot \underline{w}$$

$$= r(0) - \underline{w}^H \underline{r}_\Delta - \underline{r}_\Delta^H \underline{w} + \underline{w}^H R \underline{w}$$

where i) $\underline{r}_\Delta = E[\underline{u}(n-\Delta) \cdot u^*(n)] = \begin{bmatrix} r(-\Delta) \\ r(-\Delta-1) \\ \vdots \\ r(-\Delta-M) \end{bmatrix}$

ii) $R = E[\underline{u}(n-\Delta) \cdot \underline{u}^H(n-\Delta)] = \begin{bmatrix} r(0) & r(1) & \cdots & r(M) \\ r^*(1) & r(0) & \cdots & r(M-1) \\ \vdots & \vdots & & \vdots \\ r^*(M) & r^*(M-1) & \cdots & r(0) \end{bmatrix}$.

To minimize the mean-square value of the estimation error, we set

$$\frac{\partial J}{\partial \underline{w}} = -2 \cdot \underline{r}_\Delta + 2 R \underline{w} = \underline{0}.$$

Hence, the optimum value of $\underline{w}(n)$ is

$$\underline{w}_o = R^{-1} \underline{r}_\Delta.$$

Problem 3.6.

For the first-order AR process $u(n)$

$$u(n) = 0.9\,u(n-1) + v(n),$$

we have: $1 = \sigma_v^2 = r(0) - 0.9\,r(1)$ and $0.9\,r(0) = r(1)$ where $r(0) \triangleq E[u(n)u^*(n)]$, $r(1) \triangleq E[u(n)u^*(n-1)]$.

Then $1 = r(0) - 0.9^2 r(0) \Rightarrow r(0) = \frac{100}{19} \doteq 5.2632$

$$\Rightarrow r(1) = 0.9\,r(0) = \frac{90}{19} \approx 4.7368$$

$$\Rightarrow r(2) = 0.9\,r(1) = \frac{81}{19} \doteq 4.2632.$$

a). From the relationship of the AR modeling and the prediction-error filtering, we know that the tap weights of the forward prediction-error filter are

$$a_{2,1} = -w_{01} = -0.9$$

$$a_{22} = -w_{02} = 0.$$

where $u(n) = 0.9\,u(n-1) + v(n)$

$$\triangleq \sum_{k=1}^{2} w_{0k}^* u(n-k) + v(n).$$

We also have the average power of predition error $P_2 = \sigma_v^2 = 1$.

b). The reflection coefficient κ_2 satisfies

$$\kappa_2 = a_{22} = 0.$$

From the Levinson-Durbin recursion $a_{m,k} = a_{m-1,k} + \kappa_m a^*_{m-1,m-k}$,

we have $a_{2,1} = a_{1,1} + \kappa_2 a^*_{11}$.

Since $\kappa_1 = a_{1,1}$, we have $\kappa_1 + \kappa_2 \kappa_1^* = a_{2,1} = -0.9 \Rightarrow \kappa_1 = a_{2,1} = -0.9$.

From parts (a) and (b), we find that for the first-order AR process, it is reasonable to use a first-order predictor for the representation of this process.

Problem 3. X1:

The process $s(n)$ is an AR process of order 1 with $a_1=0.2$ and $\sigma_v=1$, that is,

$$v(n) = s(n) + a_1 s(n-1) \quad ; \quad a_1 = 0.2$$

From the observations of signal $x(n) = s(n) + w(n)$,

the estimation of $s(n)$ using Wiener filter can be expressed as

$$\hat{s}(n) = \sum_k w_k \, x(n-k) = \underline{w}^H \cdot \underline{x} \, .$$

Then, the estimation error and the mean-square value of error are

$$e(n) \triangleq s(n) - \hat{s}(n) \, ,$$

$$J \triangleq E[|e(n)|^2] = E[e(n)\, e^*(n)] \, .$$

Furthermore,

$$J = E\left[\left(s(n) - \sum_k w_k x(n-k)\right)\left(s^*(n) - \sum_k w_k^* x^*(n-k)\right)\right]$$

$$= E\left[s(n) s^*(n)\right] - \sum_k w_k E[x(n-k) s^*(n)] - \sum_k E[s(n) x^*(n-k)] w_k^* + \sum_k \sum_l w_k E[x(n-k) x^*(n-l)] w_l^*$$

$$\Rightarrow \frac{\partial J}{\partial w_k} = -2 E[x(n-k) s^*(n)] + 2 \sum_l E[x(n-k) x^*(n-l)] w_l^* \, .$$ To minimize J,

let $\frac{\partial J}{\partial w_k} = 0$. Then:

$$\sum_l E[x(n-k) x^*(n-l)] w_{0,l}^* = E[x(n-k) s^*(n)] \quad \text{for all } k,$$

$$\Leftrightarrow \sum_l r_x(l-k)\, w_{0,l}^* = p(-k) \quad \text{where } p(-k) = E[x(n-k) s^*(n)] \quad \text{------ } (*)$$

1). For non-causal case, we have

$$\sum_{l=-\infty}^{+\infty} r_x(l-k)\, w_{0,l}^* = p(-k) \quad \Leftrightarrow \quad \sum_{l=-\infty}^{+\infty} w_{0,l}^* \, r_x^*(k-l) = p(-k) \quad \text{----} (**)$$

Applying Z-transform to Eqn. (**), we have

$$\left(\sum_{l=-\infty}^{+\infty} w_{0,l}^* z^{-l}\right) \cdot \left(\sum_{j=-\infty}^{+\infty} r_x^*(j) z^{-j}\right) = \sum_{k=-\infty}^{+\infty} p(-k) z^{-k} \quad \text{--------} (***)$$

Define:

$$\begin{cases} S_x(z) = \sum_{j=-\infty}^{+\infty} r_x(j) \cdot z^{-j} & \Rightarrow S_x(\frac{1}{z}) = \sum_{j=-\infty}^{+\infty} r_x^*(j) z^{-j} \quad \text{------------ (1)} \\ P_{xs}(z) = \sum_{k=-\infty}^{+\infty} p(-k) z^{-k} \\ H_s(z) = \sum_{l=-\infty}^{+\infty} w_{o,l}^* z^{-l} \end{cases}$$

Hence, From Eqn. (***), we have

$$H_s(z) \cdot S_x(\tfrac{1}{z}) = P_{xs}(z).$$

$\Rightarrow H_s(z) = P_{xs}(z) \Big/ S_x(\frac{1}{z})$. This is the required non-causal filter.

Next, we need to compute $P_{xs}(z)$ and $S_x(\frac{1}{z})$.

- From the first-order AR process $s(n) + a_1 s(n-1) = v(n)$, $a_1 = 0.2$,

we have: $1 = \sigma_v^2 = r_s(0) + a_1 r_s(1)$ and $-a_1 \cdot r_s(0) = r_s^*(1)$.

$$\Rightarrow r_s(0) = \frac{\sigma_v^2}{1-a_1^2} = \frac{1}{1-0.2^2} = 1.041\overline{7}$$

$$r_s(1) = r_s^*(1) = r_s(-1) = -a_1 \cdot r_s(0) = (-0.2) \cdot 1.041\overline{7}$$

Recursively,

$$r_s(k) = r_s^*(k) = r_s(-k) = (-a_1) \cdot r_s(k-1) = \cdots = (-a_1)^k \cdot r_s(0) = (-0.2)^k \cdot 1.041\overline{7} \qquad k > 0$$

Hence for any $k \in \{-\infty, \cdots -1, 0, +1, \cdots +\infty\}$,

$$r_s(k) = (-0.2)^{|k|} \cdot 1.041\overline{7}.$$

From $x(n) = s(n) + w(n)$ where $w(n)$ is independent of $s(n)$ and $r_w(k) = 0.5^{|k|}$,

we have:

$$r_x(k) = E[x(n) x^*(n-k)] = E[(s(n)+w(n))(s^*(n-k) + w^*(n-k))]$$

$$= E[s(n) s^*(n-k)] + E[w(n) w^*(n-k)]$$

$$= r_s(k) + r_w(k)$$

and $p(-k) = E[x(n-k) s^*(n)] = E[(s(n-k) + w(n-k)) s^*(n)] = E[s(n-k) s^*(n)] = r_s(-k) = r_s^*(k) = r_s(k)$

$$\Rightarrow_{(1)} S_x(z) = \sum_{j=-\infty}^{+\infty} r_x(j) z^{-j} = \sum_{j=-\infty}^{+\infty} (r_s(j) + r_w(j)) z^{-j} = \sum_{j=-\infty}^{+\infty} 1.041\overline{7} \cdot (-0.2)^{|j|} \cdot z^{-j} + \sum_{j=-\infty}^{+\infty} 0.5^{|j|} \cdot z^{-j}$$

$$= 1.041\overline{7} \cdot \frac{0.96}{(1+0.2z^{-1})(1+0.2z)} + \frac{0.75}{(1-0.5z^{-1})(1-0.5z)}$$

$$\therefore \quad S_x(z) = \frac{1}{(1+0.2z^{-1})(1+0.2z)} + \frac{0.75}{(1-0.5z^{-1})(1-0.5z)} = S_x\left(\frac{1}{z}\right)$$

$$(*) \cdots\cdots = \frac{a(1-bz^{-1})(1-bz)}{(1+0.2z^{-1})(1+0.2z)(1-0.5z^{-1})(1-0.5z)}, \quad \text{where } b = \frac{29-\sqrt{141}}{10}, \quad a = \frac{0.35}{b} = \frac{3.5}{29-\sqrt{141}}$$

ii) $$P_{xs}(z) = \sum_{k=-\infty}^{+\infty} p(-k)\, z^{-k} = \sum_{k=-\infty}^{+\infty} r_s(k)\, z^{-k} = \sum_{k=-\infty}^{+\infty} 1.0417 \cdot (-0.2)^{|k|} z^{-k}$$

$$= \frac{1}{(1+0.2z^{-1})(1+0.2z)}$$

$\Rightarrow$ The non-causal filter is

$$H_s(z) = P_{xs}(z) \Big/ S_x\left(\frac{1}{z}\right) = P_{xs}(z) \Big/ S_x(z) = \frac{\dfrac{1}{(1+0.2z^{-1})(1+0.2z)}}{\dfrac{1}{(1+0.2z^{-1})(1+0.2z)} + \dfrac{0.75}{(1-0.5z^{-1})(1-0.5z)}}$$

$$= \frac{(1-0.5z^{-1})\cdot(1-0.5z)}{(1-0.5z^{-1})(1-0.5z) + 0.75(1+0.2z^{-1})(1+0.2z)} = \frac{(1-0.5z^{-1})(1-0.5z)}{a(1-bz^{-1})(1-bz)}$$

$$\left(a = \frac{3.5}{29-\sqrt{141}}, \quad b = \frac{29-\sqrt{141}}{10} \right)$$

$$\Downarrow$$

$a \approx 1.9677$

$b \approx 0.1779$

2) For the causal case:

From the discussion in Page 6, we have

$$r_x(k) = r_s(k) + r_w(k) = (-0.2)^{|k|} \cdot r_s(0) + 0.5^{|k|}$$

$$= (-0.2)^{|k|} \cdot 1.0417 + 0.5^{|k|},$$

and $$p(-k) = r_s(k) = (-0.2)^{|k|} \cdot 1.0417.$$

$\Rightarrow$ The process $\{x(n)\}$ is not white.

So we first whiten $\{x(n)\}$ as $\quad x(n) \longrightarrow \boxed{\dfrac{1}{H_{CA}(z)}} \xrightarrow{v(n)}$

where $H_{CA}(z)$ is the filter through which the process $\{x(n)\}$ is obtained by filtering the white noise $v(n)$, i.e. $\quad v(n) \longrightarrow \boxed{H_{CA}(z)} \xrightarrow{x(n)}$.

$$\Rightarrow S_x(z) = \sigma_v^2 H_{CA}(z) H_{CA}\left(\frac{1}{z^*}\right)^*.$$

From Eqn. $(*)$ in the first two lines in this page $\quad S_x(z) = \dfrac{a(1-bz^{-1})(1-bz)}{(1+0.2z^{-1})(1+0.2z)(1-0.5z^{-1})(1-0.5z)}$

Let $\sigma_v^2 = a$, $H_{CA}(z) = \dfrac{1-bz^{-1}}{(1+0.2z^{-1})(1-0.5z^{-1})}$. Then $H_{CA}\left(\dfrac{1}{z^*}\right)^* = H_{CA}\left(\dfrac{1}{z}\right) = \dfrac{1-bz}{(1+0.2z)(1-0.5z)}$.

After whitening $x(n)$, the white process $v(n)$ is applied to the filter

$$\sum_{l=0}^{\infty} r_v(l-k)\, w_{0,l}^* = p'(-k) \quad , \quad p'(-k) = E[v(n-k)\, s^*(n)] \quad , k=0,1,2,\cdots$$

$$p'(-k) = \sum_{l=0}^{\infty} r_v(l-k)\, w_{0,l}^* = r_v(0)\, w_{0,k}^* = \sigma_v^2 w_{0,k}^* \qquad (\because\ v(n) \text{ is white})$$

$$\Rightarrow w_{0,k}^* = \frac{p'(-k)}{\sigma_v^2}$$

$$\Rightarrow H_s'(z) = \frac{1}{\sigma_v^2} P_{vs}'(z) \quad \text{where} \quad \begin{cases} P_{vs}'(z) = \sum_{k=0}^{+\infty} p'(-k) z^{-k} \\ H_s'(z) = \sum_{k=0}^{\infty} w_{0,k}^* z^{-k} \end{cases}$$

similar to Eqn. (1) in Page 6.

Moreover,

$$p(k) = E[x(n-k)\, s^*(n)] = E\left[\left[\sum_l h_{CA,l} \cdot v(n-k-l)\right] \cdot s^*(n)\right] = \sum_l h_{CA,l}\, E[v(n-k-l) \cdot s^*(n)]$$

$$= \sum_l h_{CA,l} \cdot p'(-k-l) = h_{CA} * p'$$

$$\Downarrow \quad P_{xs}(z) = H_{CA}\left(\frac{1}{z^*}\right)^* \cdot P_{vs}(z)$$

where

$$\begin{cases} H_{CA}(z) = \sum_{k=-\infty}^{\infty} h_{CA,k}^* z^{-k} \quad \text{or} \quad H_{CA}\left(\frac{1}{z^*}\right)^* = \sum_{k=-\infty}^{+\infty} h_{CA,k} z^{k}, \\ P_{vs}(z) = \sum_{k=-\infty}^{\infty} p'(-k) z^{-k} \quad \left(P_{vs}'(z) = [P_{vs}(z)]_+\right) \end{cases}$$

$$\Downarrow \quad H_s'(z) = \frac{1}{\sigma_v^2} \left[\frac{P_{xs}(z)}{H_{CA}\left(\frac{1}{z^*}\right)^*}\right]_+$$

Combinining the part of whiterning, we have the final filter

$$H_s(z) = \frac{1}{H_{CA}(z)} H_s'(z) = \frac{1}{\sigma_v^2 \cdot H_{CA}(z)} \cdot \left[\frac{P_{xs}(z)}{H_{CA}\left(\frac{1}{z^*}\right)^*}\right]_+$$

$$= \frac{1}{a} \cdot \frac{(1+0.2z^{-1})(1-0.5z^{-1})}{1-bz^{-1}} \cdot \frac{1-0.5\frac{1+2b}{b+5}}{1+0.2z^{-1}} = \frac{1-0.5\frac{1+2b}{b+5}}{a} \cdot \frac{1-0.5z^{-1}}{1-bz^{-1}} \doteq 0.4417 \cdot \frac{1-0.5z^{-1}}{1-0.1779z^{-1}}.$$

$$\left(\because \frac{P_{xs}(z)}{H_{CA}\left(\frac{1}{z^*}\right)^*} = \frac{1}{(1+0.2z^{-1})(1+0.2z)} \cdot \frac{(1+0.2z)(1-0.5z)}{1-bz} = \frac{1-0.5z}{(1+0.2z^{-1})(1-bz)} = \frac{1-0.5\frac{1+2b}{b+5}}{1+0.2z^{-1}} + \frac{2.5\frac{1+2b}{b+5} \cdot z}{1-bz}\right.$$

$$= \frac{1-0.5\frac{1+2b}{b+5}}{1+0.2z^{-1}} + \frac{2.5\frac{1+2b}{b+5}}{b \cdot (1-\frac{1}{b}z^{-1})}, \quad (b<1)$$

$$\left.\Downarrow \left[\frac{P_{xs}(z)}{H_{CA}\left(\frac{1}{z^*}\right)^*}\right]_+ = \frac{1-0.5\frac{1+2b}{b+5}}{1+0.2z^{-1}}\right)$$