Report of pendulum model project

1. Abstract

In this paper, Newton mechanics, Lagrange mechanics and other physical analysis methods are used to establish the dynamic model of single pendulum system and double pendulum system, and numerical solution method or linearization method is used to solve the model. In addition, the effects of drag, initial Angle, step size and other parameters on the system dynamics performance and numerical solution accuracy are also analyzed. In this paper, phase diagram is also introduced to describe the evolution of the system directly, which can better compare the effect of parameter changes on the system. The ultimate vision of this project is to deepen our understanding of differential equations and master specific techniques for engineering implementation through a concrete and simple real-world physical model

2. Introduction

Although a simple pendulum system is a very basic physical model, it is still an excellent real-world example for getting started with differential equations. The study of simple pendulum model is of great significance for us to understand the properties of differential equations. We can deeply understand how a differential equation plays its role through the dynamic performance of the physical model. More importantly, we can also establish the connection between physical model and mathematical formula, master the basic methodology of analyzing the motion process of a system. This paper summarizes a set of analytical procedures: mechanical analysis (Newtonian mechanics or Lagrange mechanics), differential equations based on the laws of physics, and then the model is solved using linearization methods or numerical solutions. This set of processes can basically analyze most physical motion models.

In this paper, when solving the simple pendulum model, the nonlinear term is linearized in the case of small Angle, which can greatly simplify the model. Then the differential equation is transformed into algebraic equation by Laplace transform for easy solution. This paper suggests that this linearization method is a very useful simplification method in analyzing complex models. Of course, there are certain limitations. Therefore, we choose to use Euler method to solve the nonlinear model of the first order linear approximation recursive solution. Here, we will explore the influence of solving step size on the solution results of the system, and finally prove that the step size setting of the solution needs to meet the conditions of the first order approximation of Euler's method to ensure that the solution results will not diverge. We will then add damping terms to the model to see how the first order terms in the differential equation play out. More importantly, we will explore how the initial conditions and damping coefficients of different systems affect the motion performance of the system, and draw phase diagrams to visualize the changes. It gives us a new way to think about the properties of differential equations and what the parameters mean.

The double pendulum system is also a very interesting topic. We use Lagrangian mechanics to build a mathematical model of the double pendulum system and describe the changes of the system from the energy point of view, which proves to be more convenient than Newtonian mechanics analysis. Because the double pendulum system involves a lot of nonlinear terms, it needs high precision of numerical solution. In this paper, the fourth order Runge-Kutta algorithm is used to ensure the accuracy of the model. At the same time, we will also demonstrate the motion process of the double pendulum system in the way of animation, and its motion trajectory is very fascinating. Of course, we will also give the phase diagram of the system to show the dynamics of the system directly.

The purpose of this paper is to deepen the understanding of the properties of differential equations and how it functions through a simple and common physical model, and to appreciate the close connection between real physical phenomena and mathematical theoretical formulas.

3. Model Description

3.1 Dimensional analysis of period of undamped simple pendulum system

The swing period τ of a simple pendulum system involves factors such as mass, length, gravitational acceleration and other basic physical quantities, which are represented by the symbols m,l and g respectively. Assuming a relationship between them without considering the air resistance like below:

$$\tau = km^{\alpha} l^{\beta} g^{\gamma} \tag{1}$$

Where α,β,γ are undetermined constants, and k is dimensionless scaling coefficients. In addition, the dimensions of these physical quantities can be written as follows:

$$[\tau] = T, [m] = M, [g] = LT^{-2}, [l] = L \tag{2}$$

Then the dimensionality of these physical quantities is written into a dimensionality expression according to the relation of the equation above:

$$T = kM^{\alpha} L^{\beta+\gamma} T^{-2\gamma} \tag{3}$$

According to the principle of dimensional consistency, a system of linear equations with undetermined constants can be listed:

$$\begin{cases} \alpha = 0 \\ \beta + \gamma = 0 \\ -2\gamma = 1 \end{cases} \tag{4}$$

The solutions of the above equations are $\alpha = 0,\ \ \beta = 1/2,\ \ \gamma = -1/2$. Substituting these results into formula (1), the oscillation period of a simple pendulum can be obtained as follows:

$$\tau = k\sqrt{\frac{l}{g}} \tag{5}$$

3.2 Mathematical modeling of a undamped simple pendulum system

3.2.1 Establishment of model

A simple pendulum vibration system is shown like Fig.1. Let g represents the acceleration of gravity, θ represents the angle between the rope and the vertical direction, l represents the length of the rope and the motion radius of the pendulum, m represents the weight of the ball in the pendulum system, and finally, let time t represents the motion duration of the pendulum system.

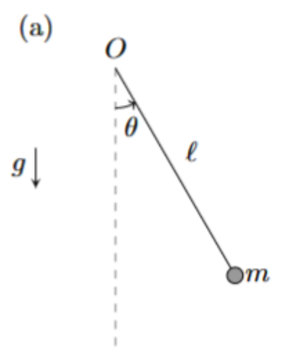

Fig.1

The kinetic equation of the system can be easily written out according to Newton's second law, where a represents the acceleration of the mass ball. The minus sign on the right side of the equation is to indicate that the restoring force is in the opposite direction of acceleration.

$$ma = -mg\sin\theta \tag{6}$$

There is the following relationship between the motion acceleration of the ball and the angular acceleration of the angle θ, which is represented by $\ddot{\theta}$.

$$\ddot{\theta}l = a \tag{7}$$

From the actual physical meaning, when a pendulum system swings, θ is related to the motion time t. θ can be considered as a function related to time t, so it is written as $\theta(t)$. Simultaneous equations (3) and (4) can lead to the nonlinear ordinary differential equations like below, which describes the simple pendulum systems.

$$\begin{gathered}\ddot{\theta}(t)l = -g\sin\theta(t)\\ \ddot{\theta}(t) = \frac{d^2\theta}{dt^2}\end{gathered} \tag{8}$$

3.2.2 Linear approximate analytical solution of the model

Equation (5) contains a trigonometric function term, and the equation with nonlinear term is not easy to be solved directly. As is known to all, in order to facilitate the solving of nonlinear ordinary differential equation models, nonlinear terms are usually approximated by linearization in a slightly varying interval. Therefore, linear approximate solution is performed for equation (5) above. It can be assumed that θ changes in a small interval when the pendulum system moves, and $\sin\theta$ can be expanded by Taylor and only its first-order term is taken to approximate the model by linearization.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots \tag{9}$$

After taking $\sin\theta = \theta$according to formula (6), the equation after linear approximation is as follows:

$$\ddot{\theta}(t)l = -g\theta(t) \tag{10}$$

Next, take the Laplace transform for both sides of the formula (7) in the time domain. Using L to represent the Laplace transform operator and s to represent the integral variable of the Laplace transform. Using Laplace transform method can directly transform differential equation into algebraic equation and simplify the solving process. Another advantage is that the transient and steady-state components of the solution can be obtained at the same time, and the solution of the equation can be obtained simply.

$$\begin{gathered}L[\ddot{\theta}(t)l] = L[-g\theta(t)]\\ [s^2\Theta(s) - s\theta(0) - \dot{\theta}(0)]l = -g\Theta(s)\end{gathered} \tag{11}$$

Given the initial conditions of the model $\theta = \theta_0$ and $\dot{\theta} = 0$, equation (9) can be obtained by substituting them into equation (8)

$$[s^2\Theta(s) - s\theta_0]l = -g\Theta(s) \tag{12}$$

The function $\Theta(s)$ can be expressed as an expression related to s by sorting the terms of equation (9)

$$\Theta(s) = \theta_0\frac{s}{s^2 - \frac{g}{l}} \tag{13}$$

By transforming $\Theta(s)$ in the s domain to $\theta(t)$ in the time domain using the inverse Laplace transform, the analytical solution of the model can be easily obtained, which is a function describing the time t and the swing Angle θ.

$$\begin{gathered}L^{-1}[\Theta(s)] = L^{-1}[\theta_0\frac{s}{s^2 - \frac{g}{l}}]\\ \theta(t) = \theta_0\cos(\sqrt{\frac{g}{l}}t)\end{gathered} \tag{14}$$

It can be easily found that $\theta(t)$ is a trigonometric function with periodicity, from which the inherent period of the pendulum oscillating system (without considering damping) can be obtained. Let τ represent the wobble period:

$$\tau = 2\pi\sqrt{\frac{l}{g}} \tag{15}$$

3.2.3 Numerical solution of the model

Because the original model is a nonlinear model, it is difficult to find its analytical solution without using linear approximation, so the solution of the original model needs to use numerical calculation method to find its numerical solution. This kind of work is very suitable for computer programs to complete. Therefore, this section uses python language to write a program for solving the original nonlinear model by Euler method, calculates the value of $\theta(t)$ corresponding to many t moments, and then connects these discrete solutions to draw a curve to intuitively show the curve of $\theta(t)$ function.

Firstly, the original second-order differential equation model is sorted into a form suitable for solving by Euler's method. Let $y_1 = \theta(t)$, $y_2 = \dot{\theta}(t)$, their first-order derivative can be calculated as follows:

$$\begin{gathered}\dot{y_1} = \dot{\theta}(t)\\ \dot{y_2} = -\frac{g}{l}\sin\theta(t)\end{gathered} \tag{16}$$

Let Δt represents the step size of Euler's method, k represents the number of iterations, y_{1k} and y_{2k} represent the solution of the kth iteration. The iterative solution process can be divided into two steps: Update the solution and update the first derivative. Every time when the numerical solution of the kth sample is completed, the numerical solution of the $k-1$th sample is used to calculate and update the first derivative of the current kth sample, which is then used to calculate the numerical solution of the $k+1$th sample. The process of iterative updating can be expressed by the formula as follows:

$$
\begin{array}{ccc}
y_{10} = \theta_0 & y_{20} = 0 & \text{The initial condition for the solution} \\
\dot{y_{10}} = y_{20} & \dot{y_{20}} = -\dfrac{g}{l}\sin y_{10} & \text{The initial condition for the first derivative} \\
y_{11} = y_{10} + \dot{y_{10}}\Delta t & y_{21} = y_{20} + \dot{y_{20}}\Delta t & \text{The first iteration to solve the problem} \\
\dot{y_{11}} = y_{21} & \dot{y_{21}} = -\dfrac{g}{l}\sin y_{11} & \text{The first iteration to update the first derivative} \\
y_{12} = y_{11} + \dot{y_{11}}\Delta t & y_{22} = y_{21} + \dot{y_{21}}\Delta t & \text{The second iteration to solve the problem} \\
\dot{y_{12}} = y_{22} & \dot{y_{22}} = -\dfrac{g}{l}\sin y_{12} & \text{The second iteration to update the first derivati} \\
 & \vdots & \\
y_{1k} = y_{1(k-1)} + \dot{y}_{1(k-1)}\Delta t & y_{2k} = y_{2(k-1)} + \dot{y}_{(k-1)}\Delta t & \text{The kth iteration to solve the problem} \\
\dot{y_{1k}} = y_{2k} & \dot{y_{2k}} = -\dfrac{g}{l}\sin y_{1k} & \text{The kth iteration to update the first derivative}
\end{array}
$$

Write a python program to complete the iterative solution process of Euler method, and encapsulate this process as a solver class. The solution of the original nonlinear second-order differential equation can be completed by putting the defined related physical quantity, initial condition and solving step into it. Here,set $\theta_0 = \pi/60$, $m = 1kg$, $l = 1$, $g = 9.8m/s^2$. Plot θ with respect to dimensionless time t/τ as Fig.2:

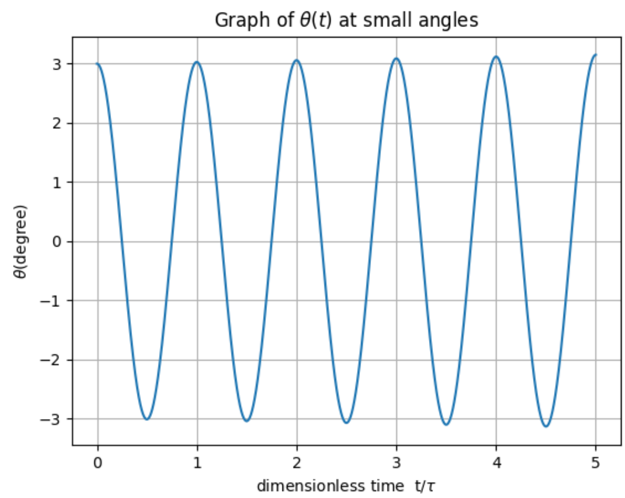

Fig.2

3.2.4 Effect of parameters on the model

Next, change the solving step size Δt of the Euler method solver, and draw the curve between θ and dimensionless time t/τ under different Δt in Fig.3:

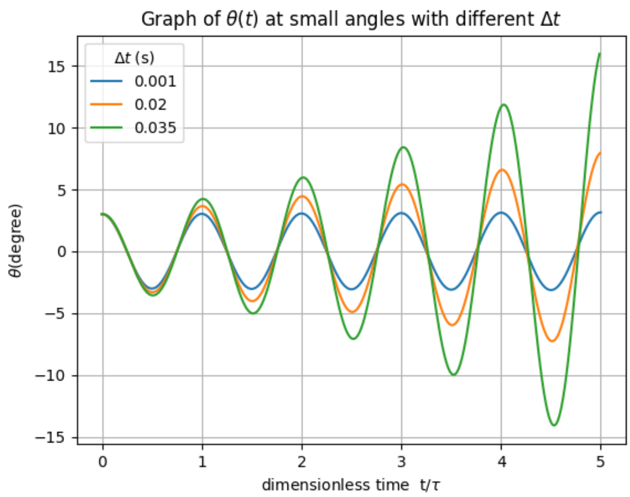

Fig.3

Then, the step size Δt is fixed as a smaller value, $\Delta t = 0.001s$ in the program. The initial value of angle θ_0 is changed, and the curve relationship between θ and dimensionless time t/τ is drawn under different initial conditions θ_0. In addition, in the case of linear approximation, the initial value of different angles θ_0 will affect the analytical solution of the equation. Finally, the relationship curve between θ and dimensionless time t/τ under different initial conditions θ_0 is drawn in Fig.4(a), and it is compared with the curve of the original nonlinear equation in Fig.4(b), which is solved numerically.

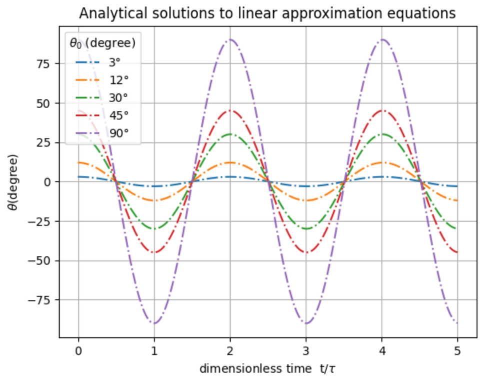

Fig.4(a)

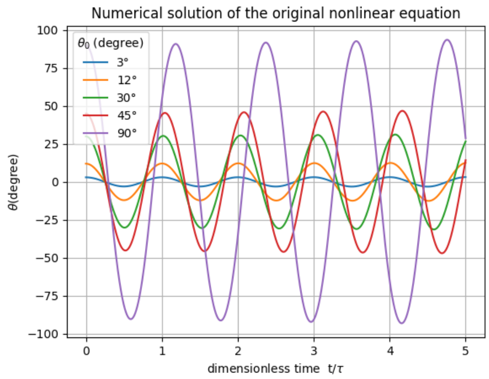

Fig.4(b)

3.2.5 Phase diagram of the system

How the system behaves in phase space can also be studied , which refers to a two-dimensional planar coordinate system with θ on the horizontal axis and $\dot{\theta(t)}$ on the vertical axis. Write a program to find $\dot{\theta(t)}$ and $\theta(t)$ at all times by Euler's method, and plot all solutions in phase space to get the phase diagram of the system. Set the initial condition of the system as $\theta = \pi/60$, $\dot{\theta(t)} = 0$, and set the step $\Delta t = 0.0001$. The resulting phase diagram is shown in Fig.5, where the red dots represent the starting point of the system, which is the initial condition of the system.

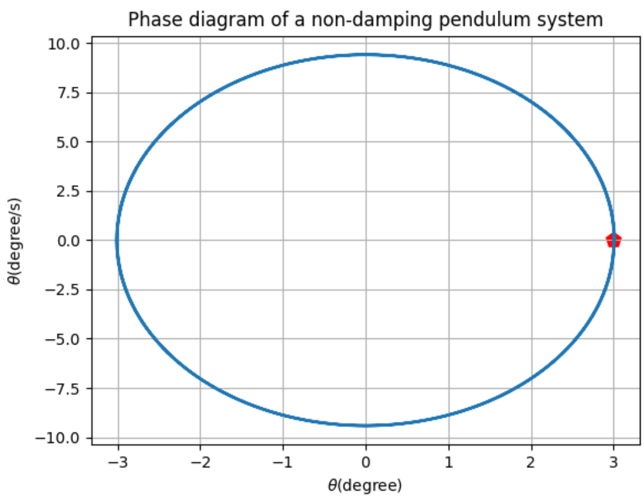

Fig.5

3.3 Mathematical modeling of a damped simple pendulum system

3.3.1 Establishment of model

Chapter 2.3 considers the effect of air resistance on the model of simple pendulum in actual situations, so we need to introduce a resistance term in formula (8) to improve the model. The resistance of the ball in a simple pendulum system is known as follows:

$$F_D = -kAv \tag{18}$$

Where k is the drag coefficient, A is the projected area, and v is the motion velocity of the ball. The projected area of the sphere can be easily obtained as follows:

$$A = \pi r^2 \tag{19}$$

Where r is the radius of the ball. And the ball's motion velocity v can be expressed by the angular velocity $\dot{\theta(t)}$:

$$v = \dot{\theta}(t)l \tag{20}$$

By substituting formula (15) and (16) into formula (14), the final resistance formula can be obtained:

$$F_D = -k\pi r^2 l\dot{\theta}(t) \tag{21}$$

The resistance term is added to the dynamic model of the pendulum system:

$$\begin{aligned} ml\ddot{\theta}(t) + mg\sin\theta + k\pi r^2 l\dot{\theta}(t) = 0 \\ \ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) + \frac{k\pi r^2}{m}\dot{\theta}(t) = 0 \end{aligned} \tag{22}$$

For the convenience of expression, the coefficient of the first order differential term in the above equation is expressed as $\mu = \frac{k\pi r^2}{m}$, and finally the complete mathematical model of the damped simple pendulum system is obtained. Compared to the undamped simple pendulum system, this is a second order differential equation with a first order differential term:

$$\ddot{\theta}(t) + \mu\dot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0 \tag{23}$$

3.3.2 Numerical solution of the model

The model of the damped simple pendulum system is a second-order nonlinear differential equation. We can still write a python program to numerically calculate the solution of the model using Euler's method. Set the system parameter $m = 1kg, l = 1, g = 9.8m/s^2$, and the initial conditions for the model is $\theta_0 = \pi/60$, $\dot{\theta(t)} = 0$. Assuming $\mu = 0.5$ and setting the solving step size $\Delta t = 0.001s$, the graph of the function of $\theta(t)$ can be drawn like Fig.6:

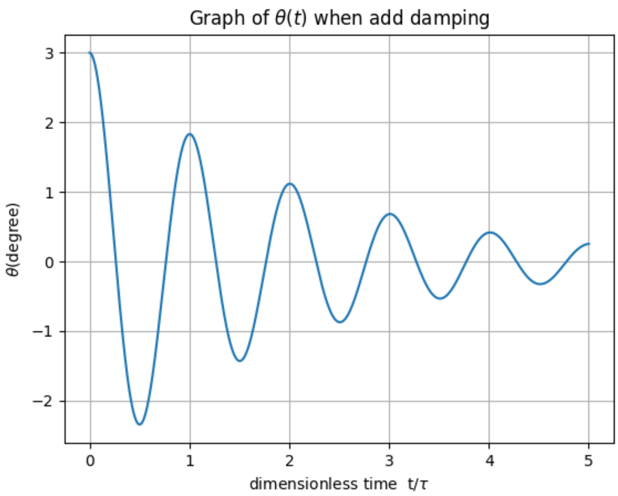

Fig.6

By observing the function image, we can intuitively feel the effect of resistance on the motion of the simple pendulum system. Resistance will cause the pendulum system to gradually stop swinging.

3.3.3 Phase diagram of the system

We can study the behavior of the system in phase space and explore the effect of drag on the motion of the system and energy conversion. Further, we can also set different μ to explore the effect of μ on the phase diagram of the pendulum system. Set the initial quantity of the system as $\theta = \pi/60$, $\dot{\theta(t)} = 0$, and set the solving step size as $\Delta t = 0.0001s$. When μ is $0.4, 0.8, 1.6, 2.4$. The phase diagram is drawn in Fig.7(a), Fig.7(b), Fig.7(c) and Fig.7(d) respectively, where the red dot represents the starting point of the system change on the phase diagram, that is, the initial condition of the system.

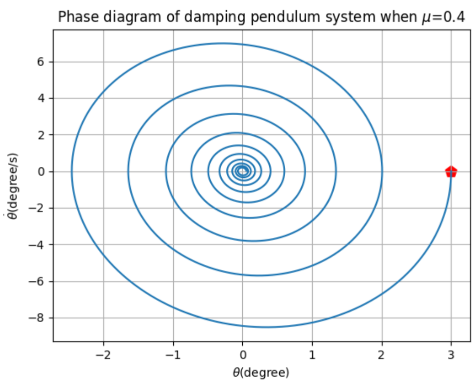

Fig.7(a)

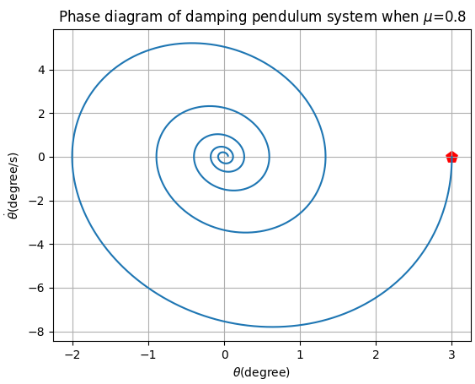

Fig.7(b)

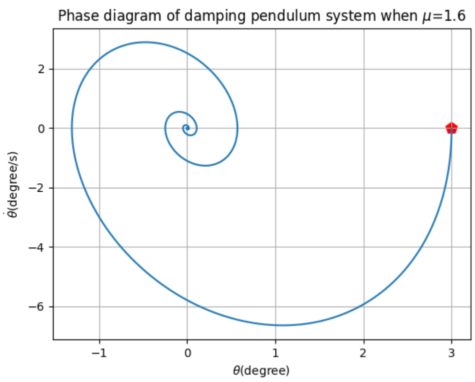

Fig.7(c)

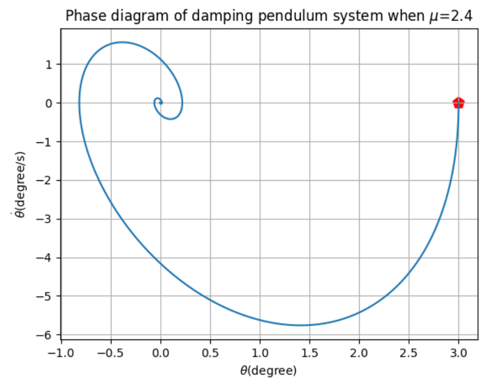

Fig.7(d)

It is easy to find that when the drag term is added to the simple pendulum system, the system phase diagram is no longer a closed curve. Instead of going back to the initial situation as in the case of no resistance, the curve will go back to the zero point of phase space. This means that when there is resistance, the motion amplitude of the simple pendulum is slowly reduced, and this process is accompanied by energy loss. The simple pendulum system with resistance cannot maintain a periodic motion like the simple pendulum system without resistance. So we can venture to conjecture that his curve in phase space is closed when the system has no energy loss, and diverges when the system has energy loss.

In addition, we find that as μ increases, the degree of divergence of the system phase space curve becomes larger, the number of "circle" formed by the curve enveloping becomes less, and the curve returns to zero earlier. This shows that the greater the air resistance, the faster the energy loss of the single pendulum system and the earlier stop of the swing motion.

What's more, we can set the initial conditions θ_0 to $30^\circ, 45^\circ, 60^\circ, 90^\circ$ to explore the effect of the initial angle on the phase map of the system. Write a python program to plot the phase map as Fig.8:

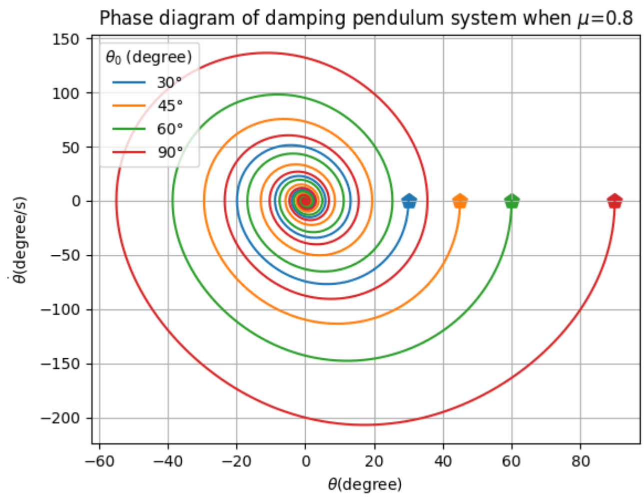

Fig.8

It is easy to draw a conclusion that the different initial angles will not affect the divergence degree of the phase diagram curve. In other words, the energy loss of the above four cases should be similar. The initial Angle only affects where the curve starts and how big it is.

3.4 Mathematical modeling of a link system with a damped end fixed

3.4.1 Establishment of model

This part replaces the previous system of a small ball on a string with a rod with diameter and mass, and then analyzes its movement around the fixed point. The essence of the system is actually the rotation of the cylinder around a fixed point in space. So we need to use the Euler equation of rigid body dynamics to model this system. First we can calculate the moment M_G of the rod under the influence of gravity, where $l$represents the length of the rod, a represents the radius of the rod, dm represents the tiny mass unit on the rod, and drrepresents the tiny length unit.

$$M_G = \int_0^l dF_G \cdot dr = \int_0^l g\sin\theta \cdot dm \cdot dr = \int_0^l \frac{m}{l} g\sin\theta \cdot dr^2 = mgl\sin\theta \tag{24}$$

The rod is also affected by air resistance, and the air resistance moment is set as M_F. The calculation formula is as follows:

$$M_F = \int_0^l dF_D \cdot dr = \int_0^l k \cdot dA \cdot dv \cdot dr = k\int_0^l 2a\dot\theta \cdot dr^3 = 2akl^3\dot\theta \tag{25}$$

$$M_F = \int_0 dF_D \cdot dr = \int_0 k \cdot dA \cdot dv \cdot dr = k\int_0 2a\dot{\theta} \cdot dr = 2akl^3\dot{\theta} \tag{25}$$

Then the moment of inertia I of the rod body is calculated

$$I = \int_0^l \frac{m}{l} r^2 dr = \frac{1}{3} ml^2 \tag{26}$$

Finally, Euler's equation is written and equations (20), (21) and (22) are substituted into Euler's equation to obtain the mathematical model of the link system with a damped end fixed

$$\begin{gathered} M_G + M_F = -I\ddot{\theta} \\ mgl\sin\theta + 2akl^3\dot{\theta} + \frac{1}{3}ml^2\ddot{\theta} = 0 \\ \ddot{\theta}(t) + \frac{6akl}{m}\dot{\theta}(t) + 3\frac{g}{l}\sin\theta(t) = 0 \end{gathered} \tag{27}$$

3.4.2 Numerical solution of the model

Write a python program to solve the model in 3.4.1 numerically using Euler's method, set the system parameters as $m = 1kg, l = 1$, $g = 9.8m/s^2$, and the initial condition of the model as $\theta_0 = \pi/60$, $\dot{\theta}(t) = 0$. Set the variable constant parameter $k = 1$i n the model, set the solving step size $\Delta t = 0.001s$, and draw the following function image in Fig.9:

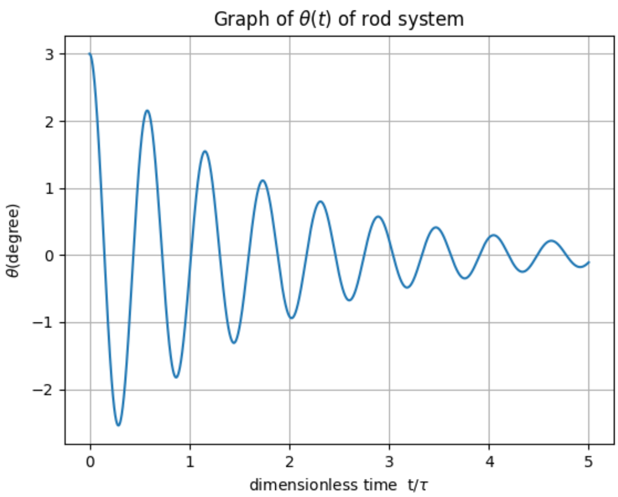

Fig.9

Look at the picture, the motion of a damped fixed link system is actually very similar to the motion of a damped simple pendulum system. We can try to keep k unchanged, set the initial Angle θ_0 as $30^\circ, 45^\circ, 60^\circ, 90^\circ$ respectively, and draw the function image as shown in Fig.10(a). Then fix the initial Angle $\theta_0 = 30^\circ$, set the constant k as $1, 5, 10, 15$, and draw the function image as shown in the following figure in Fig.10(b).

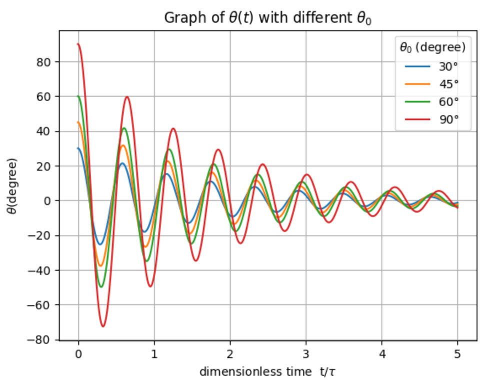

Fig.10(a)

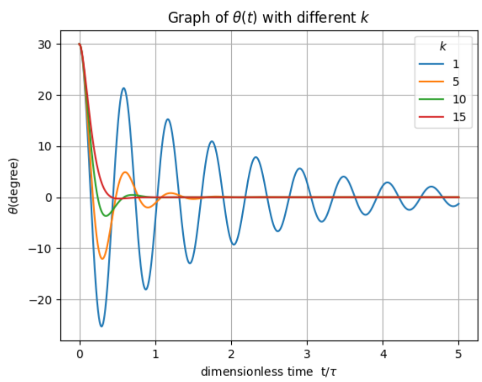

Fig.10(b)

3.5 Mathematical modeling of a undamped double pendulum system

3.5.1 Establishment of model

The composition diagram of the double pendulum system is as Fig.11, in which the physical quantities are required for modeling are marked. We need to make a dynamic analysis of the spheres m_1 and m_2, and build a model of the change of θ_1 and θ_2 over time. Thus, modeling a two-pendulum system requires two second-order differential equations. In order to facilitate analysis and simplify the model, the effect of air resistance is not considered here.

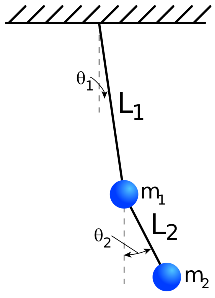

Fig.11

The Lagrange equation in analytical mechanics is used to construct the dynamic model of the system. For complex systems such as double pendulum, Lagrangian mechanics is more formalized, and the analysis of complex systems is simpler. It only needs to list the Lagrangian of the system and determine the state variables of the analysis, then equations can be written for solving. Firstly, the origin of the Cartesian coordinate system is defined as the fixed point of the rope above. The positive direction of the x axis points to the right and the positive direction of the y axis points to the top. We can list the location coordinates two balls, one (x_1, y_1) for the position of the ball m_1, (x_2, y_2) for the position of the ball m_2.

$$\begin{aligned} x_1 &= L_1 \sin\theta_1 \\ y_1 &= -L_1 \cos\theta_1 \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin\theta_2 \\ y_2 &= -L_1 \cos\theta_1 - L_2 \cos\theta_2 \end{aligned} \tag{28}$$

The gravitational potential energy P of the whole system is

$$P = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) L_1 g \cos\theta_1 - m_2 g L_2 \cos\theta_2 \tag{29}$$

Then, the velocities v_1 and v_2 of the two spheres are analyzed and expressed in terms of angles.

$$\begin{aligned} & v_1^2 = \dot{x_1}^2 + \dot{y_1}^2 = L_1 \dot\theta_1^2 \\ & v_2^2 = \dot{x_2}^2 + \dot{y_2}^2 = \left(L_1\dot\theta_1 \cos\theta_1 + L_2 \dot\theta_2 \cos\theta_2\right)^2 + \left(L_1\dot\theta_1 \sin\theta_1 + L_2\dot\theta_2 \sin\theta_2\right)^2. \end{aligned} \tag{30}$$

The kinetic energy M of the whole system is

$$M = \frac{1}{2} m_1 L_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left[L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right] \tag{31}$$

The Lagrangian $\mathcal{L}$ for the whole system is

$$\begin{aligned} \mathcal{L} &= M - P \\ &= \frac{1}{2} m_1 L_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left[L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right] + (m_1+m_2)L_1 g\cos\theta_1 + m_2 g L_2 \cos\theta_2 \end{aligned} \tag{32}$$

Next, we write Lagrangian equations for the small ballm_1 and construct differential equations for the state variable θ_1

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0 \\ (m_1+m_2)\,L_1\ddot\theta_1 + m_2L_2\ddot\theta_2\cos(\theta_1-\theta_2) + m_2L_2\dot\theta_2^2\sin(\theta_1-\theta_2) + g\,(m_1+m_2)\sin\theta_1 = 0 \end{gathered} \tag{33}$$

Similarly, the differential equation of state variable θ_2 is constructed as below:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0 \\ m_2L_2\ddot\theta_2 + m_2L_1\ddot\theta_1\cos(\theta_1-\theta_2) - m_2L_1\dot\theta_1^2\sin(\theta_1-\theta_2) + m_2 g\sin\theta_2 = 0 \end{gathered} \tag{34}$$

Finally, we have a model describing the motion of a two-pendulum system in the absence of air resistance, which is two coupled second-order nonlinear differential equations

$$\begin{gathered} (m_1+m_2)\,L_1\ddot\theta_1 + m_2L_2\ddot\theta_2\cos(\theta_1-\theta_2) + m_2L_2\dot\theta_2^2\sin(\theta_1-\theta_2) + g\,(m_1+m_2)\sin\theta_1 = 0 \\ m_2L_2\ddot\theta_2 + m_2L_1\ddot\theta_1\cos(\theta_1-\theta_2) - m_2L_1\dot\theta_1^2\sin(\theta_1-\theta_2) + m_2 g\sin\theta_2 = 0 \end{gathered} \tag{35}$$

3.5.2 Solution of the model

The model is highly coupled, with a single expression containing the second derivative of two state variables. So we need to transform the equation to get a form that's easier to solve numerically. Transform formula (31) into the following form:

$$\begin{gathered}(m_1+m_2)\,L_1\ddot{\theta}_1+m_2L_2\ddot{\theta}_2\cos(\theta_1-\theta_2)=-m_2L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)-g\,(m_1+m_2)\sin\theta_1\\ m_2L_2\ddot{\theta}_2+m_2L_1\ddot{\theta}_1\cos(\theta_1-\theta_2)=m_2L_1\dot{\theta}_1^2\sin(\theta_1-\theta_2)-m_2g\sin\theta_2\end{gathered}\tag{36}$$

We can rewrite this as a system of linear equations:

$$\begin{bmatrix}(m_1+m_2)L_1 & m_2L_2\cos(\theta_1-\theta_2)\\ m_2L_1\cos(\theta_1-\theta_2) & m_2L_2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}=\begin{bmatrix}-m_2L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)-g\,(m_1+m_2)\sin\theta_1\\ m_2L_1\dot{\theta}_1^2\sin(\theta_1-\theta_2)-m_2g\sin\theta_2\end{bmatrix}\tag{37}$$

Let's write the above equation in a simpler form:

$$A\ddot{\Theta}=b,\quad \Theta=\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\tag{38}$$

This gives us a simpler form of the differential equation, which is easier and more intuitive to solve numerically.

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{dt}}\dot{\Theta}=A^{-1}b\\ \frac{\mathrm{d}}{\mathrm{dt}}\Theta=\dot{\Theta}\end{cases}\tag{39}$$

The numerical solution of Euler's method is a first-order algorithm with a truncation error of $O(h^2)$. However, there are many nonlinear terms in this model, so the solution accuracy of Euler's method is limited. Therefore, the fourth-order Runge-Kutta method with higher accuracy is used here, and its truncation error can be reduced to $O(h^5)$. The fourth-order Runge-Kutta method can approximate complex curves within one solving step size through the four-step piecewise approximation of differentiation. The solving process is as follows:

Set the state variable of the differential equation at time $k$as y,

$$y_k=\begin{bmatrix}\dot{\Theta}_k\\ \Theta_k\end{bmatrix}\tag{40}$$

Then the formula (35) can be written as

$$\frac{d}{dt}y_k=\begin{bmatrix}A(y_k)^{-1}b(y_k)\\ \begin{bmatrix}1 & 0\end{bmatrix}y_k\end{bmatrix}=f(y_k)\tag{41}$$

Let the minute time interval from the time k to the time $k+1$ be h, and write the expression of the tiny quantity in the process of solving the algorithm of fourth-order Runge-Kutta method

$$\begin{aligned}s_1&=f(y_k)\\ s_2&=f\left(y_k+\tfrac{h}{2}s_1\right)\\ s_3&=f\left(y_k+\tfrac{h}{2}s_2\right)\\ s_4&=f\left(y_k+hs_3\right)\end{aligned}\tag{42}$$

The final update from the state variable at time k to the state variable at time $k+1$ is as follows

$$y_{k+1}=y_k+\frac{h}{6}(s_1+2s_2+2s_3+s_4)\tag{43}$$

The above fourth-order Runge-kutta method is written into a python program to solve the double pendulum system numerically. The tiny time interval h is set as $0.01s$, and the initial condition is $\theta_1=60^\circ,\theta_2=120^\circ,\dot{\theta}_1=0,\dot{\theta}_2=0$, we can draw the motion path of the two balls in $50s$ as shown Fig.12. The program can also play an animated diagram of the double pendulum motion.

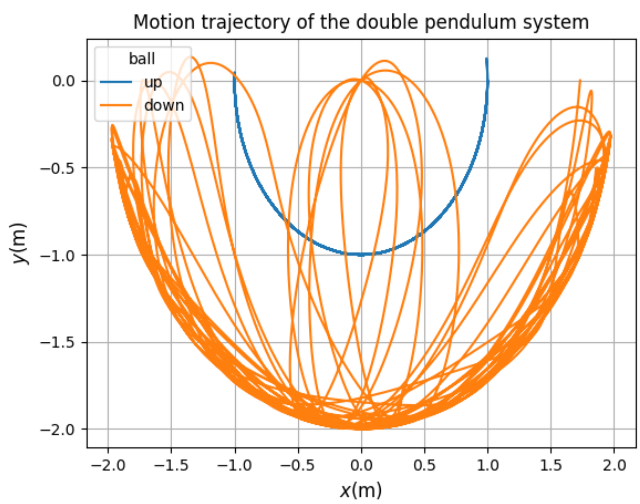

Fig.12

According to theoretical analysis, the system is not affected by air resistance, so the mechanical energy of the system is conserved. We can draw the total mechanical energy change curve of the system within $50s$ to verify the correctness of the model and the precision of numerical solution. The curve is shown in Fig.13.

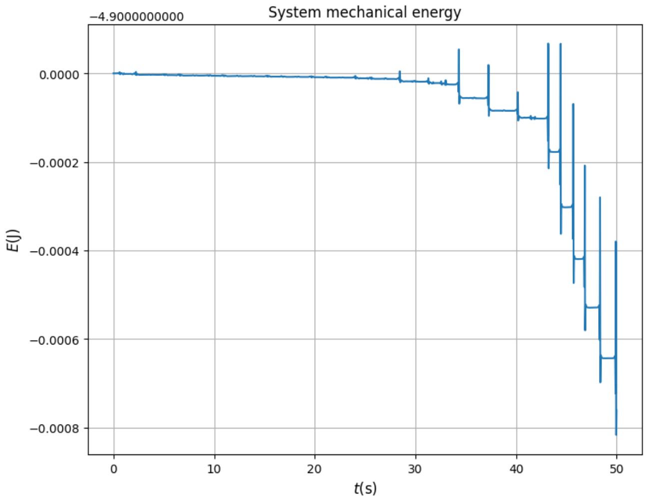

Fig.13

According to the figure above, the total mechanical energy of the system in this $50s$ is basically 0, and there is only a small divergence in the end. Therefore, it can show that our mathematical model is correct, and the precision of the fourth order Runge-Kutta method is accurate enough.

3.5.3 Phase diagram of the system

In order to visually understand the behavior and nature of the two balls in the double-pendulum system, we can also draw the phase diagram of the two balls in Fig.14(a) and Fig.14(b):

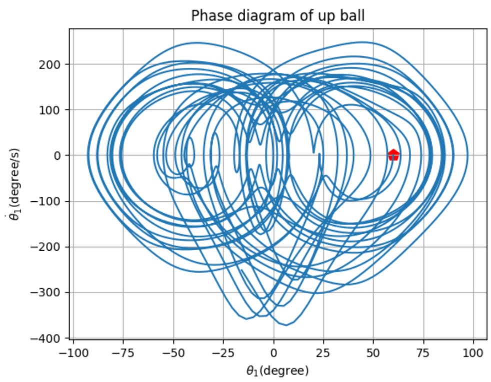

Fig.14(a)

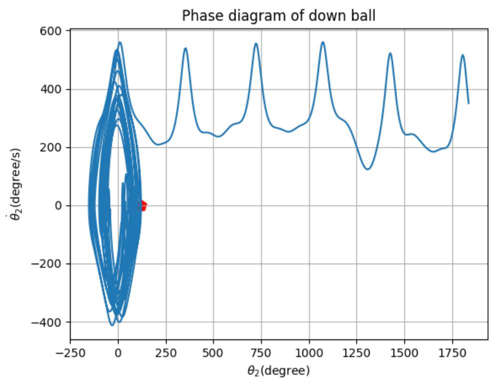

Fig.14(b)

It is not difficult to find that the motion of the two balls does not have obvious periodicity and shows very strong chaotic characteristics.

3.5.4 Effect of the initial value

In order to explore the influence of initial conditions on the motion of the two-pendulum system, we can set another set of initial conditions $\theta_1 = 60^\circ, \theta_2 = 121^\circ, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0$, Keep the micro time interval $h = 0.01s$ unchanged. Compared with 3.5.2, only the initial Angle of θ_2 increases by 1°, and no other conditions change. Fourth-order Runge-Kutta method is still used to solve the problem, and the running trajectory of the double pendulum is obtained as Fig.15:

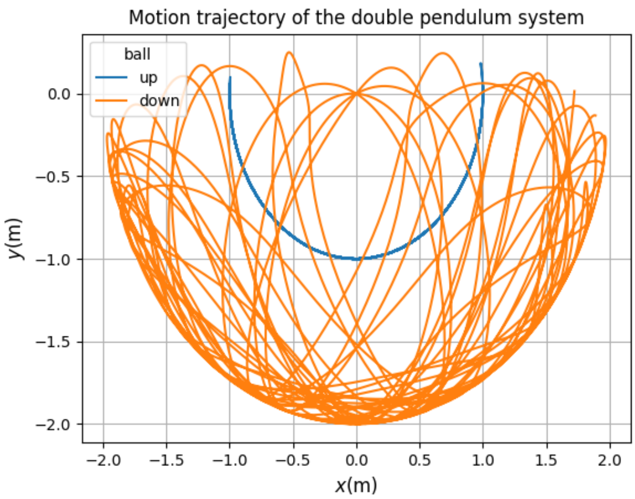

Fig.15

Although there is only a small change compared with the initial condition of the model in 3.5.2, the motion tracks of the two models are far apart in the $50s$, which is not similar or indicates that the motion conditions of the two models will eventually show the same rule. This is completely different from the case of a pendulum, which has completely different properties of motion. We can also draw the phase diagram in Fig.16(a) and Fig.16(b) to compare them with the phase diagram in 3.5.3.

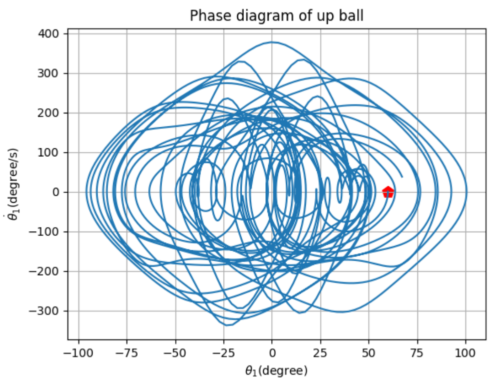

Fig.16(a)

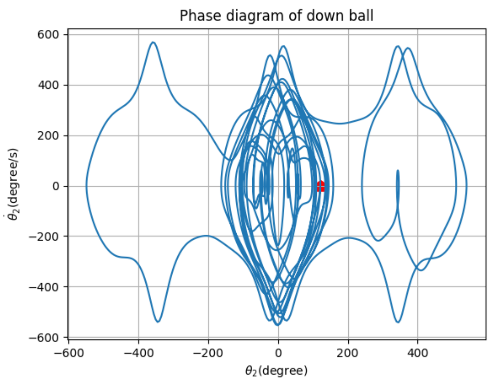

Fig.16(b)

It is easy to observe that the phase diagrams of the two initial cases, which are slightly different, are completely distinct. The two double pendulum systems evolved completely different motion conditions with no similarity and regularity over time. So we can conclude that the double pendulum system is a very sensitive mathematical model to the initial conditions, and its motion has remarkable chaotic characteristics. It can also be inferred that it is very difficult to accurately predict the motion of the double pendulum system through mathematical model in practical application. Firstly, with the passage of time, the errors of numerical solution will gradually accumulate and affect the accuracy of the model. Another important reason is that extremely small differences in initial conditions will cause the system to evolve very different motion conditions in the future, resulting in the completely different actual motion.

4. Conclusions

1. It can be seen from Section 3.2.4 that when the solving step size of Euler's method is too large, the error of each step can no longer be regarded as a first-order infinite small term, which will lead to divergence of solution results.
2. It can be seen from Section 3.2.4 that different initial angles of a pendulum system actually cause changes in the period of oscillatory motion. However, under the incorrect linearization assumption, their oscillation period remains the same.
3. It can be seen from Section 3.3.3 that the introduction of damping term will cause the phase diagram of the system to no longer be closed, and the larger the damping coefficient is, the fewer "circle" formed by the curve will be, and the curve will return to zero earlier. It can be inferred that the unclosed phase diagram curve represents energy loss in the system. The earlier the curve returns to zero, the faster the energy is lost.
4. It can be seen from Section 3.5.4 that two-pendulum system is extremely sensitive to initial value conditions. A small change in the initial value can make the system evolve very differently in the future.

5. Reference

[1] [Pendulum[Z]. wikipedia,2022.]()

[2] [Double pendulum[Z]. wikipedia,2022.]()

[2] [MATHEMATICA tutorial, Part 2.3: Double pendulum (brown.edu)]()

6. Appendix

6.1 Userguide

> This project uses python language to write programs to simulate common physical models such as single pendulum system, double pendulum system and connecting rod system. The numerical solution algorithm is used to solve the model and then the motion of the system is obtained. Meanwhile, the influence of different parameters in the system on the motion is also explored.

6.1.1 Project dependency

- The system is `Windows10 64-bit`, The compatibility on other systems has not been verified.
- Need to install `python`, the version is 3.10.2.
- Need to install `numpy`, the version is 1.24.2.
- Need to install `matplotlib`, the version is 3.5.1.

6.1.2 Use method

For example, start a terminal in the `codebase` folder directory,then type the following command to run a python program

```
python '.\undamped double pendulum\demo.py'
```

Of course, you can also choose to run in an IDE, which is `VsCode` for development

6.1.3 File schema

- `System.py` ***This file is the core code of the project. Mathematical models and numerical solutions of different systems are encapsulated into classes to provide external interfaces***
  - *Undamped_Pendulum* Describes the class of undamped simple pendulums
  - *Damping_Pendulum* Describes the class of damped simple pendulums
  - *Damping_Rod* Describes the class of damped rods
  - *Undamped_Double_Pendulum* Describes the class of undamped double pendulums
- `damping pendulum` ***Simulation of damped simple pendulum***
  - `graph of function.py` Show the curve of angle over time
  - `phase map.py` Show the phase diagram of the system
  - `theta_0 vary.py` Explore the influence of different initial values on the system motion
- `damping rod` ***Simulation of damped rods***
  - `graph of function.py` Show the curve of angle over time
  - `k vary.py` Explore the influence of different damping coefficients on the system motion
  - `theta_0 vary.py` Explore the influence of different initial values on the system motion
- `undamped double pendulum` ***Simulation of undamped double pendulums***
  - `demo.py` Show the demo animation of the double pendulum system and the phase diagram of the two balls
- `undamped pendulum` ***Simulation of undamped simple pendulums***
  - `graph of function.py` Show the curve of angle over time
  - `delta_t vary.py` Show the influence of Euler method's step size on the solution result
  - `phase map.py` Show the phase diagram of the system
  - `theta_0 vary.py` Explore the influence of different initial values on the system motion