# Solutions of HW5

## Problem 3.9:

From the definition of $\Delta_{m-1}$, we know

$$\Delta_{m-1} = \underline{r}_m^{BT} \underline{a}_{m-1} = \sum_{l=0}^{m-1} a_{m-1,l}\, r(l-m).$$

Since $r(l-m) = E[u(n-m)\, u^*(n-l)]$, we have

$$\Delta_{m-1} = \sum_{l=0}^{m-1} a_{m-1,l} \cdot E[u(n-m)\, u^*(n-l)] = E\left[u(n-m) \cdot \left(\sum_{l=0}^{m-1} a_{m-1,l} \cdot u^*(n-l)\right)\right]$$

$$= E\left[u(n-m)\, f_{m-1}^*(n)\right] \quad \left(\because f_{m-1}(n) = \sum_{l=0}^{m-1} a_{m-1,l}^*\, u(n-l)\right)$$

$$= E\left[\left(b_{m-1}(n-1) + \hat{u}(n-m \mid \mathcal{U}_{n-1})\right) f_{m-1}^*(n)\right]$$

($\because$ the delayed backward prediction error is $b_{m-1}(n-1) = u(n-m) - \hat{u}(n-m \mid \mathcal{U}_{n-1})$ where $\mathcal{U}_{n-1}$ is the ~~(n-1)-dim~~ space spanned by $u(n-1), u(n-2), \cdots, u(n-m+1)$)

$$= E\left[\left(b_{m-1}(n-1) + \sum_{k=1}^{m-1} w_{b,k}^*\, u(n-k)\right) f_{m-1}^*(n)\right]$$

$$= E[b_{m-1}(n-1) \cdot f_{m-1}^*(n)] + \sum_{k=1}^{m-1} w_{b,k}^*\, E[u(n-k)\, f_{m-1}^*(n)]$$

$$= E[b_{m-1}(n-1)\, f_{m-1}^*(n)].$$

The above last equality holds since

$$E[u(n-k) f_{m-1}^*(n)] = E\left[u(n-k) \left(\sum_{l=0}^{m-1} a_{m-1,l}\, u^*(n-l)\right)\right]$$

$$= \sum_{l=0}^{m-1} a_{m-1,l} \cdot r(l-k)$$

$$= 0 \quad \text{for } k = 1, 2, \cdots, m-1. \qquad \#$$

## Problem 3.11:

For the second-order AR process $u(n) = u(n-1) - 0.5u(n-2) + v(n)$, we have that

i) the AR coefficients are $a_1 = -1$, $a_2 = 0.5$,

ii) $0.5 = \sigma_v^2 = r(0) - r(1) + 0.5\, r(2)$

iii) Yule Walker equations are:
$$\begin{pmatrix} r(0) & r(1) \\ r^*(1) & r(0) \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \end{pmatrix} = \begin{pmatrix} r^*(1) \\ r^*(2) \end{pmatrix}$$

$\Rightarrow r(0) = 1.2$, $r(1) = \frac{2}{3} r(0) = 0.8$, $r(2) = \frac{1}{6} r(0) = 0.2$.

(a). The average power of $u(n)$ is $P_0 = r(0) = 1.2$

(b) From the relations of the AR modeling and the prediction-error filtering, we known that the tap weights (for oder 2) are

$$a_{2,0} = 1, \quad a_{2,1} = a_1 = -1, \quad a_{22} = a_2 = 0.5.$$

Using the inverse Levinson-Durbin recursion,

$$a_{m-1,k} = \frac{a_{m,k} - a_{m,m}\, a^*_{m,m-k}}{1 - |a_{m,m}|^2}, \quad k = 0, \cdots, m$$

we have:

$$a_{1,1} = \frac{a_{2,1} - a_{2,2}\, a^*_{21}}{1 - a^2_{2,2}} = \frac{-1 - 0.5\cdot(-1)}{1 - 0.5^2} = -\frac{2}{3}$$

$$a_{1,0} = 1.$$

Hence: the reflection coefficients are

$$\kappa_1 = a_{1,1} = -\frac{2}{3}$$

$$\kappa_2 = a_{2,2} = \frac{1}{2}$$

(c): From the recursion $P_m = P_{m-1}(1 - |\kappa_m|^2)$, we have the average prediction-error powers as:

$$P_0 = r(0) = 1.2$$

$$P_1 = P_0(1 - \kappa_1^2) = 1.2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = \frac{2}{3}$$

$$P_2 = P_1(1 - \kappa_2^2) = \frac{2}{3}\left[1 - \left(\tfrac{1}{2}\right)^2\right] = \frac{1}{2} \quad . \ \#$$

Problem 3.12:

Using the one-to-one correspondence

$$r(m) = -\kappa^*_m P_{m-1} - \sum_{k=1}^{m-1} a^*_{m-1,k}\, r(m-k),$$

we have

$$r(1) = -\kappa_1^* P_0 = +\frac{2}{3}\cdot 1.2 = +0.8$$

$$r(2) = -\kappa_2^* P_1 - a^*_{1,1} r(1) = -\frac{1}{2}\cdot\frac{2}{3} - \left(-\frac{2}{3}\right)\cdot(+0.8) = 0.2 \ , \ \#$$

Problem 13:

(a). From Fig. P3.2, the transfer function of the forward prediction-error filter is

$$H_{f,M}(z) = (1 - z_i z^{-1})\, C_i(z) \qquad \text{where } z_i = \rho_i e^{j\omega_i}.$$

Let $S_f(\omega)$ and $S_u(\omega)$ be the power spectral densities of the prediction-error $f_M(n)$ and the input process $u(n)$, respectively. Then

$$S_f(\omega) = \left| H_{f,M}(z) \right|^2 S_u(\omega) \qquad \text{where } z = e^{j\omega},\ j = \sqrt{-1}.$$

Hence, the mean-square value of the prediction error $f_M(n)$ is

$$\varepsilon = \int_{-\pi}^{\pi} S_f(\omega)\, d\omega = \int_{-\pi}^{\pi} \left| H_{f,M}(e^{j\omega}) \right|^2 S_u(\omega)\, d\omega$$

$$= \int_{-\pi}^{\pi} \left| 1 - \rho_i e^{j\omega_i} e^{-j\omega} \right|^2 \cdot \left| C_i(e^{j\omega}) \right|^2 S_u(\omega)\, d\omega$$

$$= \int_{-\pi}^{\pi} S_u(\omega) \left| C_i(e^{j\omega}) \right|^2 \cdot (1 - \rho_i e^{j\omega_i} e^{-j\omega})(1 - \rho_i e^{-j\omega_i} e^{j\omega})\, d\omega$$

$$= \int_{-\pi}^{\pi} S_u(\omega) \left| C_i(e^{j\omega}) \right|^2 \cdot [1 - 2\rho_i \cos(\omega - \omega_i) + \rho_i^2]\, d\omega.$$

Differentiating $\varepsilon$ w.r.t. $\rho_i$,

$$\frac{\partial \varepsilon}{\partial \rho_i} = \int_{-\pi}^{\pi} S_u(\omega) \left| C_i(e^{j\omega}) \right|^2 [-2\cos(\omega - \omega_i) + 2\rho_i]\, d\omega$$

$$= 2 \int_{-\pi}^{\pi} S_u(\omega) \left| C_i(e^{j\omega}) \right|^2 (\rho_i - \cos(\omega - \omega_i))\, d\omega. \qquad \text{----} \quad (*)$$

(b). If $\rho_i > 1$ (equivalently, $z_i = \rho_i e^{j\omega_i}$ lies outside the unit circle), we have

$$\rho_i - \cos(\omega - \omega_i) > 0 \ \Rightarrow\ \frac{\partial \varepsilon}{\partial \rho_i} > 0 \quad \text{(see Eqn. (*))}.$$

If the filter operates optimally, its parameters must be chosen in such a way that $\frac{\partial \varepsilon}{\partial \rho_i} = 0$.

Hence it is not possible for $\rho_i > 1$ ~~satisfying~~ optimality condition $\frac{\partial \varepsilon}{\partial \rho_i} = 0$.

This means that the transfer function of a forward prediction-error filter has no zeros outside the unit circle.

Problem 3.19

(a) The lower triangular matrix $L$ defined in Eqn. (3.106) is

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{1,1} & 1 & 0 & \cdots & 0 \\ a_{2,2} & a_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{M,M} & a_{M,M-1} & a_{M,M-2} & \cdots & 1 \end{bmatrix} \triangleq \begin{bmatrix} \underline{a}_0^T \\ \underline{a}_1^T \\ \vdots \\ \underline{a}_M^T \end{bmatrix}$$

The correlation matrix $R$ is

$$R = \begin{bmatrix} r(0) & r(1) & \cdots & r(M) \\ r(-1) & r(0) & \cdots & r(M-1) \\ \vdots & \vdots & \ddots & \vdots \\ r(-M) & r(-M+1) & \cdots & r(0) \end{bmatrix} \triangleq [\underline{r}_0 \ \underline{r}_1, \cdots \underline{r}_M]$$

Let $Y = LR$.

Hence, the mk th element of $Y$ is

$$y_{mk} = \underline{a}_m^T \underline{r}_k = \sum_{l=0}^{m} a_{m,m-l} \cdot r(k-l) \qquad \begin{pmatrix} m=0,1,\cdots M \\ k=0,1,\cdots M \end{pmatrix}$$

where $a_{m,0} = 1$.

$$\Rightarrow y_{mm} = \sum_{l=0}^{m} a_{m,m-l}\, r(m-l) \qquad m=0,1,\cdots M$$

According to the augmented Wiener-Hopf equation for backward prediction,

we have
$$\sum_{l=0}^{m} a^*_{m,m-l}\, r(l-i) = \begin{cases} 0, & i=0,\cdots,m-1 \\ P_m, & i=m \end{cases}$$

$$\Downarrow \quad \sum_{l=0}^{m} a_{m,m-l}\, r^*(l-i) = \sum_{l=0}^{m} a_{m,m-l}\, r(i-l) = \begin{cases} 0, & i=0,\cdots m-1, \\ P_m, & i=m \end{cases}$$

$$\Rightarrow y_{mm} = P_m \qquad m=0,1,\cdots M$$

(b)

(b). The mth column of matrix $Y$ is

$$(*) \text{---} \ \underline{y}_m = \begin{bmatrix} y_{0m} \\ y_{1m} \\ \vdots \\ y_{mm} \\ y_{m+1\,m} \\ \vdots \\ y_{Mm} \end{bmatrix} = \begin{bmatrix} r(m) \\ \sum_{l=0}^{1} a_{1,1-l}\, r(m-l) \\ \vdots \\ \sum_{l=0}^{m} a_{m,m-l}\, r(m-l) \\ \sum_{l=0}^{m+1} a_{m+1,m+1-l}\, r(m-l) \\ \vdots \\ \sum_{l=0}^{M} a_{M,M-l}\, r(m-l) \end{bmatrix} = \begin{bmatrix} r(m) \\ \sum_{l=0}^{1} a_{1,1-l}\, r(m-l) \\ \vdots \\ \sum_{l=0}^{m} a_{m,m-l}\, r(m-l) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(From the augmented Wiener-Hopf equations $\sum_{l=0}^{m+i} a_{m+i,\,m+i-l} \cdot r(m-l) = 0$, $i = 1, 2, \cdots, (M-m)$ )

The transfer functions of a backward prediction-error filter are given by

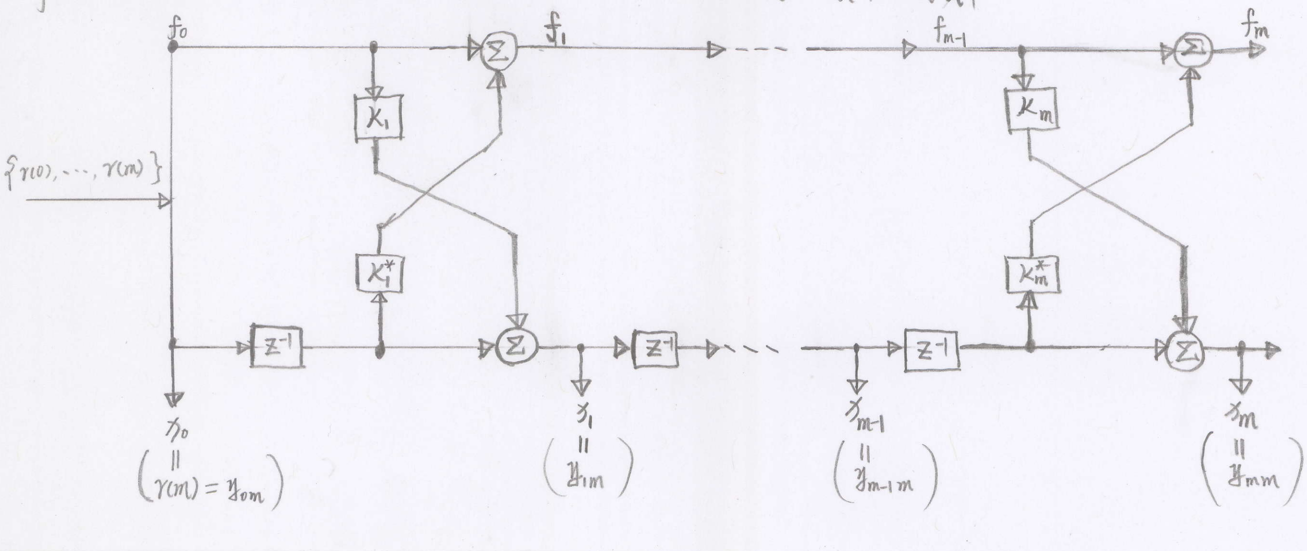
$$(**) \text{---} \ H_{b,i}(z) = \sum_{l=0}^{i} a_{i,i-l}\, z^{-l} \qquad i = 0, 1, \cdots m.$$

From $(*)$, we find that $y_{km}$ can be considered as the output produced by the backward prediction-error with tap weights $\underline{a}_k^* = [a_{k,k}^*, a_{k,k-1}^*, \cdots a_{k,0}^*]^T$ and inputs $\underline{r}_k(m) \triangleq [r(m), r(m-1), \cdots r(m-k)]^T$

And $y_{km}$ is the inner product of $\underline{a}_k$ and $\underline{r}_k(m)$.

Hence, the mth column $\underline{y}_m$ of $Y$ is obtained by passing the sequence $\{r(0), r(1), \cdots, r(m)\}$ through the sequence of the backward prediction-error filter respresented by the transfer functions $\{H_{b,0}(z), H_{b,1}(z), \cdots, H_{b,m}(z)\}$.

c) We apply the autocorrelation sequence $\{r(0), r(1), \cdots r(m)\}$ to the input of a lattice predictor of order $m$, shown as the following figure:
$$\begin{cases} \gamma_l = \gamma_{l-1} + k_l f_{l-1} \\ f_l = f_{l-1} + k_l^* \gamma_{l-1} \end{cases}$$



Let

$$\underline{c}_l^B \triangleq [a_{l,l} \quad a_{l,l-1} \quad \cdots \quad a_{l,1} \quad a_{l,0}]^T \qquad (a_{l,0}=1)$$

$$\underline{c}_l \triangleq [a_{l,0} \quad a_{l,1}, \quad \cdots \quad a_{l,l-1} \quad a_{l,l}]^T \qquad \text{and } \underline{r}_l(m) \triangleq [r(m),\ r(m-1),\ \cdots\ r(m-l)]^T$$

From parts (a) and (b), we have $y_{l,m} = \underline{c}_l^{BT} \underline{r}_l(m)$ ----- (1)

We also set $f_{l,m} \triangleq \underline{c}_l^H \underline{r}_l(m)$. ----- (2)

According to the Levinson-Durbin recursions, we have

$$\begin{cases} \underline{c}_l = \begin{bmatrix} \underline{c}_{l-1} \\ 0 \end{bmatrix} + k_l \begin{bmatrix} 0 \\ \underline{c}_l^{B*} \end{bmatrix} & \text{---- (3)} \\ \underline{c}_l^{B*} = \begin{bmatrix} 0 \\ \underline{c}_{l-1}^{B*} \end{bmatrix} + k_l^* \begin{bmatrix} \underline{c}_{l-1} \\ 0 \end{bmatrix} & \text{---- (4)} \end{cases}$$

Combining Eqns. (1), (2), (3) and (4). we have

$$\begin{cases} f_{l,m} = f_{l-1,m} + k_l^* y_{l-1,m} \\ y_{l,m} = y_{l-1,m} + k_l f_{l-1,m} \end{cases}$$

Observing the lattice predictor shown in the figure at the bottom of the previous page.
we can see that the variables appearing at the various points on the lower line of the predictor are $\gamma_l = y_{l,m}$ $(l=0, 1, 2, \cdots, m)$.

d) From Eqn. (2) in part c), we know that the upper output of stage $m$ in the predictor at time $m+1$ is:

$$f_{m,m+1} = \underline{c}_m^H \underline{r}_m(m+1) = \sum_{k=0}^{m} a_{m,k}^* \cdot r(m+1-k)$$

Recalling the definition of $\Delta_m$, we have

$$\Delta_m^* = \sum_{l=0}^{m} a_{m,l}^* \, r(m+1-l) = f_{m,m+1}.$$

From Eqn. (1) in part c), we know that the lower output of stage $m$ in the predictor at time $m$ is

$$y_{m,m} = \underline{c}_m^{BT} \underline{r}_m(m) = \sum_{l=0}^{m} a_{m,m-l}\, r(m-l) = P_m \quad \text{as shown in part (a).}$$

The ratio of $\dfrac{\Delta_m^*}{P_m}$ satisfies: $\dfrac{\Delta_m^*}{P_m} = -k_{m+1}^*$ ($P_m$ is real),

where $k_{m+1}$ is the reflection coefficient of stage $m+1$.

(e). From the result of part (d), we can compute the reflection coefficients as the following recursive procedure:

i) for $m=0$, $\quad -\kappa_1^* = \dfrac{r(1)}{r(0)}$ since the upper output at time 1 and the lower output at time 0 are $r(1)$ and $r(0)$, respectively.

ii) for $m \geq 1$. $\quad -\kappa_{m+1}^* = \dfrac{\Delta_m^*}{P_m}$ since the upper output at time $m+1$ and the lower output at time $m$ are $\Delta_m^*$ and $P_m$, respectively.

Problem 3.20:

(a) By the principle of orthogonality, we have the augmented Wiener-Hopf equations of the forward and backward prediction-error filters as

$$\sum_{l=0}^{m} a_{m,l}\, r(l-i) = \begin{cases} P_m, & i=0 \\ 0, & i=1,2,\cdots,m. \end{cases}$$

and $$\sum_{l=0}^{m} a_{m,m-l}^*\, r(l-i) = \begin{cases} 0, & i=0,\cdots,m-1 \\ P_m, & i=m \end{cases}.$$

Then

$$E[f_m(n)\, u^*(n-k)] = E\Big[\Big(\sum_{l=0}^{m} a_{m,l}\, u(n-l)\Big)\cdot u^*(n-k)\Big] = \sum_{l=0}^{m} a_{ml}^* \cdot r(k-l) = \sum_{l=0}^{m} a_{m,l}^*\, r^*(l-k)$$

$$= \Big[\sum_{l=0}^{m} a_{ml}\, r(l-k)\Big]^* = \begin{cases} P_m, & k=0 \\ 0, & k=1,2,\cdots,m. \end{cases} \quad \text{----} (1)$$

$$E[b_m(n)\, u^*(n-k)] = E\Big[\Big(\sum_{l=0}^{m} a_{m,m-l}\cdot u(n-l)\Big)\cdot u^*(n-k)\Big] = \sum_{l=0}^{m} a_{m,m-l}\, r(k-l)$$

$$= \sum_{l=0}^{m} a_{m,m-l}\, r^*(l-k) = \Big[\sum_{l=0}^{m} a_{m,m-l}^*\, r(l-k)\Big]^*$$

$$= \begin{cases} 0 & k=0,\cdots,m-1 \\ P_m & k=m \end{cases} \quad \text{------} (2)$$

(b) From part (a), we have $E[f_m(n)\, u^*(n)] = E[b_m(n)\, u^*(n-m)] = P_m$.

(c) $\because b_i(n) = \sum_{l=0}^{i} a_{i,i-l}\, u(n-l)$ (Assuming that $i \le m$ without loss of generality)

$\therefore E[b_m(n) b_i^*(n)] = E[b_m(n) \cdot (\sum_{l=0}^{i} a_{i,i-l}^* u^*(n-l))] = \sum_{l=0}^{i} a_{i,i-l}^* E[b_m(n) u^*(n-l)]$

1) If $i<m$, for all $l \in \{0, 1, \dots, i\}$ $E[b_m(n) u^*(n-l)] = 0$ (from $\overbrace{\text{part (a)}}^{\text{Eqn. (2) in}}$)

then $E[b_m(n) b_i^*(n)] = 0$ ;

2) If $i=m$, for all $l \in \{0, 1, \dots, m-1\}$ $E[b_m(n) u^*(n-l)] = 0$,

and for $l=i=m$, $E[b_m(n) u^*(n-l)] = P_m$ and $a_{m0}^* = 1$

the $E[b_m(n) b_i^*(n)] = P_m$.

Hence, generally,
$$E[b_m(n) b_i^*(n)] = \begin{cases} P_m, & m=i \\ 0, & m \ne i \end{cases}$$

(d) 1) $E[f_m(n) f_i^*(n-l)] = E[f_m(n) \cdot (\sum_{k=0}^{i} a_{i,k}^* u^*(n-l-k))]$ $\quad f_m(n) = \sum_{l=0}^{m} a_{m,l} \cdot u(n-l)$

$= \sum_{k=0}^{i} a_{i,k}^* E[f_m(n) u^*(n-l-k)]$

If $\begin{cases} i<m \\ 1 \le l \le m-i \end{cases}$, then for $k \in \{0, 1, \dots, i\}$, we have $1 \le l+k \le m$.

$\Rightarrow E[f_m(n) u^*(n-l-k)] = 0$ (from $\overbrace{\text{part (a)}}^{\text{Eqn. (1) in}}$)

Hence $E[f_m(n) f_i^*(n-l)] = 0$.

2) $E[f_m(n+l) f_i^*(n)] = E[f_m(n+l) \cdot [\sum_{k=0}^{i} a_{i,k}^* u^*(n-k))] = \sum_{k=0}^{i} a_{i,k}^* E[f_m(n+l) u^*(n+l-(l+k))]$

From the result in Eqn. (1) in part (a), for all $k=0,1,\dots,i$, we have

$E[f_m(n+l) u^*(n+l-(l+k))] = 0$ since $1 \le l+k \le m$.

Hence $E[f_m(n+l) f_i^*(n)] = 0$

3) $E[b_m(n) b_i^*(n-l)] = E[b_m(n) \cdot (\sum_{k=0}^{i} a_{i,i-k}^* u^*(n-l-k))] = \sum_{k=0}^{i} a_{i,i-k}^* E[b_m(n) u^*(n-(l+k))]$

$E[b_m(n+l) b_i^*(n)] = E[b_m(n+l) (\sum_{k=0}^{i} a_{i,i-k}^* u^*(n-k))] = \sum_{k=0}^{i} a_{i,i-k}^* E[b_m(n+l) u^*((n+l)-(l+k))]$

If $\begin{cases} i<m \\ 0 \le l \le m-i-1 \end{cases}$, then for all $k \in \{0, 1, \dots, i\}$, we have $0 \le l+k \le m-1$

$\Rightarrow E[b_m(n) u^*(n-(l+k))] = E[b_m(n+l) u^*((n+l)-(l+k))] = 0$ for all $k \in \{0, 1, \dots, i\}$ (from Eqn. (2) in part (a))

Hence $E[b_m(n) b_i^*(n-l)] = E[b_m(n+l) b_i^*(n)] = 0$.

(e)

1) Without loss of generality, we assume that $i \le m$.

$E[f_m(n+m) f_i^*(n+i)] = E[f_m(n) f_i^*(n+i-m)] = E[f_m(n) f_i^*(n-l)]$ where $l = m-i \Rightarrow 0 \le l$

If $i < m$, then $0 < l = m-i$. From results of part (d), we have

$E[f_m(n) f_i^*(n-l)] = 0$

If $i = m$, $l = 0$. Then $E[f_m(n) f_i^*(n)] = E[f_m(n) f_m^*(n)] = E[f_m(n) \sum_{l=0}^{m} a_{ml}^* u^*(n-l)]$

$$= \sum_{l=0}^{m} a_{m,l}^* \; E[f_m(n) u^*(n-l)]$$

$$= a_{m0}^* \cdot P_m + \sum_{l=1}^{m} a_{ml}^* \cdot 0 \quad \text{(from Eqn. (1) in part (a))}$$

$$= P_m$$

Hence $E[f_m(n+m) f_i^*(n+i)] = \begin{cases} P_m, & i = m \\ 0, & i \ne m \end{cases}$

2) For $i \le m$,

$E[b_m(n+m) b_i^*(n+i)] = E[b_m(n) b_i^*(n-(m-i))] = E[b_m(n) b_i^*(n-l)]$ where $l = m-i \ge 0$

$$= E\Big[b_m(n) \cdot \Big(\sum_{k=0}^{i} a_{i,i-k}^* u^*(n-l-k)\Big)\Big]$$

$$= \sum_{k=0}^{i} a_{i,i-k}^* \; E\big[b_m(n) u^*(n-(l+k))\big]$$

$$= a_{i0}^* \; E\big[b_m(n) u^*(n-(l+i))\big] + \sum_{k=0}^{i-1} a_{i,i-k}^* E\big[b_m(n) u^*(n-(l+k))\big]$$

$$= a_{i0}^* P_m + \sum_{k=0}^{i-1} a_{i,i-k}^* \cdot 0 \quad \Big(\text{From } l+i = m \text{ and Eqn. (2) in part (a)}\Big)$$

$$= P_m.$$

(f) $E[f_m(n) b_i^*(n)] = E\Big[f_m(n) \cdot \Big(\sum_{l=0}^{i} a_{i,i-l}^* u^*(n-l)\Big)\Big] = \sum_{l=0}^{i} a_{i,i-l}^* \; E[f_m(n) u^*(n-l)]$

$$= a_{ii}^* E[f_m(n) u^*(n)] + \sum_{l=1}^{i} a_{i,i-l}^* \; E[f_m(n) u^*(n-l)]$$

1) If $i \le m$, $E[f_m(n) u^*(n-l)] = 0$ for all $l \in \{1, \cdots, i\}$. (from Eqn. (1) in part (a))

Then $E[f_m(n) b_i^*(n)] = a_{ii}^* \; E[f_m(n) u^*(n)] = a_{ii}^* P_m = K_i^* P_m$

2) if $i > m$, $E[f_m(n) b_i^*(n)] = E\Big[\Big(\sum_{l=0}^{m} a_{m,l} u(n-l)\Big) b_i^*(n)\Big] = \sum_{l=0}^{m} a_{m,l} \Big[E\big[b_i(n) u^*(n-l)\big]\Big]^*$

$$= \sum_{l=0}^{m} a_{m,l} \cdot 0 \quad (\text{From Eqn. (2) in part (a) and } l = 0, 1, \cdots, m < i).$$

$$= 0$$

Problem 3.X2:

(a)

From the following recursions:

$$\begin{cases} k_m = -\frac{1}{P_{m-1}} \sum_{k=0}^{m-1} a_{m-1,k}\, r(k-m) & m=1,2,\cdots M \quad \cdots (1) \\ P_m = P_{m-1}(1-|k_m|^2) & \cdots (2) \end{cases}$$

we have: $k_1 = -\frac{1}{P_0} a_{0,0}\, r(-1) = -\frac{1}{r(0)} \cdot 1 \cdot c\, r(0) = -c \quad (\because P(0)=r(0),\ r(-1)=c^{|-1|} \cdot r(0)),\ a_{0,0}=1$

$$P_1 = P_0(1-|k_1|^2) = P_0(1-c^2) = (1-c^2)\cdot r(0)$$

$$a_{1,1} = k_1 = -c. \qquad (a_{1,0}=1)$$

(b) From the recursion $a_{m,k} = a_{m-1,k} + k_m a^*_{m-1\,m-k} \quad k=0,1,\cdots m. \quad \cdots (3)$

we have:

i) $m=2$, $a_{2,1} = a_{1,1} + k_2 a^*_{1,1}$ $\left(\text{where } k_2 = -\frac{1}{P_1}[a_{1,0} r(-2) + a_{11} r(-1)] \ (\text{using Eqn. (1)}) = -\frac{1}{(1-c^2) r(0)}[1\cdot c^2 \cdot r(0) - c\cdot c\cdot r(0)] = 0\right)$

$= a_{1,1}$
$= -c$

$a_{22} = k_2 = 0$, $a_{20} = 1$

$$P_2 = P_1(1-|k_2|^2) = P_1 = (1-c^2)\, r(0).$$

ii) $m=3$, $k_3 = -\frac{1}{P_2}[a_{20}\, r(-3) + a_{2,1}\, r(-2) + a_{22}\, r(-1)]$ (using Eqn. (1))

$= -\frac{1}{P_2}[1\cdot c^3 \cdot r(0) - c\cdot c^2 \cdot r(0)]$

$= 0$

$P_3 = P_2(1-k_3^2) = P_2$

$a_{3,1} = a_{2,1} + k_3 a^*_{22} = a_{2,1} = -c$ (using Eqn (3))

$a_{3,2} = a_{22} = 0$

$a_{30} = 1$, $a_{33} = k_3 = 0$

Similarly, we have

$$\begin{cases} k_2 = k_3 = \cdots = k_q = 0, \\ P_1 = P_2 = \cdots = P_q = (1-c^2)\, r(0) \\ a_{10} = a_{20} = \cdots = a_{q0} = 1 \\ a_{1,1} = a_{2,1} = a_{3,1} = \cdots = a_{q,1} = -c \\ a_{mk} = 0 \quad m \in \{2,3,\cdots q\},\ k \in \{2,3,\cdots q\} \end{cases}$$

(c): $m=10$:

$$K_{10} = -\frac{1}{P_9}\left[a_{90}\, r(-10) + a_{9,1}\, r(-9) + \cdots + a_{9,9}\, r(-1)\right] \quad \text{(using Eqn. (1))}$$

$$= -\frac{1}{P_9}\left[a_{90}\, r(-10) + a_{91}\, r(-9) + 0\right]$$

$$= -\frac{1}{(1-c^2)r(0)}\left[1\cdot 0 - c\cdot c^9 r(0)\right]$$

$$= \frac{c^{10}}{1-c^2}$$

$$P_{10} = P_9\,(1-K_{10}^2) = (1-c^2)\cdot r(0)\cdot\left(1-\frac{c^{20}}{(1-c^2)^2}\right)$$

$$= \frac{r(0)\cdot\left[(1-c^2)^2 - c^{20}\right]}{1-c^2}$$

$$a_{10,10} = K_{10} = \frac{c^{10}}{1-c^2} \qquad a_{10,0} = 1$$

$$a_{10,1} = a_{9,1} + K_{10}\cdot a_{9,9}^* = a_{91} + K_{10}\cdot 0 = a_{91} = -c$$

$$a_{10,2} = a_{9,2} + K_{10}\, a_{9,8}^* = 0 + K_{10}\cdot 0 = 0.$$

$$a_{10,3} = a_{10,4} = a_{10,5} = a_{10,6} = a_{10,7} = a_{10,8} = 0.$$

$$a_{10,9} = a_{9,9} + K_{10}\cdot a_{9,1}^* = 0 + K_{10}\cdot(-c) = -\frac{c^{11}}{1-c^2}.$$